Bivariate Wavelet Construction Based on Solutions of Algebraic Polynomial Identities

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Declaration

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Summary

Multi-resolution analysis (MRA) has become a very popular field of mathematical study in the past two decades, being not only an area rich in applications but one that remains filled with open problems. Building on the foundation of refinability of functions, MRA seeks to filter through levels of ever-increasing detail components in data sets – a concept enticing to an age where development of digital equipment (to name but one example) needs to capture more and more information and then store this information in different levels of detail. Except for designing digital objects such as animation movies, one of the most recent popular research areas in which MRA is applied, is inpainting, where "lost" data (in example, a photograph) is repaired by using boundary values of the data set and "smudging" these values into the empty entries. Two main branches of application in MRA are *subdivision* and *wavelet* analysis. The former uses refinable functions to develop algorithms with which digital curves are created from a finite set of initial points as input, the resulting curves (or drawings) of which possess certain levels of smoothness (or, mathematically speaking, continuous derivatives). Wavelets on the other hand, yield filters with which certain levels of detail components (or noise) can be edited out of a data set. One of the greatest advantages when using wavelets, is that the detail data is never lost, and the user can re-insert it to the original data set by merely applying the wavelet algorithm in reverse. This opens up a wonderful application for wavelets, namely that an existent data set can be edited by inserting detail components into it *that were never there*, by also using such a wavelet algorithm. In the recent book by Chui and De

Villiers (see [2]), algorithms for both subdivision and wavelet applications were developed without using Fourier analysis as foundation, as have been done by researchers in earlier years and which have left such algorithms unaccessible to end users such as computer programmers. The fundamental result of Chapter 9 on wavelets of [2] was that feasibility of wavelet decomposition is equivalent to the solvability of a certain set of identities consisting of Laurent polynomials, referred to as *Bezout identities*, and it was shown how such a system of identities can be solved in a systematic way. The work in [2] was done in the *univariate* case only, and it will be the purpose of this thesis to develop similar results in the *bivariate* case, where such a generalization is entirely non-trivial. After introducing MRA in Chapter 1, as well as discussing the refinability of functions and introducing *box splines* as prototype examples of functions that are refinable in the bivariate setting, our fundamental result will also be that wavelet decomposition is equivalent to solving a set of Bezout identities; this will be shown rigorously in Chapter 2. In Chapter 3, we give a set of Laurent polynomials of shortest possible length satisfying the system of Bezout identities in Chapter 2, for the particular case of the Courant hat function, which will have been introduced as a linear box spline in Chapter 1. In Chapter 4, we investigate an application of our result in Chapter 3 to bivariate interpolatory subdivision. With the view to establish a general class of wavelets corresponding to the Courant hat function, we proceed in the subsequent Chapters 5 – 8 to develop a general theory for solving the Bezout identities of Chapter 2 separately, before suggesting strategies for reconciling these solution classes in order to be a simultaneous solution of the system.

Opsomming

Multi-resolusie analise (MRA) het in die afgelope twee dekades toenemende gewildheid geniet as 'n veld in wiskundige wetenskappe. Nie net is dit 'n area wat ryklik toepaslik is nie, maar dit bevat ook steeds vele oop vraagstukke. MRA bou op die grondleggings van verfynbare funksies en poog om deur vlakke van data-komponente te sorteer, of te filter, 'n konsep wat aanloklik is in 'n era waar die ontwikkeling van digitale toestelle (om maar 'n enkele voorbeeld te noem) sodanig moet wees dat meer en meer inligting vasgelê en gestoor moet word. Behalwe vir die ontwerp van digitale voorwerpe, soos animasie-films, word MRA ook toegepas in 'n mees vername navorsingsgebied genaamd inverwing, waar "verlore" data (soos byvoorbeeld in 'n foto) herwin word deur data te neem uit aangrensende gebiede en dit dan oor die leë data-dele te "smeer." Twee hooftakke in toepassing van MRA is *subdivisie* en *golfie*-analise. Die eerste gebruik verfynbare funksies om algoritmes te ontwikkel waarmee digitale krommes ontwerp kan word vanuit 'n eindige aantal aanvanklike gegewe punte. Die verkrygde krommes (of sketse) kan voldoen aan verlangde vlakke van gladheid (of verlangde grade van kontinue afgeleides, wiskundig gesproke). Golfies word op hul beurt gebruik om filters te bou waarmee gewensde data- of geraas-komponente verwyder kan word uit datastelle. Een van die grootste voordeel van die gebruik van golfies bo ander soortgelyke instrumente om datafilters mee te bou, is dat die geraas-komponente wat uitgetrek word nooit verlore gaan nie, sodat die proses omkeerbaar is deurdat die gebruiker die sodanige geraas-komponente in die groter datastel kan terugbou deur die golfie-algoritme in trurat toe te pas. Hierdie eienskap van golfies

open 'n wonderlike toepassingsmoontlikheid daarvoor, naamlik dat 'n bestaande datastel verander kan word deur data-komponente daartoe te voeg *wat nooit daarin was nie*, deur so 'n golfie-algoritme te gebruik. In die onlangse boek deur Chui and De Villiers (sien [2]) is algoritmes ontwikkel vir die toepassing van subdivisie sowel as golfies, sonder om staat te maak op die grondlegging van Fourier-analise, soos wat die gebruik was in vroeëre navorsing en waardeur algoritmes wat ontwikkel is minder effektief was vir eindgebruikers. Die fundamentele resultaat oor golfies in Hoofstuk 9 in [2], verduidelik hoe suksesvolle golfie-ontbinding ekwivalent is aan die oplosbaarheid van 'n sekere versameling van identiteite bestaande uit Laurent-polinome, bekend as *Bezout-identiteite*, en dit is bewys hoedat sodanige stelsels van identiteite opgelos kan word in 'n sistematiese proses. Die werk in [2] is gedoen in die eenveranderlike geval, en dit is die doelwit van hierdie tesis om soortgelyke resultate te ontwikkel in die tweeveranderlike geval, waar sodanige veralgemening absoluut nie-triviaal is. Nadat 'n inleiding tot MRA in Hoofstuk 1 aangebied word, terwyl die verfynbaarheid van funksies, met *boks-latfunksies* as prototipes van verfynbare funksies in die tweeveranderlike geval, bespreek word, word ons fundamentele resultaat gegee en bewys in Hoofstuk 2, naamlik dat golfie-ontbinding in die tweeveranderlike geval ook ekwivalent is aan die oplos van 'n sekere stelsel van Bezout-identiteite. In Hoofstuk 3 word 'n versameling van Laurent-polinome van korste moontlike lengte gegee as illustrasie van 'n oplossing van 'n sodanige stelsel van Bezout-identiteite in Hoofstuk 2, vir die besondere geval van die Courant hoedfunksie, wat in Hoofstuk 1 gedefinieer word. In Hoofstuk 4 ondersoek ons 'n toepassing van die resultaat in Hoofstuk 3 tot tweeveranderlike interpolerende subdivisie. Met die oog op die ontwikkeling van 'n algemene klas van golfies verwant aan die Courant hoedfunksie, brei ons vervolglik in Hoofstukke 5 – 8 'n algemene teorie uit om die oplossing van die stelsel van Bezout-identiteite te ondersoek, elke identiteit apart, waarna ons moontlike strategieë voorstel vir die versoening van hierdie klasse van gelyktydige oplossings van die Bezout stelsel.

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of awe to have classes to teach that you simply adore, and my work with students has played a great role in inspiring me while completing the research for this work. Thanks to all my first-years.

Contents

List of Symbols

I_n	the identity matrix of order n, $n \in \mathbb{N}$
1-D	one-dimensional or univariate
$\sum_{i,j}$	$\sum_{(i,j)\in\mathbb{Z}^2}$
$M(\mathbb{R}^s)$	the set of functions on $\mathbb{R}^s$
$M(\mathbb{C}^s)$	the set of functions on $\mathbb{C}^s$
$M_0(\mathbb{R}^s)$	the set of functions that are compactly supported on $\mathbb{R}^s$
$M_0(\mathbb{C}^s)$	the set of functions that are compactly supported on $\mathbb{C}^s$
$C(\mathbb{R}^s)$	the set of functions that are continuous on $\mathbb{R}^s$
$C_0(\mathbb{R}^s)$	the set of functions that are compactly supported and continuous on $\mathbb{R}^s$
$C^k(\mathbb{R}^s)$	the set of functions in $C(\mathbb{R}^2)$ with k continuous derivatives
$C_0^k(\mathbb{R}^s)$	the set of functions in $C_0(\mathbb{R}^2)$ with k continuous derivatives
$M(\mathbb{Z}^s)$	the set of sequences $c = \{c_i\}_{i\in\mathbb{Z}^s} \subset \mathbb{R}$
$M_0(\mathbb{Z}^s)$	the set of sequences $c = \{c_i\}_{i\in\mathbb{Z}^s} \subset \mathbb{R}$ that are finitely supported
$L^2(\mathbb{R})$	the set of square-integrable functions on $\mathbb{R}$
$L^2(\mathbb{R}^2)$	the set of square-integrable functions on $\mathbb{R}^2$
$\langle f, g\rangle$	the convolution of the functions f and g
N_m	Cardinal B-spline of order m

$\mathcal{D}$	direction matrix
$\mathcal{D}_k$	direction matrix consisting of k direction vectors
	(except where explicitly stated otherwise)
B_k, $B_{\mathcal{D}_k}$	box spline associated with $\mathcal{D}_k$
B_{n_1,n_2,n_3,n_4},	box spline associated to 4-directional direction matrix
ϕ	refinable function
p	refinement mask
P	2-refinement mask symbol
P_k	2-refinement mask symbol associated with the pair (p_k, B_k)
$\ell^\infty(\mathbb{Z}^2)$	the set $\left\{c \in M(\mathbb{Z}^2) \subset \mathbb{R} : \sup_{i,j} \lvert c_{i,j}\rvert < \infty\right\}$ of bounded sequences
$\lvert\lvert a\rvert\rvert_\infty$	the infinity-norm of a sequence, $\lvert\lvert a\rvert\rvert_\infty = \sup_{i,j\in\mathbb{Z}} \lvert a_{i,j}\rvert,\ a \in M(\mathbb{Z}^2)$

Chapter 1

Multi-resolution Analysis: From Univariate to Bivariate

1.1 Introduction

It is no secret that the twentieth century has brought forth technological developments greater than had ever been anticipated. Where computers were initially utilized to do the simplest of computations and contribute to secure storage of data, they were soon put to use for not only capturing data, but also modifying or, if you will, manipulating it. Photographs, sound files and even videos can nowadays be stored and improved to the user's will to such an extent that the only limits to digital design seem to be the creator's imagination. On the one side, digitalization has changed the existence of the design industry, for instance, making the work of automobile and aircraft designers more efficient and economical, while broadening the limits of design vastly. Companies working with massive sets of data on a daily basis now use computer systems to equip them in their processing needs, and even in medical practice specialized equipment along with high-end computer programs are employed in pursuit of efficiency and attempts at decreasing discomfort in the treatment of patients in various ways. On the other hand, the entertainment world has witnessed the advances made by the digital development of computers with great awe, and gradually seized the opportunity to utilize these developments for the creation of movies, computer games, and more. Computer graphics are made to look just as real as ordinary life and details to videos and sound can be added and adjusted at will.

Of course, even artifical intelligence still remains exactly that, even with the means of all the technological and digital developments of the last decade. Whether in the world of computerized design, or in the entertainment mecca, the best performing equipment is useless without the corresponding underlying software. And at the heart of these software developments, lie the mathematical algorithms that eventually steer the use of the machine.

In earlier years, most of the mathematical foundation for these kind of algorithms relied on data sampling and Fourier analysis (see [17]), which in turn depended strongly on the Complex Analysis branch of Mathematics, and therefore resulting in these algorithms being rather inaccessible to most people without the proper mathematical background. Moreover, some of the earlier theory might no longer be helpful or relevant for the design of current computer algorithms, due to the fact that the available hardwares have outdated the available mathematical theory.

A recent mathematical tool to employ in computerized utilities, is a mathematical function called a *wavelet*. They yield conceptual filters since they can be used to decompose data sets into their "high-level" and "low-level" components, therefore making it possible for users to literally filter through data sets, or to remove unwanted "noise" from image or sound files, or to detect irregularities in industrial designs. For example, it is helpful when searching for cracks on a wing of an aircraft that may not be detectable by the human eye, to have scanners available that can examine the aircraft's wing for specified levels of regularity or smoothness. Such scanners also work on the foundations of wavelet filters. What makes wavelets such an attractive tool when compared to other filter-yielding mathematical tools (such as discrete Fourier transforms or discrete cosine transforms), is that this process of filtering out some noise or artifact features, can, in fact, be reversed. In other words, in addition to removing some unwanted bits of data from, say, an image, wavelets can also be used to *insert* some required bits of detail into the image. This latter property of wavelet theory is what makes it such a sophisticated tool in especially animated movie design, where the goal has always been to improve the appearance of images, in order to make them look more realistic, by including sufficient amounts of detail.

The theory of wavelets and their decomposition techniques form part of a mathematical field that is known as multi-resolution analysis (MRA), wherein one studies how data sets are decomposed into different levels of frequencies using filter-like mathematical tools, such as wavelets (see e.g. [1]), and where the principle of such decomposition is built on some given basis function. Such a function must be *refinable* for the decomposition process to be successful, and so, underneath all the mathematics of wavelet and MRA theory, lies the fundamental theory and analysis of refinable functions.

Another mathematical tool that relies on refinable functions is known as *subdivision*, which is particularly useful in computer-aided geometric design (CAGD), both for undustrial and entertainment uses. Plainly said, a given subdivision algorithm "completes" a user's design of a particularly chosen object after being given as input a finite number of initial coordinate points, the "outline" of the design. This is very satisfactory not only because of the ease with which these initial points can be input into the computer algorithm and the time-efficiency of the algorithm, but especially because of the fact that the algorithm can be specifically designed, by specifying the refinable function on which the algorithm is based, so that the produced output graphics exhibit smoothness criteria of the user's own discretion. Another positive property of subdivision is that the user can specify that his design should contain the original points of input, in which case the corresponding subdivision algorithm is described as being *interpolatory*.

Due to recentness of the technological advances and corresponding developments of the underlying mathematical theory, much improvement still remains to be done in order to develop algorithms that are efficient and user-friendly. For example, the original development of MRA and wavelet theory depended mainly on Fourier analysis. Recently, it was shown that in fact wavelets can be constructed from the underlying basis (refinable) functions by a method based on finding solutions for required Laurent polynomials in certain sets of corresponding Bezout identities, therefore not having to work in the Complex Analysis realm anymore and as such being much more accessible to computer programmers who want to design and customize algorithms for wavelet construction. (See [2].) While the corresponding Bezout identities for the construction of wavelets were designed in [2] only for the univariate case, it will be the purpose of this thesis to develop analogous results for the bivariate case, a generalization which is non-trivial in nature as can already be understood when looking at the differences between the refinable functions themselves in the univariate and the bivariate case, respectively.

These functions will be studied in detail in this chapter, where emphasis will be given to *box spline* functions, which are the prototype of refinable functions in the bivariate case. The concept of a *direction matrix* will be established in general with examples to show the relationship between direction matrices and their corresponding box splines, and after which it will be noted that the main results of this thesis will build on particular cases of direction matrices and their corresponding box splines.

1.2 Multi-resolution Analysis (MRA) and Refinability

As mentioned above, refinable functions are the instruments that lie at the heart of all of MRA, wavelet, and subdivision theories. In general, they are functions that satisfy the identity

$$\phi(\mathbf{x}) = \sum_{\mathbf{j}} p_{\mathbf{j}} \phi(M\mathbf{x} - \mathbf{j}), \quad \mathbf{x} \in \mathbb{R}^s, \tag{1.2.1}$$

where $s \in \mathbb{N}$, $\{p_{\mathbf{j}}\} \in M_0(\mathbb{R}^s)$, and M is some $s \times s$ matrix, referred to as the *dilation matrix*. Here and everywhere in this work, $\sum_{\mathbf{j}} := \sum_{\mathbf{j} \in \mathbb{Z}^s}$, while $\mathbb{Z}^s$ and $\mathbb{R}^s$ denote the sets of s-dimensional vectors in $\mathbb{Z}$ (the integers) and $\mathbb{R}$ (the real numbers), respectively, and $\mathbb{N}$ indicate the natural numbers. Also, $\mathbb{Z} := \mathbb{Z}^1$ and $\mathbb{R} := \mathbb{R}^1$. Furthermore, the notation $M_0(\mathbb{R}^s)$ will denote arrays with all entries in $\mathbb{R}$, but where only a finite number of entries are non-zero.

The work in this thesis is based on the case where $M = 2I_s$ in (1.2.1), where I_s is the $s \times s$ identity matrix. Some results regarding the refinability of functions with respect to general dilation matrices have been developed in [18]. In the case where $M = 2I_s$, (1.2.1) becomes

$$\phi(\mathbf{x}) = \sum_{\mathbf{j}} p_{\mathbf{j}} \phi(2I_s\mathbf{x} - \mathbf{j}), \quad \mathbf{x} \in \mathbb{R}^s. \tag{1.2.2}$$

An important role is played by the array (or sequence, if $s = 1$) $\{p_{\mathbf{j}}\} = \{p_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^s\}$, which is called the corresponding *refinement sequence* or the *mask*, while the equation (1.2.2) is known as the *refinement equation*. It was shown by De Wet in [5] that the correspondence between a given refinable function and "its" refinement sequence is unique; that is, if a function ϕ and an array $\{p_{\mathbf{j}}\}$ are related by means of (1.2.2), then the function ϕ satisfies

$$\phi(\mathbf{x}) = \sum_{\mathbf{j}} \tilde{p}_{\mathbf{j}} \phi(2I_s\mathbf{x} - \mathbf{j}), \quad \mathbf{x} \in \mathbb{R}^s,$$

for a sequence $\{\tilde{p}_{\mathbf{j}}\} \in M_0(\mathbb{R}^s)$, if and only if $\{\tilde{p}_{\mathbf{j}}\} = \{p_{\mathbf{j}}\}$. Furthermore, the (unique) refinement sequence $\{p_{\mathbf{j}}\}$ is used to construct the Laurent polynomial

$$P(\mathbf{z}) := \frac{1}{2^s} \sum_{\mathbf{j}} p_{\mathbf{j}} \mathbf{z}^{\mathbf{j}}, \quad \mathbf{z} \in \mathbb{C}^s \backslash \{(0, 0, \ldots, 0)\}, \tag{1.2.3}$$

which is known as the *refinement mask symbol* of ϕ. Here, $\mathbb{C}^s$ denotes the set of s-dimensional vectors in set of the complex numbers $\mathbb{C}$. In the following, the role of refinable functions in univariate MRA will be discussed.

Let ϕ be the univariate refinable function on $\mathbb{R}$ given by

$$\phi(x) := \begin{cases} 1, & x \in [0, 1); \\ 0, & x \in \mathbb{R}, \ x \notin [0, 1). \end{cases} \tag{1.2.4}$$

(The reason for the refinability of ϕ and the implications thereof are not important here – the meaning of refinability will be discussed later in this section.) Define the function ψ (as will be referred to as the *mother wavelet function*) on $\mathbb{R}$, by

$$\psi(x) := \begin{cases} 1, & x \in [0, \frac{1}{2}); \\ -1, & x \in [\frac{1}{2}, 1); \\ 0, & x \in \mathbb{R},\ x \notin [0, 1). \end{cases} \tag{1.2.5}$$

Based on the function ψ, next define a class of *wavelet* functions $\psi_{k,n}$, in terms of the integer shifts of dilated versions of ψ; in particular,

$$\psi_{k,n}(x) := 2^{k/2}\psi(2^k x - n), \quad k \in \mathbb{Z}_+,\ n \in \mathbb{Z}, \tag{1.2.6}$$

where $\mathbb{Z}_+$ indicates the set of non-negative integers. It is shown in [1] that any function $f \in L^2(\mathbb{R})$, where $L^2(\mathbb{R})$ denotes the square-integrable functions on $\mathbb{R}$, can be written in the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi(x-n) + \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} c_{k,n}\psi_{k,n}(x), \quad x \in \mathbb{R}, \tag{1.2.7}$$

where

$$c_n := \langle f, \phi(\cdot - n)\rangle = \int_n^{n+1} f(x)\,dx, \tag{1.2.8}$$

and

$$c_{k,n} := \langle f, \psi_{k,n}\rangle = 2^{k/2}\left[\int_{n/2^k}^{(2n+1)/2^{k+1}} f(x)\,dx - \int_{(2n+1)/2^{k+1}}^{(n+1)/2^k} f(x)\,dx\right]. \tag{1.2.9}$$

Here, as usual, $\langle g, h\rangle$ means the convolution on $L^2(\mathbb{R})$ of two functions g and h, i.e.,

$$\langle g, h\rangle(x) := \int_{\mathbb{R}} g(t)h(x-t)\,dt, \quad x \in \mathbb{R}. \tag{1.2.10}$$

In fact, if one defines, for $N = 1, 2, \dots$,

$$S_N := \{\phi(\cdot - n) : n \in \mathbb{Z}\} \bigcup \{\psi_{k,n} : k = 0, \dots, N-1,\ n \in \mathbb{Z}\}, \tag{1.2.11}$$

and

$$V_N := \mathrm{Span}(S_N) = \left\{ \sum_{\ell} a_\ell g : \sum_{\ell} a_\ell^2 < \infty, \; g \in S_N \right\}, \tag{1.2.12}$$

then S_N is an orthonormal basis for V_N for all $N \geq 1$. Also, $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$ is an orthonormal basis for the space V_0 defined as the set of all piecewise constant functions on the integer intervals. Then, any given function $f \in L^2(\mathbb{R})$ can be best approximated by its projection into any one of the spaces V_N, by means of the formula

$$f_N(x) := \sum_n \langle f, \phi(\cdot - n)\rangle \phi(x - n) + \sum_{k=0}^{N-1} \sum_n \langle f, \psi_{k,n}\rangle \psi_{k,n}(x), \quad x \in \mathbb{R}. \tag{1.2.13}$$

What is more, is that, for each $N = 0, 1, \ldots$, the relationship $f_{N+1} = f_N + g_N$ holds, where

$$g_N(x) := \sum_n \langle f, \psi_{N,n}\rangle \psi_{N,n}(x), \quad x \in \mathbb{R}. \tag{1.2.14}$$

By repeated application of this relationship, one gets

$$\begin{aligned} f_N &= f_{N-1} + g_{N-1} \\ &= f_{N-2} + g_{N-2} + g_{N-1} \\ &= \ldots \\ &= f_0 + g_0 + g_1 + \ldots + g_{N-1}. \end{aligned} \tag{1.2.15}$$

Note from (1.2.6) that the wavelet functions $\psi_{k,n}$ possess increasing levels of frequency for fixed n and increasing values of the first index k. Therefore, from (1.2.14), for increasing values of the index N in the functions g_N, $N = 0, 1, \ldots$, more "texture" is included. This implies that for increasing values of N, a better, more detailed approximation of the function f is obtained by its projection f_N. What is more, is that these "texture levels" can be *separated* from f by means of the relationship (1.2.15); i.e., the wavelets $\psi_{k,n}$ serve as a *filter* to decompose f into its low-frequency (namely f_0) and high-frequency (namely $g_0, \ldots, g_{N-1}$) parts. This decomposition of a function into its different levels of resolution, is known as *multi-resolution analysis,* and is elaborated on in more detail in [1].

Now, note from (1.2.4) and (1.2.5) that the wavelet ψ can be written in terms of the refinable function ϕ, by

$$\psi(x) = \phi(2x) - \phi(2x-1), \quad x \in \mathbb{R}, \tag{1.2.16}$$

whereas, according to (1.2.14), the functions g_N, $N \in \mathbb{Z}_+$, are dependent only on $\psi_{N,n}$, $n \in \mathbb{Z}$, being linear combinations of the small wavelet functions. Therefore, after defining the vector space

$$W_N := \text{Span}\left(\{\psi_{N,n} : n \in \mathbb{Z}\}\right), \tag{1.2.17}$$

the inclusions $V_N \subset V_{N+1}$ and $W_N \subset V_{N+1}$ hold, together with the fact that $W_N \cap V_N = \{0\}$.

The fact that ψ and $\psi_{k,n}$ are functions of the refinable function ϕ (as follows from (1.2.16) and (1.2.6)), clearly emphasizes the significance of in-depth study of refinable functions in general, in order to establish similar decomposition results for a function f as above, with different refinable functions than the one defined in Equation (1.2.4). Note that ϕ itself can be written in terms of its own dilated integer shifts, namely

$$\phi(x) = \phi(2x) + \phi(2x-1), \quad x \in \mathbb{R}; \tag{1.2.18}$$

that is, ϕ satisfies the equation (1.2.2) for the case $s = 1$, with $p_0 = p_1 = 1$ and $p_j = 0$ for $j \neq 0, 1$, so that ϕ is indeed refinable.

1.3 Basis Functions for MRA: From B-splines to Box splines

In the univariate case (i.e., $s = 1$ in (1.2.2)), the refinement equation becomes

$$\phi(x) = \sum_j p_j \phi(2x - j), \quad x \in \mathbb{R}, \tag{1.3.1}$$

with corresponding refinement mask symbol

$$P(z) = \frac{1}{2}\sum_j p_j z^j, \quad z \in \mathbb{C}\backslash\{0\}. \tag{1.3.2}$$

An important class of functions that satisfy (1.3.1), are the cardinal B-splines. They are essentially particularly designed piecewise polynomials that satisfy elegant smoothness and symmetry properties and are used in vast numbers of applications as primary examples of refinable functions, mainly for their attractive characteristics as well as their ease of computation and the fact that they are explicitly known refinable functions (whereas all refinable functions other than the B-splines are only known *implicitly*, to date). Formally, for $m = 2, 3, \dots$, the cardinal B-spline N_m of *order* m is defined inductively by the following:

$$N_m(x) := \int_0^1 N_{m-1}(x-t)\,dt, \quad x \in \mathbb{R}, \tag{1.3.3}$$

where

$$N_1(x) := \begin{cases} 1, & x \in [0,1); \\ 0, & x \in \mathbb{R},\ x \notin [0,1), \end{cases} \tag{1.3.4}$$

i.e., N_1 is the same function as the refinable function ϕ in the illustration of MRA in the previous section. This inductive definition can be employed (see e.g. Chapter 4 in [2]) to deduce several properties of the cardinal B-splines in Theorem 1.1 below. Here, $C^k(\mathbb{R})$ denotes all functions on $\mathbb{R}$ that have k'th order continuous derivatives, and $C_0^k(\mathbb{R})$ are those functions in $C^k(\mathbb{R})$ of which the support (i.e., non-zero part) is compact. Naturally, $C(\mathbb{R}) := C^0(\mathbb{R})$ is the set of continuous functions on $\mathbb{R}$. Also, the standard notation for the binomial coefficients $\binom{m}{j} := \dfrac{m!}{j!(m-j)!}$ is used.

Theorem 1.1 *For each $m = 1, 2, \dots$, the cardinal B-spline N_m of order m defined by* (1.3.3) *and* (1.3.4) *satisfies the following properties:*

(a) *N_m satisfies the refinement equation* (1.3.1) *with refinement sequence $\{p_j\} = \{p_{m,j}\}$ given by*

$$p_{m,j} = \frac{1}{2^{m-1}}\binom{m}{j}, \quad j \in \mathbb{Z}; \tag{1.3.5}$$

(b) N_m *has support interval* $[0, m]$; *that is,* $suppN_m = [0, m]$;

(c) N_m *is strictly positive inside its support; that is,*

$$N_m(x) > 0, \quad x \in (0, m); \tag{1.3.6}$$

(d) *For increasing values of* m, N_m *has increasing levels of smoothness. Specifically,*

$$N_m \in C_0^{m-2}. \tag{1.3.7}$$

As it is the intention of this thesis to study wavelet decomposition theory in the bivariate case, we will proceed to establish some definitions and basic results regarding box splines, which, as mentioned in the introduction part of this chapter, are an extension of the cardinal B-splines from the univariate to the bivariate setting. Box splines are the prototype example of refinable functions in the bivariate case, i.e., with $s = 2$ in (1.2.2). Similar to the inductive definition (1.3.3) and (1.3.4) for cardinal B-splines above, box splines can also be defined inductively. However, whereas any given cardinal B-spline N_m is characterized by its order m, box splines are characterized by what will be called *direction matrices*. In particular, a direction matrix is one of the form $\mathcal{D}_m = [\mathbf{d}_1 \ \mathbf{d}_2 \ \ldots \ \mathbf{d}_m]$, where $\mathbf{d}_i \in \mathbb{Z}^2 \backslash \{(0,0)\}$, $i = 1, \ldots, m$, for some $m \in \mathbb{N}$, $m \geq 2$. Given a direction matrix $\mathcal{D}_m = [\mathbf{d}_1 \ \mathbf{d}_2 \ \ldots \ \mathbf{d}_m]$, the "sub"-matrix $[\mathbf{d}_1 \ \mathbf{d}_2]$ will be referred to as the *initial direction matrix,* and it is required that its determinant be non-zero for the corresponding box spline to be well defined (see the recursive definition for box splines corresponding to an initial box spline given in (1.3.8) and (1.3.9) below).

For $k = 3, 4, \ldots$, and corresponding to a given direction matrix $\mathcal{D}_k = [\mathbf{d}_1 \ \mathbf{d}_2 \ \ldots \ \mathbf{d}_{k-1} \ \mathbf{d}_k]$, the box spline $B_k := B_{\mathcal{D}_k}$ is defined (see e.g. Prautzch and Boehm, [16]) as

$$B_k(x, y) := \int_0^1 B_{k-1}\left((x, y) - t\mathbf{d}_k\right) dt, \quad (x, y) \in \mathbb{R}^2, \tag{1.3.8}$$

where the box spline corresponding to a general initial direction matrix $\mathcal{D}_2 = [\mathbf{d}_1\ \mathbf{d}_2] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is given by

$$B_2(x,y) := \begin{cases} \dfrac{1}{ad-bc}, & (x,y) \in \mathcal{D}_2[0,1)^2; \\ 0, & (x,y) \in \mathbb{R}^2,\ (x,y) \notin \mathcal{D}_2[0,1)^2. \end{cases} \tag{1.3.9}$$

Here, $\mathcal{D}_2[0,1)^2$ means the parallelogram-shaped region defined by

$$\mathcal{D}_2[0,1)^2 := \left\{ \begin{bmatrix} a \\ c \end{bmatrix} t + \begin{bmatrix} b \\ d \end{bmatrix} s :\ t,s \in [0,1] \right\},$$

and is illustrated more elaborately in [18].

In the remainder of this work, the direction vectors will more often than not be denoted by *row* vectors instead of *column* vectors for simplicity, i.e., the notation $(a,b)^T$ will have the same meaning as $\begin{bmatrix} a \\ b \end{bmatrix}$. For further simplicity, the transpose symbol will be omitted, so that, in the context of direction vectors, the vector (a,b) will have the same meaning as $(a,b)^T$, even though one is a row and the other a column vector. It can be assumed that in the strict proofs of the necessary results, the mathematically correct notation will have always been used. Note that the direction matrix is not order-specific; that is, the direction matrix $[\mathbf{d}_1\ \mathbf{d}_2\ \dots\ \mathbf{d}_k]$ gives rise to the same box spline regardless of the ordering of the columns $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_k$. Also, in this thesis, note that only the vectors $\mathbf{e}_1 := (1,0)$, $\mathbf{e}_2 := (0,1)$, $\mathbf{e}_3 := (1,1)$, and $\mathbf{e}_4 := (-1,1)$ will be included in any direction matrix. We call a box spline corresponding to such a direction matrix, a *four-directional* box spline, whereas one corresponding to a direction matrix which only includes the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, will be called a *three-directional* box spline. A box spline corresponding to a direction matrix with only the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, is called a *two-directional* box spline. In the light of this, and to simplify notation, let $\mathcal{D} := \mathcal{D}_k$ be a direction matrix consisting of a finite number of each of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_4$, and define

$$n_j := \text{number of times } \mathbf{e}_j \text{ is in } \mathcal{D}, \quad j = 1, \ldots, 4. \tag{1.3.10}$$

Then $k = n_1 + \ldots + n_4$, and, following [2], the notation B_{n_1,n_2,n_3,n_4} will be used instead of B_k. Furthermore, if $n_4 = 0$, we shall write $B_{n_1,n_2,n_3} := B_{n_1,n_2,n_3,0}$, and if $n_3 = n_4 = 0$, we shall write $B_{n_1,n_2} := B_{n_1,n_2,0,0}$. Before continuing to a few examples of four-directional box splines, the following result on properties of box splines is given. We shall use standard matrix notation when a refinement mask is given in the bivariate case; that is, if ϕ is a bivariate refinable function with corresponding refinement mask $\mathbf{p} = \{p_{i,j}\}$, then the element $p_{i,j}$ will correspond to the entry in the i^{th} row and the j^{th} column of the *refinement mask matrix* (or refinement matrix) $\mathbf{p}$. Note that we use the same symbol $\mathbf{p}$ to denote the matrix consisting of the refinement mask entries of the refinable function.

Theorem 1.2 *For $k = 2, 3, \ldots$, let B_k be defined by* (1.3.8) *and* (1.3.9)*. Then B_k satisfies the refinement equation* (1.2.2)*. Specifically, the corresponding refinement masks $\mathbf{p}_{1,1} = \mathbf{p}_{1,1,0,0}$ and $\mathbf{p}_{1,1,1} = \mathbf{p}_{1,1,1,0}$ corresponding to, respectively, the box splines $B_{1,1}$ and $B_{1,1,1}$, are given by, respectively,*

$$\mathbf{p}_{1,1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \tag{1.3.11}$$

$$\mathbf{p}_{1,1,1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}. \tag{1.3.12}$$

Furthermore, for $n_1, n_2, n_3, n_4 \in \mathbb{N}$, $n_1 \neq 0$, $n_2 \neq 0$, the four-directional box spline $B_n = B_{n_1,n_2,n_3,n_4}$ satisfies the following:

(a) *B_n is a piecewise polynomial on some 4-directional domain in $\mathbb{Z}^2$;*

(b) *B_n is compactly supported, with rm supp$^c B_n = \mathcal{D}_n[0,1)^2$; that is, $B_n(x,y) = 0$, $(x,y) \in \mathbb{R}^2 \backslash \mathcal{D}_n[0,1)^2$;*

(c) *B_n is strictly positive inside of its support; that is,*

$$\begin{cases} B_n(x,y) > 0, & (x,y) \in \mathcal{D}_n[0,1)^2; \\ B_n(x,y) = 0, & (x,y) \in \mathbb{R}^2, \ (x,y) \notin \mathcal{D}_n[0,1)^2; \end{cases}$$

(d) $\displaystyle\sum_{i,j} B_n(x-i, y-j) = 1, \quad (x,y) \in \mathbb{R}^2;$

(e) $\displaystyle\int_{\mathbb{R}^2} B_n(x,y)\, dx\, dy = 1.$

Proof. It can be easily verified that, for $\phi = B_{1,1}$ and $\mathbf{p}_{1,1}$ given by (1.3.11), then the refinement equation (1.2.2) is satisfied; similarly, if $\phi = B_{1,1,1}$ and $\mathbf{p}_{1,1,1}$ is given by (1.3.12), then (1.2.2) holds. Furthermore, it was shown in [18] that the inductive definition (1.3.8) preserves refinability, in the sense that, for an initial direction matrix that corresponds to a refinable function and refinement mask, refinability is preserved for any non-zero vector to be included in the direction matrix. Further emphasis was placed in [18] on the fact that certain choices of combinations of direction vectors to be included in the direction matrix, yield box splines that possess prescribed orders of smoothness. Properties (a) through (c) follow directly from the inductive definition in (1.3.8) and (1.3.9), whereas properties (d) and (e) were proved rigorously in [18]. ■

Example 1.1 The two-directional box spline $B_{1,1} = B_2$ corresponding to the direction matrix $\mathcal{D}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, is given, according to (1.3.9), by

$$B_{1,1}(x,y) = \begin{cases} 1, & (x,y) \in [0,1)^2; \\ 0, & (x,y) \in \mathbb{R}^2 \backslash [0,1)^2. \end{cases} \tag{1.3.13}$$

The box spline $B_{1,1}$ is known as the bivariate Haar function and is illustrated in Figure 1.3.1.

Example 1.2 Let $n_1 = n_2 = n_3 = 1$ and $n_4 = 0$ in equation (1.3.10), i.e., with the direction matrix given by $\mathcal{D}_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Then the three-directional box spline $B_{1,1,1} =$

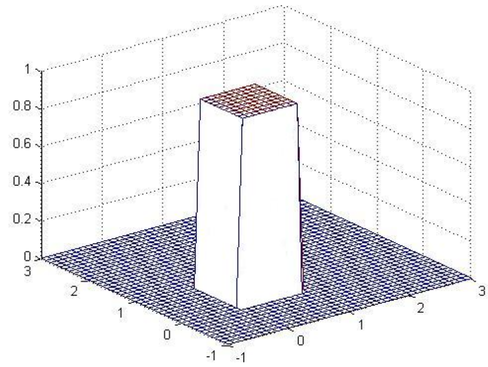

Figure 1.3.1: The Haar function $B_{1,1}$

B_3 is given, after applying (1.3.13) and the inductive definition (1.3.8), by

$$B_{1,1,1}(x,y) = \begin{cases} y, & x \in [0,1),\ y \in [0,1),\ y < x; \\ x, & x \in [0,1),\ y \in [0,1),\ y \geq x; \\ 1+x-y, & x \in [0,1),\ y \in [1,2),\ y < x+1; \\ 2-y, & x \in [1,2),\ y \in [1,2),\ y \geq x; \\ 2-x, & x \in [1,2),\ y \in [1,2),\ y < x; \\ 1+y-x, & x \in [1,2),\ y \in [0,1),\ y \geq x-1; \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.14}$$

The box spline $B_{1,1,1}$ is known as the *Courant hat function* and is illustrated in Figure 1.3.2.

Example 1.3 For $n_1 = n_2 = n_3 = n_4 = 1$, the four-directional box spline $B_{1,1,1,1} = B_4$ is known as the *Zwart-Powell element*, and is illustrated in Figure 1.3.3.

Note that, in the bivariate case, the refinement equation (1.2.2) becomes:

$$\phi(x,y) = \sum_{(i,j)\in\mathbb{Z}^2} p_{i,j}\phi(2x-i, 2y-j), \quad (x,y) \in \mathbb{R}^2, \tag{1.3.15}$$

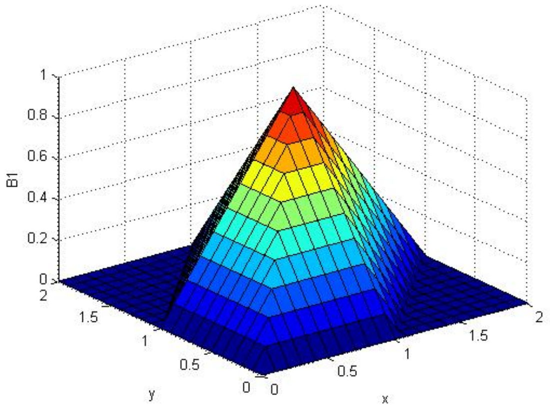

Figure 1.3.2: The Courant hat function $B_{1,1,1}$

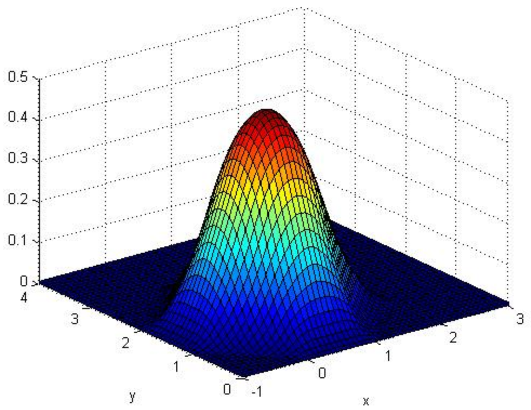

Figure 1.3.3: The Zwart-Powell element $B_{1,1,1,1}$

and with corresponding refinement mask symbol

$$P(z_1, z_2) = \frac{1}{4} \sum_{i,j} p_{i,j} z_1^i z_2^j, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{1.3.16}$$

It follows directly from (1.3.12) in Theorem 1.2 that the Courant hat function in Example 1.2 is refinable with corresponding refinement mask symbol

$$\begin{aligned} P_{1,1,1}(z_1, z_2) &= \frac{1}{4}\sum_{i,j} p_{i,j} z_1^i z_2^j \\ &= \frac{1}{8}(1 + z_1 + z_2 + 2z_1z_2 + z_1^2 z_2 + z_1 z_2^2 + z_1^2 z_2^2) \\ &= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right), \quad (z_1, z_2) \in \mathbb{C}^2. \end{aligned} \tag{1.3.17}$$

Note that, since $P_{1,1,1}$ is in fact a *polynomial* that posesses no negative powers of z_1 or z_2, the origin $(0, 0)$ need not be excluded from its domain $\mathbb{C}^2$. It is further remarked that the inductive definition for box splines in (1.3.8) and (1.3.9) above is one of three equivalent ways in which box splines are obtained in the literature. Since it is not the intention of this thesis to give a detailed historic study of box splines, it suffices to mention that the *geometric* and *analytic* definitions of box splines are given and studied in e.g. [4], [12] [15], and [16].

Chapter 2

Wavelets Decomposition Results: Necessary and Sufficient Conditions

2.1 Wavelet Decomposition in the Univariate Case

Following the discussion about Multi-resolution analysis in Section 1.2, we proceed to establish a general univariate wavelet decomposition technique based on a given refinable function, i.e., a function satisfying the refinement relation (1.2.2) with $s = 1$. We first assume that ϕ is a univariate refinable function with refinement sequence $\{p_j\}$ and corresponding refinement mask symbol

$$P(z) := \frac{1}{2}\sum_j p_j z^j, \quad z \in \mathbb{C}\backslash\{0\}. \tag{2.1.1}$$

Our purpose is to find a function (called a wavelet) ψ satisfying a similar relationship in terms of the refinable function ϕ than the relationship in (1.2.16), and such that an MRA scheme similar to the one in Section 1.2 exists. To this end, define the vector spaces

$$S_\phi^r := \left\{ \sum_j c_j \phi(2^r \cdot -j) : \ \{c_j\} \in M(\mathbb{Z}) \right\}, \quad r \in \mathbb{Z}; \tag{2.1.2}$$

$$W_{\phi,\mathbf{q}}^r := \left\{ \sum_j d_j \psi_{\phi,\mathbf{q}}(2^r \cdot -j) : \ \{d_j\} \in M(\mathbb{Z}) \right\}, \quad r \in \mathbb{Z}, \tag{2.1.3}$$

for $\mathbf{q} = \{q_j\} \in M_0(\mathbb{Z})$, and where

$$\psi_{\phi,\mathbf{q}}(x) := \sum_j q_j \phi(2x - j), \quad x \in \mathbb{R}. \tag{2.1.4}$$

In [2] it is shown that, for $r \in \mathbb{Z}$, $S_\phi^r \subset S_\phi^{r+1}$ and $W_{\phi,\mathbf{q}}^r \subset S_\phi^{r+1}$. The following fundamental result is also proved in [2].

Theorem 2.1 *Let ϕ be a refinable function with linearly independent integer shifts, and-with refinement mask symbol as in* (2.1.1)*, and let $\mathbf{q} = \{q_j\}$, $\{a_j\}$, and $\{b_j\}$ be sequences in $M_0(\mathbb{Z})$, with corresponding Laurent polynomial symbols*

$$\left.\begin{array}{c} Q(z) := \dfrac{1}{2}\displaystyle\sum_j q_j z^j, \\ A(z) := \displaystyle\sum_j a_j z^j, \quad B(z) := \displaystyle\sum_j b_j z^j, \end{array}\right\} z \in \mathbb{C}\backslash\{0\}, \tag{2.1.5}$$

and define the function $\psi_{\phi,\mathbf{q}}$ as in (2.1.4)*. Then the wavelet decomposition relation*

$$\phi(2x - j) = \sum_k a_{2k-j}\phi(x - k) + \sum_k b_{2k-j}\psi_{\phi,\mathbf{q}}(x - k), \quad x \in \mathbb{R}, \ j \in \mathbb{Z}, \tag{2.1.6}$$

holds, if and only if the Laurent polynomials P, Q, A, and B satisfy the following identities:

$$\left.\begin{array}{rcl} P(z)A(z) + P(-z)A(-z) & = & 1, \\ Q(z)A(z) + Q(-z)A(-z) & = & 0, \\ P(z)B(z) + P(-z)B(-z) & = & 0, \\ Q(z)B(z) + Q(-z)B(-z) & = & 1, \end{array}\right\}, \ z \in \mathbb{C}\backslash\{0\}. \tag{2.1.7}$$

It is further shown in [2] that the decomposition relation in (2.1.6) can be extended to an arbitrary level r of resolution; that is, if (2.1.7) is satisfied, then, for any $r \in \mathbb{Z}$ and $\{c_j\} \in M(\mathbb{Z})$,

$$\begin{aligned} \sum_j c_j \phi(2^{r+1}x - j) \;&=\; \sum_j \left[\sum_k a_{2j-k}c_k\right]\phi(2^r x - j) \\ &\quad + \sum_j \left[\sum_k b_{2j-k}c_k\right]\psi_{\phi,\mathbf{q}}(2^r x - j), \quad x \in \mathbb{R}. \end{aligned} \tag{2.1.8}$$

Also, for $r \in \mathbb{Z}$, it holds that $S_\phi^r \cap W_{\phi,\mathbf{q}}^r = \{0\}$, a result from which it follows that the decomposition in (2.1.8) is in fact unique. The function $\psi_{\phi,\mathbf{q}}$ is called a synthesis wavelet, and it is further shown in [2] how the system of Bezout identities in (2.1.7) can systematically be solved, a method that relies, amongst other things, on the assumption that the wavelet ψ to be constructed must possess a prescribed number of *vanishing moments*. This thesis will not study the definition of vanishing moments of a function in detail. It is shown in [2] that a univariate wavelet $\psi(x)$ satisfying the vanishing moment condition of order $\ell \in \mathbb{N}$, intersects the x-axis at least ℓ times inside of its support. Therefore, for a given refinable function, wavelets with higher frequencies can be obtained by increasing the prescribed orders of the vanishing moments to be satisfied by the wavelets.

Example 2.1 For the quadratic B-spline N_3 in (1.3.3), we have, from (1.3.5), $\{p_{-1}, p_0, p_1, p_2\} = \left\{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right\}$ and $p_j = 0,\ j \in \mathbb{Z},\ j \notin \{-1, 0, 1, 2\}$. In the case where the order of vanishing moments is specified as $\ell = 0$, and after solving the Bezout identities in (2.1.7), the sequences $\{q_j\}$, $\{a_j\}$ and $\{b_j\}$ are obtained to be

$$\{q_{-1}, q_0\} = \{-3, -1\}; \quad q_j = 0, \quad j \in \mathbb{Z},\ j \notin \{-1, 0\};$$

$$\{a_0, a_1\} = \left\{\frac{3}{2}, -\frac{1}{2}\right\}; \quad a_j = 0, \quad j \in \mathbb{Z},\ j \notin \{0, 1\};$$

$$\{b_0, b_1, b_2, b_3\} = \left\{\frac{1}{8}, -\frac{3}{8}, \frac{3}{8}, -\frac{1}{8}\right\}; \quad b_j = 0, \quad j \in \mathbb{Z},\ j \notin \{0, 1, 2, 3\}.$$

The resulting wavelet is then given by

$$\psi_{N_3,\mathbf{q}}(x) = -3N_3(x+1) - N_3(x), \quad x \in \mathbb{R},$$

and is illustrated in Figure 2.1.1

It is the purpose of our work to generalize the results in [2] to the bivariate case without relying on conditions regarding vanishing moments. In the next section, it will be shown that, given a bivariate refinable function ϕ (that is, a function satisfying (1.3.15)) with its corresponding mask symbol P, then a wavelet decomposition relationship similar to the one in (2.1.6) holds if and only if a system of bivariate Bezout identities similar to the one in (2.1.7) can be solved.

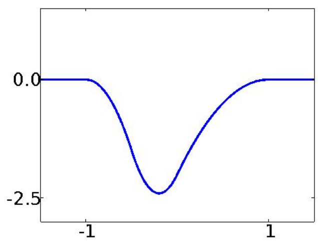

Figure 2.1.1: The wavelet $\psi_{N_{3,\mathbf{q}}}(x) = -3N_3(x+1) - N_3(x)$

2.2 Wavelet Decomposition in the Bivariate Case

Our fundamental result in this section gives a necessary and sufficient condition to hold for wavelet decomposition to be feasible in two variables, based on a refinement mask symbol P as input. In the result below and everywhere in this thesis, the standard notation for the *Kronecker delta* will be used:

$$\delta_\alpha := \begin{cases} 1, & \alpha = 0; \\ 0, & \alpha \in \mathbb{Z}\backslash\{0\}; \end{cases} \tag{2.2.1}$$

$$\delta_{\alpha,\beta} := \begin{cases} 1, & \alpha = \beta = 0; \\ 0, & (\alpha,\beta) \in \mathbb{Z}^2\backslash\{(0,0)\}. \end{cases} \tag{2.2.2}$$

Our result below will rely on a given refinable function having linearly independent integer shifts. We use the standard definition of linearly independent integer shifts, namely that a function f posesses linearly independent integer shifts if, for any sequence $\{c_{i,j}\}$, if $\sum_{i,j} c_{i,j} f(x-i, y-j) = 0, \quad (x,y) \in \mathbb{R}$, then $c_{i,j} = 0$ for all $(i,j) \in \mathbb{Z}^2$. It was shown in Chapter 2 in [4] that all box splines possess this property. In particular, the Courant hat function, which will be our primary object of study, possesses linearly independent integer shifts on $\mathbb{R}^2$.

Theorem 2.2 *Let ϕ be a bivariate refinable function satisfying*

$$\phi(x,y) = \sum_{k,l} p_{k,\ell}\phi(2x-k, 2y-\ell), \quad (x,y) \in \mathbb{R}^2, \tag{2.2.3}$$

such that ϕ possesses linearly independent integer shifts on $\mathbb{R}^2$, and where $\{p_{k,\ell}\} \in M_0(\mathbb{Z}^2)$, with corresponding refinement mask symbol

$$P(z_1,z_2) = \frac{1}{4}\sum_{k,\ell} p_{k,\ell} z_1^k z_2^\ell, \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{2.2.4}$$

For $\alpha \in \{1,2,3\}$ and finitely supported sequences $\mathbf{q}_\alpha = \{q_{k,\ell}^{[\alpha]}\}$, $\{a_{k,\ell}\}$, and $\{b_{k,\ell}^{[\alpha]}\}$, let

$$Q_\alpha(z_1,z_2) := \frac{1}{4}\sum_{k,\ell} q_{k,\ell}^{[\alpha]} z_1^k z_2^\ell, \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}; \tag{2.2.5}$$

$$A(z_1,z_2) := \sum_{k,\ell} a_{k,\ell} z_1^k z_2^\ell, \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}; \tag{2.2.6}$$

$$B_\alpha(z_1,z_2) := \sum_{k,\ell} b_{k,\ell}^{[\alpha]} z_1^k z_2^\ell, \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{2.2.7}$$

Also, for $\alpha \in \{1,2,3\}$, let $\psi_\alpha := \psi_{\phi,\mathbf{q},\alpha}$ be defined by

$$\psi_\alpha(x,y) := \sum_{k,\ell} q_{k,\ell}^{[\alpha]} \phi(2x-k, 2y-\ell), \quad (x,y) \in \mathbb{R}^2. \tag{2.2.8}$$

Then the decomposition relation

$$\phi(2x-i, 2y-j) = \sum_{k,\ell} a_{2k-i,2\ell-j}\phi(x-k, y-\ell) + \sum_{\alpha=1}^{3}\left[\sum_{k,\ell} b_{2k-i,2\ell-j}^{[\alpha]}\psi_\alpha(x-k,y-\ell)\right], \quad (x,y) \in \mathbb{R}^2, \tag{2.2.9}$$

holds if and only if the Laurent polynomials A, B_α, P, and Q_α, satisfy the following system *of identities for all* $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$:

$$\begin{aligned} P(z_1,z_2)A(z_1,z_2) &+ P(-z_1,z_2)A(-z_1,z_2) + P(z_1,-z_2)A(z_1,-z_2) \\ &+ P(-z_1,-z_2)A(-z_1,-z_2) = 1; \end{aligned} \tag{2.2.10}$$

$$\begin{aligned} Q_\alpha(z_1,z_2)A(z_1,z_2) &+ Q_\alpha(-z_1,z_2)A(-z_1,z_2) + Q_\alpha(z_1,-z_2)A(z_1,-z_2) \\ &+ Q_\alpha(-z_1,-z_2)A(-z_1,-z_2) = 0, \quad \alpha \in \{1,2,3\}; \end{aligned} \tag{2.2.11}$$

$$\begin{aligned} P(z_1,z_2)B_\beta(z_1,z_2) &+ P(-z_1,z_2)B_\beta(-z_1,z_2) + P(z_1,-z_2)B_\beta(z_1,-z_2) \\ &+ P(-z_1,-z_2)B_\beta(-z_1,-z_2) = 0, \quad \beta \in \{1,2,3\}; \end{aligned} \tag{2.2.12}$$

$$\begin{aligned} Q_\alpha(z_1,z_2)B_\beta(z_1,z_2) &+ Q_\alpha(-z_1,z_2)B_\beta(-z_1,z_2) + Q_\alpha(z_1,-z_2)B_\beta(z_1,-z_2) \\ &+ Q_\alpha(-z_1,-z_2)B_\beta(-z_1,-z_2) = \delta_{\alpha-\beta}, \quad \alpha \in \{1,2,3\}, \ \beta \in \{1,2,3\}. \end{aligned} \tag{2.2.13}$$

Remark. Note that the decomposition relation in (2.2.9) has the same form as in (2.1.6) for the univariate case, whereas, for the bivariate case, one needs to construct *three* wavelets ψ_1, ψ_2, and ψ_3, and therefore needs to find seven Laurent polynomials A, B_1, B_2, B_3, Q_1, Q_2, and Q_3, to satisfy (2.2.10) through (2.2.13), for a given Laurent polynomial P as input.

In [13], attention is focussed on refinable functions with general dilation matrices in (1.2.1), and it is shown that, for any dilation matrix M of nonzero determinant in (1.2.1), the number of wavelets to participate in the decomposition algorithm is equal to $|\det M - 1|$. Note that this agrees with the fact that there are (as will be shown) *three* wavelet "generators" in (2.2.9), whereas $M = 2I$ implies $|\det M - 1| = 4 - 1 = 3$. While it is not the purpose of this thesis to work with general dilation matrices M, it is remarked how, with the aim to accomplish wavelet decomposition techniques for which the number of wavelet generators is as low as possible, it is suggested in [13] to work with dilation matrices M that satisfy

$$|\det M| = 2; \quad M^s = \pm 2I_s, \tag{2.2.14}$$

where s is the number of variables in play, and M^s means the power of the matrix M. It is noted that the well-known *Quincunx* matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, satisfies the conditions in (2.2.14), for the case $s = 2$. However, the work in [13] still relies strongly on the foundations of Fourier analysis. The Quincunx condition was studied somewhat in [18], where certain refinement preservation results were established.

Proof of Theorem 2.2. From the refinability of ϕ and by using (2.2.8), we have, for $(x, y) \in \mathbb{R}^2$,

$$\sum_{k,\ell} a_{2k-i,2\ell-j}\phi(x-k, y-\ell) + \sum_{\alpha=1}^{3}\left[\sum_{k,\ell} b^{[\alpha]}_{2k-i,2\ell-j}\psi_\alpha(x-k, y-\ell)\right]$$

$$\begin{aligned}
&= \sum_{k,\ell} a_{2k-i,2\ell-j}\left[\sum_{u,v} p_{u,v}\phi(2x-2k-u, 2y-2\ell-v)\right] \\
&\qquad + \sum_{\alpha=1}^{3}\left\{\sum_{k,\ell} b^{[\alpha]}_{2k-i,2\ell-j}\left[\sum_{u,v} q^{[\alpha]}_{u,v}\phi(2x-2k-u, 2y-2\ell-v)\right]\right\} \\
&= \sum_{k,\ell} a_{2k-i,2\ell-j}\left[\sum_{u,v} p_{u-2k,v-2\ell}\phi(2x-u, 2y-v)\right] \\
&\qquad + \sum_{\alpha=1}^{3}\left\{\sum_{k,\ell} b^{[\alpha]}_{2k-i,2\ell-j}\left[\sum_{u,v} q^{[\alpha]}_{u-2k,v-2\ell}\phi(2x-u, 2y-v)\right]\right\}
\end{aligned}$$

$$= \sum_{u,v} \left[\sum_{k,\ell} p_{u-2k,v-2\ell} a_{2k-i,2\ell-j} \right] \phi(2x-u, 2y-v)$$

$$+ \sum_{u,v} \left[\sum_{\alpha=1}^{3} \sum_{k,\ell} q^{[\alpha]}_{u-2k,v-2\ell} b^{[\alpha]}_{2k-i,2\ell-j} \right] \phi(2x-u, 2y-v).$$

$$= \sum_{u,v} \left\{ \sum_{k,\ell} p_{u-2k,v-2\ell} a_{2k-i,2\ell-j} + \sum_{\alpha=1}^{3} \sum_{k,\ell} q^{[\alpha]}_{u-2k,v-2\ell} b^{[\alpha]}_{2k-i,2\ell-j} \right\} \phi(2x-u, 2y-v).$$

Since ϕ has linearly independent integer shifts, it follows that (2.2.9) holds if and only if

$$\sum_{k,\ell} p_{u-2k,v-2\ell} a_{2k-i,2\ell-j} + \sum_{\alpha=1}^{3} \sum_{k,\ell} q^{[\alpha]}_{u-2k,v-2\ell} b^{[\alpha]}_{2k-i,2\ell-j} = \delta_{i-u,j-v}, \qquad (u,v),(i,j) \in \mathbb{Z}^2, \tag{2.2.15}$$

which, after also using (2.2.4) and (2.2.5), holds if and only if, for every $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$ and $(i,j) \in \mathbb{Z}^2$,

$$z_1^i z_2^j = \sum_{u,v} \delta_{i-u,j-v} z_1^u z_2^v$$

$$= \sum_{u,v} \left[\sum_{k,\ell} p_{u-2k,v-2\ell} a_{2k-i,2\ell-j} + \sum_{\alpha=1}^{3} \sum_{k,\ell} q^{[\alpha]}_{u-2k,v-2\ell} b^{[\alpha]}_{2k-i,2\ell-j} \right] z_1^u z_2^v$$

$$= z_1^i z_2^j \left\{ \sum_{k,\ell} a_{2k-i,2\ell-j} z_1^{2k-i} z_2^{2\ell-j} \left[\sum_{u,v} p_{u-2k,v-2\ell} z_1^{u-2k} z_2^{v-2\ell} \right] \right.$$

$$\left. + \sum_{\alpha=1}^{3} \left(\sum_{k,\ell} b^{[\alpha]}_{2k-i,2\ell-j} z_1^{2k-i} z_2^{2\ell-j} \left[\sum_{u,v} q^{[\alpha]}_{u-2k,v-2\ell} z_1^{u-2k} z_2^{v-2\ell} \right] \right) \right\}$$

$$
\begin{aligned}
&= z_1^i z_2^j \left\{ \left[\sum_{k,\ell} a_{2k-i,2\ell-j} z_1^{2k-i} z_2^{2\ell-j} \right] \left[\sum_{u,v} p_{u,v} z_1^u z_2^v \right] \right. \\
&\qquad \left. + \sum_{\alpha=1}^{3} \left[\sum_{k,\ell} b^{[\alpha]}_{2k-i,2\ell-j} z_1^{2k-i} z_2^{2\ell-j} \right] \left[\sum_{u,v} q^{[\alpha]}_{u,v} z_1^u z_2^v \right] \right\} \\
&= 4 z_1^i z_2^j \left\{ P(z_1,z_2) \sum_{k,\ell} a_{2k-i,2\ell-j} z_1^{2k-i} z_2^{2\ell-j} \right. \\
&\qquad \left. + \sum_{\alpha=1}^{3} \left[Q_\alpha(z_1,z_2) \sum_{k,\ell} b^{[\alpha]}_{2k-i,2\ell-j} z_1^{2k-i} z_2^{2\ell-j} \right] \right\},
\end{aligned}
$$

which is equivalent to

$$
P(z_1,z_2) \sum_{k,\ell} a_{2k-i,2\ell-j} z_1^{2k-i} z_2^{2\ell-j} \;+\; \sum_{\alpha=1}^{3} \left[Q_\alpha(z_1,z_2) \sum_{k,\ell} b^{[\alpha]}_{2k-i,2\ell-j} z_1^{2k-i} z_2^{2\ell-j} \right] = \frac{1}{4},
$$

$$
(z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \ (i,j) \in \mathbb{Z}^2. \tag{2.2.16}
$$

Note that (2.2.16) is equivalent to the following, for all $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$:

$$\begin{cases} P(z_1,z_2)\displaystyle\sum_{k,\ell} a_{2k-2i,2\ell-2j} z_1^{2k-2i} z_2^{2\ell-2j} + \sum_{\alpha=1}^{3}\left[Q_\alpha(z_1,z_2)\sum_{k,\ell} b^{[\alpha]}_{2k-2i,2\ell-2j} z_1^{2k-2i} z_2^{2\ell-2j}\right] = \dfrac{1}{4}; \\[2ex] P(z_1,z_2)\displaystyle\sum_{k,\ell} a_{2k-2i,2\ell-2j-1} z_1^{2k-2i} z_2^{2\ell-2j-1} \\ \qquad + \displaystyle\sum_{\alpha=1}^{3}\left[Q_\alpha(z_1,z_2)\sum_{k,\ell} b^{[\alpha]}_{2k-2i,2\ell-2j-1} z_1^{2k-2i} z_2^{2\ell-2j-1}\right] = \dfrac{1}{4}; \\[2ex] P(z_1,z_2)\displaystyle\sum_{k,\ell} a_{2k-2i-1,2\ell-2j} z_1^{2k-2i-1} z_2^{2\ell-2j} \\ \qquad + \displaystyle\sum_{\alpha=1}^{3}\left[Q_\alpha(z_1,z_2)\sum_{k,\ell} b^{[\alpha]}_{2k-2i-1,2\ell-2j} z_1^{2k-2i-1} z_2^{2\ell-2j}\right] = \dfrac{1}{4}; \\[2ex] P(z_1,z_2)\displaystyle\sum_{k,\ell} a_{2k-2i-1,2\ell-2j-1} z_1^{2k-2i-1} z_2^{2\ell-2j-1} \\ \qquad + \displaystyle\sum_{\alpha=1}^{3}\left[Q_\alpha(z_1,z_2)\sum_{k,\ell} b^{[\alpha]}_{2k-2i-1,2\ell-2j-1} z_1^{2k-2i-1} z_2^{2\ell-2j-1}\right] = \dfrac{1}{4}, \end{cases}$$

which, for $(z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$, is equivalent to the following set of identities:

$$P(z_1,z_2)\sum_{k,\ell} a_{2k,2\ell} z_1^{2k} z_2^{2\ell} + \sum_{\alpha=1}^{3}\left[Q_\alpha(z_1,z_2)\sum_{k,\ell} b_{2k,2\ell}^{[\alpha]} z_1^{2k} z_2^{2\ell}\right] = \frac{1}{4}; \quad \text{[a]}$$

$$P(z_1,z_2)\sum_{k,\ell} a_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1} + \sum_{\alpha=1}^{3}\left[Q_\alpha(z_1,z_2)\sum_{k,\ell} b_{2k,2\ell+1}^{[\alpha]} z_1^{2k} z_2^{2\ell+1}\right] = \frac{1}{4}; \quad \text{[b]}$$

$$P(z_1,z_2)\sum_{k,\ell} a_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell} + \sum_{\alpha=1}^{3}\left[Q_\alpha(z_1,z_2)\sum_{k,\ell} b_{2k+1,2\ell}^{[\alpha]} z_1^{2k+1} z_2^{2\ell}\right] = \frac{1}{4}; \quad \text{[c]}$$

$$P(z_1,z_2)\sum_{k,\ell} a_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1} + \sum_{\alpha=1}^{3}\left[Q_\alpha(z_1,z_2)\sum_{k,\ell} b_{2k+1,2\ell+1}^{[\alpha]} z_1^{2k+1} z_2^{2\ell+1}\right] = \frac{1}{4}. \quad \text{[d]} \tag{2.2.17}$$

Next, the identities in (2.2.17) are added together in all possible groups of two identities at a time, so as to preserve equivalence. More particularly, in (2.2.17), we take, respectively, the following linear combinations of the identities in (2.2.17):

[A] := [a] + [d], [B] := [b] + [c], [C] := [a] + [b], [D] := [c] + [d], [E] := [a] + [c], and [F] := [b] + [d].

Since the inverse operations, namely,

$\text{[a]} = \dfrac{\text{[A]}+\text{[C]}+\text{[E]}}{2}$, $\text{[b]} = \dfrac{-\text{[A]}+2\text{[B]}+\text{[C]}-\text{[E]}}{2}$, $\text{[c]} = \dfrac{\text{[A]}+\text{[E]}-\text{[C]}}{2}$, and $\text{[d]} = \dfrac{\text{[A]}-\text{[E]}-\text{[C]}}{2}$,

hold, it follows that (2.2.17) is equivalent to the set of identities [A] through [F] given in the following, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$:

$$\begin{aligned}
&P(z_1,z_2)\left[2\sum_{k,\ell} a_{2k,2\ell} z_1^{2k} z_2^{2\ell} + 2\sum_{k,\ell} a_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right] \\
&\quad + \sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[2\sum_{k,\ell} b^{[\alpha]}_{2k,2\ell} z_1^{2k} z_2^{2\ell} + 2\sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right]\right) = 1; \qquad [A]
\end{aligned}$$

$$\begin{aligned}
&P(z_1,z_2)\left[2\sum_{k,\ell} a_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1} + 2\sum_{k,\ell} a_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell}\right] \\
&\quad + \sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[2\sum_{k,\ell} b^{[\alpha]}_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1} + 2\sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell}\right]\right) = 1; \qquad [B]
\end{aligned}$$

$$\begin{aligned}
&P(z_1,z_2)\left[2\sum_{k,\ell} a_{2k,2\ell} z_1^{2k} z_2^{2\ell} + 2\sum_{k,\ell} a_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1}\right] \\
&\quad + \sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[2\sum_{k,\ell} b^{[\alpha]}_{2k,2\ell} z_1^{2k} z_2^{2\ell} + 2\sum_{k,\ell} b^{[\alpha]}_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1}\right]\right) = 1; \qquad [C]
\end{aligned}$$

$$\begin{aligned}
&P(z_1,z_2)\left[2\sum_{k,\ell} a_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell} + 2\sum_{k,\ell} a_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right] \\
&\quad + \sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[2\sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell} + 2\sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right]\right) = 1; \qquad [D]
\end{aligned}$$

$$\begin{aligned}
&P(z_1,z_2)\left[2\sum_{k,\ell} a_{2k,2\ell} z_1^{2k} z_2^{2\ell} + 2\sum_{k,\ell} a_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell}\right] \\
&\quad + \sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[2\sum_{k,\ell} b^{[\alpha]}_{2k,2\ell} z_1^{2k} z_2^{2\ell} + 2\sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell}\right]\right) = 1; \qquad [E]
\end{aligned}$$

$$\begin{aligned}
&P(z_1,z_2)\left[2\sum_{k,\ell} a_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1} + 2\sum_{k,\ell} a_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right] \\
&\quad + \sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[2\sum_{k,\ell} b^{[\alpha]}_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1} + 2\sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right]\right) = 1. \qquad [F]
\end{aligned}$$

(2.2.18)

Next, observe that the first identity in (2.2.18) holds if and only if, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$,

$$P(z_1,z_2)\left[A(z_1,z_2)+A(-z_1,-z_2)\right]+\sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[B_\alpha(z_1,z_2)+B_\alpha(-z_1,-z_2)\right]\right)$$

$$\begin{aligned}
= \;& P(z_1,z_2)\left\{\left[\sum_{k,\ell} a_{2k,2\ell} z_1^{2k} z_2^{2\ell} + \sum_{k,\ell} a_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1}\right.\right.\\
&\qquad \left. + \sum_{k,\ell} a_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell} + \sum_{k,\ell} a_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right]\\
&\qquad + \left[\sum_{k,\ell} a_{2k,2\ell} z_1^{2k} z_2^{2\ell} - \sum_{k,\ell} a_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1}\right.\\
&\qquad \left.\left. - \sum_{k,\ell} a_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell} + \sum_{k,\ell} a_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right]\right\}\\
& + \sum_{\alpha=1}^{3} Q_\alpha \left\{\left[\sum_{k,\ell} b^{[\alpha]}_{2k,2\ell} z_1^{2k} z_2^{2\ell} + \sum_{k,\ell} b^{[\alpha]}_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1}\right.\right.\\
&\qquad \left. + \sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell} + \sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right]\\
&\qquad + \left[\sum_{k,\ell} b^{[\alpha]}_{2k,2\ell} z_1^{2k} z_2^{2\ell} - \sum_{k,\ell} b^{[\alpha]}_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1}\right.\\
&\qquad \left.\left. - \sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell} z_1^{2k+1} z_2^{2\ell} + \sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right]\right\}\\
= \;& P(z_1,z_2)\left\{2\sum_{k,\ell} a_{2k,2\ell} z_1^{2k} z_2^{2\ell} + 2\sum_{k,\ell} a_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right\}\\
&\qquad + \sum_{\alpha=1}^{3} Q_\alpha(z_1,z_2)\left\{2\sum_{k,\ell} b^{[\alpha]}_{2k,2\ell} z_1^{2k} z_2^{2\ell} + 2\sum_{k,\ell} b^{[\alpha]}_{2k+1,2\ell+1} z_1^{2k+1} z_2^{2\ell+1}\right\}\\
= \;& 1.
\end{aligned}$$

In a similar way, all the identities in (2.2.18) hold if and only if, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$,

$$\begin{aligned}
P(z_1,z_2)\left[A(z_1,z_2)+A(-z_1,-z_2)\right]+\sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[B_\alpha(z_1,z_2)+B_\alpha(-z_1,-z_2)\right]\right) &= 1; \quad [1]\\
P(z_1,z_2)\left[A(z_1,z_2)-A(-z_1,-z_2)\right]+\sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[B_\alpha(z_1,z_2)-B_\alpha(-z_1,-z_2)\right]\right) &= 1; \quad [2]\\
P(z_1,z_2)\left[A(z_1,z_2)+A(-z_1,z_2)\right]+\sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[B_\alpha(z_1,z_2)+B_\alpha(-z_1,z_2)\right]\right) &= 1; \quad [3]\\
P(z_1,z_2)\left[A(z_1,z_2)-A(-z_1,z_2)\right]+\sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[B_\alpha(z_1,z_2)-B_\alpha(-z_1,z_2)\right]\right) &= 1; \quad [4]\\
P(z_1,z_2)\left[A(z_1,z_2)+A(z_1,-z_2)\right]+\sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[B_\alpha(z_1,z_2)+B_\alpha(z_1,-z_2)\right]\right) &= 1; \quad [5]\\
P(z_1,z_2)\left[A(z_1,z_2)+A(z_1,-z_2)\right]+\sum_{\alpha=1}^{3}\left(Q_\alpha(z_1,z_2)\left[B_\alpha(z_1,z_2)-B_\alpha(z_1,-z_2)\right]\right) &= 1. \quad [6]
\end{aligned} \tag{2.2.19}$$

Note that, in (2.2.19), the simultaneous identities [1] and [2] are equivalent to

$$\begin{cases} P(z_1,z_2)A(z_1,z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(z_1,z_2) &= 1; \\ P(z_1,z_2)A(-z_1,-z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(-z_1,-z_2) &= 0. \end{cases} \tag{2.2.20}$$

Similarly, the simultaneous identities [3] and [4] are equivalent to

$$\begin{cases} P(z_1,z_2)A(z_1,z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(z_1,z_2) &= 1; \\ P(z_1,z_2)A(-z_1,z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(-z_1,z_2) &= 0, \end{cases} \tag{2.2.21}$$

whereas the simultaneous identities [5] and [6] are equivalent to

$$\begin{cases} P(z_1,z_2)A(z_1,z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(z_1,z_2) &= 1; \\ \\ P(z_1,z_2)A(z_1,-z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(z_1,-z_2) &= 0. \end{cases} \tag{2.2.22}$$

It follows from (2.2.20) – (2.2.22) that the process of simultaneously solving the six identities in (2.2.19) is equivalent to solving the following system of identities, for $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$:

$$\begin{cases} P(z_1,z_2)A(z_1,z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(z_1,z_2) &= 1; \\ \\ P(z_1,z_2)A(-z_1,-z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(-z_1,-z_2) &= 0; \\ \\ P(z_1,z_2)A(-z_1,z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(-z_1,z_2) &= 0; \\ \\ P(z_1,z_2)A(z_1,-z_2)+\sum_{\alpha=1}^{3}Q_\alpha(z_1,z_2)B_\alpha(z_1,-z_2) &= 0. \end{cases} \tag{2.2.23}$$

By applying the transformations $(z_1,z_2)\to(-z_1,-z_2)$, $(z_1,z_2)\to(-z_1,z_2)$, and $(z_1,z_2)\to(z_1,-z_2)$ to each of the equations in (2.2.23), we deduce that (2.2.23) holds if and only if, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$,

$$\left\{\begin{array}{lcl}
P(-z_1,-z_2)A(-z_1,-z_2) + \sum_{\alpha=1}^{3} Q_\alpha(-z_1,-z_2)B_\alpha(-z_1,-z_2) & = & 1; \\
P(-z_1,-z_2)A(z_1,z_2) + \sum_{\alpha=1}^{3} Q_\alpha(-z_1,-z_2)B_\alpha(z_1,z_2) & = & 0; \\
P(-z_1,z_2)A(-z_1,z_2) + \sum_{\alpha=1}^{3} Q_\alpha(-z_1,z_2)B_\alpha(-z_1,z_2) & = & 1; \\
P(-z_1,z_2)A(z_1,-z_2) + \sum_{\alpha=1}^{3} Q_\alpha(-z_1,z_2)B_\alpha(z_1,-z_2) & = & 0; \\
P(z_1,-z_2)A(z_1,-z_2) + \sum_{\alpha=1}^{3} Q_\alpha(z_1,-z_2)B_\alpha(z_1,-z_2) & = & 1; \\
P(z_1,-z_2)A(-z_1,z_2) + \sum_{\alpha=1}^{3} Q_\alpha(z_1,-z_2)B_\alpha(-z_1,z_2) & = & 0; \\
P(-z_1,-z_2)A(z_1,-z_2) + \sum_{\alpha=1}^{3} Q_\alpha(-z_1,-z_2)B_\alpha(z_1,-z_2) & = & 0; \\
P(-z_1,z_2)A(z_1,z_2) + \sum_{\alpha=1}^{3} Q_\alpha(-z_1,z_2)B_\alpha(z_1,z_2) & = & 0; \\
P(z_1,-z_2)A(-z_1,-z_2) + \sum_{\alpha=1}^{3} Q_\alpha(z_1,-z_2)B_\alpha(-z_1,-z_2) & = & 0; \\
P(-z_1,-z_2)A(-z_1,z_2) + \sum_{\alpha=1}^{3} Q_\alpha(-z_1,-z_2)B_\alpha(-z_1,z_2) & = & 0; \\
P(-z_1,z_2)A(-z_1,-z_2) + \sum_{\alpha=1}^{3} Q_\alpha(-z_1,z_2)B_\alpha(-z_1,-z_2) & = & 0; \\
P(z_1,-z_2)A(z_1,z_2) + \sum_{\alpha=1}^{3} Q_\alpha(z_1,-z_2)B_\alpha(z_1,z_2) & = & 0.
\end{array}\right. \tag{2.2.24}$$

The matrix identity $MN = I_4$ now follows from the simultaneous identities in (2.2.23) and (2.2.24), where, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$,

$$M := M(z_1,z_2) = \begin{bmatrix} P(z_1,z_2) & Q_1(z_1,z_2) & Q_2(z_1,z_2) & Q_3(z_1,z_2) \\ P(-z_1,-z_2) & Q_1(-z_1,-z_2) & Q_2(-z_1,-z_2) & Q_3(-z_1,-z_2) \\ P(-z_1,z_2) & Q_1(-z_1,z_2) & Q_2(-z_1,z_2) & Q_3(-z_1,z_2) \\ P(z_1,-z_2) & Q_1(z_1,-z_2) & Q_2(z_1,-z_2) & Q_3(z_1,-z_2) \end{bmatrix};$$

$$N := N(z_1,z_2) = \begin{bmatrix} A(z_1,z_2) & A(-z_1,-z_2) & A(-z_1,z_2) & A(z_1,-z_2) \\ B_1(z_1,z_2) & B_1(-z_1,-z_2) & B_1(-z_1,z_2) & B_1(z_1,-z_2) \\ B_2(z_1,z_2) & B_2(-z_1,-z_2) & B_2(-z_1,z_2) & B_2(z_1,-z_2) \\ B_3(z_1,z_2) & B_3(-z_1,-z_2) & B_3(-z_1,z_2) & B_3(z_1,-z_2) \end{bmatrix},$$

and where I_4 is the 4×4 identity matrix. According to a standard result in Linear Algebra, the identity

$$MN = I_4, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{2.2.25}$$

holds if and only if $NM = I_4$, with I_4 denoting the 4×4 identity matrix. Hence, with the 4×4 matrices M and N defined above, it follows from (2.2.24) that

$$NM = I_4, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{2.2.26}$$

The latter matrix identity is equivalent to the identities in (2.2.10) through (2.2.13). ■

The identities (2.2.10) through (2.2.13) of which the existence of a solution is equivalent to feasibility of wavelet decomposition for a given refinement mask symbol according to

Theorem 2.2, are known as *Bezout identities*. Throughout the rest of this thesis, attention will be restricted to the case where P is the refinement mask symbol corresponding to the Courant hat function of Example 1.2, as given by (1.3.17). In the next chapter, a particular solution set to the identities (2.2.10) through (2.2.13) will be given, and in the subsequent chapters a more general approach for solving these identities will be suggested. First, we conclude this chapter with some background about different approaches to wavelet construction in the bivariate setting.

2.3 Other Approaches to Wavelet Construction

2.3.1 Wavelet Construction With a Dual Chain Method

In [3], a different approach is suggested for constructing wavelets in the univariate case, and which is applicable to refinable functions with arbitrary dilation factor (whereas our work is focussed on the case where the dilation factor is 2 in the univariate case, or $2I$ in the bivariate case, in (1.2.2)). By finding an initial *dual* refinement mask corresponding to a given refinement mask, the *dual chain method* produces a chain of successive refinement masks, where at each step the support of the given mask is strictly smaller than the support of its predecessor. Finally, a mask with only one element is obtained, for which wavelet construction is trivial, and an analogous chain of dual masks is then constructed so as to be certain of an end product comprising a set of wavelets corresponding to the original refinement mask. The title of this method, "dual chain," is therefore signifying of both the fact that there are two chains (a "downward chain", as well as an "upward chain"), and the fact that all of the refinement masks in the chain are dual to each other, in a sense made precise in e.g., [3] and [8]. (The main idea of duality in this context is that the basic properties of multi-resolution analysis must be preserved from one function to the next for them to be dual; the precise definition relies, as expected, on the foundation of refinability of the functions, but will not be discussed here.) The results in [3] have not yet been proved for the bivariate case, and would certainly provide an elegant method for constructing bivariate wavelets if it can be achieved. Some work in the univariate and bivariate settings of *dual refinable functions* has been done by Han in, e.g., [8].

The crucial idea behind the dual chain method, as mentioned above, is that, given a refinement mask (i.e., the sequence or its corresponding Laurent polynomial symbol), another mask needs to be constructed with support smaller than that of the original one, and such that these two masks satisfy a certain duality condition. And since wavelet construction is the ultimate goal, the Bezout identities in Chapter 2 remain of the essence. Particularly, one is interested in finding a set of Laurent polynomials A (in the bivariate case) such that (2.2.10) is to be satisfied, and moreover, such that each Laurent polynomial A_i has support smaller than that of A_j, $i < j$. Now, note from (2.2.10) that, given an "initial" Laurent polynomial A that satisfies (2.2.10), any Laurent polynomial B (say, as the solution (6.2.14) that will be derived in Chapter 6) that satisfies (2.2.12) lets the identity (2.2.10) remain solved when B is added to A. Hence, if one can find a systematic approach of finding a set of Laurent polynomials B such that the polynomial $A+B$ is of smaller support than A, then one will have established a "chain" of Laurent polynomials, all of which satisfy the identity (2.2.10), and of subsequently decreasing support. One would attempt to bring the support of a Laurent polynomial in this chain to an absolute minimum, so as to ensure a systematic approach for constructing a wavelet (known as a *lazy* wavelet in the univariate case) from this "smallest Laurent polynomial" A.

At this stage, the literature is not yet sufficiently developed in order to complete this process in the bivariate setting, one of the reasons being that there is not yet a consistent definition of the lazy wavelet for the bivariate case. Another reason is that, whereas the support of a univariate refinement sequence is always an interval and therefore has a fixed "length," in the bivariate case the "support" of a refinement sequence is on the two-dimensional plane (or grid), and one has to establish what one means by "smallest support." These technical obstacles contribute greatly to the level of difficulty of constructing a dual chain, and the challenge of constructing an efficient dual chain method for the bivariate setting therefore remains an open problem; the work in this thesis will continue with the Bezout identity approach as described in Section 2.2.

2.3.2 Tensor Product Wavelets

Let $\phi^{[1]}$ and $\phi^{[2]}$ be univariate refinable functions with corresponding refinement mask symbols $P^{[1]}(z) = \frac{1}{2}\sum_k p_k^{[1]} z^k$ and $P^{[2]}(z) = \frac{1}{2}\sum_k p_k^{[2]} z^k$, $z \in \mathbb{C}\backslash\{0\}$, respectively. Then, a bivariate function can be constructed from $\phi^{[1]}$ and $\phi^{[2]}$ by

$$\phi(x,y) := \phi^{[1]}(x)\phi^{[2]}(y), \quad (x,y) \in \mathbb{R}^2. \tag{2.3.1}$$

It is shown in [2] that the function ϕ is a refinable function with respect to the dilation matrix $2I_2$, and with corresponding mask sequence $\mathbf{p}$ obtained by

$$p_{k,\ell} = p_k^{[1]} p_\ell^{[2]}, \quad (k,\ell) \in \mathbb{Z}^2. \tag{2.3.2}$$

Note that, in terms of the corresponding mask symbol P, (2.3.2) is equivalent to

$$\begin{aligned}
P(z_1,z_2) &= \frac{1}{4}\sum_{k,\ell} p_{k,\ell} z_1^k z_2^\ell \\
&= \frac{1}{4}\sum_{k,\ell} p_k^{[1]} p_\ell^{[2]} z_1^k z_2^\ell \\
&= \frac{1}{2}\sum_{\ell}\left[\frac{1}{2}\sum_k p_k^{[1]} z_1^k\right] p_\ell^{[2]} z_2^\ell \\
&= P^{[1]}(z_1)\left[\frac{1}{2}\sum_\ell p_\ell^{[2]} z_2^\ell\right] \\
&= P^{[1]}(z_1)P^{[2]}(z_2), \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}.
\end{aligned} \tag{2.3.3}$$

If, moreover, there exists for each respective refinable function $\phi^{[\alpha]}$ a corresponding univariate synthesis wavelet $\psi^{[\alpha]}$, where $\alpha \in \{1,2\}$, then it is also shown in [2] that a bivariate wavelet system consisting of three wavelet generators can be formed corresponding to the refinable function ϕ, as follows:

$$\left.\begin{array}{rcl}\psi_1(x,y) & := & \phi^{[1]}(x)\psi^{[2]}(y);\\ \psi_2(x,y) & := & \psi^{[1]}(x)\phi^{[2]}(y);\\ \psi_3(x,y) & := & \psi^{[1]}(x)\psi^{[2]}(y),\end{array}\right\} (x,y)\in\mathbb{R}^2. \tag{2.3.4}$$

The function ϕ is called a *tensor-product refinable function,* whereas the set of functions ψ_1, ψ_2, and ψ_3, in (2.3.4), is called the corresponding *tensor-product bivariate wavelet system*. In the following, we briefly explain how the wavelets $\psi_1 - \psi_3$ indeed satisfy the wavelet decomposition result in (2.2.9).

Note that, for $\alpha \in \{1, 2\}$, the existence of a synthesis wavelet $\psi^{[\alpha]}$ implies the existence of corresponding univariate Laurent polynomials $A^{[\alpha]}$, $B^{[\alpha]}$, and $Q^{[\alpha]}$, such that the univariate Bezout identities in (2.1.7) are satisfied. The corresponding bivariate Laurent polynomials corresponding to the bivariate wavelet decomposition scheme, are now defined as follows:

$$\left.\begin{array}{rcl}A(z_1,z_2) & := & A^{[1]}(z_1)A^{[2]}(z_2);\\ B_1(z_1,z_2) & := & A^{[1]}(z_1)B^{[2]}(z_2);\\ B_2(z_1,z_2) & := & B^{[1]}(z_1)A^{[2]}(z_2);\\ B_3(z_1,z_2) & := & B^{[1]}(z_1)B^{[2]}(z_2);\\ Q_1(z_1,z_2) & := & P^{[1]}(z_1)Q^{[2]}(z_2);\\ Q_2(z_1,z_2) & := & Q^{[1]}(z_1)P^{[2]}(z_2);\\ Q_3(z_1,z_2) & := & Q^{[1]}(z_1)Q^{[2]}(z_2),\end{array}\right\} (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{2.3.5}$$

It follows from (2.3.5) and (2.1.7) that

$$\begin{aligned}
& P(z_1,z_2)A(z_1,z_2)+P(z_1,-z_2)A(z_1,-z_2)+P(-z_1,z_2)A(-z_1,z_2) \\
& \qquad +P(-z_1,-z_2)A(-z_1,-z_2) \\
&= P_1(z_1)P_2(z_2)A_1(z_1)A_2(z_2)+P_1(z_1)P_2(-z_2)A_1(z_1)A_2(-z_2) \\
& \qquad +P_1(-z_1)P_2(z_2)A_1(-z_1)A_2(z_2)+P_1(-z_1)P_2(-z_2)A_1(-z_1)A_2(-z_2) \\
&= P_1(z_1)A_1(z_1)\left[P_2(z_2)A_2(z_2)+P_2(-z_2)A_2(-z_2)\right] \\
& \qquad +P_1(-z_1)A_1(-z_1)\left[P_2(z_2)A_2(z_2)+P_2(-z_2)A_2(-z_2)\right] \\
&= P_1(z_1)A_1(z_1)\,[1]+P_1(-z_1)A_1(-z_1)\,[1] \\
&= 1, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\},
\end{aligned}$$

so that the first identity (2.2.10) in the bivariate Bezout system is indeed satisfied. In a similar way, it follows that all of the bivariate Laurent polynomials in (2.3.5) satisfy the Bezout identities in (2.2.10) through (2.2.13), so that the wavelet decomposition (2.2.9) indeed holds.

Example 2.2 Let N_1 be the cardinal B-spline given by (1.3.4), and of which the corresponding wavelet (with no vanishing moments) is the mother wavelet function given by (1.2.5). Let $\phi^{[1]}=\phi^{[2]}:=N_1$. It then follows from (1.2.18), (1.2.16), (2.1.1), and (2.1.5), that the refinement mask symbols $P^{[1]}$ and $P^{[2]}$, and the Laurent polynomials $Q^{[1]}$ and $Q^{[2]}$, are given by

$$\left.\begin{aligned}
P^{[1]}(z)=P^{[2]}(z) &= \frac{1}{2}(1+z); \\
Q^{[1]}(z)=Q^{[2]}(z) &= \frac{1}{2}(1-z),
\end{aligned}\right\} z\in\mathbb{C}. \tag{2.3.6}$$

The tensor product refinable function $\phi(x,y):=\phi^{[1]}(x)\phi^{[2]}(y),\ \ (x,y)\in\mathbb{R}^2$, is then the Haar box spline in (1.3.13) of Example 1.1, and, as follows from (2.3.3) and (2.3.5), with corresponding refinement mask symbol

$$P(z_1, z_2) = P^{[1]}(z_1)P^{[2]}(z_2) = \frac{1}{4}(1 + z_1)(1 + z_2) = \frac{1}{4}(1 + z_1 + z_2 + z_1 z_2), \quad (z_1, z_2) \in \mathbb{C}^2, \tag{2.3.7}$$

which corresponds with (1.3.11). It also follows from (2.3.5) and (2.3.6) that the Laurent polynomials Q_1 through Q_3 that govern construction of the three corresponding wavelets ψ_1 through ψ_3 in the bivariate wavelet system, are given by

$$\left.\begin{aligned} Q_1(z_1, z_2) &= P^{[1]}(z_1)Q^{[2]}(z_2) = \frac{1}{4}(1 + z_1)(1 - z_2) = \frac{1}{4}(1 + z_1 - z_2 - z_1 z_2); \\ Q_2(z_1, z_2) &= Q^{[1]}(z_1)P^{[2]}(z_2) = \frac{1}{4}(1 - z_1)(1 + z_2) = \frac{1}{4}(1 - z_1 + z_2 - z_1 z_2); \\ Q_3(z_1, z_2) &= Q^{[1]}(z_1)Q^{[2]}(z_2) = \frac{1}{4}(1 - z_1)(1 - z_2) = \frac{1}{4}(1 - z_1 - z_2 + z_1 z_2), \end{aligned}\right\} (z_1, z_2) \in \mathbb{C}^2; \tag{2.3.8}$$

that is, the corresponding wavelet coefficients $\{\mathbf{q}_\alpha\} := \{q_{(\alpha),k,\ell}\}_{k,\ell\in\mathbb{Z}}$, are given for $\alpha = 1, 2,$ and 3, by

$$\begin{cases} \{q_{(1),0,0}, q_{(1),0,1}, q_{(1),1,0}, q_{(1),1,1}\} = \{1, -1, 1, -1\}\,; \\ q_{(1)k,\ell} = 0, \quad (k, \ell) \in \mathbb{Z}^2 \backslash \{(0, 0), (0, 1), (1, 0), (1, 1)\}; \end{cases} \tag{2.3.9}$$

$$\begin{cases} \{q_{(2),0,0}, q_{(2),0,1}, q_{(2),1,0}, q_{(2),1,1}\} = \{1, 1, -1, -1\}\,; \\ q_{(2)k,\ell} = 0, \quad (k, \ell) \in \mathbb{Z}^2 \backslash \{(0, 0), (0, 1), (1, 0), (1, 1)\}; \end{cases} \tag{2.3.10}$$

$$\begin{cases} \{q_{(3),0,0}, q_{(3),0,1}, q_{(3),1,0}, q_{(3),1,1}\} = \{1, -1, -1, 1\}\,; \\ q_{(3)k,\ell} = 0, \quad (k, \ell) \in \mathbb{Z}^2 \backslash \{(0, 0), (0, 1), (1, 0), (1, 1)\}, \end{cases} \tag{2.3.11}$$

where we use the usual definition $Q_\alpha(z_1, z_2) := \frac{1}{2}\sum_{k,\ell} q_{(\alpha),k,\ell} z_1^k z_2^\ell$. It finally follows from (2.2.8) that the three wavelets ψ_1, ψ_2, and ψ_3, corresponding to the Haar function, are given by

$$\left.\begin{aligned} \psi_1(x, y) &= \phi(2x, 2y) + \phi(2x - 1, 2y) - \phi(2x, 2y - 1) - \phi(2x - 1, 2y - 1); \\ \psi_2(x, y) &= \phi(2x, 2y) - \phi(2x - 1, 2y) + \phi(2x, 2y - 1) - \phi(2x - 1, 2y - 1); \\ \psi_3(x, y) &= \phi(2x, 2y) - \phi(2x - 1, 2y) - \phi(2x, 2y - 1) + \phi(2x - 1, 2y - 1), \end{aligned}\right\} (x, y) \in \mathbb{R}^2. \tag{2.3.12}$$

The Haar function was illustrated in Figure 1.3.1. We illustrate the three wavelets ψ_1, ψ_2, and ψ_3, in Figures 2.3.1 – 2.3.3 below.

Not all bivariate refinable functions can be written as the tensor product of two univari-

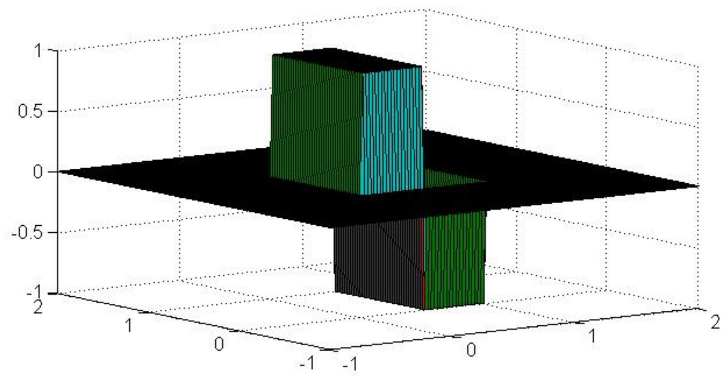

Figure 2.3.1: The Haar wavelet ψ_1

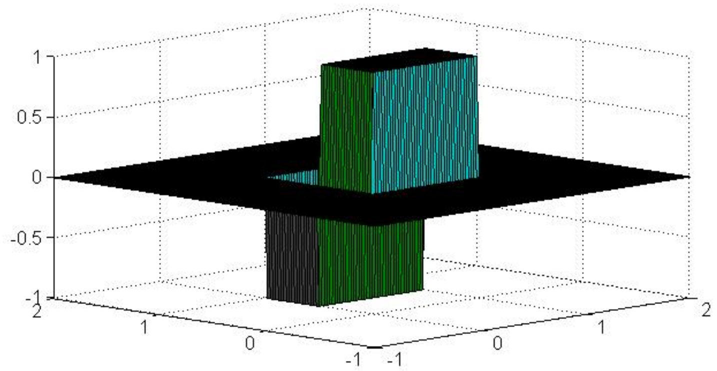

Figure 2.3.2: The Haar wavelet ψ_2

ate refinable functions, which makes it necessary to investigate construction of bivariate wavelets without the use of tensor products. A more intuitive reason is that tensor product wavelets act as filters that only filter out noise in the two partial directions of a data set defined on the two-directional plane. The purpose is to construct wavelets that do not depend on two univariate functions defined on orthogonal domains in the two-directional plane, in order to filter through noise on the plane in other directions as well. The rest of this work will revisit the Bezout system in Section 2.2 with the purpose of finding solutions to the identities (2.2.10) through (2.2.13). The next chapter will give a particular solution to the Bezout system for the case where the refinable function is the Courant hat function of Example 1.2.

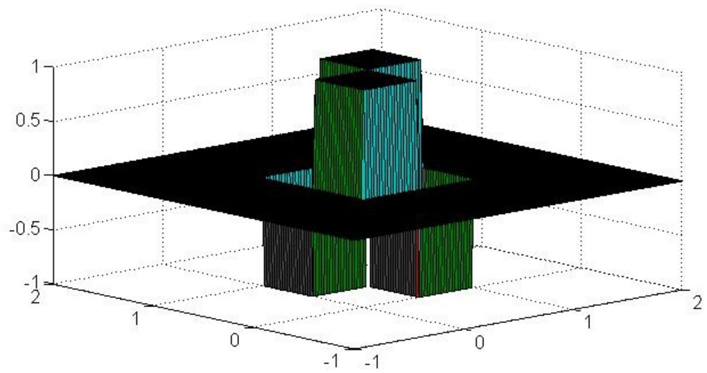

Figure 2.3.3: The Haar wavelet ψ_3

Chapter 3

Constructing a Courant Hat Function Wavelet: A Particular Solution to the Bezout Identities

3.1 Deriving a Solution to the Bezout Identities

It will be the purpose of this chapter to provide a particular solution to the Bezout identities equivalent to wavelet decomposition, as in (2.2.10) through (2.2.13), for the case where the refinable function ϕ is the Courant hat function $B_{1,1,1}$ in Example 1.2, with corresponding refinement mask symbol as given by (1.3.17).

To this end, note that, after setting P fixed as in (1.3.17), and by choosing

$$A(z_1, z_2) := \frac{1}{z_1 z_2}, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{3.1.1}$$

we have

$$
\begin{aligned}
&P(z_1,z_2)A(z_1,z_2)+P(z_1,-z_2)A(z_1,-z_2)+P(-z_1,z_2)A(-z_1,z_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad +P(-z_1,-z_2)A(-z_1,-z_2)\\
&= \frac{1}{8}\frac{1}{z_1z_2}\left[(1+z_1)(1+z_2)(1+z_1z_2)-(1+z_1)(1-z_2)(1-z_1z_2)\right.\\
&\qquad\qquad \left.-(1-z_1)(1+z_2)(1-z_1z_2)+(1-z_1)(1-z_2)(1+z_1z_2)\right]\\
&= \frac{1}{8}\frac{1}{z_1z_2}\left[2(1+z_1z_2)^2-2(1-z_1z_2)^2\right]\\
&= \frac{1}{4}\frac{1}{z_1z_2}\left[4z_1z_2\right]=1, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}.
\end{aligned}
$$

Hence, the Laurent polynomial $A(z_1,z_2)$ given by (3.1.1) is a particular solution to (2.2.10) in the Courant hat function case.

Next, choose the Laurent polynomial $B_1(z_1,z_2) := \frac{1}{z_1}(1-z_1)(1-z_1z_2), \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$, in which case, with P as in (1.3.17), we have

$$
B_1(z_1,z_2)P(z_1,z_2)=\frac{1}{8}\frac{1}{z_1}(1-z_1^2)(1-z_1^2z_2^2)(1+z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\},
$$

so that

$$
\begin{aligned}
&P(z_1,z_2)B_1(z_1,z_2)+P(z_1,-z_2)B_1(z_1,-z_2)+P(-z_1,z_2)B_1(-z_1,z_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad +P(-z_1,-z_2)B_1(-z_1,-z_2)\\
&= \frac{1}{8}\frac{1}{z_1}(1-z_1^2)(1-z_1^2z_2^2)\left[(1+z_2)+(1-z_2)-(1+z_2)-(1-z_2)\right]\\
&= 0, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\},
\end{aligned}
$$

i.e., the Laurent polynomial B_1 is a particular solution to (2.2.12) in the Courant hat function case. Similarly, choose the Laurent polynomials $B_2(z_1, z_2) := \frac{1}{z_1}(1-z_1)(1-z_2)$ and $B_3(z_1, z_2) := \frac{1}{z_1 z_2}(1-z_2)(1-z_1 z_2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, so that

$$B_2(z_1, z_2)P(z_1, z_2) = \frac{1}{8}\frac{1}{z_1}(1-z_1^2)(1-z_2^2)(1+z_1 z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\};$$

$$B_3(z_1, z_2)P(z_1, z_2) = \frac{1}{8}\frac{1}{z_1 z_2}(1-z_2^2)(1-z_1^2 z_2^2)(1+z_1), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\},$$

and in which case we get

$$\begin{aligned} & P(z_1, z_2)B_2(z_1, z_2) + P(z_1, -z_2)B_2(z_1, -z_2) + P(-z_1, z_2)B_2(-z_1, z_2) \\ & \qquad + P(-z_1, -z_2)B_2(-z_1, -z_2) \\ = {} & \frac{1}{8}\frac{1}{z_1}(1-z_1^2)(1-z_2^2)\left[(1+z_1 z_2) + (1-z_1 z_2) - (1-z_1 z_2) - (1+z_1 z_2)\right] \\ = {} & 0, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \end{aligned}$$

and

$$\begin{aligned} & P(z_1, z_2)B_3(z_1, z_2) + P(z_1, -z_2)B_3(z_1, -z_2) + P(-z_1, z_2)B_3(-z_1, z_2) \\ & \qquad + P(-z_1, -z_2)B_3(-z_1, -z_2) \\ = {} & \frac{1}{8}\frac{1}{z_1 z_2}(1-z_2^2)(1-z_1^2 z_2^2)\left[(1+z_1) - (1+z_1) - (1-z_1) + (1-z_1)\right] \\ = {} & 0, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \end{aligned}$$

It follows that the Laurent polynomials B_2 and B_3 are both also particular solutions to the Bezout identity (2.2.12) in the Courant hat function case.

We proceed to show how a particular set of solutions Q_1, Q_2, and Q_3, exists such that both the identities (2.2.11) and (2.2.13) are satisfied. Let us first choose $Q_1(z_1, z_2) :=$

$-\frac{1}{8}\frac{1}{z_2}(1+z_2)$, $(z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$. Then, with P, A, B_1, B_2, and B_3 as above, we obtain the following:

$$\begin{aligned}
&Q_1(z_1,z_2)A(z_1,z_2)+Q_1(z_1,-z_2)A(z_1,-z_2)+Q_1(-z_1,z_2)A(-z_1,z_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad +Q_1(-z_1,-z_2)A(-z_1,-z_2)\\
&= -\frac{1}{8}\frac{1}{z_1z_2^2}\left[(1+z_2)+(1-z_2)-(1+z_2)-(1-z_2)\right]\\
&= 0, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\};
\end{aligned}$$

$$\begin{aligned}
&Q_1(z_1,z_2)B_1(z_1,z_2)+Q_1(z_1,-z_2)B_1(z_1,-z_2)+Q_1(-z_1,z_2)B_1(-z_1,z_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad +Q_1(-z_1,-z_2)B_1(-z_1,-z_2)\\
&= -\frac{1}{8}\frac{1}{z_1z_2}\left[(1-z_1)(1+z_2)(1-z_1z_2)-(1-z_1)(1-z_2)(1+z_1z_2)\right.\\
&\qquad\qquad \left.-(1+z_1)(1+z_2)(1+z_1z_1)+(1+z_1)(1-z_2)(1-z_1z_2)\right]\\
&= -\frac{1}{8}\frac{1}{z_1z_2}\left[-8z_1z_2\right]\\
&= 1=\delta_{1-1}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\};
\end{aligned}$$

$$\begin{aligned}
&Q_1(z_1,z_2)B_2(z_1,z_2)+Q_1(z_1,-z_2)B_2(z_1,-z_2)+Q_1(-z_1,z_2)B_2(-z_1,z_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad +Q_1(-z_1,-z_2)B_2(-z_1,-z_2)\\
&= -\frac{1}{8}\frac{1}{z_1z_2}(1-z_2^2)\left[(1-z_1)-(1-z_1)-(1+z_1)+(1+z_1)\right]\\
&= 0=\delta_{1-2}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\};
\end{aligned}$$

$$\begin{aligned} & Q_1(z_1,z_2)B_3(z_1,z_2)+Q_1(z_1,-z_2)B_3(z_1,-z_2)+Q_1(-z_1,z_2)B_3(-z_1,z_2) \\ & \qquad +Q_1(-z_1,-z_2)B_3(-z_1,-z_2) \\ &= -\frac{1}{8}\frac{1}{z_1z_2^2}(1-z_2^2)\left[(1-z_1z_2)+(1+z_1z_2)-(1+z_1z_2)-(1-z_1z_2)\right] \\ &= 0=\delta_{1-3}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \end{aligned}$$

It follows that the Laurent polynomial Q_1 satisfies both of the identities (2.2.11) and (2.2.13), for every choice of B_1, B_2, and B_3.

Similarly, with the choice $Q_2(z_1,z_2):=\frac{1}{8}\frac{1}{z_2}(1+z_1z_2)$, and P, A, B_1, B_2, and B_3 as above, we obtain the following:

$$\begin{aligned} & Q_2(z_1,z_2)A(z_1,z_2)+Q_2(z_1,-z_2)A(z_1,-z_2)+Q_2(-z_1,z_2)A(-z_1,z_2) \\ & \qquad +Q_2(-z_1,-z_2)A(-z_1,-z_2) \\ &= \frac{1}{8}\frac{1}{z_1z_2^2}\left[(1+z_1z_2)+(1-z_1z_2)-(1-z_1z_2)-(1+z_1z_2)\right] \\ &= 0, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}; \end{aligned}$$

$$\begin{aligned} & Q_2(z_1,z_2)B_1(z_1,z_2)+Q_2(z_1,-z_2)B_1(z_1,-z_2)+Q_2(-z_1,z_2)B_1(-z_1,z_2) \\ & \qquad +Q_2(-z_1,-z_2)B_1(-z_1,-z_2) \\ &= \frac{1}{8}\frac{1}{z_1z_2}(1-z_1^2z_2^2)\left[(1-z_2)-(1-z_1)-(1+z_1)+(1+z_1)\right] \\ &= 0=\delta_{2-1}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}; \end{aligned}$$

$$
\begin{aligned}
& Q_2(z_1,z_2)B_2(z_1,z_2)+Q_2(z_1,-z_2)B_2(z_1,-z_2)+Q_2(-z_1,z_2)B_2(-z_1,z_2) \\
& \qquad +Q_2(-z_1,-z_2)B_2(-z_1,-z_2) \\
= \; & \frac{1}{8}\frac{1}{z_1z_2}\left[(1-z_1)(1-z_2)(1+z_1z_2)-(1-z_1)(1+z_2)(1-z_1z_2)\right. \\
& \qquad \left.-(1+z_1)(1-z_2)(1-z_1z_2)+(1+z_1)(1+z_2)(1+z_1z_2)\right] \\
= \; & \frac{1}{8}z_1z_2\left[8z_1z_2\right] \\
= \; & 1=\delta_{2-2}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\};
\end{aligned}
$$

$$
\begin{aligned}
& Q_2(z_1,z_2)B_3(z_1,z_2)+Q_2(z_1,-z_2)B_3(z_1,-z_2)+Q_2(-z_1,z_2)B_3(-z_1,z_2) \\
& \qquad +Q_2(-z_1,-z_2)B_3(-z_1,-z_2) \\
= \; & \frac{1}{8}\frac{1}{z_1z_2^2}(1-z_1^2z_2^2)\left[(1-z_2)+(1+z_2)-(1-z_2)-(1+z_2)\right] \\
= \; & 0=\delta_{2-3}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}.
\end{aligned}
$$

It follows that the Laurent polynomial Q_2 also satisfies both of the identities (2.2.11) and (2.2.13), for every choice of B_1, B_2, and B_3.

Finally, for the choice $Q_3(z_1,z_2):=-\frac{1}{8}(1+z_1)$, and P, A, B_1, B_2, and B_3 as above, we similarly obtain the following:

$$\begin{aligned}
& Q_3(z_1,z_2)A(z_1,z_2)+Q_3(z_1,-z_2)A(z_1,-z_2)+Q_3(-z_1,z_2)A(-z_1,z_2) \\
& \qquad +Q_3(-z_1,-z_2)A(-z_1,-z_2) \\
= \; & -\frac{1}{8}\frac{1}{z_1z_2}\left[(1+z_1)-(1+z_1)-(1-z_1)+(1-z_1)\right] \\
= \; & 0, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\};
\end{aligned}$$

$$\begin{aligned}
& Q_3(z_1,z_2)B_1(z_1,z_2)+Q_3(z_1,-z_2)B_1(z_1,-z_2)+Q_3(-z_1,z_2)B_1(-z_1,z_2) \\
& \qquad +Q_3(-z_1,-z_2)B_1(-z_1,-z_2) \\
= \; & -\frac{1}{8}\frac{1}{z_1}(1-z_1^2)\left[(1-z_1z_2)+(1+z_1z_2)-(1+z_1z_2)-(1-z_1z_2)\right] \\
= \; & 0=\delta_{3-1}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\};
\end{aligned}$$

$$\begin{aligned}
& Q_3(z_1,z_2)B_2(z_1,z_2)+Q_3(z_1,-z_2)B_2(z_1,-z_2)+Q_3(-z_1,z_2)B_2(-z_1,z_2) \\
& \qquad +Q_3(-z_1,-z_2)B_2(-z_1,-z_2) \\
= \; & -\frac{1}{8}\frac{1}{z_1}(1-z_1^2)\left[(1-z_2)+(1+z_2)-(1-z_2)-(1+z_2)\right] \\
= \; & 0=\delta_{3-2}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\};
\end{aligned}$$

$$\begin{aligned} & Q_3(z_1,z_2)B_3(z_1,z_2)+Q_3(z_1,-z_2)B_3(z_1,-z_2)+Q_3(-z_1,z_2)B_3(-z_1,z_2) \\ & \qquad +Q_3(-z_1,-z_2)B_3(-z_1,-z_2) \\ = \; & -\frac{1}{8}\frac{1}{z_1z_2}\left[(1+z_1)(1-z_2)(1-z_1z_2)-(1+z_1)(1+z_2)(1+z_1z_2)\right. \\ & \qquad \left. -(1-z_1)(1-z_2)(1+z_1z-2)+(1-z_1)(1+z_2)(1-z_1z_2)\right] \\ = \; & -\frac{1}{8}\frac{1}{z_1z_2}\left[-4z_1z_2\right] \\ = \; & 1=\delta_{3-3}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \end{aligned}$$

Therefore, the Laurent polynomial Q_3 also satisfies both of the identities (2.2.11) and (2.2.13), for every choice of B_1, B_2, and B_3. We thus have the following result.

Theorem 3.1 *For* $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, *the Laurent polynomials*

$$\left.\begin{aligned} A(z_1,z_2) &= \frac{1}{z_1z_2}; \\ B_1(z_1,z_2) &= \frac{1}{z_1}(1-z_1)(1-z_1z_2); \\ B_2(z_1,z_2) &= \frac{1}{z_1}(1-z_1)(1-z_2); \\ B_3(z_1,z_2) &= \frac{1}{z_1z_2}(1-z_2)(1-z_1z_2); \\ Q_1(z_1,z_2) &= -\frac{1}{8}\frac{1}{z_2}(1+z_2); \\ Q_2(z_1,z_2) &= \frac{1}{8}\frac{1}{z_2}(1+z_1z_2); \\ Q_3(z_1,z_2) &= -\frac{1}{8}(1+z_1), \end{aligned}\right\} \tag{3.1.2}$$

satisfy the system of Bezout identities in (2.2.10) *through* (2.2.13).

3.2 The Courant Hat Function Wavelet

We proceed use the work in the previous section to show to show how a system of wavelets corresponding to the refinement mask symbol of the Courant hat function $B_{1,1,1}$ in Example 1.2 is constructed. It follows from (2.2.4) through (2.2.7), as well as (1.3.17) and (3.1.2), that, for $\alpha \in \{1,2,3\}$, the sequences $\{p_{k,\ell}\}$, $\mathbf{q}_\alpha = \{q^{[\alpha]}_{k,\ell}\}$, $\{a_{k,\ell}\}$, and $\{b^{[\alpha]}_{k,\ell}\}$, corresponding to the Laurent polynomials P, Q_α, A, and B_α, respectively, are given by:

$$\left.\begin{aligned}
\{p_{0,0}, p_{0,1}, p_{1,0}, p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}\} &= \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\};\\
\left\{q^{[1]}_{0,-1}, q^{[1]}_{0,0}\right\} &= \left\{-\frac{1}{2}, -\frac{1}{2}\right\};\\
\left\{q^{[2]}_{0,-1}, q^{[2]}_{1,0}\right\} &= \left\{\frac{1}{2}, \frac{1}{2}\right\};\\
\left\{q^{[3]}_{0,0}, q^{[3]}_{1,0}\right\} &= \left\{-\frac{1}{2}, -\frac{1}{2}\right\};\\
\{a_{-1,-1}\} &= \{1\};\\
\left\{b^{[1]}_{-1,0}, b^{[1]}_{0,0}, b^{[1]}_{1,0}, b^{[1]}_{0,1}\right\} &= \{1, -1, 1, 1\};\\
\left\{b^{[2]}_{-1,0}, b^{[2]}_{0,0}, b^{[2]}_{-1,1}, b^{[2]}_{0,1}\right\} &= \{1, -1, -1, 1\};\\
\left\{b^{[3]}_{-1,-1}, b^{[3]}_{-1,0}, b^{[3]}_{0,0}, b^{[3]}_{0,1}\right\} &= \{1, -1, -1, 1\}.
\end{aligned}\right\} \tag{3.2.1}$$

(It is understood that the entries of these sequences are zero everywhere else.) The following result is now a direct consequence of Theorems 2.2 and 3.1, as well as (3.2.1).

Theorem 3.2 *Let $\phi := B_{1,1,1}$ be the Courant hat function as in Example* 1.2, *with corresponding refinement mask symbol $P := P_{1,1,1}$ as given by* (1.3.17). *Let the bivariate Laurent polynomials A, B_1, B_2, B_3, Q_1, Q_2, and Q_3, be defined as in* (3.1.2). *Then the* *synthesis wavelets in* (2.2.8) *corresponding to the refinable function* ϕ *are given by*

$$\left.\begin{aligned}\psi_1(x,y) &= -\frac{1}{2}\phi(2x,2y)-\frac{1}{2}\phi(2x,2y+1);\\ \psi_2(x,y) &= \frac{1}{2}\phi(2x-1,2y)+\frac{1}{2}\phi(2x,2y+1);\\ \psi_3(x,y) &= -\frac{1}{2}\phi(2x-1,2y)-\frac{1}{2}\phi(2x,2y),\end{aligned}\right\}\quad (x,y)\in\mathbb{R}^2, \tag{3.2.2}$$

and the following wavelet decomposition holds, for $(x,y)\in\mathbb{R}^2$ *and* $(i,j)\in\mathbb{Z}^2$:

$$\phi(2x-i,2y-j) = \sum_{k,\ell} a_{2k-i,2\ell-j}\phi(x-k,y-\ell)+\sum_{\alpha=1}^{3}\left[\sum_{k,\ell} b^{[\alpha]}_{2k-i,2\ell-j}\psi_\alpha(x-k,y-\ell)\right], \tag{3.2.3}$$

where the sequences $\{p_{k,\ell}\}$, $\mathbf{q}_\alpha=\{q^{[\alpha]}_{k,\ell}\}$, $\{a_{k,\ell}\}$, *and* $\{b^{[\alpha]}_{k,\ell}\}$, *are as in* (3.2.1).

The refinable Courant hat function was illustrated in Figure 1.3.2. We illustrate the three wavelets ψ_1, ψ_2, and ψ_3, corresponding to the Courant hat function, in Figures 3.2.1 through 3.2.3 below.

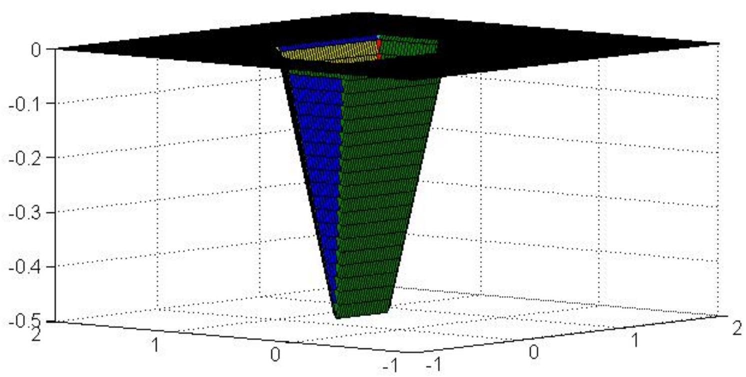

Figure 3.2.1: The Courant hat function wavelet ψ_1

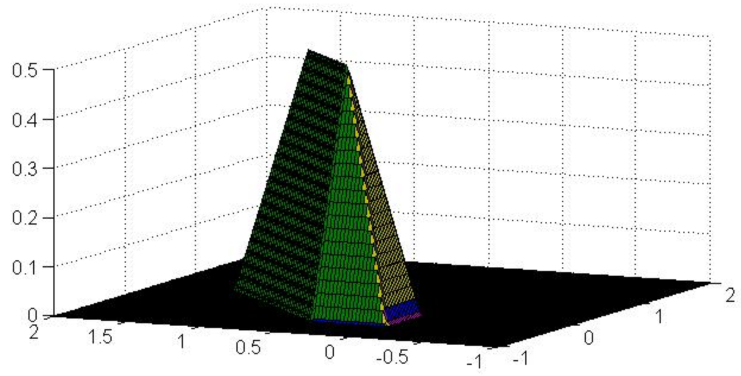

Figure 3.2.2: The Courant hat function wavelet ψ_2

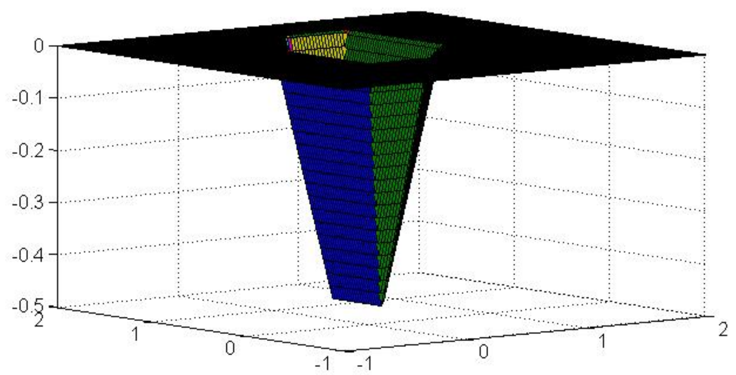

Figure 3.2.3: The Courant hat function wavelet ψ_3

Chapter 4

Interpolatory Subdivision

4.1 An Interpolatory Subdivision Scheme Based on a Solution to the Bezout Identity System

It was mentioned in Chapter 1 that subdivision is a popular tool used by computer programmers with tasks like designing objects or images on a digital interface. Subdivision was in its raw form introduced by Chaikin in 1974, when he addressed the problem of producing mathematical curves from a finite set of points in a plane (referred to commonly as *control points*), by simply taking weighted averages of the given points according to a specific formula. Applying such a rule (called a *subdivision algorithm*) repeatedly, produces, under certain conditions to be fulfilled by the subdivision algorithm, a mathematical curve with very pleasing properties. Not only can the smoothness (or the number of continuous derivatives) of the curve be determined by the subdivision algorithm, but these algorithms are easy to visualize, effective to apply, and time-economical in practice. Moreover, a class of these subdivision algorithms exhibit the fundamental property that the end curve runs through all of the original points (referred to as *interpolatory subdivision*), which for many practical applications means no loss of data or structure. In the recent book by Chui and De Villiers (see [2]), it was shown how the mathematical ideas of subdivision and wavelets can in fact be combined; that is, when a user creates an image with some subdivision algorithm that he has chosen and from some fixed set of control points, wavelet theory adds to his artillary a tool that allows him to modify his end image further by inserting minor detail components as needed. This opens up a plethora of possibilities for users in all ranges of design industries, to construct objects and images and further modify these images at will.

This section will give a brief example of an interpolatory subdivision scheme in the bivariate case, making use of the fact that Section 3.1 obtained a particular solution to the Bezout identity in (2.2.10) in Chapter 2. First, we establish the definition of an interpolatory refinable function. A refinable function ϕ^I with corresponding refinement mask symbol P^I is defined to be interpolatory if

$$\phi^I(i,j) = \delta_{i,j}, \quad (i,j) \in \mathbb{Z}^2, \tag{4.1.1}$$

where we use the usual Kronecker delta:

$$\delta_{i,j} := \begin{cases} 1, & i = j = 0; \\ 0, & i \neq 0 \text{ or } j \neq 0. \end{cases} \tag{4.1.2}$$

Lemma 4.1 *Let ϕ^I be a refinable function satisfying* (1.3.15), *with corresponding refinement mask symbol*

$$P^I(z_1, z_2) := \sum_{k,\ell} p_{k,\ell} z_1^k z_2^\ell, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{4.1.3}$$

and suppose

$$\sum_{k,\ell} p_{k,\ell} = 1. \tag{4.1.4}$$

Then ϕ^I is interpolatory if and only if

$$P^I(z_1, z_2) + P^I(z_1, -z_2) + P^I(-z_1, z_2) + P^I(-z_1, -z_2) = 1, \quad (z_1, z_2) \in \mathbb{R}^2. \tag{4.1.5}$$

Proof. It follows from (4.1.3) that, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$,

$$\begin{aligned} P^I(z_1, z_2) + P^I(-z_1, z_2) &= \frac{1}{4}\left[\sum_{k,\ell} p^I_{2k,\ell} z_1^{2k} z_2^{\ell} + \sum_{k,\ell} p^I_{2k+1,\ell} z_1^{2k+1} z_2^{\ell}\right. \\ &\qquad \left. + \sum_{k,\ell} p^I_{2k,\ell} z_1^{2k} z_2^{\ell} - \sum_{k,\ell} p^I_{2k+1,\ell} z_1^{2k+1} z_2^{\ell}\right] \\ &= \frac{1}{2}\sum_{k,\ell} p^I_{2k,\ell} z_1^{2k} z_2^{\ell} \\ &= \frac{1}{2}\left[\sum_{k,\ell} p^I_{2k,2\ell} z_1^{2k} z_2^{2\ell} + \sum_{k,\ell} p^I_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1}\right]. \end{aligned}$$

Similarly, it follows that, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$,

$$P^I(z_1, -z_2) + P^I(-z_1, -z_2) = \frac{1}{2}\left[\sum_{k,\ell} p^I_{2k,2\ell} z_1^{2k} z_2^{2\ell} - \sum_{k,\ell} p^I_{2k,2\ell+1} z_1^{2k} z_2^{2\ell+1}\right],$$

and it therefore follows that

$$P^I(z_1, z_2) + P^I(-z_1, z_2) + P^I(z_1, -z_2) + P^I(-z_1, -z_2) = \sum_{k,\ell} p^I_{2k,2\ell} z_1^{2k} z_2^{2\ell}, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{4.1.6}$$

Next, note that ϕ^I is interpolatory if and only if

$$\begin{aligned} \delta_{k,\ell} = \phi^I(k,\ell) &= \sum_{i,j} p^I_{i,j} \phi^I(2k-i, 2\ell-j) \\ &= \sum_{i,j} p^I_{i,j} \delta_{2k-i,2\ell-j} = p^I_{2k,2\ell}, \end{aligned}$$

and it thus follows from (4.1.6) that ϕ^I is interpolatory if and only if

$$P^I(z_1, z_2) + P^I(-z_1, z_2) + P^I(z_1, -z_2) + P^I(-z_1, -z_2) = \sum_{k,\ell} \delta_{k,\ell} z_1^{2k} z_2^{2\ell},$$

from which, along with the assumption (4.1.4), the result follows. ■

The result in Lemma 4.1, as well as more in-depth investigation into interpolatory refinement masks, were studied in [5]. Recall from (3.1.2) that $A(z_1, z_2) := \frac{1}{z_1 z_2}$ is a particular solution to the identity (2.2.10) for the case where P is the refinement mask symbol corresponding to the Courant hat function as given by (1.3.17). Further, it is easy to verify that, similar to our work in Section 3.1, the Laurent polynomials

$$\left.\begin{aligned}
\tilde{B}_1(z_1, z_2) &:= \frac{(1-z_1)(1+z_2)(1-z_1z_2)}{z_1z_2};\\
\tilde{B}_2(z_1, z_2) &:= \frac{(1+z_1)(1-z_2)(1-z_1z_2)}{z_1z_2};\\
\tilde{B}_3(z_1, z_2) &:= \frac{(1-z_1)(1-z_2)(1-z_1z_2)}{z_1z_2};\\
\tilde{B}_4(z_1, z_2) &:= \frac{(1-z_1)(1+z_2)(1-z_1z_2)}{z_1^3z_2^3};\\
\tilde{B}_5(z_1, z_2) &:= \frac{(1+z_1)(1-z_2)(1-z_1z_2)}{z_1^3z_2^3};\\
\tilde{B}_6(z_1, z_2) &:= \frac{(1-z_1)(1-z_2)(1-z_1z_2)}{z_1^3z_2^3},
\end{aligned}\right\} (z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\}, \tag{4.1.7}$$

all satisfy (2.2.12) for the Courant hat function case. It follows that, for any constants α, β, and γ, the Laurent polynomial

$$\begin{aligned}
\hat{A}(z_1, z_2) &:= A(z_1, z_2) + \alpha\tilde{B}_1(z_1, z_2) + \beta\tilde{B}_2(z_1, z_2) + \gamma\tilde{B}_3(z_1, z_2) + \alpha\tilde{B}_4(z_1, z_2)\\
&\quad + \beta\tilde{B}_5(z_1, z_2) - \gamma\tilde{B}_6(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\},
\end{aligned} \tag{4.1.8}$$

is a solution to the identity (2.2.10), and it therefore follows that the polynomial

$$P^I(z_1, z_2) := \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right)\hat{A}(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\}, \tag{4.1.9}$$

satisfies the condition (4.1.5); that is, P^I is an interpolatory refinement mask symbol. Observe from (4.1.9), (4.1.8), (4.1.7), together with $A(z_1, z_2) = \frac{1}{z_1 z_2}$, that P^I satisfies the symmetry condition

$$P^I\left(\frac{1}{z_1},\frac{1}{z_2}\right) = P^I(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}.$$

We proceed to simplify the Laurent polynomial P^I. For $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, we have

$$\begin{aligned}
P^I(z_1,z_2) &= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right)\left[\frac{1}{z_1z_2}+\frac{\alpha(1-z_1)(1+z_2)(1-z_1z_2)}{z_1z_2}\right.\\
&\quad +\frac{\beta(1+z_1)(1-z_2)(1-z_1z_2)}{z_1z_2}+\frac{\gamma(1-z_1)(1-z_2)(1-z_1z_2)}{z_1z_2}\\
&\quad +\frac{\alpha(1-z_1)(1+z_2)(1-z_1z_2)}{z_1^3z_2^3}+\frac{\beta(1+z_1)(1-z_2)(1-z_1z_2)}{z_1^3z_2^3}\\
&\quad \left.-\frac{\gamma(1-z_1)(1-z_2)(1-z_1z_2)}{z_1^3z_2^3}\right]\\
&= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1^{-1}z_2^{-1}}{2}\right)\Big[1+\alpha(1-z_1)(1+z_2)(1-z_1z_2)\\
&\quad +\beta(1+z_1)(1-z_2)(1-z_1z_2)+\gamma(1-z_1)(1-z_2)(1-z_1z_2)\\
&\quad +\alpha\left(1-\frac{1}{z_1}\right)\left(1+\frac{1}{z_2}\right)\left(1-\frac{1}{z_1z_2}\right)+\beta\left(1+\frac{1}{z_1}\right)\left(1-\frac{1}{z_2}\right)\left(1-\frac{1}{z_1z_2}\right)\\
&\quad +\gamma\left(1-\frac{1}{z_1}\right)\left(1-\frac{1}{z_2}\right)\left(1-\frac{1}{z_1z_2}\right)\Big].
\end{aligned}$$

Under the preference of a refinement mask with smallest support, we set $\gamma=\alpha+\beta$, and it follows that

$$\begin{aligned}
P^I(z_1,z_2) &= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1^{-1}z_2^{-1}}{2}\right)\Big[1+2(1-z_1z_2)(\alpha+\beta-\alpha z_1-\beta z_2)\\
&\quad +2\left(1-\frac{1}{z_1z_2}\right)\left(\alpha+\beta-\frac{\alpha}{z_1}-\frac{\beta}{z_2}\right)\Big]. \qquad (4.1.10)
\end{aligned}$$

Under further preference of a symmetric refinement mask, we require that P^I must satisfy $P^I(z_1,z_2) = P^I(z_2,z_1),\ (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$, so that, from (4.1.10), we have the further restriction that $\alpha = \beta$, and after which (4.1.10) becomes

$$\begin{aligned} P^I(z_1,z_2) &= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1^{-1}z_2^{-1}}{2}\right) \\ &\qquad \cdot \left[1 + 2\alpha(1-z_1z_2)(2-z_1-z_2) + 2\alpha\left(1-\frac{1}{z_1z_2}\right)\left(2-\frac{1}{z_1}-\frac{1}{z_2}\right)\right] \\ &= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1^{-1}z_2^{-1}}{2}\right)\left[1-\omega C(z_1,z_2)\right], \end{aligned}$$

where C is the Laurent polynomial

$$\begin{aligned} C(z_1,z_2) \;:=\; & z_1^{-1}z_2^{-2} + z_1^{-2}z_2^{-1} - 2z_1^{-1}z_2^{-1} - z_1^{-1} - z_2^{-1} + 4 - z_1 - z_2 - 2z_1z_2 + z_1^2z_2 + z_1z_2^2, \\ & (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}. \end{aligned} \tag{4.1.11}$$

The subdivision scheme corresponding to the mask symbol P^I is similar to the *Butterfly scheme* which is studied in more depth in, e.g., [6] and [7].

Chapter 5

A General Approach to the First Identity

Although a particular solution was given in Chapter 3 for the system of Bezout identities in (2.2.10) through (2.2.13) for the Courant hat function case, it would be beneficial to develop a systematic approach with the view to obtaining the general solution of the system of Bezout identites (2.2.10) through (2.2.13) of Theorem 2.2, by means of which the user can then choose at will which particular solution he prefers. One preference for a user might be a solution set in which the Laurent polynomials have the smallest possible support, as the particular solution derived in Theorem 3.1, whereas another preference might be that he is looking for the Laurent polynomial solutions (especially the solutions for Q_1, Q_2, and Q_3) that are symmetric around certain points, since then the constructed wavelet as end result will possess similar symmetry properties. Finally, as was seen at the end of the previous chapter, the process of solving the Bezout identities (2.2.10) and (2.2.12) also yields a systematic approach for constructing interpolatory subdivision schemes, and thus a more general approach to solving these identities should positively contribute to the enhancement of the theory of interpolatory subdivision.

5.1 The First Identity

In this chapter, we show how the system of sixteen Bezout identities in (2.2.10) through (2.2.13) can be solved in the particular case where ϕ is the Courant hat function $B_{1,1,1}$ in (1.3.14), with corresponding refinement mask symbol

$$P(z_1, z_2) = \left(\frac{1+z_2}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right), \quad (z_1, z_2) \in \mathbb{C}^2. \tag{5.1.1}$$

In particular, this chapter will focus on the first of the sixteen Bezout identities, namely:

$$\begin{aligned} P(z_1, z_2)A(z_1, z_2) \quad &+ \quad P(-z_1, z_2)A(-z_1, z_2) + P(z_1, -z_2)A(z_1, -z_2) \\ &+ \quad P(-z_1, -z_2)A(-z_1, -z_2) = 1, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0, 0)\}. \end{aligned} \tag{5.1.2}$$

With the symbol P fixed as in (5.1.1), a general class of solutions for the Laurent polynomial A in (5.1.2) will be derived. This class of solutions will be characterized by four Laurent polynomial "degrees of freedom."

5.1.1 A Class of Solutions for the Three-Directional Box Spline Case

Before proceeding to derive the general solution of (5.1.2) for the Courant hat function case, it is remarked that, in [11], an efficient method is derived for obtaining a particular solution for A in (5.1.2), in the case where P is the refinement symbol of *any* three-directional box spline. In particular, for the Courant hat function case in (5.1.1), the method in [11] depends on writing $A(z_1, z_2) := H(z_1, z_2)D(z_1z_2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0, 0)\}$, where H is some bivariate Laurent polynomial and D a univariate one, so that it remains to solve for H and D such that, from (5.1.2),

$$\begin{aligned}
&\left(\frac{1+z_1z_2}{2}\right)\left[\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)H(z_1,z_2)\right.\\
&\qquad\left.+\left(\frac{1-z_1}{2}\right)\left(\frac{1-z_2}{2}\right)H(-z_1,-z_2)\right]D(z_1z_2)\\
&+\left(\frac{1-z_1z_2}{2}\right)\left[\left(\frac{1-z_1}{2}\right)\left(\frac{1+z_2}{2}\right)H(-z_1,z_2)\right.\\
&\qquad\left.+\left(\frac{1+z_1}{2}\right)\left(\frac{1-z_2}{2}\right)H(z_1,-z_2)\right]D(-z_1z_2)\\
&=1,\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}.
\end{aligned}\tag{5.1.3}$$

Note that the expression

$$\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)H(z_1,z_2)+\left(\frac{1-z_1}{2}\right)\left(\frac{1-z_2}{2}\right)H(-z_1,-z_2)=\left(\frac{1+z_1z_2}{2}\right),$$
$$(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\},$$

holds if $H(z_1,z_2)=1,\ (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, in which case, also,

$$\left(\frac{1-z_1}{2}\right)\left(\frac{1+z_2}{2}\right)H(-z_1,z_2)+\left(\frac{1+z_2}{2}\right)\left(\frac{1-z_2}{2}\right)H(z_1,-z_2)=\left(\frac{1-z_1z_2}{2}\right),$$
$$(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}.$$

Hence, if $H(z_1,z_2)=1,\ (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, then (5.1.3) becomes

$$\left(\frac{1+z_1z_2}{2}\right)^2D(z_1z_2)+\left(\frac{1-z_1z_2}{2}\right)^2D(-z_1z_2)=1,\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}.\tag{5.1.4}$$

Let $u := \dfrac{1+z_1z_2}{2}$, so that $1-u = \dfrac{1-z_1z_2}{2}$, and thus also $z_1z_2 = 2u-1$. Then it follows from (5.1.4) and the binomial theorem that

$$\begin{aligned} u^2D(2u-1)+(1-u)^2D(-2u+1) &= 1 \\ &= (1-u+u)^3 \\ &= (1-u)^3+3(1-u)^2u+3(1-u)u^2+u^3 \\ &= u^2\left[u+3(1-u)\right]+(1-u)^2\left[(1-u)+3u\right] \\ &= u^2\left[2-(2u-1)\right]+(1-u)^2\left[2-(1-2u)\right], \\ &\qquad u \in \mathbb{C}\backslash\{0\}, \end{aligned}$$

which is satisfied by $D(2u-1) := 2-(2u-1),\ u \in \mathbb{C}\backslash\{0\}$. Thus,

$$D(z_1z_2) = 2 - z_1z_2, \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.5}$$

is a particular solution to (5.1.4). It then follows that the polynomial

$$A(z_1,z_2) = H(z_1,z_2)D(z_1z_2) = 1(2-z_1z_2) = 2-z_1z_2, \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.6}$$

is a particular solution to (5.1.2).

Remark. One can generalize the work done in [11] to a certain extent for the Courant hat function case, by letting $v := z_1z_2$ in (5.1.4), which then becomes

$$\left(\frac{1+v}{2}\right)^2 D(v) + \left(\frac{1-v}{2}\right)^2 D(-v) = 1, \quad v \in \mathbb{C}\backslash\{0\}, \tag{5.1.7}$$

and for which a particular solution is given by $\tilde{D}(v) := \dfrac{1}{v},\ v \in \mathbb{C}\backslash\{0\}$, i.e.,

$$\left(\frac{1+v}{2}\right)^2 \tilde{D}(v) + \left(\frac{1-v}{2}\right)^2 \tilde{D}(-v) = 1, \quad v \in \mathbb{C}\backslash\{0\}. \tag{5.1.8}$$

Subtracting (5.1.8) from (5.1.7), yields

$$\left(\frac{1+v}{2}\right)^2 \left[D(v)-\tilde{D}(v)\right] = -\left(\frac{1-v}{2}\right)^2 \left[D(-v)-\tilde{D}(-v)\right], \quad v \in \mathbb{C}\backslash\{0\}, \tag{5.1.9}$$

so that there exists a Laurent polynomial $\tilde{J}$ such that

$$D(v) - \tilde{D}(v) = \left(\frac{1-v}{2}\right)^2 \tilde{J}(v), \quad v \in \mathbb{C}\backslash\{0\}. \tag{5.1.10}$$

Observe from (5.1.9) and (5.1.10) that $\tilde{J}$ is odd in v, so that there exists a Laurent polynomial J such that $\tilde{J}(v) = \frac{1}{v}J(v^2),\ v \in \mathbb{C}\backslash\{0\}$. It follows from (5.1.10) and the definition of $\tilde{D}$ that a class of solutions of the identity (5.1.4) is given, in terms of a general Laurent polynomial J, by

$$D(z_1 z_2) = \frac{1 + (1 - z_1 z_2)^2 J(z_1^2 z_2^2)}{z_1 z_2}, \quad (z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.11}$$

Notice further that, with the choice $J(v) = 1,\ v \in \mathbb{C}\backslash\{0\}$, one obtains

$$D(z_1 z_2) = \frac{1 + (1 - z_1 z_2)^2}{z_1 z_2} = \frac{2 z_1 z_2 - z_1^2 z_2^2}{z_1 z_2} = 2 - z_1 z_2, \quad (z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\},$$

which is the particular solution (5.1.5) given in [11]. However, the class of solutions (5.1.11) based on the work in [11], and which relies on the setting $v = z_1 z_2$, does not yield a systematic approach for finding the Laurent polynomial solutions for the identities in (2.2.11) through (2.2.13). In the following, it will be shown how a general class of solutions for (5.1.2) can be derived which can be used instead of the method for finding a particular solution as in [11]. In particular, our work is based on a result that was proved by Rabarison and De Villiers in [7], where (5.1.2) is solved (most) generally for the case where P is any *two-directional* box spline. We will proceed to show how this general class of solutions can be used to solve (5.1.2) for the case where P is as in (5.1.1).

5.1.2 The Haar Function Case

Before continuing our work on the Courent hat function case, we study the following result, which is given and proved in the more general setting in [7] for general two-directional box splines. We only give the result with corresponding proof for the Haar function case of Example 1.2, which we will subsequently use to develop a result on the Courant hat function case.

Theorem 5.1 *The general Laurent polynomial solution to the identity*

$$\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\tilde{A}(z_1,z_2)+\left(\frac{1-z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\tilde{A}(-z_1,z_2)$$
$$+\left(\frac{1+z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\tilde{A}(z_1,-z_2)+\left(\frac{1-z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\tilde{A}(-z_1,-z_2)=1,$$
$$(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.12}$$

is given by

$$\begin{aligned}\tilde{A}(z_1,z_2) &= 4z_1^{-2}z_2^{-2}\left[T(z_1,z_2)(1-z_1)(1-z_2)\right.\\ &\quad \left.+\left\{\frac{z_2}{2}+T_1(z_1,z_2)(1-z_1)\right\}\left\{\frac{z_1}{2}+T_2(z_1,z_2)(1-z_2)\right\}\right],\end{aligned}$$
$$(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.13}$$

where $\tilde{T}_1$ is a Laurent polynomial that is even in z_1 but odd in z_2, $\tilde{T}_2$ is one that is even in z_2 but odd in z_1, and $\tilde{T}$ is one that is odd in both z_1 and z_2.

Proof. Define the Laurent polynomial C by

$$C(z_1,z_2):=(1+z_1)(1+z_2)\tilde{A}(z_1,z_2),\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.14}$$

so that

$$C(z_1,z_2)+C(z_1,-z_2)=(1+z_1)D_1(z_1,z_2),\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.15}$$

where

$$D_1(z_1,z_2):=(1+z_2)\tilde{A}(z_1,z_2)+(1-z_2)\tilde{A}(z_1,-z_2),\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.16}$$

Then, the identity (5.1.12) will hold if and only if

$$(1+z_1)(1+z_2)\tilde{A}(z_1,z_2)+(1-z_1)(1+z_2)\tilde{A}(-z_1,z_2)$$
$$+(1+z_1)(1-z_2)\tilde{A}(z_1,-z_2)+(1-z_1)(1-z_2)\tilde{A}(-z_1,-z_2)=4,$$
$$(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.17}$$

which is equivalent to

$$C(z_1,z_2)+C(z_1,-z_2)+C(-z_1,z_2)+C(-z_1,z_2)=4, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\},$$

i.e.,

$$(1+z_1)D_1(z_1,z_2)+(1-z_1)D_1(-z_1,z_2)=4, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.18}$$

Let the Laurent polynomial E_1 be defined by

$$E_1(z_1,z_2):=\frac{1}{4}z_1z_2D_1(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.19}$$

Then (5.1.18) is equivalent to

$$(1+z_1)E_1(z_1,z_2)-(1-z_1)E_1(-z_1,z_2)=z_1z_2, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.20}$$

Note that the Laurent polynomial $E_1^*(z_1,z_2):=\frac{1}{2}z_2,\ (z_1,z_2)\in\mathbb{C}^2$, is a particular solution for (5.1.20), i.e.,

$$(1+z_1)E_1^*(z_1,z_2)-(1-z_1)E_1^*(-z_1,z_2)=z_1z_2, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.21}$$

Subtracting (5.1.21) from (5.1.20), one arrives at

$$(1+z_1)[E_1(z_1,z_2)-E_1^*(z_1,z_2)]=(1-z_1)[E_1(-z_1,z_2)-E_1^*(-z_1,z_2)], \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.22}$$

Hence, there must exist a Laurent polynomial $\tilde{T}_1$ such that

$$E_1(z_1,z_2)=\frac{1}{2}z_2+(1-z_1)\tilde{T}_1(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.23}$$

It is useful to note, after substituting (5.1.23) into (5.1.22), that $\tilde{T}_1$ is even in z_1. Moreover, recall from (5.1.16) that the Laurent polynomial D_1 is even in z_2. Therefore, from (5.1.19), E_1 is in fact odd in z_2, and it follows from (5.1.23) that $\tilde{T}_1$ is odd in z_2. Therefore, one is able to write $\tilde{T}_1(z_1,z_2)=z_2T_1(z_1^2,z_2^2),\ (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, and it follows that

$$E_1(z_1,z_2)=\frac{1}{2}z_2+z_2(1-z_1)T_1(z_1^2,z_2^2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.24}$$

i.e., from (5.1.19),

$$D_1(z_1,z_2) = \frac{2}{z_1} + \frac{4}{z_1}(1-z_1)T_1(z_1^2,z_2^2), \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.25}$$

In a similar way, it follows from (5.1.14) that

$$C(z_1,z_2) + C(-z_1,z_2) = (1+z_2)D_2(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{5.1.26}$$

where

$$D_2(z_1,z_2) := (1+z_1)\tilde{A}(z_1,z_2) + (1-z_1)\tilde{A}(-z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.27}$$

Similar to before, the identity (5.1.12) is satisfied if and only if

$$(1+z_2)D_2(z_1,z_2) + (1-z_2)D_2(z_1,-z_2) = 4, \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.28}$$

Let the Laurent polynomial E_2 be defined by

$$E_2(z_1,z_2) := \frac{1}{4}z_1z_2D_2(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.29}$$

Then (5.1.28) is equivalent to

$$(1+z_2)E_2(z_1,z_2) - (1-z_2)E_2(z_1,-z_2) = z_1z_2, \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.30}$$

Note that the Laurent polynomial $E_2^*(z_1,z_2) := \frac{1}{2}z_1$, $(z_1,z_2) \in \mathbb{C}^2$, is a particular solution for (5.1.30), i.e.,

$$(1+z_2)E_2^*(z_1,z_2) - (1-z_2)E_2^*(z_1,-z_2) = z_1z_2, \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.31}$$

Subtracting (5.1.31) from (5.1.30), one arrives at

$$(1+z_2)[E_2(z_1,z_2)-E_2^*(z_1,z_2)] = (1-z_2)[E_2(z_1,-z_2)-E_2^*(z_1,-z_2)], \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.32}$$

Hence, there must exist a Laurent polynomial $\tilde{T}_2$ such that

$$E_2(z_1,z_2) = \frac{1}{2}z_1 + (1-z_2)\tilde{T}_2(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.33}$$

It is useful to note, after substituting (5.1.33) into (5.1.32), that $\tilde{T}_2$ is even in z_2. Moreover, recall from (5.1.27) that the Laurent polynomial D_2 is even in z_1. Therefore, as follows from (5.1.29), E_2 is in fact odd in z_1, and it follows from (5.1.33) that $\tilde{T}_2$ is odd in z_1. Therefore, one is able to write $\tilde{T}_2(z_1, z_2) = z_1 T_2(z_1^2, z_2^2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, and it follows that

$$E_2(z_1, z_2) = \frac{1}{2} z_1 + z_1(1 - z_2) T_2(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{5.1.34}$$

i.e., from (5.1.29),

$$D_2(z_1, z_2) = \frac{2}{z_2} + \frac{4}{z_2}(1 - z_2) T_2(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.35}$$

It remains to find the solution $\tilde{A}$ in (5.1.27), which we do by recalling that $\tilde{A}$ also satisfies (5.1.16). It follows from (5.1.27) and (5.1.29) that $\tilde{A}$ must satisfy

$$(1 + z_1)\tilde{A}(z_1, z_2) + (1 - z_1)\tilde{A}(-z_1, z_2) = \frac{4}{z_1 z_2} E_2(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.36}$$

For the Laurent polynomial

$$\hat{A}(z_1, z_2) := \frac{z_1^2 z_2^2}{4} \tilde{A}(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{5.1.37}$$

(5.1.36) is equivalent to

$$(1 + z_1)\hat{A}(z_1, z_2) + (1 - z_1)\hat{A}(-z_1, z_2) = z_1 z_2 E_2(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.38}$$

It follows from (5.1.20) as well as the fact that E_2 is odd in z_1, that the Laurent polynomial $\hat{A}^*(z_1, z_2) := E_1(z_1, z_2) E_2(z_1, z_2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, is a particular solution for (5.1.38), i.e.,

$$(1 + z_1)\hat{A}^*(z_1, z_2) + (1 - z_1)\hat{A}^*(-z_1, z_2) = z_1 z_2 E_2(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{5.1.39}$$

Subtracting (5.1.39) from (5.1.38) yields

$$(1+z_1)[\hat{A}(z_1,z_2)-\hat{A}^*(z_1,z_2)] = -(1-z_1)[\hat{A}(-z_1,z_2)-\hat{A}^*(-z_1,z_2)], \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.40}$$

so there exists a Laurent polynomial $\tilde{T}_3$ such that

$$\hat{A}(z_1,z_2) - \hat{A}^*(z_1,z_2) = (1-z_1)\tilde{T}_3(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.41}$$

Notice from substituting (5.1.41) back into (5.1.40) that $\tilde{T}_3$ is odd in z_1, so that $\tilde{T}_3(z_1,z_2) = z_1\hat{T}_3(z_1^2,z_2)$, $(z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$. This fact, along with (5.1.41) and (5.1.37), gives

$$\tilde{A}(z_1,z_2) = \frac{4}{z_1^2 z_2^2}\left[E_1(z_1,z_2)E_2(z_1,z_2) + z_1(1-z_1)\hat{T}_3(z_1^2,z_2)\right], \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}. \tag{5.1.42}$$

Finally, we recall from (5.1.15) that

$$C(z_1,z_2) + C(z_1,-z_2) = (1+z_1)D_1(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\},$$

whereas, for $(z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$, it follows from (5.1.14) and (5.1.42), as well as the fact that E_1 is odd in z_2, and (5.1.30), that

$$\begin{aligned}
&C(z_1,z_2) + C(z_1,-z_2)\\
=\;& \frac{4}{z_1^2 z_2^2}(1+z_1)\,[(1+z_2)E_1(z_1,z_2)E_2(z_1,z_2) + (1-z_2)E_1(z_1,-z_2)E_2(z_1,-z_2)\\
&\qquad + z_1(1-z_1)(1+z_2)\hat{T}_3(z_1^2,z_2) + z_1(1-z_1)(1-z_2)\hat{T}_3(z_1^2,-z_2)\Big]\\
=\;& \frac{4}{z_1^2 z_2^2}(1+z_1)\,[E_1(z_1,z_2)\,\{(1+z_2)E_2(z_1,z_2) - (1-z_2)E(z_1,-z_2)\}\\
&\qquad + z_1(1-z_1)\,\Big\{(1+z_2)\hat{T}_3(z_1^2,z_2) + (1-z_2)\hat{T}_3(z_1^2,-z_2)\Big\}\Big]\\
=\;& \frac{4}{z_1^2 z_2^2}(1+z_1)\,\Big[z_1z_2E_1(z_1,z_2) + z_1(1-z_1)\,\Big\{(1+z_2)\hat{T}_3(z_1^2,z_2) + (1-z_2)\hat{T}_3(z_1^2,-z_2)\Big\}\Big].
\end{aligned} \tag{5.1.43}$$

Hence, it is concluded from (5.1.15), (5.1.43), and (5.1.19), that

$$(1+z_2)\hat{T}_3(z_1^2,z_2) = -(1-z_2)\hat{T}_3(z_1^2,-z_2), \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.44}$$

i.e.,

$$\hat{T}_3(z_1^2,z_2) = (1-z_2)T_3(z_1^2,z_2), \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}, \tag{5.1.45}$$

for some Laurent polynomial T_3. Again, by substituting (5.1.45) into (5.1.44), it is noted that T_3 is odd in z_2, which implies that $T_3(z_1,z_2) = z_2T(z_1,z_2^2)$, $(z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$, for some Laurent polynomial T. It finally follows from (5.1.42), (5.1.24), (5.1.34), and

(5.1.45), that the general Laurent polynomial solution to the identity (5.1.12) is given by

$$\begin{aligned}\tilde{A}(z_1,z_2) \;=\; & \frac{4}{z_1z_2}\Big[(1-z_1)(1-z_2)T(z_1^2,z_2^2)\\ & +\left\{\frac{1}{2}+(1-z_1)T_1(z_1^2,z_2^2)\right\}\left\{\frac{1}{2}+(1-z_2)T_2(z_1^2,z_2^2)\right\}\Big],\\ & (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \qquad (5.1.46)\end{aligned}$$

■

5.1.3 The Courant Hat Function Case

Theorem 5.1 gives the general solution $\tilde{A}$ to the identity (2.2.10) in the case where P is the two-directional refinement mask symbol corresponding to the Haar function in Example 1.1. We proceed to show how the solution (5.1.46) can be used to find the general solution A in (5.1.2), with P as in (5.1.1). To this end, we note that a Laurent polynomial A satisfies (5.1.2) if and only if the Laurent polynomial solution $\tilde{A}$ in (5.1.46) of (5.1.12) satisfies $\tilde{A}(z_1,z_2) = \left(\frac{1+z_1z_2}{2}\right)A(z_1,z_2),\;\;(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$. To find such a Laurent polynomial A, we will rely on the following lemma, in which the notation $M(\mathbb{C}^2)$ is used to donate functions with domain in $\mathbb{C}^2$.

Lemma 5.1 *If* $f(z_1,z_2)\in M(\mathbb{C}^2)$, *then*

$$f(z_1,z_2) = \left(\frac{z_1+z_2}{2}\right)\tilde{f}(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \qquad (5.1.47)$$

if and only if

$$f(z_1,-z_1) = 0, \quad z_1\in\mathbb{C}^2\backslash\{0\}. \qquad (5.1.48)$$

In order to find the solution A to (5.1.2) for the Courent hat function case, i.e., where P is given by (5.1.1), we will first find $\tilde{A}$ in (5.1.46) that satisfies $\tilde{A}(z_1,-z_1)=0,\;(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, after which a simple transformation will show how to obtain A from the Laurent polynomial $\tilde{A}$. To this end, note from (5.1.46) that

$$\begin{aligned}\tilde{A}(z_1,z_2) &= 4z_1^{-1}z_2^{-1}\left\{\left[T(z_1^2,z_2^2)+\frac{1}{4}+\frac{1}{2}T_2(z_1^2,z_2^2)+\frac{1}{2}T_1(z_1^2,z_2^2)+T_1(z_1^2,z_2^2)T_2(z_1^2,z_2^2)\right]\right.\\
&\quad +z_1z_2\left[T(z_1^2,z_2^2)+T_1(z_1^2,z_2^2)T_2(z_1^2,z_2^2)\right]\\
&\quad +z_1\left[-T(z_1^2,z_2^2)-\frac{1}{2}T_1(z_1^2,z_2^2)-T_1(z_1^2,z_2^2)T_2(z_1^2,z_2^2)\right]\\
&\quad \left.+\,z_2\left[-T(z_1^2,z_2^2)-\frac{1}{2}T_2(z_1^2,z_2^2)-T_1(z_1^2,z_2^2)T_2(z_1^2,z_2^2)\right]\right\},\\
&\qquad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}.\end{aligned}$$

Setting $\tilde{A}(z_1,-z_1)=0,\ z_1\in\mathbb{C}\backslash\{0\}$, it follows that

$$\begin{aligned}0 &= \left[T(z_1^2,z_1^2)+\frac{1}{4}+\frac{1}{2}T_2(z_1^2,z_1^2)+\frac{1}{2}T_1(z_1^2,z_1^2)+T_1(z_1^2,z_1^2)T_2(z_1^2,z_1^2)\right]\\
&\quad -z_1^2\left[T(z_1^2,z_1^2)+T_1(z_1^2,z_1^2)T_2(z_1^2,z_1^2)\right]\\
&\quad +z_1\left[-T(z_1^2,z_1^2)-\frac{1}{2}T_1(z_1^2,z_1^2)-T_1(z_1^2,z_1^2)T_2(z_1^2,z_1^2)\right]\\
&\quad -z_1\left[-T(z_1^2,z_1^2)-\frac{1}{2}T_2(z_1^2,z_1^2)-T_1(z_1^2,z_1^2)T_2(z_1^2,z_1^2)\right],\quad z_1\in\mathbb{C}\backslash\{0\},\end{aligned}\tag{5.1.49}$$

which is equivalent to

$$\begin{aligned} 0 &= \left[T(z_1^2,z_1^2)+\frac{1}{2}T_1(z_1^2,z_1^2)+\frac{1}{2}T_2(z_1^2,z_1^2)+T_1(z_1^2,z_1^2)T_2(z_1^2,z_1^2)+\frac{1}{4}\right] \\ &\quad +z_1\left[-\frac{1}{2}T_1(z_1^2,z_1^2)+\frac{1}{2}T_2(z_1^2,z_1^2)\right] \\ &\quad +z_1^2\left[-T(z_1^2,z_1^2)-T_1(z_1^2,z_1^2)T_2(z_1^2,z_1^2)\right], \quad z_1\in\mathbb{C}\backslash\{0\}. \end{aligned} \tag{5.1.50}$$

Since the only odd powers of z_1 on the right-hand-side of (5.1.50) are contained in the terms $z_1\left[-\frac{1}{2}T_1(z_1^2,z_1^2)+\frac{1}{2}T_2(z_1^2,z_1^2)\right]$, while there are no odd powers of z_1 on the left-hand-side, it follows that

$$T_1(z_1^2,z_1^2)=T_2(z_1^2,z_1^2), \quad z_1\in\mathbb{C}\backslash\{0\}. \tag{5.1.51}$$

Substituting (5.1.51) into (5.1.50), yields

$$(1-z_1^2)T(z_1^2,z_1^2)+\frac{1}{2}(1-z_1)T_1(z_1^2,z_1^2)+\frac{1}{2}(1+z_1)T_1(z_1^2,z_1^2)+(1-z_1^2)(T_1(z_1^2,z_1^2))^2+\frac{1}{4}=0, \quad z_1\in\mathbb{C}\backslash\{0\},$$

which is equivalent to

$$(1-z_1^2)\left[T(z_1^2,z_1^2)+(T_1(z_1^2,z_1^2))^2\right]+T_1(z_1^2,z_1^2)+\frac{1}{4}=0, \quad z_1\in\mathbb{C}\backslash\{0\},$$

i.e.,

$$(1-z_1)\left[T(z_1,z_1)+(T_1(z_1,z_1))^2\right]+T_1(z_1,z_1)+\frac{1}{4}=0, \quad z_1\in\mathbb{C}\backslash\{0\}.$$

Hence, it follows that

$$T_1(z_1,z_1)+\frac{1}{4}=-(1-z_1)\left[T(z_1,z_1)+(T_1(z_1,z_1))^2\right], \quad z_1\in\mathbb{C}\backslash\{0\}. \tag{5.1.52}$$

It follows that there exists a bivariate Laurent polynomial U such that

$$T_1(z_1,z_1)+\frac{1}{4}=(1-z_1)U(z_1,z_1), \quad z_1\in\mathbb{C}\backslash\{0\}. \tag{5.1.53}$$

Further, by substituting (5.1.53) into (5.1.52), it follows that

$$(1-z_1)U(z_1,z_1) = -(1-z_1)T(z_1,z_1) - (1-z_1)\left[(1-z_1)U(z_1,z_1) - \frac{1}{4}\right]^2, \quad z_1 \in \mathbb{C}\backslash\{0\},$$

from which it follows that

$$T(z_1,z_1) = -(1-z_1)^2(U(z_1,z_1))^2 - \frac{1}{2}(1+z_1)U(z_1,z_1) - \frac{1}{16}, \quad z_1 \in \mathbb{C}\backslash\{0\}. \qquad (5.1.54)$$

It follows from Lemma 5.1 that, if the bivariate Laurent polynomial "degrees of freedom" T, T_1, and T_2 satisfy the conditions

$$\left.\begin{aligned} T(z_1,z_1) &= -(1-z_1)^2(U(z_1,z_1))^2 - \frac{1}{2}(1+z_1)U(z_1,z_1) - \frac{1}{16}, \quad z_1 \in \mathbb{C}\backslash\{0\}; \\ T_1(z_1,z_1) &= (1-z_1)U(z_1,z_1) - \frac{1}{4}, \quad z_1 \in \mathbb{C}\backslash\{0\}; \\ T_2(z_1,z_1) &= (1-z_1)U(z_1,z_1) - \frac{1}{4}, \quad z_1 \in \mathbb{C}\backslash\{0\}, \end{aligned}\right\} \qquad (5.1.55)$$

in terms of a single bivariate Laurent polynomial degree of freedom U, then the Laurent polynomial $\tilde{A}$ in (5.1.46) satisfies (5.1.2) as well as

$$\tilde{A}(z_1,z_2) = \left(\frac{z_1+z_2}{2}\right)\tilde{\tilde{A}}(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}, \qquad (5.1.56)$$

for some Laurent polynomial $\tilde{\tilde{A}}$. It follows that (5.1.55) and (5.1.56) provide a characteristic for the general solution $\tilde{\tilde{A}}$ to the identity

$$\left(\frac{z_1+z_2}{2}\right)\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\tilde{\tilde{A}}(z_1,z_2) + \left(\frac{-z_1+z_2}{2}\right)\left(\frac{1-z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\tilde{\tilde{A}}(-z_1,z_2)$$

$$\left(\frac{z_1-z_2}{2}\right)\left(\frac{1+z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\tilde{\tilde{A}}(z_1,-z_2) + \left(\frac{-z_1-z_2}{2}\right)\left(\frac{1-z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\tilde{\tilde{A}}(-z_1,-z_2)$$

$$= 1, \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}. \qquad (5.1.57)$$

We propose the following choice for the bivariate Laurent polynomials T, T_1, and T_2, in (5.1.55), in terms of a total number of *four* degrees of freedom Laurent polynomials U, V, W, and Z :

$$\left.\begin{aligned} T(z_1,z_2) &= -(1-z_1)(1-z_2)U^2(z_1,z_2) - \frac{1}{2}\left(1+\frac{1}{2}(z_1+z_2)\right)U(z_1,z_2) \\ &\qquad -\frac{1}{16} + (z_1-z_2)Z(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}; \\ T_1(z_1,z_2) &= (1-z_2)U(z_1,z_2) - \frac{1}{4} + (z_1-z_2)V(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}; \\ T_2(z_1,z_2) &= (1-z_1)U(z_1,z_2) - \frac{1}{4} + (z_1-z_2)W(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \end{aligned}\right\} \tag{5.1.58}$$

Note that we use the notation $U^2(z_1,z_2)$ for $(U(z_1,z_2))^2$, and it is easily verified that this choice for T, T_1, and T_2, does adhere to the conditions in (5.1.55). By this choice for T, T_1, and T_2, the Laurent polynomial $\tilde{A}$ becomes, for $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$,

$$\begin{aligned} \tilde{A}(z_1,z_2) &= \frac{4}{z_1z_2}\Bigg\{(1-z_1)(1-z_2)\Big[-(1-z_1^2)(1-z_2^2)U^2(z_1^2,z_2^2) \\ &\qquad -\frac{1}{2}\left(1+\frac{1}{2}(z_1^2+z_2^2)\right)U(z_1^2,z_2^2) - \frac{1}{16} + (z_1^2-z_2^2)Z(z_1^2,z_2^2)\Big] \\ &\qquad +\left[\frac{1}{2}+(1-z_1)\left[(1-z_2^2)U(z_1^2,z_2^2) - \frac{1}{4} + (z_1^2-z_2^2)V(z_1^2,z_2^2)\right]\right] \\ &\qquad \times\left[\frac{1}{2}+(1-z_2)\left[(1-z_1^2)U(z_1^2,z_2^2) - \frac{1}{4} + (z_1^2-z_2^2)W(z_1^2,z_2^2)\right]\right]\Bigg\} \end{aligned}$$

$$\begin{aligned}
= \quad & \frac{8}{z_1 z_2}\left(\frac{z_1+z_2}{2}\right)\left[\frac{1}{2}(1-z_1)(1-z_2)U(z_1^2,z_2^2) + (1-z_2)(1-z_2)(z_1-z_2)Z(z_1^2,z_2^2)\right. \\
& +(1-z_1)(1-z_2^2)(1-z_1)(z_1-z_2)U(z_1^2,z_2^2)W(z_1^2,z_2^2) \\
& +(1-z_1^2)(1-z_1)(1-z_2)(z_1-z_2)U(z_1^2,z_2^2)V(z_1^2,z_2^2) \\
& +\frac{1}{4}(1-z_1)(1+z_2)(z_1-z_2)V(z_1^2,z_2^2) \\
& +\frac{1}{4}(1+z_1)(1-z_2)(z_1-z_2)W(z_1^2,z_2^2) \\
& \left.+(1-z_1)(1-z_2)(z_1^2-z_2^2)(z_1-z_2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)+\frac{1}{8}\right], \qquad (5.1.59)
\end{aligned}$$

so that the Laurent polynomial $\tilde{\tilde{A}}$ in (5.1.56) can easily be found from (5.1.59). Let A^* be defined, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, by

$$
\begin{aligned}
A^*(z_1,z_2) \;&:=\; \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{z_1+z_2}{2}\right)\tilde{\tilde{A}}(z_1,z_2) \\
&= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{z_1+z_2}{2}\right)\frac{8}{z_1z_2} \\
&\quad \times \Bigg[\frac{1}{2}(1-z_1)(1-z_2)U(z_1^2,z_2^2) + (1-z_1)(1-z_2)(z_1-z_2)Z(z_1^2,z_2^2) \\
&\quad +(1-z_1)(1-z_2^2)(1-z_2)(z_1-z_2)U(z_1^2,z_2^2) \\
&\quad +(1-z_1^2)(1-z_1)(1-z_2)(z_1-z_2)U(z_1^2,z_2^2)V(z_1^2,z_2^2) \\
&\quad +\frac{1}{4}(1-z_1)(1+z_2)(z_1-z_2)V(z_1^2,z_2^2) \\
&\quad +\frac{1}{4}(1+z_1)(1-z_2)(z_1-z_2)W(z_1^2,z_2^2) \\
&\quad +(1-z_1)(1-z_2)(z_1^2-z_2^2)(z_1-z_2)V(z_1^2,z_2^2)W(z_1^2,z_2^2) + \frac{1}{8}\Bigg].
\end{aligned}
\tag{5.1.60}
$$

Hence, the Laurent polynomial

$$\begin{aligned}
A^*(z_1, z_1z_2) \quad = \quad & \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right)\frac{8}{z_1z_2} \\
& \times \left[\frac{1}{2}(1-z_1)(1-z_1z_2)U(z_1^2, z_1^2z_2^2)\right. \\
& +(1-z_1)(1-z_1z_2)(z_1-z_1z_2)Z(z_1^2, z_1^2z_2^2) \\
& +(1-z_1)(1-z_1^2z_2^2)(1-z_1z_2)(z_1-z_1z_2)U(z_1^2, z_1^2z_2^2) \\
& +(1-z_1^2)(1-z_1)(1-z_1z_2)(z_1-z_1z_2)U(z_1^2, z_1^2z_2^2)V(z_1^2, z_1^2z_2^2) \\
& +\frac{1}{4}(1-z_1)(1+z_1z_2)(z_1-z_1z_2)V(z_1^2, z_1^2z_2^2) \\
& +\frac{1}{4}(1+z_1)(1-z_1z_2)(z_1-z_1z_2)W(z_1^2, z_1^2z_2^2) \\
& +(1-z_1)(1-z_1z_2)(z_1^2-z_1^2z_2^2)(z_1-z_1z_2)V(z_1^2, z_1^2z_2^2)W(z_1^2, z_1^2z_2^2) \\
& \left.+\frac{1}{8}\right]. \qquad (5.1.61)
\end{aligned}$$

It finally follows that a class of solutions A for the identity (5.1.2) is given, in terms of arbitrary bivariate Laurent polynomials U, V, W, and Z, by

$$\begin{aligned}
A(z_1,z_2) \quad = \quad & \frac{8}{z_1z_2}\bigg[\frac{1}{2}(1-z_1)(1-z_1z_2)U(z_1^2,z_2^2)+z_1(1-z_1)(1-z_2)(1-z_1z_2)Z(z_1^2,z_2^2) \\
& +z_1(1-z_1)(1-z_2)(1-z_1^2z_2^2)(1-z_1z_2)U(z_1^2,z_2^2)W(z_1^2,z_2^2) \\
& +z_1(1-z_1^2)(1-z_1)(1-z_2)(1-z_1z_2)U(z_1^2,z_2^2)V(z_1^2,z_2^2) \\
& +\frac{1}{4}z_1(1-z_1)(1-z_2)(1+z_1z_2)V(z_1^2,z_2^2) \\
& +\frac{1}{4}z_1(1+z_1)(1-z_2)(1-z_1z_2)W(z_1^2,z_2^2) \\
& +z_1^3(1-z_1)(1-z_2^2)(1-z_2)(1-z_1z_2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)+\frac{1}{8}\bigg], \\
& \qquad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \qquad (5.1.62)
\end{aligned}$$

We therefore have the following result.

Theorem 5.2 *Let P be the refinement mask symbol corresponding to the Courant hat function of Example* 1.2, *as given by* (1.3.17). *A class of solutions to the identity* (2.2.10) *is then given by* (5.1.62), *where U, V, W, and Z, are arbitrary bivariate Laurent polynomials.*

Remark. Note that, in (5.1.62), the choice $U(z_1,z_2)=V(z_1,z_2)=W(z_1,z_2)=Z(z_1,z_2)=0$, $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, yields

$$A(z_1,z_2)=\frac{1}{z_1z_2}, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\},$$

which is the same particular solution that was given in (3.1.2) in Theorem 3.1, and is the solution A to the identity (2.2.10) with minimum support.

Chapter 6

The Homogenous Case: The Third Class of Identities

6.1 Solving the Third Class of Identities for the Two-Directional Box Spline Case

In this chapter, we focus on finding a general class of solutions for the Laurent polynomial B in the identity (2.2.12), where we are still interested in the case where P is the refinement symbol of the Courant hat function, as in (1.3.17). Towards that end, it is first shown, in a similar approach as in Chapter 5, how the identity (2.2.12) is solved in the case of any two-directional box spline, i.e., a solution $\tilde{B}$ is deduced for the identity

$$\left(\frac{1+z_1}{2}\right)^k \left(\frac{1+z_2}{2}\right)^\ell \tilde{B}(z_1,z_2) + \left(\frac{1-z_1}{2}\right)^k \left(\frac{1+z_2}{2}\right)^\ell \tilde{B}(-z_1,z_2)$$

$$+\left(\frac{1+z_1}{2}\right)^k \left(\frac{1-z_2}{2}\right)^\ell \tilde{B}(z_1,-z_2) + \left(\frac{1-z_1}{2}\right)^k \left(\frac{1-z_2}{2}\right)^\ell \tilde{B}(-z_1,-z_2) = 0,$$

$$(z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \qquad (6.1.1)$$

First, note that (6.1.1) holds if and only if

$$(1+z_1)^k(1+z_2)^\ell \tilde{B}(z_1,z_2) + (1-z_1)^k(1+z_2)^\ell \tilde{B}(-z_1,z_2)$$

$$+(1+z_1)^k(1-z_2)^\ell \tilde{B}(z_1,-z_2) + (1-z_1)^k(1-z_2)^\ell \tilde{B}(-z_1,-z_2) = 0,$$

$$(z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \qquad (6.1.2)$$

In (6.1.2), let $F(z_1,z_2)$ be the Laurent polynomial defined by

$$F(z_1,z_2) := (1+z_1)^k(1+z_2)^\ell \tilde{B}(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \qquad (6.1.3)$$

and define

$$\begin{aligned} F_1(z_1,z_2) &:= F(z_1,z_2) + F(z_1,-z_2) \\ &= (1+z_1)^k G_1(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \qquad (6.1.4) \end{aligned}$$

where

$$G_1(z_1,z_2) = (1+z_2)^\ell \tilde{B}(z_1,z_2) + (1-z_2)^\ell \tilde{B}(z_1,-z_2), \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \qquad (6.1.5)$$

It follows from (6.1.2) and (6.1.4) that $F_1(z_1,z_2) + F_1(-z_1,z_2) = 0,\ (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, i.e.,

$$(1+z_1)^k G_1(z_1,z_2) + (1-z_1)^k G_1(-z_1,z_2) = 0, \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \qquad (6.1.6)$$

It follows that there exists a Laurent polynomial $\tilde{I}_1$ such that

$$G_1(z_1,z_2) = (1-z_1)^k \tilde{I}_1(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \qquad (6.1.7)$$

Notice, by substituting (6.1.7) into (6.1.6), that the Laurent polynomial $\tilde{I}_1$ is odd in z_1. Also, since, from (6.1.4), F_1 is even in z_2, it follows from the definition of G_1 and (6.1.7) that $\tilde{I}_1$ is even in z_2. It follows that there exists a Laurent polynomial I_1 such that

$\tilde{I}_1(z_1, z_2) = z_1 I_1(z_1^2, z_2^2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, i.e.,

$$G_1(z_1, z_2) = z_1(1-z_1)^k I_1(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{6.1.8}$$

Similarly, let

$$\begin{aligned} F_2(z_1, z_2) &:= F(z_1, z_2) + F(-z_1, z_2) \\ &= (1+z_2)^\ell G_2(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \end{aligned} \tag{6.1.9}$$

where

$$G_2(z_1, z_2) = (1+z_1)^k \tilde{B}(z_1, z_2) + (1-z_1)^k \tilde{B}(-z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{6.1.10}$$

It follows from (6.1.2) and (6.1.9) that $F_2(z_1, z_2) + F_2(z_1, -z_2) = 0$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, i.e.,

$$(1+z_2)^\ell G_2(z_1, z_2) + (1-z_2)^\ell G_2(z_1, -z_2) = 0, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{6.1.11}$$

This implies that there exists a Laurent polynomial $\tilde{I}_2$ such that

$$G_2(z_1, z_1) = (1-z_2)^\ell \tilde{I}_2(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{6.1.12}$$

Notice, by substituting (6.1.12) into (6.1.11), that the Laurent polynomial $\tilde{I}_2$ is odd in z_2. Also, since, from (6.1.9), F_2 is even in z_1, it follows from the definition of G_2 and (6.1.12) that $\tilde{I}_2$ is even in z_1. Hence there exists a Laurent polynomial I_2 such that $\tilde{I}_2(z_1, z_2) = z_2 I_2(z_1^2, z_2^2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, i.e.,

$$G_2(z_1, z_2) = z_2(1-z_2)^\ell I_2(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{6.1.13}$$

Next, recall from (6.1.10) and (6.1.13) that

$$(1+z_1)^k \tilde{B}(z_1, z_2) + (1-z_1)^k \tilde{B}(-z_1, z_2) = z_2(1-z_2)^\ell I_2(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{6.1.14}$$

Note that a particular solution to (6.1.14) is given by

$$\tilde{B}^*(z_1, z_2) := \frac{1}{2} z_2 (1 - z_2)^\ell I_2(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{6.1.15}$$

i.e.,

$$(1 + z_1)^k \tilde{B}^*(z_1, z_2) + (1 - z_1)^k \tilde{B}^*(-z_1, z_2) = z_2 (1 - z_2)^\ell I_2(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{6.1.16}$$

Subtracting (6.1.16) from (6.1.14), yields

$$(1 + z_1)^k \left[\tilde{B}(z_1, z_2) - \tilde{B}^*(z_1, z_2)\right] = -(1 - z_1)^k \left[\tilde{B}(-z_1, z_2) - \tilde{B}^*(-z_1, z_2)\right],$$

$$(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{6.1.17}$$

which implies the existence of a Laurent polynomial $\tilde{I}_3$ such that

$$\tilde{B}(z_1, z_2) - \tilde{B}^*(z_1, z_2) = (1 - z_1)^k \tilde{I}_3(z_1, z_2). \tag{6.1.18}$$

By substituting (6.1.18) into (6.1.17), it follows that $\tilde{I}_3$ is odd in z_1, so that there exists a Laurent polynomial I_3 such that $\tilde{I}_3(z_1, z_2) = z_1 I_3(z_1^2, z_2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, i.e., as follows further from (6.1.18) and (6.1.15),

$$\tilde{B}(z_1, z_2) = \frac{1}{2} z_2 (1 - z_2)^\ell I_2(z_1^2, z_2^2) + z_1 (1 - z_1)^k I_3(z_1^2, z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{6.1.19}$$

We next substitute (6.1.19) into (6.1.5) and use (6.1.8) to obtain

$$(1 + z_2)^\ell \left(\frac{1}{2} z_2\right) (1 - z_2)^\ell I_2(z_1^2, z_2^2) + (1 + z_2)^\ell z_1 (1 - z_1)^k I_3(z_1^2, z_2)$$

$$+(1 - z_2)^\ell \left(-\frac{1}{2} z_2\right) (1 + z_2)^\ell I_2(z_1^2, z_2^2) + (1 - z_2)^\ell z_1 (1 - z_1)^k I_3(z_1^2, -z_2)$$

$$= z_1 (1 - z_1)^k I_1(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\},$$

i.e.,

$$z_1(1-z_1)^k\left[(1+z_2)^\ell I_3(z_1^2,z_2)+(1-z_2)^\ell I_3(z_1^2,-z_2)\right] \;=\; z_1(1-z_1)^k I_1(z_1^2,z_2^2),$$

$$(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\},$$

which is equivalent to

$$(1+z_2)^\ell I_3(z_1^2,z_2)+(1-z_2)^\ell I_3(z_1^2,-z_2)=I_1(z_1^2,z_2^2),\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{6.1.20}$$

It is noted that a particular solution to (6.1.20) is given by

$$I_3^*(z_1,z_2):=\frac{1}{2}I_1(z_1,z_2^2),\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{6.1.21}$$

i.e.,

$$(1+z_2)^\ell I_3^*(z_1^2,z_2)+(1-z_2)^\ell I_3^*(z_1^2,-z_2)=I_1(z_1^2,z_2^2),\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{6.1.22}$$

Subtracting (6.1.22) from (6.1.20), yields

$$(1+z_2)^\ell\left[I_3(z_1^2,z_2)-I_3^*(z_1^2,z_2)\right] \;=\; -(1-z_2)^\ell\left[I_3(z_1^2,-z_2)-I_3^*(z_1^2,-z_2)\right],$$
$$(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{6.1.23}$$

which implies the existence of a Laurent polynomial $\tilde{I}_4$ such that

$$I_3(z_1^2,z_2)-I_3^*(z_1^2,z_2)=(1-z_2)^\ell\tilde{I}_4(z_1^2,z_2),\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{6.1.24}$$

Notice, by substituting (6.1.24) into (6.1.23), that $\tilde{I}_4$ is odd in z_2, so that there exists a Laurent polynomial I_4 such that $\tilde{I}_4(z_1,z_2)=z_2I_4(z_1,z_2^2)$, $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, i.e., from (6.1.24) and (6.1.21),

$$I_3(z_1,z_2)=\frac{1}{2}I_1(z_1,z_2^2)+z_2(1-z_2)^\ell I_4(z_1,z_2^2),\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{6.1.25}$$

Finally, it follows from (6.1.19) and (6.1.25) that the general Laurent polynomial solution to the homogenous identity (6.1.2) is given, in terms of three *degrees of freedom* Laurent polynomials I_1, I_2, and I_4, by

$$\tilde{B}(z_1, z_2) = \frac{1}{2}z_1(1-z_1)I_1(z_1^2, z_2^2) + \frac{1}{2}z_2(1-z_2)I_2(z_1^2, z_2^2) + z_1z_2(1-z_1)(1-z_2)I_4(z_2^2, z_2^2),$$

$$(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \qquad (6.1.26)$$

Since, as in Chapter 3, we are interested in the case of the Courant hat function $B_{1,1,1}$, we set $k = \ell = 1$ in (6.1.26) and conclude the following result.

Theorem 6.1 *The general solution to the homogenous identity*

$$\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\tilde{B}(z_1, z_2) + \left(\frac{1-z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\tilde{B}(-z_1, z_2)$$
$$+\left(\frac{1+z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\tilde{B}(z_1, -z_2) + \left(\frac{1-z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\tilde{B}(-z_1, -z_2),$$
$$= 0, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \qquad (6.1.27)$$

is given by the Laurent polynomial

$$\tilde{B}(z_1, z_2) = z_1(1-z_1)M_1(z_1^2, z_2^2) + z_2(1-z_2)M_2(z_1^2, z_2^2) + z_1z_2(1-z_1)(1-z_2)M_3(z_1^2, z_2^2),$$

$$(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \qquad (6.1.28)$$

where M_1, M_2, and M_3 are arbitrary Laurent polynomials.

6.2 The Homogenous Identity for the Three-Directional Box Spline Case

We proceed to show how the result in Theorem 6.1 can be used to derive a class of solutions to the identities in (2.2.12) for the case where P is the mask symbol corresponding to the Courant hat function, i.e., $B_{1,1,1}$. We will again rely on the result in Lemma 5.1 in Chapter 5. With this result in mind, we proceed to find conditions on the Laurent polynomials M_1,

M_2, and M_3 in (6.1.28), such that the solution $\tilde{B}$ given by (6.1.28) satisfies the property

$$\tilde{B}(z_1, -z_1) = 0, \quad z_1 \in \mathbb{C}\backslash\{0\}. \tag{6.2.1}$$

From (6.1.28), it follows that (6.2.1) holds if and only if

$$z_1(1-z_1)M_1(z_1^2, z_1^2) - z_1(1+z_1)M_2(z_1^2, z_1^2) - z_1^2(1-z_1^2)M_3(z_1^2, z_1^2) = 0, \quad z_1 \in \mathbb{C}\backslash\{0\},$$

which is equivalent to

$$z_1\left[M_1(z_1^2, z_1^2) - M_2(z_1^2, z_1^2)\right] - z_1^2\left[M_1(z_1^2, z_1^2) + M_2(z_1^2, z_1^2)\right] = z_1^2(1-z_1^2)M_3(z_1^2, z_1^2), \quad z_1 \in \mathbb{C}\backslash\{0\}, \tag{6.2.2}$$

from which it follows that

$$M_1(z_1^2, z_1^2) = M_2(z_1^2, z_1^2), \quad z_1 \in \mathbb{C}\backslash\{0\},$$

i.e.,

$$M_1(z_1, z_1) = M_2(z_1, z_1), \quad z_1 \in \mathbb{C}\backslash\{0\}. \tag{6.2.3}$$

Substituting (6.2.3) into (6.2.2) yields

$$M_1(z_1^2, z_1^2) = -\frac{1}{2}(1-z_1)^2 M_3(z_1^2, z_1^2), \quad z_1 \in \mathbb{C}\backslash\{0\},$$

and it is concluded that a necessary condition for the Laurent polynomial solution $\tilde{B}$ in (6.1.28) to satisfy the condition in (6.2.1), is given by

$$\left.\begin{aligned} M_1(z_1, z_1) &= -\frac{1}{2}(1-z_1)M_3(z_1, z_1), \quad z_1 \in \mathbb{C}\backslash\{0\}; \\ M_2(z_1, z_1) &= M_1(z_1, z_1), \quad z_1 \in \mathbb{C}\backslash\{0\}, \end{aligned}\right\} \tag{6.2.4}$$

where M_3 is any Laurent polynomial. It also follows immediately that, if M_1 and M_2 satisfy (6.2.4), then $\tilde{B}$ in (6.1.28) indeed satisfies (6.2.1), so that (6.2.4) is also a sufficient condition for the property (6.2.1) to hold. Thus, by using the result in Lemma 5.1, we have proved the following result.

Theorem 6.2 *The general solution $\tilde{B}$ for the homogenous identity*

$$\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{z_1+z_2}{2}\right)\tilde{B}(z_1,z_2)+\left(\frac{1-z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{-z_1+z_2}{2}\right)\tilde{B}(-z_1,z_2)$$
$$\left(\frac{1+z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\left(\frac{z_1-z_2}{2}\right)\tilde{B}(z_1,-z_2)+\left(\frac{1-z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\left(\frac{-z_1-z_2}{2}\right)\tilde{B}(-z_1,-z_2)$$
$$=0, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{6.2.5}$$

is given by (6.1.28)*, where M_1, M_2, and M_3, are any Laurent polynomials such that M_1 and M_2 satisfy the condition* (6.2.4)*.*

Observe that the second condition in (6.2.4) is equivalent to the existence of a bivariate Laurent polynomial N_1 such that

$$M_2(z_1,z_2)=M_1(z_1,z_2)-(z_1-z_2)N_1(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{6.2.6}$$

Also, by taking a bivariate Laurent polynomial $\tilde{M}$ that satisfies

$$\tilde{M}(z_1,z_1):=-\frac{1}{2}(1-z_1)M_3(z_1,z_1), \quad z_1\in\mathbb{C}\backslash\{0\}, \tag{6.2.7}$$

for an arbitrary Laurent polynomial M_3, a class of bivariate Laurent polynomials M_1 that satisfy the first condition in (6.2.4) is characterized by

$$M_1(z_1,z_2)=\tilde{M}(z_1,z_2)+(z_1-z_2)N_2(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{6.2.8}$$

for a degree of freedom Laurent polynomial N_2. We therefore have the following result.

Theorem 6.3 *Let M_3 be an arbitrarily chosen bivariate Laurent polynomial, and let $\tilde{M}$ be any bivariate Laurent polynomial that satisfies $\tilde{M}(z_1,z_1)=-\frac{1}{2}(1-z_1)M_3(z_1,z_1)$, $z_1\in\mathbb{C}\backslash\{0\}$. Define the Laurent polynomials M_1 and M_2 by*

$$\left.\begin{aligned} M_1(z_1,z_2) &:= \tilde{M}(z_1,z_2)+(z_1-z_2)N_2(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\};\\ M_2(z_1,z_2) &:= \tilde{M}(z_1,z_2)+(z_1-z_2)\left[N_2(z_1,z_2)-N_1(z_1,z_2)\right], \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \end{aligned}\right\} \tag{6.2.9}$$

in terms of arbitrary Laurent polynomials N_1 and N_2. Then the conditions in (6.2.4) *are satisfied, and hence the Laurent polynomial $\tilde{B}$ in* (6.1.28) *satisfies the homogenous identity* (6.2.5)*.*

In the light of Theorem 6.3, let us choose

$$\left.\begin{aligned} M_1(z_1,z_2) &= -\frac{1}{2}(1-z_2)M_3(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}; \\ M_2(z_1,z_2) &= -\frac{1}{2}(1-z_1)M_3(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \end{aligned}\right\} \tag{6.2.10}$$

where M_3 is arbitrary. Then the Laurent polynomial $\tilde{B}$ in (6.1.28) is given, for $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, by

$$\begin{aligned} \tilde{B}(z_1,z_2) &= -\frac{1}{2}z_1(1-z_1)(1-z_2^2)M_3(z_1^2,z_2^2)-\frac{1}{2}z_2(1-z_1^2)(1-z_2)M_3(z_1^2,z_2^2) \\ &\quad +z_1z_2(1-z_1)(1-z_2)M_3(z_1^2,z_2^2) \\ &= -\left(\frac{z_1+z_2}{2}\right)(1-z_1)(1-z_2)M_3(z_1^2,z_2^2). \end{aligned}$$

It follows from (6.2.5) that the Laurent polynomial $\tilde{B}^*$ defined, for $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, by

$$\begin{aligned} \tilde{B}^*(z_1,z_2) &:= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\tilde{B}(z_1,z_2) \\ &= -\frac{1}{8}(1-z_1^2)(1-z_2^2)(z_1+z_2)M_3(z_1^2,z_2^2), \end{aligned} \tag{6.2.11}$$

satisfies the identity

$$\tilde{B}^*(z_1,z_2)+\tilde{B}^*(-z_1,z_2)+\tilde{B}^*(z_1,-z_2)+\tilde{B}^*(-z_1,-z_2)=0, \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{6.2.12}$$

It also follows from (6.2.11) that, for $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$,

$$\begin{aligned} \tilde{B}^*(z_1,z_1z_2) &= -\frac{1}{8}(1-z_1^2)(1-z_1^2z_2^2)(z_1+z_1z_2)M_3(z_1^2,z_1^2z_2^2) \\ &= (1+z_1)(1+z_2)(1+z_1z_2)\left[z_1(1-z_1)(1-z_1z_2)M(z_1^2,z_2^2)\right] \end{aligned} \tag{6.2.13}$$

for some Laurent polynomial M. The following result is now a direct consequence of (6.2.11) and (6.2.13),

Theorem 6.4 *A class of solutions to the Bezout identity* (2.2.12) *is given by*

$$B(z_1, z_2) = z_1(1 - z_1)(1 - z_1 z_2)M(z_1^2, z_2^2), \quad z_1, z_2 \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{6.2.14}$$

for any bivariate Laurent polynomial M.

Remark. Note that the choice $M(z_1, z_2) := \frac{1}{z_1^2}$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, gives the same particular solution to the identity (2.2.12) as the Laurent polynomial B_1 in (3.1.2) in Theorem 3.1. The Laurent polynomials B_2 and B_3 in (3.1.2), which are both also particular solutions to (2.2.12), can not be obtained from (6.2.14) by any choice of the Laurent polynomial M. One would however be able to obtain B_1 and B_2 by different choices of the Laurent polynomial $\tilde{M}$ in Theorem 6.3. The particular choice (see also (6.2.10)) that finally yielded the result in Theorem 6.4, was to show how at least one of the particular Laurent polynomial solutions B_1 in (3.1.2) can be obtained in a systematic approach. The result in Theorem 6.4 also paves the way for systematically solving the two remaining Bezout identities (2.2.11) and (2.2.13) in the wavelet decomposition system, as will be seen in the next two chapters.

Chapter 7

Solving the Fourth Class of Identities

In the previous two chapters, it was shown how the general solutions to the sets of bivariate Bezout identities in (2.2.10) and (2.2.12) were characterized in terms of certain conditions to be satisfied by their Laurent polynomial solutions in one variable alone, and in terms of a number of arbitrary bivariate Laurent polynomials satisfying some constraints. At some point, one had to impose certain assumptions on the required Laurent polynomial solution, and continue to deal with a "general class" of solutions in terms of corresponding degrees of freedom, instead of working with the entire most general solution class simultaneously. For example, in the previous chapter we discussed one way of choosing the two Laurent polynomials M_1 and M_2 of (6.2.4) in terms of one Laurent polynomial, M_3, the specific choices as were given in (6.2.10), and which yielded the Laurent polynomial B in (6.2.14) as a solution to the identity (2.2.12) for the Courent hat function case. In this chapter, the aim will be to work with this particular choice for B and to derive the general solution Q corresponding to it, as a solution for (2.2.13) in Chapter 2. Hence, the Laurent polynomial solutions derived in this chapter will *not* be the *most* general solutions to (2.2.13), but instead they will represent a class of "most general solutions corresponding to the solution B in (6.2.14)."

The remainder of this chapter will thus be based on the particular choice of the bivariate Laurent polynomial B given by (6.2.14). This chapter will be presented in two parts: the first part will work with the case $\delta = 0$ in (2.2.13) (that is, $\alpha = \beta$), whereas the second part will work with the case $\delta = 1$ (that is, $\alpha \neq \beta$). Note that throughout this chapter and the next, we will, for the sake of simplicity of writing and to emphasize the pursuit of general solutions B and Q to the identities (2.2.13) and (2.2.11) respectively, write B and Q instead of B_β and Q_α, respectively.

7.1 Solving the Fourth Class of Identities for the Three-Directional Box Spline Case: $\delta = 0$

Recall from (2.2.13), where $\delta = 0$ is fixed and where B is given by (6.2.14) in terms of some fixed Laurent polynomial M, that we are to solve the following identity for Q :

$$\begin{aligned} z_1 M(z_1^2, z_2^2)\,[(1-z_1)(1-z_1z_2)Q(z_1,z_2) + (1-z_1)(1+z_1z_2)Q(z_1,-z_2) \\ - (1+z_1)(1+z_1z_2)Q(-z_1,z_2) - (1+z_1)(1-z_1z_2)Q(-z_1,-z_2)] = 0, \\ (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \qquad (7.1.1) \end{aligned}$$

Since we are not interested in the trivial case where $M(z_1, z_2) = 0,\ (z_1, z_2) \in \mathbb{C}^2\backslash\{(0, 0)\}$, (7.1.1) is equivalent to

$$\begin{aligned} &(1-z_1)\,[(1-z_1z_2)Q(z_1,z_2) + (1+z_1z_2)Q(z_1,-z_2)] \\ &= (1+z_1)\,[(1+z_1z_2)Q(-z_1,z_2) + (1-z_1z_2)Q(-z_1,-z_2)] \\ &\qquad (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \qquad (7.1.2) \end{aligned}$$

which implies the existence of a Laurent polynomial $\tilde{J}$ such that

$$(1-z_1z_2)Q(z_1,z_2) + (1+z_1z_2)Q(z_1,-z_2) = (1+z_1)\tilde{J}(z_1,z_2), \quad (z_1,z_2) \in \mathbb{C}^2\backslash\{(0,0)\}. \qquad (7.1.3)$$

By substituting (7.1.3) into (7.1.2), it holds that $\tilde{J}$ is even in z_1, whereas it is clear from (7.1.3) that $\tilde{J}$ is also even in z_2. This implies the existence of a Laurent polynomial $\hat{J}$

such that $\tilde{J}(z_1, z_2) = \hat{J}(z_1^2, z_2^2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, i.e.,

$$(1 - z_1z_2)Q(z_1, z_2) + (1 + z_1z_2)Q(z_1, -z_2) = (1 + z_1)\hat{J}(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{7.1.4}$$

Note that the Laurent polynomial

$$Q^*(z_1, z_2) := \frac{1}{2}(1 + z_1)\hat{J}(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{7.1.5}$$

is a particular solution to (7.1.4), i.e.,

$$(1 - z_1z_2)Q^*(z_1, z_2) + (1 + z_1z_2)Q^*(z_1, -z_2) = (1 + z_1)\hat{J}(z_1^2, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{7.1.6}$$

Subtracting (7.1.6) from (7.1.4), yields

$$\begin{aligned} (1 - z_1z_2)\left[Q(z_1, z_2) - Q^*(z_1, z_2)\right] &= -(1 + z_1z_2)\left[Q(z_1, -z_2) - Q^*(z_1, -z_2)\right], \\ & \qquad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \end{aligned} \tag{7.1.7}$$

and it follows that there is a Laurent polynomial K such that

$$Q(z_1, z_2) - Q^*(z_1, z_2) = (1 + z_1z_2)\tilde{K}(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{7.1.8}$$

Substituting (7.1.8) into (7.1.7), yields $\tilde{K}$ to be odd in z_2, implying the existence of a Laurent polynomial K such that $\tilde{K}(z_1, z_2) = z_2K(z_1, z_2^2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, i.e., by using also (7.1.8) and (7.1.5),

$$Q(z_1, z_2) = \frac{1}{2}(1 + z_1)\hat{J}(z_1^2, z_2^2) + z_2(1 + z_1z_2)K(z_1, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\},$$

or equivalently, for some bivariate Laurent polynomial J,

$$Q(z_1, z_2) = (1 + z_1)\hat{J}(z_1^2, z_2^2) + z_2(1 + z_1z_2)K(z_1, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{7.1.9}$$

We therefore have the following result.

Theorem 7.1 *Let M be an arbitrary bivariate Laurent polynomial, and let the bivariate Laurent polynomial B be given as in* (6.2.14). *Then the general solution to the homogenous* *identity* (7.1.1)*, and hence also the identity (2.2.13) for the case* $\delta = 0$*, is given by* (7.1.9)*, where* J *and* K *are arbitrary bivariate Laurent polynomials.*

Our next goal is to place certain restrictions on the arbitrary Laurent polynomials in the solution Q in (7.1.9) such that, moreover, the Laurent polynomial Q also satisfies the final identity (2.2.11) in our wavelet decomposition system, with the Laurent polynomial A given by (5.1.62). We shall investigate this in the following chapter. First, we shall focus our attention for the remainder of this chapter on the case where $\delta = 1$ in the set of identities (2.2.13).

7.2 Solving the Fourth Class of Identities for the Three-Directional Box Spline Case: $\delta = 1$

With the aim to find the solution Q for the identity in (2.2.13), we now fix $\delta = 1$ as well as the Laurent polynomial B as in (6.2.14), and note that (2.2.13) becomes

$$M(z_1^2, z_2^2)\hat{M}(z_1, z_2) = 1, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{7.2.1}$$

where M is the Laurent polynomial in (6.2.14), and

$$\begin{aligned}\hat{M}(z_1, z_2) \ :=\ & z_1(1-z_1)(1-z_1z_2)Q(z_1,z_2) + z_1(1-z_1)(1+z_1z_2)Q(z_1,-z_2) \\ & -z_1(1+z_1)(1+z_1z_2)Q(-z_1,z_2) - z_1(1+z_1)(1-z_1z_2)Q(-z_1,-z_2), \\ & (z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}.\end{aligned} \tag{7.2.2}$$

Note that, since M and Q, and hence also $\hat{M}$, are all Laurent polynomials, it follows from (7.2.1) that both M and $\hat{M}$ have to in fact be *monomials* on $\mathbb{C}^2 \backslash \{(0,0)\}$ (that is, they must each have the form $cz_1^k z_2^\ell$ for some $(k,\ell) \in \mathbb{Z}^2$ and some coefficient c), so that we can write

$$M(z_1^2, z_2^2) = z_1^{2u} z_2^{2v}, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{7.2.3}$$

for some $u, v \in \mathbb{Z}$ in (7.2.1), and hence it remains to solve for Q from

$$z_1^{2u} z_2^{2v} \hat{M}(z_1, z_2) = 1, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\},$$

with $\hat{M}$ defined by (7.2.2), or equivalently, by writing $\tilde{Q}(z_1, z_2) := z_1^{2u} z_2^{2v} Q(z_1, z_2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, and after also using (7.2.2),

$$\begin{aligned} z_1(1-z_1)(1-z_1z_2)\tilde{Q}(z_1, z_2) + z_1(1-z_1)(1+z_1z_2)\tilde{Q}(z_1, -z_2) \\ -z_1(1+z_1)(1+z_1z_2)\tilde{Q}(-z_1, z_2) - z_1(1+z_1)(1-z_1z_2)\tilde{Q}(-z_1, -z_2) = 1, \\ (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \end{aligned} \tag{7.2.4}$$

Let the Laurent polynomials L and L_1 be defined, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, by

$$L(z_1, z_2) := z_1(1-z_1)(1-z_1z_2)\tilde{Q}(z_1, z_2); \tag{7.2.5}$$

$$L_1(z_1, z_2) := L(z_1, z_2) + L(z_1, -z_2) = z_1(1-z_1)R(z_1, z_2), \tag{7.2.6}$$

where

$$R(z_1, z_2) := (1-z_1z_2)\tilde{Q}(z_1, z_2) + (1+z_1z_2)\tilde{Q}(z_1, -z_2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{7.2.7}$$

It follows from (7.2.6) that (7.2.4) is equivalent to

$$z_1(1-z_1)R(z_1, z_2) - z_1(1+z_1)R(-z_1, z_2) = 1, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{7.2.8}$$

Note that a particular solution to (7.2.8) is given by

$$R^*(z_1, z_2) := -\frac{1}{2z_1^2}, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, \tag{7.2.9}$$

i.e.,

$$z_1(1-z_1)R^*(z_1, z_2) - z_1(1+z_1)R^*(-z_1, z_2) = 1, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{7.2.10}$$

Subtracting (7.2.10) from (7.2.8), yields

$$z_1(1-z_1)\left[R(z_1,z_2)-R^*(z_1,z_2)\right]=z_1(1+z_1)\left[R(-z_1,z_2)-R^*(-z_1,z_2)\right], \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{7.2.11}$$

so that there exists a Laurent polynomial $\tilde{S}$ such that

$$R(z_1,z_2)-R^*(z_1,z_2)=(1+z_1)\tilde{S}(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{7.2.12}$$

Note that substituting (7.2.12) into (7.2.11) yields $\tilde{S}$ to be even in z_1. Furthermore, since, from (7.2.7) and (7.2.9), both R and R^* are even in z_2, it follows immediately from (7.2.12) that $\tilde{S}$ is even in z_2. Hence there exists a Laurent polynomial S such that $\tilde{S}(z_1,z_2)=S(z_1^2,z_2^2)$, $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, and it follows from (7.2.12) and (7.2.9) that the general solution to (7.2.8) is given by

$$R(z_1,z_2)=-\frac{1}{2z_1^2}+(1+z_1)S(z_1^2,z_2^2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{7.2.13}$$

It further follows from (7.2.7) and (7.2.13) that

$$(1-z_1z_2)\tilde{Q}(z_1,z_2)+(1+z_1z_2)\tilde{Q}(z_1,-z_2)=-\frac{1}{2z_1^2}+(1+z_1)S(z_1^2,z_2^2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{7.2.14}$$

for which we proceed to find the general solution $\tilde{Q}$. We observe that the Laurent polynomial

$$\tilde{Q}^*(z_1,z_2):=-\frac{1}{4z_1^2}+\frac{1}{2}(1+z_1)S(z_1^2,z_2^2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \tag{7.2.15}$$

is a particular solution to the identity (7.2.14), i.e.,

$$(1-z_1z_2)\tilde{Q}^*(z_1,z_2)+(1+z_1z_2)\tilde{Q}^*(z_1,-z_2)=-\frac{1}{2z_1^2}+(1+z_1)S(z_1^2,z_2^2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{7.2.16}$$

Subtracting (7.2.16) from (7.2.14), yields

$$(1-z_1z_2)\left[\tilde{Q}(z_1,z_2)-\tilde{Q}^*(z_1,z_2)\right]=-(1+z_1z_2)\left[\tilde{Q}(z_1,-z_2)-\tilde{Q}^*(z_1,-z_2)\right],$$
$$(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \qquad (7.2.17)$$

which implies the existence of a Laurent polynomial $\tilde{N}$ such that

$$\tilde{Q}(z_1,z_2)-\tilde{Q}^*(z_1,z_2)=(1+z_1z_2)\tilde{N}(z_1,z_2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \qquad (7.2.18)$$

Substituting (7.2.18) into (7.2.17) yields $\tilde{N}$ to be odd in z_2, so that $\tilde{N}$ has the form $\tilde{N}(z_1,z_2)=z_2N(z_1,z_2^2)$, $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, for some Laurent polynomial N. It finally follows from (7.2.18) and (7.2.15) that the general solution to (7.2.14) is given by

$$\tilde{Q}(z_1,z_2)=-\frac{1}{4z_1^2}+\frac{1}{2}(1+z_1)S(z_1^2,z_2^2)+z_2(1+z_1z_2)N(z_1,z_2^2), \quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}. \qquad (7.2.19)$$

Finally, by recalling the relationship $\tilde{Q}(z_1,z_2)=z_1^{2u}z_2^{2v}Q(z_1,z_2)$, $(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, as well as (7.2.3), we therefore have the following result.

Theorem 7.2 *Let the Laurent polynomial B be as in* (6.2.14), *in terms of an arbitrary Laurent polynomial M. The Laurent polynomial solution to the identity* (7.2.1), *where $\hat{M}$ is defined by* (7.2.2), *and hence also the Laurent polynomial solution to the identity* (2.2.13), *where $\delta=1$, is given by*

$$Q(z_1,z_2) = \frac{1}{M(z_1^2,z_2^2)}\left[-\frac{1}{4z_1^2}+\frac{1}{2}(1+z_1)S(z_1^2,z_2^2)+z_2(1+z_1z_2)N(z_1,z_2^2)\right],$$
$$(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \qquad (7.2.20)$$

where S and N are arbitrary Laurent polynomials.

Remark. Note that, for the case $\delta=1$, the work in the beginning of this section has also stipulated a further condition on the Laurent polynomial M in (6.2.14) in Chapter 4, namely that it must be a monomial. Also note that, by the particular choice $M(z_1,z_2)=\frac{1}{z_1^2}$, that is needed to obtain the one particular solution B_1 in (3.1.2) to the identity (2.2.12) (see also the remark after Theorem 6.4), the Laurent polynomial M does indeed satisfy the property of being a monomial.

Chapter 8

Reconciling Classes of Solutions: The Second Class of Identities

In the previous chapters it has been shown how classes of Laurent polynomial solutions to all of the identities in (2.2.10), (2.2.12), and (2.2.13) exist, and can be expressed in terms of arbitrary Laurent polynomials. The work in this chapter will aim to reconcile these classes of solutions, in the sense that (2.2.11) is also satisfied. Since the previous chapter has shown how Laurent polynomial solutions for Q in the identity (2.2.13) exist in the case where $\delta = 0$ and $\delta = 1$, respectively, we will similarly conduct our investigation in this chapter in two parts.

8.1 Solving the Second Class of Identities for the Three-Directional Box Spline Case: $\delta = 0$

Recall that the Laurent polynomials A in (5.1.62) and Q in (7.1.9) satisfy the identities (2.2.10) and (2.2.13), respectively, where the Laurent polynomials U, V, W, Z, J, and K, are arbitrary, and where we keep in mind that the refinement mask symbol P is fixed as the symbol of the Courant hat function $B_{1,1,1}$, and B, as given by (6.2.14), belongs to a class of Laurent polynomial solutions to the identity (2.2.12), in terms of a bivariate Laurent polynomial M. For this case, the identity (2.2.11) becomes, for $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$,

$$
\begin{aligned}
0 \;=\;& \frac{2}{z_1 z_2}(1-z_1^2)J(z_1^2,z_2^2)U(z_1^2,z_2^2)\left[(1-z_1z_2)-(1+z_1z_2)-(1+z_1z_2)+(1-z_1z_2)\right] \\[2ex]
&+\frac{4}{z_2}(1-z_1^2)J(z_1^2,z_2^2)Z(z_1^2,z_2^2)\left[(1-z_2)(1-z_1z_2)\right. \\
&\qquad\left.-(1+z_2)(1+z_1z_2)+(1-z_2)(1+z_1z_2)-(1+z_2)(1-z_1z_2)\right] \\[2ex]
&+\frac{4}{z_2}(1-z_1^2)(1-z_1^2z_2^2)J(z_1^2,z_2^2)U(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_2)(1-z_1z_2)\right. \\
&\qquad\left.-(1+z_2)(1+z_1z_2)+(1-z_2)(1+z_1z_2)-(1+z_2)(1-z_1z_2)\right] \\[2ex]
&+\frac{4}{z_2}(1-z_1^2)^2J(z_1^2,z_2^2)U(z_1^2,z_2^2)V(z_1^2,z_2^2)\left[(1-z_2)(1-z_1z_2)\right. \\
&\qquad\left.-(1+z_2)(1+z_1z_2)+(1-z_2)(1+z_1z_2)-(1+z_2)(1-z_1z_2)\right] \\[2ex]
&+\frac{1}{z_2}(1-z_1^2)J(z_1^2,z_2^2)V(z_1^2,z_2^2)\left[(1-z_2)(1+z_1z_2)\right. \\
&\qquad\left.-(1+z_2)(1-z_1z_2)+(1-z_2)(1-z_1z_2)-(1+z_2)(1+z_1z_2)\right] \\[2ex]
&+\frac{1}{z_2}J(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1+z_1)^2(1-z_2)(1-z_1z_2)-(1+z_1)^2(1+z_2)(1+z_1z_2)\right. \\
&\qquad\left.-(1-z_1)^2(1-z_2)(1+z_1z_2)+(1-z_1)^2(1+z_2)(1-z_1z_2)\right] \\[2ex]
&+\frac{4z_1^2}{z_2}(1-z_1^2)(1-z_2^2)J(z_1^2,z_2^2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_2)(1-z_1z_2)\right. \\
&\qquad\left.-(1+z_2)(1+z_1z_2)+(1-z_2)(1+z_1z_2)-(1+z_2)(1-z_1z_2)\right] \\[2ex]
&+\frac{1}{2z_1z_2}J(z_1^2,z_2^2)\left[(1+z_1)-(1+z_1)-(1-z_1)+(1-z_1)\right]
\end{aligned}
$$

$$+\frac{4}{z_1}(1-z_1^2z_2^2)U(z_1^2,z_2^2)\left[(1-z_1)K(z_1,z_2^2)\right.$$
$$\left.+(1-z_1)K(z_1,z_2^2)-(1+z_1)K(-z_1,z_2^2)-(1+z_1)K(-z_1,z_2^2)\right]$$

$$+8(1-z_1^2z_2^2)Z(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)K(z_1,z_2^2)+(1-z_1)(1+z_2)K(z_1,z_2^2)\right.$$
$$\left.+(1+z_1)(1-z_2)K(-z_1,z_2^2)+(1+z_1)(1+z_2)K(-z_1,z_2^2)\right]$$

$$+8(1-z_1^2z_2^2)^2U(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)K(z_1,z_2^2)+(1-z_1)(1+z_2)K(z_1,z_2^2)\right.$$
$$\left.+(1+z_1)(1-z_2)K(-z_1,z_2^2)+(1+z_1)(1+z_2)K(-z_1,z_2^2)\right]$$

$$+8(1-z_1^2)(1-z_1^2z_2^2)U(z_1^2,z_2^2)V(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)K(z_1,z_2^2)\right.$$
$$\left.+(1-z_1)(1+z_2)K(z_1,z_2^2)+(1+z_1)(1-z_2)K(-z_1,z_2^2)+(1+z_1)(1+z_2)K(-z_1,z_2^2)\right]$$

$$+2V(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)(1+z_1z_2)^2K(z_1,z_2^2)+(1-z_1)(1+z_2)(1-z_1z_2)^2K(z_1,z_2^2)\right.$$
$$\left.+(1+z_1)(1-z_2)(1-z_1z_2)^2K(-z_1,z_2^2)+(1+z_1)(1+z_2)(1+z_1z_2)^2K(-z_1,z_2^2)\right]$$

$$+2(1-z_1^2z_2^2)W(z_1^2,z_2^2)\left[(1+z_1)(1-z_2)K(z_1,z_2^2)+(1+z_1)(1+z_2)K(z_1,z_2^2)\right.$$
$$\left.+(1-z_1)(1-z_2)K(-z_1,z_2^2)+(1-z_1)(1+z_2)K(-z_1,z_2^2)\right]$$

$$+8z_1^2(1-z_1^2z_2^2)(1-z_2^2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)K(z_1,z_2^2)\right.$$
$$\left.+(1-z_1)(1+z_2)K(z_1,z_2^2)+(1+z_1)(1-z_2)K(-z_1,z_2^2)+(1+z_1)(1+z_2)K(-z_1,z_2^2)\right]$$

$$+\frac{1}{z_1}\left[(1+z_1z_2)K(z_1,z_2^2)+(1-z_1z_2)K(z_1,z_2^2)\right.$$
$$\left.-(1-z_1z_2)K(-z_1,z_2^2)-(1+z_1z_2)K(-z_1,z_2^2)\right]$$

$$\begin{aligned}
= \quad & -4(1-z_1^2)\left[2J(z_1^2,z_2^2)U(z_1^2,z_2^2)+4J(z_1^2,z_2^2)Z(z_1^2,z_2^2)\right.\\
&\qquad +4(1-z_1^2z_2^2)J(z_1^2,z_2^2)U(z_1^2,z_2^2)W(z_1^2,z_2^2)\\
&\qquad +4(1-z_1^2)J(z_1^2,z_2^2)U(z_1^2,z_2^2)V(z_1^2,z_2^2)\\
&\qquad +1J(z_1^2,z_2^2)V(z_1^2,z_2^2)\\
&\qquad \left.+4z_1^2(1-z_2^2)J(z_1^2,z_2^2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)\right]\\
& -2(1+z_1)^3J(z_1^2,z_2^2)W(z_1^2,z_2^2)\\
& +\frac{8}{z_1}(1-z_1^2z_2^2)U(z_1^2,z_2^2)\left[(1-z_1)K(z_1,z_2^2)-(1+z_1)K(-z_1,z_2^2)\right]\\
& +16(1-z_1^2z_2^2)\left[(1-z_1)K(z_1,z_2^2)+(1+z_1)K(-z_1,z_2^2)\right]\\
&\quad \cdot\left\{Z(z_1^2,z_2^2)+z_1^2(1-z_2^2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)+(1-z_1^2z_2^2)U(z_1^2,z_2^2)W(z_1^2,z_2^2)\right.\\
&\qquad \left.+(1-z_1^2)U(z_1^2,z_2^2)V(z_1^2,z_2^2)\right\}\\
& +4V(z_1^2,z_2^2)\left[(1-z_1)(1+z_1^2z_2^2-2z_1z_2^2)K(z_1,z_2^2)\right.\\
&\qquad \left.+(1+z_1)(1+z_1^2z_2^2+2z_1z_2^2)K(-z_1,z_2^2)\right]\\
& +4(1-z_1^2z_2^2)W(z_1^2,z_2^2)\left[(1+z_1)K(z_1,z_2^2)+(1-z_1)K(-z_1,z_2^2)\right]\\
& +\frac{2}{z_1}\left[K(z_1,z_2^2)-K(-z_1,z_2^2)\right]. \qquad (8.1.1)
\end{aligned}$$

It is not a trivial matter to place restrictions on the Laurent polynomials U, V, W, Z, J, and K, such that (8.1.1) is satisfied. It would be ideal to develop a systematic way of finding such restrictions on the Laurent polynomials, but for now it remains an open question to be further investigated in future work. Note however that, under the assumption $U(z_1,z_2)=V(z_1,z_2)=W(z_1,z_2)=Z(z_1,z_2)=0,\;\;(z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, as can be made to obtain the particular solution for $A(z_1, z_2) = \dfrac{1}{z_1 z_2}$ in (3.1.2) from the general solution in (5.1.62), the identity (8.1.1) becomes

$$\frac{2}{z_1}\left[K(z_1, z_2^2) - K(-z_1, z_2^2)\right] = 0, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\},$$

from which it follows that

$$K(z_1, z_2^2) = K(-z_1, z_2^2), \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}; \tag{8.1.2}$$

that is, K is even in z_1, whereas no restrictions are to be put on the other degree of freedom Laurent polynomial J. For example, the choice $J(z_1, z_2) = 0$ and $K(z_1, z_2) = \dfrac{1}{8}\dfrac{1}{z_2^2}$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, yields the particular choice $Q_2(z_1, z_2) = \dfrac{1}{8}\dfrac{1}{z_2}(1 + z_1 z_2)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, in (3.1.2) of Theorem 3.1, whereas the choice $J(z_1, z_2) = -\dfrac{1}{8}$ and $K(z_1, z_2) = 0$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, yields the particular choice $Q_3(z_1, z_2) = -\dfrac{1}{8}(1 + z_1)$, $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$, in (3.1.2). Indeed, the Laurent polynomials A and B_1 (as were obtained from our general solutions A in (5.1.62) and B in (6.2.14)), as well as Q_2 and Q_3, satisfy the identities (2.2.10) through (2.2.13).

8.2 Solving the Second Class of Identities for the Three-Directional Box Spline Case: $\delta = 1$

Recall from (5.1.62) and (7.2.20) the general classes of solutions A and Q for the identities (2.2.10) and (2.2.13), respectively, where, as before, the Laurent polynomial solution B for the identity (2.2.12) is fixed as in (6.2.14), and P is the refinement mask symbol corresponding to the three-directional box spline $B_{1,1,1}$, and where $\delta = 1$ in (2.2.13). This section uses the solutions (5.1.62) and (7.2.20) in terms of the arbitrary Laurent polynomials U, V, W, Z, S, and N, in an attempt to impose restrictions on those Laurent polynomials, in such a way that the identity in (2.2.11) is satisfied.

Before continuing, note that, with Q as in (7.2.20) and A as given by (5.1.62), (2.2.11) holds if and only if, for $(z_1, z_2) \in \mathbb{C}^2\{(0,0)\}$,

$$\begin{aligned}
0 \;=\; & -\frac{1}{z_1^3 z_2}U(z_1^2, z_2^2)\,[(1-z_1)(1-z_1z_2)-(1-z_1)(1+z_1z_2) \\
& \qquad -(1+z_1)(1+z_1z_2)+(1+z_1)(1-z_1z_2)] \\
& -\frac{2}{z_1^2 z_2^2}Z(z_1^2, z_2^2)\,[(1-z_1)(1-z_2)(1-z_1z_2)-(1-z_1)(1+z_2)(1+z_1z_2) \\
& \qquad +(1+z_1)(1-z_2)(1+z_1z_2)-(1+z_1)(1+z_2)(1-z_1z_2)] \\
& -\frac{2}{z_1^2 z_2}(1-z_1^2z_2^2)U(z_1^2, z_2^2)W(z_1^2, z_2^2)\,[(1-z_1)(1-z_2)(1-z_1z_2) \\
& -(1-z_1)(1+z_2)(1+z_1z_2)+(1+z_1)(1-z_2)(1+z_1z_2)-(1+z_1)(1+z_2)(1-z_1z_2)] \\
& -\frac{2}{z_1^2 z_2}(1-z_1^2)U(z_1^2, z_2^2)V(z_1^2, z_2^2)\,[(1-z_1)(1-z_2)(1-z_1z_2) \\
& -(1-z_1)(1+z_2)(1+z_1z_2)+(1+z_1)(1-z_2)(1+z_1z_2)-(1+z_1)(1+z_2)(1-z_1z_2)] \\
& -\frac{1}{2z_1^2 z_2}V(z_1^2, z_2^2)\,[(1-z_1)(1-z_2)(1+z_1z_2)-(1-z_1)(1+z_2)(1-z_1z_2) \\
& \qquad +(1+z_1)(1-z_2)(1-z_1z_2)-(1+z_1)(1+z_2)(1+z_1z_2)] \\
& -\frac{1}{2z_1^2 z_2}W(z_1^2, z_2^2)\,[(1+z_1)(1-z_2)(1-z_1z_2)-(1+z_1)(1+z_2)(1+z_1z_2) \\
& \qquad (1-z_1)(1-z_2)(1+z_1z_2)-(1-z_1)(1+z_2)(1-z_1z_2)] \\
& -\frac{2}{z_2}(1-z_2^2)V(z_1^2, z_2^2)W(z_1^2, z_2^2)\,[(1-z_1)(1-z_2)(1-z_1z_2) \\
& -(1-z_1)(1+z_2)(1+z_1z_2)+(1+z_1)(1-z_2)(1+z_1z_2)-(1+z_1)(1+z_2)(1-z_1z_2)] \\
& -\frac{1}{4z_1^3}\,[1+1-1-1]
\end{aligned}$$

$$+\frac{2}{z_1z_2}(1-z_1^2)S(z_1^2,z_2^2)U(z_1^2,z_2^2)\left[(1-z_1z_2)-(1+z_1z_2)-(1+z_1z_2)+(1-z_1z_2)\right]$$

$$+\frac{4}{z_2}(1-z_1^2)S(z_1^2,z_2^2)Z(z_1^2,z_2^2)\left[(1-z_2)(1-z_1z_2)-(1+z_2)(1+z_1z_2)\right.$$
$$\left.+(1-z_2)(1+z_1z_2)-(1+z_2)(1-z_1z_2)\right]$$

$$+\frac{4}{z_2}(1-z_1^2)(1-z_1^2z_2^2)S(z_1^2,z_2^2)U(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_2)(1-z_1z_2)-(1+z_2)(1+z_1z_2)\right.$$
$$\left.+(1-z_2)(1+z_1z_2)-(1+z_2)(1-z_1z_2)\right]$$

$$+\frac{4}{z_2}(1-z_1^2)^2S(z_1^2,z_2^2)U(z_1^2,z_2^2)V(z_1^2,z_2^2)\left[(1-z_2)(1-z_1z_2)-(1+z_2)(1+z_1z_2)\right.$$
$$\left.+(1-z_2)(1+z_1z_2)-(1+z_2)(1-z_1z_2)\right]$$

$$+\frac{1}{z_1}(1-z_1^2)S(z_1^2,z_2^2)V(z_1^2,z_2^2)\left[(1-z_2)(1+z_1z_2)-(1+z_2)(1-z_1z_2)\right.$$
$$\left.+(1-z_2)(1-z_1z_2)-(1+z_2)(1+z_1z_2)\right]$$

$$+\frac{1}{z_2}S(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1+z_1)^2(1-z_2)(1-z_1z_2)-(1+z_1)^2(1+z_2)(1+z_1z_2)\right.$$
$$\left.+(1-z_1)^2(1-z_2)(1+z_1z_2)-(1-z_1)^2(1+z_2)(1-z_1z_2)\right]$$

$$+\frac{4z_1^2}{z_2}(1-z_1^2)(1-z_2^2)S(z_1^2,z_2^2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_2)(1-z_1z_2)\right.$$
$$\left.-(1+z_2)(1+z_1z_2)+(1-z_2)(1+z_1z_2)-(1+z_2)(1-z_1z_2)\right]$$

$$+\frac{1}{2z_1z_2}S(z_1^2,z_2^2)\left[(1+z_1)-(1+z_1)-(1-z_1)+(1-z_1)\right]$$

$$+\frac{4}{z_1}(1-z_1^2z_2^2)U(z_1^2,z_2^2)\left[(1-z_1)N(z_1,z_2^2)+(1-z_1)N(z_1,z_2^2)\right.$$
$$\left.-(1+z_1)N(-z_1,z_2^2)-(1+z_1)N(-z_1,z_2^2)\right]$$

$$+8(1-z_1^2z_2^2)Z(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)N(z_1,z_2^2)+(1-z_1)(1+z_2)N(z_1,z_2^2)\right.$$
$$+(1+z_1)(1-z_2)N(-z_1,z_2^2)+(1+z_1)(1+z_2)N(-z_1,z_2^2)$$

$$+8(1-z_1^2z_2^2)^2U(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)N(z_1,z_2^2)+(1-z_1)(1+z_2)N(z_1,z_2^2)\right.$$
$$+(1+z_1)(1-z_2)N(-z_1,z_2^2)+(1+z_1)(1+z_2)N(-z_1,z_2^2)$$

$$+8(1-z_1^2)(1-z_1^2z_2^2)U(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)N(z_1,z_2^2)+(1-z_1)(1+z_2)N(z_1,z_2^2)\right.$$
$$+(1+z_1)(1-z_2)N(-z_1,z_2^2)+(1+z_1)(1+z_2)N(-z_1,z_2^2)$$

$$+2V(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)(1+z_1z_2)N(z_1,z_2^2)+(1-z_1)(1+z_2)(1-z_1z_2)N(z_1,z_2^2)\right.$$
$$\left.+(1+z_1)(1-z_2)(1-z_1z_2)N(-z_1,z_2^2)+(1+z_1)(1+z_2)(1+z_1z_2)N(-z_1,z_2^2)\right]$$

$$+2(1-z_1^2z_2^2)W(z_1^2,z_2^2)\left[(1+z_1)(1-z_2)N(z_1,z_2^2)+(1+z_1)(1+z_2)N(z_1,z_2^2)\right.$$
$$\left.+(1-z_1)(1-z_2)N(-z_1,z_2^2)+(1-z_1)(1+z_2)N(-z_1,z_2^2)\right]$$

$$+8z_1^2(1-z_2^2)(1-z_1^2z_2^2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_1)(1-z_2)N(z_1,z_2^2)+(1-z_1)(1+z_2)N(z_1,z_2^2)\right.$$
$$+(1+z_1)(1-z_2)N(-z_1,z_2^2)+(1+z_1)(1+z_2)N(-z_1,z_2^2)$$

$$+\frac{1}{z_1}\left[(1+z_1z_2)N(z_1,z_2^2)+(1-z_1z_2)N(z_1,z_2^2)\right.$$
$$\left.-(1-z_1z_2)N(-z_1,z_2^2)-(1+z_1z_2)N(-z_1,z_2^2)\right]$$

$$
\begin{aligned}
= \quad & \frac{4}{z_2^2}U(z_1^2,z_2^2) \\
& +\frac{1}{z_1^2}(1+z_2^2)\left[8Z(z_1^2,z_2^2)+8(1-z_1^2z_2^2)U(z_1^2,z_2^2)W(z_1^2,z_2^2)\right. \\
& \qquad +8(1-z_1^2)U(z_1^2,z_2^2)V(z_1^2,z_2^2)+V(z_1^2,z_2^2) \\
& \qquad\qquad \left.+W(z_1^2,z_2^2)+8z_1^2(1-z_2^2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)\right] \\
& -4(1-z_1^2)\left[2S(z_1^2,z_2^2)U(z_1^2,z_2^2)+4S(z_1^2,z_2^2)Z(z_1^2,z_2^2)\right. \\
& \qquad +4(1-z_1^2z_2^2)U(z_1^2,z_2^2)W(z_1^2,z_2^2)S(z_1^2,z_2^2)+4(1-z_1^2)S(z_1^2,z_2^2)U(z_1^2,z_2^2)V(z_1^2,z_2^2) \\
& \qquad \left.+\frac{1}{2}V(z_1^2,z_2^2)S(z_1^2,z_2^2)+4(1+z_1^2)(1+z_1)(1-z_2^2)S(z_1^2,z_2^2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)\right] \\
& +\frac{8}{z_1}(1-z)1^2z_2^2)U(z_1^2,z_2^2)\left[(1-z_1)N(z_1,z_2^2)-(1+z_1)N(-z_1,z_2^2)\right] \\
& +16(1-z_1^2z_2^2)Z(z_1^2,z_2^2)\left[(1-z_1)N(z_1,z_2^2)+(1+z_1)N(-z_1,z_2^2)\right] \\
& +16(1-z_1^2z_2^2)^2U(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_1)N(z_1,z_2^2)+(1+z_1)N(-z_1,z_2^2)\right] \\
& +16(1-z_1^2)(1-z_1^2z_2^2)U(z_1^2,z_2^2)V(z_1^2,z_2^2)\left[(1-z_1)N(z_1,z_2^2)+(1+z_1)N(-z_1,z_2^2)\right] \\
& +4V(z_1^2,z_2^2)\left[(1-z_1)(1-z_1z_2^2)N(z_1,z_2^2)+(1+z_1)(1+z_1z_2^2)N(-z_1,z_2^2)\right] \\
& +4(1-z_1^2z_2^2)W(z_1^2,z_2^2)\left[(1+z_1)N(z_1,z_2^2)+(1-z_1)N(-z_1,z_2^2)\right] \\
& +16z_1^2(1-z_2^2)(1-z_1^2z_2^2)V(z_1^2,z_2^2)W(z_1^2,z_2^2)\left[(1-z_1)N(z_1,z_2^2)+(1+z_1)N(-z_1,z_2^2)\right] \\
& +\frac{2}{z_1}\left[N(z_1,z_2^2)+N(-z_1,z_2^2)\right] \qquad (8.2.1)
\end{aligned}
$$

As in the previous section, it is still an open question to impose conditions on the Laurent polynomials U, V, W, Z, S, and N, such that (8.2.1) is satisfied. However, with the assumption $U(z_1, z_2) = V(z_1, z_2) = W(z_1, z_2) = Z(z_1, z_2) = 0, \ \ (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0, 0)\}$, we obtain the condition that the Laurent polynomial N must be even in z_1.

It remains an open question of how to pick Laurent polynomials from the classes of solutions (7.1.9) and (7.2.20) under the restrictions of K and N to both be even in z_1, and such that any such Laurent polynomial must, for a given choice of the solution B obtained from a certain choice of M in (6.2.14), be of the form (7.1.9), whereas, for a different choice of B, the same Laurent polynomial Q should be of the form (7.2.20). This thesis will not further pursue how this selection should be done – it is our belief that a deeper investigation into how different classes of solutions B from choices of the Laurent polynomials M_1 and M_2 in Theorem 5.3 can result in different classes of solutions Q as in (7.1.9) and (7.2.20), can shed more light on this problem. It is a positive progression that we have developed ways of finding classes of solutions to the identities (2.2.10) – (2.2.13) such that parts of our particular solution in (3.1.2) can be obtained from these general classes of solutions. This observation gives us confidence that further research into the final systematic solution of the Bezout identities can be very efficient.

Chapter 9

Conclusions

Refinable functions have earned their reputation for being fundamental in the investigation of multi-resolution analysis and have proved to be of indispensable use in the development of efficient algorithms for use in the digital design and data filtering industries. Whereas the use of refinable functions in the study of subdivision and wavelet theories has in the past relied heavily on the foundation of Fourier analysis, it has been the purpose of this thesis to develop, along similar lines than the univariate case investigated by Chui and De Villiers in [2], a strategic way of finding wavelets in the bivariate setting. It was shown in Chapter 2 how the reality of wavelet decomposition is equivalent to the existence of a (simultaneous) solution to a large system of Bezout identities in bivariate Laurent polynomials, where the only given input is the refinement mask symbol P corresponding to a given refinable function, chosen in particular ways to an end-user's will. As pointed out in Chapter 1, the prototype examples of refinable functions in the bivariate setting are box splines, and the rest of this work continued on the restriction of the refinable function to the linear box spline case, known as the Courant hat function.

Chapter 3 saw a particular solution to the system of Bezout identities in Chapter 2, which established that the solution set to the Bezout system is non-empty, a crucial fact of which suggests feasibility of research into the solutions of the Bezout identities. The rest of the thesis took a more general approach to solving the Bezout system, one identity at a time. It remains an open question how to reconcile the general classes of solutions obtained for the Laurent polynomials B and Q in Chapters 6 and 7, as discussed in Chapter 8, and this issue opens up possibilities for further research, with the promise of obtaining efficient and elegant algorithms for fast wavelet construction. More possible future research questions involve, e.g., how similar results can be obtained regarding solutions to the Bezout identities in Chapter 2, for the case where the refinable function is a different box spline than the Courant hat function in this thesis. Our positive remark in this direction is that, with the more general result of Theorem 5.1 given in [7] for the *general* two-directional box spline case, a similar approach to ours in Chapter 5 could be used to again "force" a third, and perhaps a fourth, direction into the associated direction matrix. In this regard, we mention that it was shown in [18] how, with respect to desired levels of smoothness of the resulting refinable function, the two *primary* directions $(1+z_1)$ and $(1+z_2)$ in the direction matrix of the box spline, are the crucial ones, in the sense that higher orders of the combination of these two direction vectors included in the direction matrix alone result in a smoother corresponding refinable function (where by "smooth", it is meant that the function in question possesses higher-order continuous derivatives – see [18] for a discussion on higher-order derivatives in terms of partial derivatives of bivariate refinable functions). Similarly, recall that (6.1.1) was also solved in the general two-directional box spline setting, in Section 6.1. Hence, it is our belief that the third Bezout identity (2.2.12) can also be attempted to solve by using the two-directional solution (6.1.26) and following a similar approach than in Section 6.2 to "force" a third (and fourth) direction vector into the direction matrix.

A further research problem also remains, namely whether wavelet decomposition results such as in Chapter 2 can be obtained in the case where the *dilation matrix* M in the refinement equation (1.2.1) is a general matrix instead of the matrix $M = 2I$, as has been the case in this work. Results on refinement preservation upon inclusion of certain vectors in the direction matrix for the case where M is a general direction matrix, are given to some extent in [18]. Work on wavelet construction in the general dilation matrix case has also been done in, e.g., [9] and [10]. It would be beneficial if work can be done to construct wavelets in the bivariate, general dilation matrix setting, without relying on the foundations of Fourier analysis, with the purpose of also developing efficient algorithms for useful applications. Then further research can show when certain dilation matrices are preferable above other ones, a question which, to a great extent except for a few remarks on the relationship between the number of wavelet "generators" and the dilation matrix that were made in [13], have been unanswered so far.

"... The wavelets arrive in succession, and each wavelet eventually dies out. The wavelets all have the same basic form and shape, but the strength or impetus of each wavelet is random and uncorrelated with the strength of the other wavelets... Despite the foreordained death of any individual wavelet, the time-series does not die. The reason is that a new wavelet is born each day to take the place of the one that does die. On any given day, the time-series is composed of many living wavelets, all of a different age, some young, others old." – Enders A. Robinson.

Bibliography

[1] S.A. Broughton & K. Bryan. *Discrete Fourier Analysis and Wavelets: Applications to signal and image processing*, John Wiley & Sons, Inc., 2009.

[2] C.K. Chui & J.M. de Villiers. *Wavelet Subdivision Methods: GEMS for Rendering Curves and Surfaces*, Taylor & Francis, 2010.

[3] C.K. Chui & B. Han & X Zhuang. *A Dual-Chain Approach for Bottom-Up Construction of Wavelet Filters With Any Integer Dilation*, Applied and Computational Harmonic Analysis, to appear.

[4] C. de Boor, K. Höllig & S. Riemenschneider. *Box Splines*, Springer-Verlag, 1993.

[5] W.D.V. de Wet. *On the analysis of refinable functions with respect to mask factorisation, regularity and corresponding subdivision convergence*, Ph.D thesis, University of Stellenbosch, 2007.

[6] N Dyn & D Levin. *A Butterfly Subdivision Scheme for Surface Interpolation with Tension Control*, ACM Transactions on Graphics, Vol. 9, No. 2, 1990.

[7] A.F. Rabarison & J.M. de Villiers. *On Bivariate Interpolatory Mask Symbols, Subdivision and Refinable Functions*, Approximation Algorithms for Complex Systems, Springer Proceedings in Mathematics, Vol. 3, Part 2, pp. 275-292, 2011.

[8] B. Han. *Analysis and Construction of Optimal Multivariate Biorthogonal Wavelets with Compact Support*, SIAM Journal of Mathematical Analysis, Vol. 1, No. 2, pp. I-XX, 1988.

[9] B. Han. *Symmetry Property and Construction of Wavelets With a General Dilation Matrix*, Linear Algebra and its Applications, Vol. 353, Issues 1-3, pp. 207-225, 2002.

[10] B. Han & R.Q. Jia. *Quincunx Fundamental Refinable Functions and Quincunx Biorthogonal Wavelets*, Mathematics of Computation, Vol. 71, No. 237, pp. 165-196, 2000.

[11] W. He & M.J. Lai. *Construction of bivariate compactly supported biorthogonal box spline wavelets with arbitrarily high regularities*, Journal of Applied and Computational Harmonic Analyis, Vol. 6, pp. 53-74, 1999.

[12] T. Kunkle. *Multivariate Differences, Polynomials, and Splines*, Journal of Approximation Theory, Vol. 84, pp. 290-314, 1996.

[13] J. Lian. *On Bivariate Scaling Functions and Wavelets*, Wavelets and Splines, Athens 2005.

[14] D. Mackenzie, I. Daubechies, D. Kleppner, S. Mallat, Y. Meyer, M. B. Ruskai, G. Weiss. *Wavelets: Seeing the forest - and the trees*, Beyond Discovery™: The Path from Research to Human Benefit, a project of the National Academy of Sciences, National Academy of Sciences, 2001.

[15] T.I. Mueller. *Geometric modelling with multivariate B-splines*, Ph.D thesis, Department of Computer Science, University of Utah, 1986.

[16] H. Prautzch & W. Boehm. *Box splines*, appears as Chapter 10 in the Handbook of Computer Aided Geometric Design, Farin, Hoschek, Kim, eds, March 2002.

[17] P. Schröder. *Subdivision as a fundamental building block of digital geometry processing algorithms*, Journal of Computational and Applied Mathematics 149, pp. 207-219, 2002.

[18] R. van der Bijl. *Multivariate refinable functions with emphasis on box splines*, Masters thesis, Stellenbosch University, 2009.

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