Refinable functions with prescribed values at the integers

by

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Declaration

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Summary

Refinable functions and their corresponding refinement sequences play a fundamental role in the mathematical analysis of wavelets and subdivision. For a given refinable function ϕ, its values at the integers, together with its corresponding refinement sequence $\{p_j\}$, determine ϕ everywhere on its support. Moreover, a certain eigenvector obtained from a matrix $\mathscr{P}$ based on $\{p_j\}$, yields precisely the values of ϕ at the integers.

In this thesis, we investigate the following inverse problem: for prescribed values of ϕ at the integers, find a family of finitely-supported sequences $\{p_j\}$, for which there does indeed exist corresponding refinable function ϕ attaining these prescribed values at the integers.

This problem was first considered in De Villiers, Micchelli and Sauer (2000), where a necessary condition on $\{p_j\}$ was derived, and applied for specific cases. In Section 2.3 of this thesis, it is shown rigorously that the additional demand of subdivision convergence guarantees that the corresponding limit (scaling) function ϕ, which is also a refinable function, does indeed have the prescribed values at the integers.

The above-mentioned necessary condition on $\{p_j\}$ is formulated in terms of a certain polynomial identity, and we proceed in Chapter 3 to establish a general result for a class of such identities, as will prove useful also in different contexts throughout the rest of the thesis, and which generalizes an existence result for algebraic polynomial identities in Chui and de Villiers (2010, Chapter 7). Subject to certain constraints on the prescribed values of ϕ at the integers, we proceed in Chapter 3 to establish a one-parameter class of refinement sequences $\{p_j\}$, with corresponding refinable functions ϕ attaining the required values at the integers.

After presenting, in Section 3.3, an application of our theory, we proceed in Chapter 4, as a further application, to establish a one-parameter family of perturbed cardinal B-splines, which are refinable functions co-inciding with the car-

dinal B-splines N_m at the integers.

Next, in Chapter 5, we apply the results from Chapter 3 and 4 to obtain the Pascal refinable function ϕ_ν^*, with normalized binomial coefficient values at the integers, by means of an approach that is more structured than the one used in De Villiers, Micchelli and Sauer (2000). Moreover, as also in Chapter 4, the Hölder regularity of the refinable functions obtained is investigated by means of general results from Chui and de Villiers (2010, Chapter 6).

In Chapter 6, we present a construction method, based on the values at the integers of a refinable function ϕ, for a local interpolation operator with polynomial exactness, as could prove useful in the first step of any wavelet decomposition algorithm based on ϕ. In conjunction with Chapters 2 and 3, the work of Chapter 6 demonstrates the advantage of having refinable function with prescribed values at the integers, in that these prescribed values may be chosen in such a way to facilitate an efficient construction of our local interpolation operator with polynomial exactness. Also, in the cardinal B-spline case, our construction is shown to yield an alternative construction method for certain known local spline interpolation operators. In particular, the local interpolation operator based on the Pascal refinable function is carefully studied, and provides a case in point to illustrate the advantage, in this context, of a refinable function ϕ with prescribed values of simple structure at the integers.

Finally, in Chapter 7, we follow the wavelet construction method proposed in Chui and de Villiers (2010, Chapter 9) to construct minimally- supported synthesis wavelets for the specific refinable functions obtained in this thesis.

Opsomming

Verfynbare funksies en hullle ooreenkomstige verfyningsrye speel 'n fundamentele rol in die wiskundige analise van golfies ("wavelets") en subdivisie. Vir 'n gegewe verfynbare funksie ϕ, word die waarde van ϕ orals op die steungebied van ϕ volledig bepaal deur die waardes van ϕ by die heelgetalle, tesame met die ooreenkomstige verfyningsry $\{p_j\}$. Boonop verskaf 'n sekere eievektor, soos verkry van 'n matriks $\mathscr{P}$ gebaseer op $\{p_j\}$, presies die waardes van ϕ by die heelgetalle.

In hierdie tesis ondersoek ons die volgende inverse probleem: vir voorgeskewe waardes van ϕ by die heelgetalle, bepaal 'n familie van eindig-ondersteunde rye $\{p_j\}$, waaroor daar inderdaad ooreenkomstige verfynbare funksies ϕ bestaan wat daardie voorgeskrewe waardes by die heelgetalle aanneem.

Hierdie problem is eerste beskou in De Villiers, Micchelli and Sauer (2000), waar 'n nodige voorwaarde op $\{p_j\}$ afgelei is, en toegepas is vir spesifieke gevalle. In Afdeling 2.3 van hierdie tesis word dit streng aangetoon dat die bykomende eis van subdivisie konvergensie dit waarborg dat die ooreenkomstige limietfunksie (of skaalfunksie), wat ook 'n verfynbare funksie is, inderdaad die voorgeskrewe waardes by die heelgetalle het.

Die bogenoemde nodige voorwaarde op $\{p_j\}$ is geformuleer in terme van 'n sekere polinoomidentiteit, en ons gaan in Hoofstuk 3 voort om 'n algemene resultaat daar te stel vir 'n klas van sulke identiteite, wat dan ook nuttig blyk te wees in verskillende kontekste in die res van die tesis, en wat 'n bestaansresultaat vir algebraïese polinoomidentiteite in Chui en De Villiers (2010, Hoofstuk 7) veralgeneem. Onderhewig aan sekere beperkings op die voorgeskrewe waardes van ϕ by die heelgetalle, gaan ons in Hoofstuk 3 voort om 'n een-parameter klas van verfyningsrye $\{p_j\}$ daar te stel, met ooreenkomstige verfynbare funksies ϕ wat die aangewese waardes by die heelgetalle aaneem.

Na 'n aanbieding in Afdeling 3.3 van 'n toepassing van ons teorie, gaan ons in Hoofstuk 4 voort om, as 'n verdere toepassing, 'n een-parameter familie van

geperturbeerde kardinale B-latfunksies ("B-splines") daar te stel, wat verfynbare funksies is wat ooreenstem met die kardinale B-latfunksies N_m by die heelgetalle.

Vervolgens, in Hoofstuk 5, pas ons die resultate van Hoofstukke 3 en 4 toe om die Pascal verfynbare funksies ϕ_ν^*, met genormaliseerde binomiaalkoëffisiënt waardes by die heelgetalle, te verkry met behulp van 'n metode wat meer gestruktureerd is as die een wat gebruik is in De Villiers, Micchelli en Sauer (2000). Verder, ook in Hoofstuk 4, word die Hölder regulariteit van die verfynbare funksies wat verkry is, ondersoek met behulp van algemene resultate uit Chui en De Villiers (2010, Hoofstuk 6).

In Hoofstuk 6 gee ons 'n konstruksiemetode, gebaseer op die waardes by die heelgetalle van 'n verfynbare funksie ϕ, vir 'n lokale interpolasie-operator met polinoomeksaktheid, soos wat nuttig kan wees in die eerste stap van enige golfie dekomposisie-algoritme gebaseer op ϕ. Tesame met Hoofstukke 2 en 3, demonstreer die werk van Hoofstuk 6 die voordeel daarvan om 'n verfynbare funksie met die voorgeskrewe waardes by die heelgetalle beskikbaar te hê, in die sin dat hierdie voorgeskrewe waardes sodanig gekies kan word dat 'n effektiewe en ekonomiese konstruksie van ons lokale interpolasie-operator met polinoom eksaktheid in die hand gewerk word. Daarbenewens word daar in die kardinale B-latfunksie geval aangetoon dat ons metode 'n alternatiewe konstruksiemetode vir sekere bekende lokale latfunksie-operatore lewer. In die besonder word die lokale interpolasie-operator gebaseer op die Pascal verfynbare funksie versigtig bestudeer, en wat dan die voordeel daarvan illustreer om beskikbaar te hê 'n verfynbare ϕ met voorgeskrewe waardes van 'n eenvoudige struktuur by die heelgetalle.

Laastens, in Hoofstuk 7, volg ons die golfie konstruksiemetode voorgestel in Chui en de Villiers (2010, Hoofstuk 9) om minimaal-ondersteunde sintese-golfies te konstrueer vir die spesifieke verfynbare funksies van hierdie tesis.

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Dedications

This dissertation is dedicated, with much love, to my mom, my late grandma and grandpa. *Ndiri kha mme anga Vho-Gloria ndi khou livhuwa lufuno, thuthuwedzo, dzithabelo na zwothe zwe vha nnyitela u swika ndi tshivha mungafha.*

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Nomenclature

The following symbols will be used throughout this thesis. If not below, their definitions appear at first place where they are introduced.

General notations

$\mathbb{N}$ The set of natural numbers

$\mathbb{Z}$ The set of integers

$\mathbb{Z}_+$ The set non-negative integers

$\mathbb{R}$ The set of real numbers

$\mathbb{C}$ The set of complex numbers

$\ell(\mathbb{Z})$ The linear space of bi-infinite real-valued sequences, i.e., $\boldsymbol{c} \in \ell(\mathbb{Z})$ if and only if $\boldsymbol{c} = \{c_j : j \in \mathbb{Z}\} \subset \mathbb{R}$

$\ell_0(\mathbb{Z})$ The subspace of $\ell(\mathbb{Z})$ consisting of those sequences in $\ell(\mathbb{Z})$ with finite support, i.e., a sequence $\boldsymbol{c} \in \ell(\mathbb{Z})$ is called finitely-supported if the set $\{j : c_j \neq 0,\ j \in \mathbb{Z}\}$ has a finite number of elements

$\mathscr{C}(\mathbb{R})$ The space of all real-valued continuous functions on $\mathbb{R}$

$\mathscr{C}_0(\mathbb{R})$ The subspace of $\mathscr{C}(\mathbb{R})$ consisting of all finitely supported functions, i.e., there exists a bounded interval $[\alpha,\beta] \subset \mathbb{R}$, such that $f(x) = 0$, $x \notin [\alpha,\beta]$

$\operatorname{supp}^c f$ $=[\alpha,\beta]$ The convex-hull of the support of f, i.e., if $f \in \mathscr{C}_0(\mathbb{R})$ is such that $f(x) = 0$ for $x \leq \alpha$ or $x \geq \beta$, with also $\alpha := \inf\{x : f(x) \neq 0\}$, $\beta := \sup\{x : f(x) \neq 0\}$

$\mathscr{C}^k(\mathbb{R})$ The collection of functions which, together with their derivatives up to order k, are in $\mathscr{C}(\mathbb{R})$, where $\mathscr{C}^0(\mathbb{R}) = \mathscr{C}(\mathbb{R})$

$\mathscr{H}^\alpha(\mathbb{R})$ For $\alpha \in (0,1]$, the Lipschitz space $\mathscr{H}^\alpha(\mathbb{R}) := \big\{f : \mathbb{R} \longrightarrow \mathbb{R} : |f(x) - f(y)| \leq c|x-y|^\alpha, x, y \in \mathbb{R}$, for some constant $c \in [0,\infty)\big\}$

$\mathscr{H}_0^\alpha(\mathbb{R})$ $\mathscr{H}^\alpha(\mathbb{R}) \cap \mathscr{C}_0(\mathbb{R})$, is a subspace of the space $\mathscr{C}_0(\mathbb{R})$ of compactly supported continuous functions. For $\alpha = 1$, we also call $\mathrm{Lip}(\mathbb{R}) := \mathscr{H}^1(\mathbb{R})$ the class of all Lipschitz continuous functions on $\mathbb{R}$

$\mathscr{C}^{k,\alpha}(\mathbb{R})$ For $k = 0, 1, \ldots,$ and $\alpha \in (0,1]$, the $\mathscr{H}$ölder space of order k with $\mathscr{H}$ölder continuity exponent α is defined by $\mathscr{C}^{k,\alpha}(\mathbb{R}) := \left\{f \in \mathscr{C}^k(\mathbb{R}) : f^{(k)} \in \mathscr{H}^\alpha(\mathbb{R})\right\}$

$\mathscr{C}_0^{k,\alpha}(\mathbb{R})$ $\mathscr{C}^{k,\alpha}(\mathbb{R}) \cap \mathscr{C}_0(\mathbb{R})$, and observe that $\mathscr{C}^{0,\alpha}(\mathbb{R}) = \mathscr{H}^\alpha(\mathbb{R})$ and $\mathscr{C}_0^{0,\alpha}(\mathbb{R}) = \mathscr{H}_0^\alpha(\mathbb{R})$

$\sum\limits_j$ The summation over all the integer $j \in \mathbb{Z}$

$\sup\limits_j$ The supremum over all $j \in \mathbb{Z}$

$\sup\limits_x$ The supremum over all $x \in \mathbb{R}$

$\ell^\infty(\mathbb{Z})$ The subspace of bounded sequences in $\ell(\mathbb{Z})$, i.e., if $\boldsymbol{c} \in \ell(\mathbb{Z})$, and $\sup\limits_j |c_j| < \infty$

$||\cdot||_\infty$ The sup norm for the linear space $\ell^\infty(\mathbb{Z})$

π_k The linear space of polynomials of degree $\leq k$, where $k \in \mathbb{Z}_+$

Refinable functions and subdivision

$\boldsymbol{p} = \{p_j\}$ Refinement sequence, in $\ell_0(\mathbb{Z})$

$\mathrm{supp}\{p_j\}$ The support of refinement sequence $\{p_j\}$, i.e., $\mathrm{supp}\{p_j\} = [\mu,\nu]\big|_{\mathbb{Z}}$, where $\mu := \min\{j : p_j \neq 0\}$ and $\nu := \max\{j : p_j \neq 0\}$

P The two-scale symbol associated with the refinement sequence $\{p_j\} \in \ell_0(\mathbb{Z})$, i.e., the polynomial $\frac{1}{2}\sum\limits_j p_j(\cdot)^j$

m The order of the zero at -1 of P

$\tilde{P}$ The polynomial satisfying $P = \left(\frac{1+\cdot}{2}\right)^m \tilde{P}$

$S_{\boldsymbol{p}}$ The subdivision operator mapping $\boldsymbol{c} \in \ell(\mathbb{Z})$ to $S_{\boldsymbol{p}}\boldsymbol{c} \in \ell(\mathbb{Z})$, with $(S_{\boldsymbol{p}}\boldsymbol{c})_j = \sum\limits_k p_{j-2k}\boldsymbol{c}_k, \quad j \in \mathbb{Z}$

$S_{\boldsymbol{p}}^r$ The subdivision operator $S_{\boldsymbol{p}}$ applied r-times, with the convention that $S_{\boldsymbol{p}}^0$ is the identity operator

$\boldsymbol{c}^r$ The sequence $S_{\boldsymbol{p}}^r\boldsymbol{c}$, where $\boldsymbol{c} \in \ell(\mathbb{Z})$

$\Delta\boldsymbol{c}$ The backward difference sequence defined by $(\Delta\boldsymbol{c})_j = \boldsymbol{c}_j - \boldsymbol{c}_{j-1}, \quad j \in \mathbb{Z}$, if $\boldsymbol{c} \in \ell(\mathbb{Z})$

ϕ	A refinable function, i.e., a function satisfying the refinement equation $\phi = \sum_j p_j \phi(2\cdot - j)$

Synthesis wavelets

ψ	A wavelet function, i.e., a function defined by $\psi = \sum_j q_j \phi(2\cdot - j)$
S^r_ϕ	The vector space $S^r_\phi = \left\{ \sum_j c_j \phi(2^r \cdot - j) : \{c_j\} \in \ell(\mathbb{Z}) \right\}$ at resolution level $r \in \mathbb{Z}$
$\mathscr{L}_r$	The linear operator mapping $\mathscr{C}(\mathbb{R})$ into S^{r+1}_ϕ
W^r_ϕ	The wavelet space $\left\{ \sum_j d_j \psi(2^r \cdot - j) : \{d_j\} \in \ell(\mathbb{Z}) \right\}$,
$\{d^r_j\}$	The wavelet decomposition coefficient sequence at resolution level r

Miscellaneous

$\lfloor x \rfloor$	The largest integer less $\leq x$
$\oplus$	The direct sum of two linear spaces
δ_j	The Kronecker delta: 0 for all $j \in \mathbb{Z}$, except for $\delta_0 = 1$
$\delta_{j,k}$	The Kronecker delta: 0 for all $j, k \in \mathbb{Z}$, except for $\delta_{j,j} = 1$
$\boldsymbol{\delta}$	The bi-infinite sequence $\{\delta_{j,0} : j \in \mathbb{Z}\}$
$\binom{j}{k}$	The binomial coefficient defined for $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+$ by $\binom{j}{k} = \frac{j!}{k!(j-k)!}$ if $k \in \{0, 1, \ldots, j\}$, and otherwise, with the convention that $\binom{0}{0} = 1$
$R^{(e)}$	The even part $R = \sum_j r_{2j} (\cdot)^{2j}$ of a polynomial $R = \sum_j r_j (\cdot)^j$
$R^{(o)}$	The odd part $R = \sum_j r_{2j+1} (\cdot)^{2j+1}$ of a polynomial $R = \sum_j r_j (\cdot)^j$
deg	The degree of a polynomial
gcd	The greatest common divisor

Introduction

Increasingly over the last two decades, the construction of refinement sequences and their corresponding refinable functions has developed into an important issue in the analysis of wavelets (see, e.g., Mallat (1989); Chui (1992); Daubechies (1992); Micchelli (1995); Chui and de Villiers (2010)), and subdivision (see, e.g., Cavaretta, Dahmen and Micchelli (1991); Daubechies and Lagarias (1991); Dyn (1987); Micchelli (1995)).

A refinable function, which we will denote by ϕ, is a compactly supported continuous real-valued function on the real line, and which satisfies the refinement equation

$$\phi = \sum_{j=-\infty}^{\infty} p_j \phi(2\cdot - j), \tag{0.0.1}$$

where the finitely supported real-valued sequence $\{p_j : j \in \mathbb{Z}\}$ in (0.0.1) is called the corresponding refinement (mask) sequence.

Some of the fundamental questions of interest that one could ask regarding refinable functions, particularly with respect to the two areas of analysis mentioned above, are the following:

A. *The existence and uniqueness of* ϕ: Given a finitely-supported refinement sequence $\{p_j : j \in \mathbb{Z}\}$, for what conditions on $\{p_j\}$ does a compactly-supported continuous function ϕ exist? If it exists, is it unique?

B. *The numerical evaluation of* ϕ: If we do not have a simple closed formula for ϕ, what algorithms can be used to approximate ϕ?

C. *The regularity (smoothness) of* ϕ: How regular (smooth) is ϕ? For instance, how many continuous derivatives does ϕ possess? In computer-aided geometric design (CAGD), it is desirable in many applications to have a high order of smoothness, so that the design will appear smooth.

These problems have attracted the interest of many mathematicians, and have been intensively investigated in the literature over the last two decades (see, e.g., Cavaretta, Dahmen and Micchelli (1991), Daubechies (1988, 1992), Daubechies and Lagarias (1991), Chui and de Villiers (2010), Eirola (1992), Strang (1989)).

It is important to note that, with the exception of the special case of cardinal B-splines, it is not possible to find, for a given refinement sequence $\{p_j\}$, an explicit (closed) formula for its corresponding refinable function ϕ. Hence the analysis of refinable function ϕ and its properties is therefore usually based on an explicitly known refinement sequence with finitely many non-zero coefficients. In fact, ϕ can be completely determined by the refinement sequence $\{p_j\}$, as will be shown in this thesis.

Another interesting question with respect to refinable functions is:

D. *From* $\{p_j\}$ *to* $\{\phi(j)\}$*:* Given a refinement sequence $\{p_j\}$, how can we determine the values of ϕ at the integers?

It is already pointed out in the literature (see, e.g., Cavaretta, Dahmen and Micchelli (1991) and Micchelli and Prautzsch (1987)), that an eigenvalue problem based on the refinement sequence $\{p_j\}$ for a given compactly supported continuous refinable function ϕ can be used to determine the values of ϕ at integers, which can then be used, together with the sequence $\{p_j\}$, to determine ϕ everywhere on the real line.

The main emphasis of this thesis is to investigate the following inverse question with respect to **D**:

E. *From* $\{\phi(j)\}$ *to* $\{p_j\}$*:* Can we find a refinement sequence $\{p_j\}$ such that ϕ attains prescribed values at the integers?

In their paper, De Villiers, Micchelli and Sauer (2000), derive necessary conditions for an affirmative answer to question **E** by solving an algebraic polynomial identity. In subsequent work, Micchelli and Sun (2002) introduced a recursive method to generate new refinable functions from their values at integers.

In this thesis, following De Villiers, Micchelli and Sauer (2000), we derive necessary and sufficient conditions for an affirmative answer to question **E** by solving algebraic polynomial identities by means of polynomial algebra methods, thereby yielding a one-parameter family of refinable functions possessing prescribed values at the integers. Specific examples are investigated, and graphically illustrated, in particular for the case where the values at the integers are normalized binomial coefficients. Furthermore, we show how explicit knowledge of the values of ϕ at the integers can be exploited to yield an efficient construction method for a local interpolation operator with optimal polynomial exactness.

Chapter 1

Refinable functions and subdivision

In this chapter, we introduce several notations that are used throughout this thesis. We also review some definitions and reformulate some basic results on the convergence of subdivision schemes and their corresponding limit functions. Some regularity results will be stated.

1.1 Notation

By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{Z}$ and $\mathbb{Z}_+$ the set of integers and non-negative integers respectively, by $\mathbb{R}$ the set of real numbers, and by $\mathbb{C}$ the set of complex numbers. For $s = 1, 2$, or 3, we use the notation $\mathbb{R}^s$ to denote the s-dimensional Euclidean space. Note that $\mathbb{R}^1 = \mathbb{R}$.

We denote by $\ell(\mathbb{Z})$ the space of all real-valued bi-infinite sequences defined on $\mathbb{Z}$. We denote by $\ell_0 := \ell_0(\mathbb{Z})$ the subspace of sequences in $\ell(\mathbb{Z})$ with only finitely many non-zero elements, and a sequence $\boldsymbol{c} = \{c_j\} \in \ell_0$ will be called a *finitely supported* sequence, in which case the support of $\{c_j\}$ is denoted by

$$\operatorname{supp}\{c_j\} := [\mu, \nu] \cap \mathbb{Z} = [\mu, \nu]\big|_{\mathbb{Z}},$$

where μ and ν denote the largest and smallest integers, respectively, for which $c_j = 0$ for all $j < \mu$ or $j > \nu$.

The subspace ℓ^∞ of $\ell(\mathbb{Z})$ is defined as those sequences $\{c_j\} \in \ell(\mathbb{Z})$ satisfying the boundedness condition

$$\sup_j |c_j| < \infty,$$

where $\sup\limits_j := \sup\limits_{j\in\mathbb{Z}}$. Observe that ℓ_0 is a subspace of ℓ^∞. The symbols $\ell(\mathbb{Z})$, ℓ_0 and ℓ^∞ will in fact be used to denote bi-infinite sequences $\{\boldsymbol{c}_j\}$ with $\boldsymbol{c}_j \in \mathbb{R}^s$, $j \in \mathbb{Z}$, for $s = 1,2$, or 3, where the specific value of the dimension s will always be clear from the context. We denote by $\Delta\boldsymbol{c}$ the backward difference sequence defined by $(\Delta\boldsymbol{c})_j = \boldsymbol{c}_j - \boldsymbol{c}_{j-1}$, $j \in \mathbb{Z}$, if $\boldsymbol{c} \in \ell(\mathbb{Z})$.

We write $\mathscr{C}(\mathbb{R})$ for the space of all real-valued continuous functions on $\mathbb{R}$. The symbol $\mathscr{C}_0 := \mathscr{C}_0(\mathbb{R})$ will denote the subspace of functions in $\mathscr{C}(\mathbb{R})$ that vanish identically outside some bounded intervals, and a function $f \in \mathscr{C}_0$ will be called a *compactly supported* continuous function on $\mathbb{R}$, in which case the closure of the convex hull of the support of f will be denoted by

$$\operatorname{supp}^c f = [\alpha,\beta],$$

if $f(x) = 0$ for $x \le \alpha$ or $x \ge \beta$, with also

$$\alpha := \inf\{x : f(x) \neq 0\} \qquad \text{and} \qquad \beta := \sup\{x : f(x) \neq 0\}.$$

Moreover, for $k \in \mathbb{Z}_+$ we denote by $\mathscr{C}^k := \mathscr{C}^k(\mathbb{R})$ the collection of functions which, together with their derivatives up to order k, are in $\mathscr{C}(\mathbb{R})$, and where $\mathscr{C}^0(\mathbb{R}) := \mathscr{C}(\mathbb{R})$. We write $\sum\limits_j := \sum\limits_{j\in\mathbb{Z}}$.

1.2 Refinability and refinement sequences

Definition 1.2.1 *Let $\phi \in \mathscr{C}_0$ and $\boldsymbol{p} = \{p_j\} \in \ell_0$ satisfy the equation*

$$\phi(x) = \sum_j p_j \phi(2x - j), \quad x \in \mathbb{R}. \tag{1.2.1}$$

*Then ϕ is called a **refinable function** and $\{p_j\}$ is called the corresponding **refinement sequence** or **mask sequence**. Equation* (1.2.1) *is called the **refinement relation** or the **two-scale lattice difference equation**.*

As an example of a refinable function, consider the "hat function" h defined by

$$h(x) := \begin{cases} x, & 0 < x < 1; \\ 2 - x, & 1 \le x < 2; \\ 0, & x \in \mathbb{R}\backslash(0,2). \end{cases} \tag{1.2.2}$$

Note from (1.2.2) that h can be refined as

$$h(x) = \frac{1}{2}h(2x) + h(2x-1) + \frac{1}{2}h(2x-2), \quad x \in \mathbb{R}, \tag{1.2.3}$$

a graphical illustration of which is provided in Figure 1.2.1.

Hence, according to (1.2.2) and (1.2.3), h is a refinable function with respect to a refinement sequence $\{p_j\}$ given by

$$\begin{cases} \{p_0, p_1, p_2\} = \{\frac{1}{2}, 1, \frac{1}{2}\}; \\ \\ \text{with} \quad p_j = 0, \quad j \notin \{0,1,2\}. \end{cases} \tag{1.2.4}$$

Observe that $h \in \mathscr{C}_0(\mathbb{R}) \backslash \mathscr{C}^1(\mathbb{R})$.

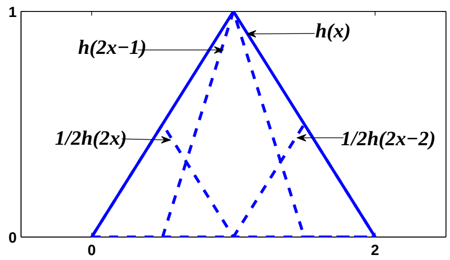


Figure 1.2.1: *The hat function and its refinement equation given by* (1.2.3).

In this thesis, we shall restrict attention to refinement sequences $\{p_j\}$ satisfying

$$\text{supp}\,\{p_j\} = [0, \nu]|_{\mathbb{Z}}, \tag{1.2.5}$$

for an integer $\nu \geq 2$.

Next, we show that the support of refinement sequence $\{p_j\}$ is preserved by its corresponding refinable function ϕ, as made precise in the next result, as was proved by Chui and de Villiers (2010, Theorem 2.1.1).

Theorem 1.2.1 *For $\nu \in \mathbb{N}$, let ϕ be a refinable function with refinement sequence $\{p_j\}$ that satisfies property* (1.2.5) *for an integer $\nu \geq 2$. Then ϕ satisfies the support*

$$supp^c \phi = [0, \nu]. \tag{1.2.6}$$

Proof. Let

$$\alpha := \inf\{x : \phi(x) \neq 0\}; \qquad \beta := \sup\{x : \phi(x) \neq 0\}.$$

Since $p_0 \neq 0$ and $p_\nu \neq 0$, and since ϕ is compactly supported, it follows from (1.2.1) and (1.2.5) that

$$\begin{aligned} \alpha = \inf\left\{x : \sum_{j=0}^{\nu} p_j \phi(2x-j) \neq 0\right\} &= \inf\{x : p_0\phi(2x) \neq 0\} \\ &= \inf\{x : \phi(2x) \neq 0\} = \frac{\alpha}{2}, \end{aligned}$$

and thus $\alpha = \frac{\alpha}{2}$, that is, $\alpha = 0$, whereas similarly

$$\beta = \sup\left\{x : \sum_{j=0}^{\nu} p_j \phi(2x-j) \neq 0\right\} = \sup\{x : p_\nu\phi(2x-\nu) \neq 0\} = \frac{\beta+\nu}{2},$$

and thus $\beta = \frac{\beta+\nu}{2}$, that is, $\beta = \nu$, an thereby completing the proof of (1.2.6). ■

A very useful tool in the study of refinable functions is the polynomial with coefficients given by refinement sequence $\{p_j\}$, as in the following definition.

Definition 1.2.2 *For a given sequence $\{p_j\} \in \ell_0$ satisfying the support property* (1.2.5) *for an integer $\nu \geq 2$, the polynomial P defined by*

$$P(z) := \frac{1}{2}\sum_j p_j z^j = \frac{1}{2}\sum_{j=0}^{\nu} p_j z^j, \tag{1.2.7}$$

*is called the corresponding **two-scale symbol** or **mask symbol**. Furthermore, if $\{p_j\}$ is the refinement sequence of some refinable function ϕ, then P is called the two-scale symbol of the refinable function ϕ.*

Definition 1.2.3 *A sequence $\{p_j\} \in \ell_0$ is said to satisfy the **sum-rule condition** if it satisfies*

$$\sum_j p_{2j} = 1 \quad ; \quad \sum_j p_{2j-1} = 1. \tag{1.2.8}$$

Observe from (1.2.4) that the refinement sequence $\{p_j\}$ for the hat function h satisfied the sum-rule condition (1.2.8). The sum-rule condition (1.2.8) plays an important role in our study of refinable functions. We proceed to describe an important relationship between a sequence $\{p_j\} \in \ell_0$ satisfying (1.2.8) and its corresponding polynomial P as in (1.2.7).

Since, according to (1.2.7), we have

$$\begin{cases} P(1) = \dfrac{1}{2}\displaystyle\sum_j p_j = \dfrac{1}{2}\sum_j p_{2j} + \dfrac{1}{2}\sum_j p_{2j-1}; \\ P(-1) = \dfrac{1}{2}\displaystyle\sum_j (-1)^j p_j = \dfrac{1}{2}\sum_j p_{2j} - \dfrac{1}{2}\sum_j p_{2j-1}; \end{cases}$$

we observe that the sum-rule condition (1.2.8) is equivalent to the two conditions

$$P(1) = 1 \quad ; \quad P(-1) = 0, \tag{1.2.9}$$

which, in turn, are satisfied if and only if there exist an integer $m \in \mathbb{N}$ and a polynomial $\tilde{P}$, with $\tilde{P}(1) = 1$, such that

$$P(z) = \left(\frac{1+z}{2}\right)^m \tilde{P}(z), \qquad z \in \mathbb{C}. \tag{1.2.10}$$

If m is chosen as the largest integer for which (1.2.10) holds, then the polynomial factor $\tilde{P}(z)$ in (1.2.10) satisfies:

$$\tilde{P}(1) = 1 \quad ; \quad \tilde{P}(-1) \neq 0. \tag{1.2.11}$$

For the refinement sequence $\{p_j\}$ of the hat function h, as given by (1.2.4), the corresponding mask symbol is given by

$$P(z) = \frac{1}{4}\left(1 + 2z + z^2\right) = \left(\frac{1+z}{2}\right)^2, \quad z \in \mathbb{C}, \tag{1.2.12}$$

so that (1.2.10) is satisfied with $m = 2$ and $\tilde{P}(z) = 1$.

The following results were proved in Chui and de Villiers (2010, Theorems 4.5.1, 4.3.2, Corollary 4.5.1, 4.5.1).

Theorem 1.2.2 *Suppose $\tilde{\phi}$ is a refinable function with refinement sequence $\{p_j\}$ satisfying the sum-rule condition* (1.2.8)*. Then:*

(a)

$$\sum_j \tilde{\phi}(x - j) = \sum_j \tilde{\phi}(j) = \int_{-\infty}^{\infty} \tilde{\phi}(s)\,ds \neq 0, \quad x \in \mathbb{R}. \tag{1.2.13}$$

(b) The function $\phi \in \mathscr{C}_0$ defined by

$$\phi := \frac{\tilde{\phi}}{\int_{-\infty}^{\infty} \tilde{\phi}(t)\,dt} = \frac{\tilde{\phi}}{\sum_j \tilde{\phi}(j)} \tag{1.2.14}$$

is refinable with refinement sequences $\{p_j\}$, and ϕ is the only solution in $\mathscr{C}_0$ of the refinement equation (1.2.1)*, together with the condition*

$$\sum_j \phi(j) = 1. \tag{1.2.15}$$

We proceed to give a precise formulation of the sum-rule property in (1.2.8), by introducing the notion of sum-rule order.

Definition 1.2.4 *Let $m \in \mathbb{N}$. A sequence $\{p_j\} \in \ell_0$ is said to satisfy the* ***sum-rule of order*** *m, if m is the largest integer for which*

$$\beta_\ell := \sum_j (2j)^\ell p_{2j} = \sum_j (2j-1)^\ell p_{2j-1}, \qquad \ell = 0,\ldots,m-1, \qquad \textit{with} \qquad \beta_0 = 1, \tag{1.2.16}$$

where $0^0 := 1$.

It was proved in Chui and de Villiers (2010, Theorem 5.3.1) that, in general, the sum-rule of order m condition can be characterized in terms of polynomial factorization, as follows.

Theorem 1.2.3 *A sequence $\{p_j\} \in \ell_0$ satisfies the sum-rule condition of order $m \in \mathbb{N}$, as given in* (1.2.16)*, if and only if its two-scale symbol P in* (1.2.7) *satisfies*

$$P(z) = \left(\frac{1+z}{2}\right)^m \tilde{P}(z), \tag{1.2.17}$$

for a polynomial $\tilde{P}$ such that

$$\tilde{P}(1) = 1 \quad ; \quad \tilde{P}(-1) \neq 0. \tag{1.2.18}$$

With the refinement sequence $\{p_j\}$ for the hat function h given in (1.2.4), and for which we easily verify that $\{p_j\}$ satisfies the sum-rule of order 2, and with corresponding two-scale symbol $P(z)$ given in (1.2.12), we see that Theorem 1.2.3 is satisfied with $m = 2$ and $\tilde{P}(z) = 1$.

We proceed in the next section to introduce the concept of subdivision, which we shall show to be closely related to refinable functions.

1.3 Subdivision

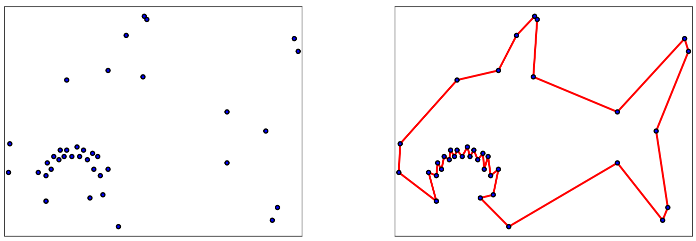
Subdivision is an efficient tool for the rendering of smooth curves and surfaces in Computer Aided Geometric Design (see, e.g., Cavaretta, Dahmen and Micchelli (1991); Dyn (1987); Micchelli (1995) and Chui and de Villiers (2010)). The basic idea behind a subdivision scheme is as follows: consider the simple iterative procedure where we start with a sequence of points $\boldsymbol{c}_j^0 = \{\boldsymbol{c}_j : j \in \mathbb{Z}\} \in \ell(\mathbb{Z})$, with

$\boldsymbol{c}_j \in \mathbb{R}^2$, $j \in \mathbb{Z}$, called the *initial control point* sequence, and generate a new sequence of control points $\{\boldsymbol{c}_j^1 : j \in \mathbb{Z}\}$ by taking a linear combination of the initial control points $\{\boldsymbol{c}_j^0\}$. Provided that the particular linear combination is chosen appropriately, we can repeat this process until the desired density is achieved. If, as will specifically be focused on in this thesis, the initial control points and the newly generated points after subdivision are not interpolated by the limit curve, then the scheme is called *approximating*, or *corner cutting* as opposed to *interpolating*.

As an example, if we generate the new sequence using

$$\left.\begin{array}{lcl} \boldsymbol{c}_{2j}^1 & = & \boldsymbol{c}_j^0, \\ \boldsymbol{c}_{2j-1}^1 & = & \frac{1}{2}\left(\boldsymbol{c}_j^0 + \boldsymbol{c}_{j+1}^0\right), \end{array}\right\} \quad j \in \mathbb{Z}; \qquad (1.3.1)$$

then the odd-indexed new points are generated halfway between the old ones, while the even-indexed new points are simply the initial points, indicated by $(\bullet)$ in Figure 1.3.1. This step can of course be repeated indefinitely, roughly "doubling" the number of points at each step. In this case the points fill in or converge to the straight lines connecting the initial control points (see Figure 1.3.1). This interpolating subdivision scheme provides us with the familiar piecewise linear curve.



(a) Initial control points $(\bullet)$ (b) Limit curve(-)

Figure 1.3.1: *Graphical illustration of the convergence of subdivision scheme* (1.3.1).

In general, we introduce the notion of a subdivision operator associated with the refinement sequence $\{p_j\}$ to describe the subdivision process and the concept of subdivision convergence.

Definition 1.3.1 *For a given non-trivial sequence $\boldsymbol{p} = \{p_j\} \in \ell_0$, the **subdivision operator** $S_{\boldsymbol{p}}$ corresponding to $\boldsymbol{p}$ is defined by*

$$(S_{\boldsymbol{p}}\boldsymbol{c})_j := \sum_k p_{j-2k}\boldsymbol{c}_k, \quad j \in \mathbb{Z}, \tag{1.3.2}$$

*where $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in \ell(\mathbb{Z})$, with $\{\boldsymbol{c}_j\} \in \mathbb{R}^s$, $j \in \mathbb{Z}$, for $s = 1, 2$ or 3. For a given initial control point sequence $\boldsymbol{c} \in \ell(\mathbb{Z})$, the operator $S_{\boldsymbol{p}}$ provides the **subdivision scheme** which generates the sequence $\{\boldsymbol{c}^{(r)} : r \in \mathbb{Z}_+\} \subset \ell(\mathbb{Z})$ recursively by means of*

$$\boldsymbol{c}_j^0 := \boldsymbol{c}_j \quad ; \quad \boldsymbol{c}_j^r := \left(S_{\boldsymbol{p}}\boldsymbol{c}^{r-1}\right)_j = \left(S_{\boldsymbol{p}}^r\boldsymbol{c}\right)_j, \quad j \in \mathbb{Z}, \tag{1.3.3}$$

where $S_{\boldsymbol{p}}^r := S_{\boldsymbol{p}}S_{\boldsymbol{p}}^{r-1}$, with $S_{\boldsymbol{p}}^0$ denoting the identity operator on $\ell(\mathbb{Z})$, and $S_{\boldsymbol{p}}^1 = S_{\boldsymbol{p}}$.

Note that the subdivision scheme (1.3.3) can be formulated as

$$\boldsymbol{c}_j^0 := \boldsymbol{c}_j; \qquad \left.\begin{aligned} \boldsymbol{c}_{2j}^r &:= \sum_k p_{2k}\boldsymbol{c}_{j-k}^{r-1}, \\ \boldsymbol{c}_{2j-1}^r &:= \sum_k p_{2k-1}\boldsymbol{c}_{j-k}^{r-1}, \end{aligned}\right\} \quad j \in \mathbb{Z}. \tag{1.3.4}$$

Following Chui and de Villiers (2010, Definition 4.1.2), we now define the concept of subdivision convergence as follows.

Definition 1.3.2 *The subdivision operator $S_{\boldsymbol{p}}$, as defined in (1.3.2) for a non–trivial sequence $\boldsymbol{p} = \{p_j\} \in \ell_0$, is said to provide a **convergent subdivision scheme**, if there exists a non-trivial function $\phi \in \mathscr{C}(\mathbb{R})$, such that*

$$\sup_j \left|\phi\left(\frac{j}{2^r}\right) - p_j^{[r]}\right| \longrightarrow 0, \quad r \longrightarrow \infty, \tag{1.3.5}$$

where, for $r = 1, 2, \ldots$,

$$p_j^{[r]} := \left(S_{\boldsymbol{p}}^r\boldsymbol{\delta}\right)_j, \quad j \in \mathbb{Z}, \tag{1.3.6}$$

with $\boldsymbol{\delta} := \{\delta_j\}$ denoting the delta sequence defined by

$$\delta_j := \begin{cases} 1, & j = 0; \\ 0, & j \in \mathbb{Z}\backslash\{0\}. \end{cases} \tag{1.3.7}$$

We call ϕ the limit function corresponding to $S_{\boldsymbol{p}}$.

Note in Definition 1.3.2 that, for every $x \in \mathbb{R}$, since the set $\{\frac{j}{2^r} : j \in \mathbb{Z}, r \in \mathbb{Z}_+\}$ of dyadic numbers is dense in $\mathbb{R}$, there exists a sequence $\{j_r : r \in \mathbb{Z}_+\} \subset \mathbb{Z}$ such that

$$\left| x - \frac{j_r}{2^r} \right| \to 0, \quad r \to \infty,$$

and thus

$$\left|\phi(x) - p^{[r]}_{j_r}\right| \leq \left|\phi(x) - \phi\left(\frac{j_r}{2^r}\right)\right| + \left|\phi\left(\frac{j_r}{2^r}\right) - p^{[r]}_{j_r}\right| \to 0 + 0 = 0, \quad r \to \infty, \tag{1.3.8}$$

from (1.3.5) and the fact that ϕ is continuous at x, so that

$$p^{[r]}_{j_r} \to \phi(x), \quad r \to \infty.$$

We proceed to state important implications of subdivision convergence, for the proof of which we refer to Cavaretta, Dahmen and Micchelli (1991, Proposition 3.1) and Chui and de Villiers (2010, Theorems 4.3.1 and 4.3.3).

Theorem 1.3.1 *Let $\{p_j\} \in \ell_0$ be such that the corresponding subdivision operator $S_{\boldsymbol{p}}$ provides a convergent subdivision scheme with limit function ϕ. Then:*

(a) *The sequence $\{p_j\}$ satisfies the sum-rule condition* (1.2.8)*;*

(b) *The limit function ϕ satisfies the following:*

(i) *ϕ is a refinable function with refinement sequence $\{p_j\}$;*

(ii)

$$\sum_j \phi(x-j) = 1, \quad x \in \mathbb{R}; \tag{1.3.9}$$

(iii)

$$\int_{-\infty}^{\infty} \phi(x)\,dx = 1. \tag{1.3.10}$$

(c) *For any non-trivial control point sequence $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in \ell(\mathbb{Z})$ with $\Delta\boldsymbol{c} \in \ell^\infty$, the subdivision scheme* (1.3.3) *converges uniformly, in the sense that*

$$\sup_k \left| \sum_j \boldsymbol{c}_j \phi\left(\frac{k}{2^r} - j\right) - \boldsymbol{c}^r_k \right| \leq e_r ||\Delta\boldsymbol{c}||_\infty \left|\phi\left(\frac{j}{2^r}\right) - p^{[r]}_j\right| \longrightarrow 0, \ r \longrightarrow \infty, \tag{1.3.11}$$

where $e_r \longrightarrow \nu^2$ for $r \longrightarrow \infty$, with the integer ν defined by supp$\{p_j\} = [0,\nu]|_{\mathbb{Z}}$.

Note that while subdivision convergence implies the existence of a refinable function, the converse of this statement is not necessarily true, as was pointed out in Neamtu (1999) (see also Chui and de Villiers (2010, Section 4.3)).

Definition 1.3.3 *A refinable function ϕ satisfying the condition*

$$\int_{-\infty}^{\infty} \phi(x)\,dx = 1, \tag{1.3.12}$$

will be called a ***scaling function****.*

Observe from Theorem 1.3.1, together with Definition 1.3.2, that, if $\boldsymbol{p} = \{p_j\} \in \ell_0$ is such that the subdivision operator $S_{\boldsymbol{p}}$ provides a convergent subdivision scheme, then the corresponding limit function ϕ is a scaling function.

Definition 1.3.4 *For a non-negative integer k, the space of* ***discrete polynomials of degree*** *$\leq k$ is defined by*

$$\pi_k^d := \left\{\{c_j\} \in \ell(\mathbb{Z}) : c_j = f(j),\ \ j \in \mathbb{Z},\ \ \textit{where}\ \ f \in \pi_k\right\}.$$

As proved in Chui and de Villiers (2010, Theorem 5.1.1, 5.2.1 and 5.3.1), the m^{th} order sum rule has the following implications.

Theorem 1.3.2 *Let ϕ be a refinable function with refinement sequence $\{p_j\}$ that satisfies the sum-rule condition of order $m \in \mathbb{N}$. Then*

(a)

$$S_{\boldsymbol{p}} : \pi_{m-1}^d \to \pi_{m-1}^d; \tag{1.3.13}$$

(b) for any polynomial $f \in \pi_{m-1}$,

$$\sum_j f(j)\phi(x-j) = \sum_j \phi(j) f(x-j), \quad x \in \mathbb{R}; \tag{1.3.14}$$

(c)

$$\pi_{m-1} \subset S_\phi := \left\{\sum_j c_j \phi(\cdot - j) : \{c_j\} \in \ell(\mathbb{Z})\right\}. \tag{1.3.15}$$

1.4 Convergence and regularity results

In this section, we state results providing sufficient condition on a sequence $\{p_j\} \in \ell_0$ for its subdivision operator $S_{\boldsymbol{p}}$ to provide a convergent subdivision scheme, as well as results on the regularity (smoothness) of the corresponding limit (scaling) function ϕ.

Definition 1.4.1 *A refinement sequence* $\{p_j\}$, *such that also the support property* (1.2.5) *is satisfied, is called a* ***positive*** *refinement sequence if*

$$p_j > 0, \quad j = 0, \ldots, \nu. \tag{1.4.1}$$

The following subdivision convergence result, for the proof of which we refer to Chui and de Villiers (2010, Theorems 6.4.1 and 6.4.2), shows that subdivision convergence is obtained in the case of a positive refinement sequence, at a geometric convergence rate, and with a corresponding refinable function that is (strictly) positive in the interior of its support, (see also Cavaretta, Dahmen and Micchelli (1991, Theorem 2.1), Micchelli (1995, Theorem 2.5), Micchelli and Prautzsch (1987) and Micchelli and Pinkus (1991)). The property of positive refinement sequence is possessed by many practical schemes in geometric modelling, for example the Chaikin algorithm, Chaikin (1974) (see also Lane and Riesenfeld (1980)).

Theorem 1.4.1 *For a given integer* $\nu \geq 2$, *let* $\boldsymbol{p} = \{p_j\}$ *be a refinement sequence that satisfies the support condition* (1.2.5), *the sum-rule condition* (1.2.8), *and the positivity condition* (1.4.1). *Then:*

(a) The subdivision operator $S_{\boldsymbol{p}}$ *provides a convergent subdivision scheme where the limit (scaling) function* ϕ *satisfies the geometric estimate*

$$\sup_j \left| \phi\left(\frac{j}{2^r}\right) - p_j^{[r]} \right| \leq \frac{\gamma^r}{1-\gamma}, \quad r = 1, 2, \ldots, \tag{1.4.2}$$

with $\{p_j^{[r]}\}$ *defined as in* (1.3.6), *and where the positive constant* $\gamma := \gamma_{\boldsymbol{p}}$ *is defined by*

$$\gamma := \frac{1}{2} \max \left\{ \sum_{\ell} |p_{j-2\ell} - p_{k-2\ell}| : j, k \in \mathbb{Z}; \ |j-k| \leq \nu - 1 \right\}, \tag{1.4.3}$$

and satisfies the inequalities

$$\frac{1}{2} \leq \gamma \leq 1 - \min\{p_0, \ldots, p_\nu\} < 1. \tag{1.4.4}$$

(b) *The limit (scaling) function ϕ is (strictly) positive in the interior of its support, that is,*

$$\phi(x) > 0, \quad x \in (0, \nu). \tag{1.4.5}$$

As mentioned in the introduction, the regularity of a refinable function is of great importance in the study of subdivision and in the construction of compactly supported wavelets. The result below in Theorem 1.4.2, as proved by Chui and de Villiers (2010, Theorem 6.4.3) gives a result on the regularity of the finitely supported refinable function ϕ for the class of sequences $\{p_j\}$ that satisfy the conditions stated in of Theorem 1.4.1, and in addition, a higher order sum-rule condition. To this end, we first introduce the concept of Hölder regularity, which is an extension of the notion of continuity, and which could also be useful for the study of fractal-like refinable functions.

Definition 1.4.2 *For a function $f : \mathbb{R} \to \mathbb{R}$, if there exist constants $c \in [0, \infty)$ and $\alpha \in (0, 1]$ such that*

$$|f(x) - f(y)| \leq c|x - y|^{\alpha}, \quad x, y \in \mathbb{R},$$

then f is said to be ***Hölder continuous*** *on $\mathbb{R}$, with* ***Hölder continuity exponent*** *α. The class of all such functions is denoted by $\mathscr{H}^{\alpha} := \mathscr{H}^{\alpha}(\mathbb{R})$.*

We define $\mathscr{H}_0^{\alpha} := \mathscr{H}^{\alpha} \cap \mathscr{C}_0$.

The Hölder continuity exponent $\alpha \in (0, 1]$ of a function $f \in \mathscr{H}_0^{\alpha}$ can be interpreted as a measure of the regularity of f, in the sense of the embedding

$$\mathscr{C}_0^1 \subset \mathscr{H}_0^{\alpha} \subset \mathscr{H}_0^{\tilde{\alpha}} \subset \mathscr{C}_0, \quad \text{for } 0 < \tilde{\alpha} \leq \alpha \leq 1. \tag{1.4.6}$$

Definition 1.4.3 *For $k \in \{0, 1, \ldots\}$, and $\alpha \in (0, 1]$, the function space*

$$\mathscr{C}^{k,\alpha} := \mathscr{C}^{k,\alpha}(\mathbb{R}) = \left\{ f \in \mathscr{C}^k : f^{(k)} \in \mathscr{H}^{\alpha} \right\} \tag{1.4.7}$$

is called the ***Hölder space of order*** *k with* ***Hölder continuity exponent*** *α. Also, we set*

$$\mathscr{C}_0^{k,\alpha} := \mathscr{C}^{k,\alpha} \cap \mathscr{C}_0, \tag{1.4.8}$$

and observe that $\mathscr{C}^{0,\alpha} = \mathscr{H}^{\alpha}$ and $\mathscr{C}_0^{0,\alpha} = \mathscr{H}_0^{\alpha}$.

According to Chui and de Villiers (2010, Theorem 6.4.3) the regularity of the limit (scaling) function ϕ of Theorem 1.4.1, satisfies the following result.

Theorem 1.4.2 *For a positive refinement sequence $\{p_j\}$ as in Theorem1.4.1, suppose that, moreover, $\{p_j\}$ satisfies the sum-rule condition of order $m \in \mathbb{N}$, and let n be the smallest integer in the set $\{1,\ldots,m\}$ such that the sequence $\{\hat{p}_j\} \in \ell_0$ given by*

$$\frac{1}{2}\sum_j \hat{p}_j z^j := \left(\frac{1+z}{2}\right)^n \tilde{P}(z), \tag{1.4.9}$$

with $\tilde{P}$ denoting the polynomial defined by (1.2.17), (1.2.18) *in Theorem 1.2.3, satisfies the condition*

$$\hat{p}_j > 0, \quad j = 0,\ldots,\nu - m + n. \tag{1.4.10}$$

Then the limit (scaling) function ϕ of Theorem1.4.1 satisfies the Hölder regularity result

$$\phi \in \mathscr{C}_0^{m-n,\alpha}, \tag{1.4.11}$$

where

$$\alpha := \log_2 \left[\left(1 - \min\{\hat{p}_0,\ldots,\hat{p}_\nu\}\right)^{-1} \right]. \tag{1.4.12}$$

Chapter 2

Refinable function values at the integers

In this chapter, after presenting a method for calculating the values of a refinable function ϕ at the integers, we proceed to introduce and analyze a method of constructing refinable functions with prescribed values at the integers.

2.1 From $\{p_j\}$ to $\{\phi(j)\}$

Let ϕ denote a refinable function with refinement sequence $\{p_j\}$ satisfying the sum-rule condition (1.2.8), and such that supp $\{p_j\} = [0,\nu]|_{\mathbb{Z}}$ for an integer $\nu \geq 2$, that is,

$$\phi(x) = \sum_j p_j \phi(2x - j), \quad x \in \mathbb{R}, \tag{2.1.1}$$

with also, from Theorem 1.2.1,

$$\mathrm{supp}^c \phi(x) = [0,\nu]. \tag{2.1.2}$$

The two finitely supported sequences $\{p_j\}$ and $\{\phi(j)\}$ can be used to compute the function value $\phi(x)$ for any $x \in (0,\nu)$, as follows. By using the refinement equation (2.1.1), we first obtain the values of ϕ at the dyadic points by means of

$$\phi\left(\frac{k}{2^r}\right) = \sum_j p_j \phi\left(\frac{k}{2^{r-1}} - j\right), \quad k \in \mathbb{Z}, \quad r \in \mathbb{N}, \tag{2.1.3}$$

which yields a recursive algorithm for the computation of the set

$$\left\{\phi\left(\tfrac{k}{2^r}\right) : k \in \mathbb{Z}, \quad r \in \mathbb{Z}_+\right\}.$$

Let $x \in \mathbb{R}$. Since the dyadic set $\left\{\frac{k}{2^r} : k \in \mathbb{Z}, \quad r \in \mathbb{Z}_+\right\}$ is dense in $\mathbb{R}$, we know that there exists a sequence $\{k_r : r \in \mathbb{Z}_+\} \subset \mathbb{R}$ such that $\frac{k_r}{2^r} \to x$, as $r \to \infty$. But then

$$\phi(x) = \phi\left(\lim_{r\to\infty} \frac{k_r}{2^r}\right) = \lim_{r\to\infty} \phi\left(\frac{k_r}{2^r}\right),$$

since $\phi \in \mathscr{C}_0$.

We proceed to show how, for a given refinement sequences $\{p_j\}$, the sequence $\{\phi(j)\}$ can be computed as an eigenvector corresponding to the eigenvalue $\lambda = 1$ of a matrix $\mathscr{P}$ built from $\{p_j\}$, under the assumption that $\lambda = 1$ is a simple eigenvalue of $\mathscr{P}$.

From the refinement equation (2.1.1), together with (2.1.2), we have

$$\phi(k) = \sum_j p_j \phi(2k - j) = \sum_j p_{2k-j}\phi(j) = \sum_{j=1}^{\nu-1} p_{2k-j}\phi(j), \quad k = 1, \ldots, \nu - 1, \tag{2.1.4}$$

or equivalently, in matrix-vector form,

$$\mathscr{P}\boldsymbol{\phi} = \boldsymbol{\phi}, \tag{2.1.5}$$

where the $(\nu - 1) \times (\nu - 1)$ matrix $\mathscr{P}$ has elements

$$(\mathscr{P})_{kj} = p_{2k-j}, \quad 1 \le k, j \le \nu - 1,$$

and where the column vector $\boldsymbol{\phi} \in \mathbb{R}^{\nu-1}$ is given by

$$\boldsymbol{\phi} = \left(\phi(1), \ldots, \phi(\nu - 1)\right)^T.$$

It follows from (2.1.4) that the matrix $\mathscr{P}$ has the eigenvalue $\lambda = 1$, which we shall assume here to be a simple eigenvalue, that is, the corresponding eigenspace has a dimension= 1, according to which the values of ϕ at the integers $\{1, \ldots, \nu-1\}$ may be obtained by normalizing the corresponding eigenvector by means of the condition

$$\sum_{j=1}^{\nu-1} \phi(j) = \sum_j \phi(j) = 1,$$

as deduced from Theorem 1.2.2.

Observe that, written out in full, the equation (2.1.5) has the form

$$\begin{bmatrix} p_1 & p_0 & 0 & \ldots & 0 & 0 \\ p_3 & p_2 & p_1 & \ldots & . & . \\ p_5 & p_4 & p_3 & \ldots & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & . \\ 0 & 0 & 0 & \ldots & p_\nu & p_{\nu-1} \end{bmatrix} \begin{bmatrix} \phi(1) \\ \phi(2) \\ \phi(3) \\ \vdots \\ \phi(\nu - 1) \end{bmatrix} = \begin{bmatrix} \phi(1) \\ \phi(2) \\ \phi(3) \\ \vdots \\ \phi(\nu - 1) \end{bmatrix}. \tag{2.1.6}$$

Hence we have developed a method for calculating ϕ everywhere on $\mathbb{R}$ from the refinement sequence $\{p_j\}$, by first calculating the sequence $\{\phi(j) : j = 1, \ldots, \nu - 1\}$ by means of (2.1.5), and then using (2.1.3) to inductively compute ϕ at the dyadic numbers in $(0, \nu)$. Illustrating examples are as follows.

Example 2.1.1

As our first example, we consider the refinement sequence $\{p_j\}$ defined by (1.2.4), according to which $\{p_j\}$ satisfies the conditions (1.2.5), (1.4.1) and (1.2.8) with $\nu = 2$, so that we may deduce from Theorems 1.2.2 and 1.4.1 the existence of a unique $\phi \in \mathscr{C}_0$ such that the properties (2.1.1), (1.2.15) and (1.4.5) are satisfied. In fact, according to (1.2.3), $\phi = h$ is the hat function defined by (1.2.2).

Here the matrix $\mathscr{P}$ defined by (2.1.6) is the 1×1 matrix $\mathscr{P} = [p_1] = [1]$, so that, using (2.1.5), together with the normalizing condition $\sum_j \phi(j) = 1$, we obtain $\phi(1) = 1$, which is in agreement with (1.2.2). A graphical illustration is provided in Figure 2.1.1. ■

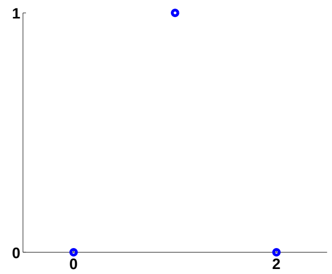


Figure 2.1.1: *Refinable function values at integers given as in Example 2.1.1.*

Example 2.1.2

Consider next the sequence $\{p_j\}$ given by

$$\begin{cases} \{p_0, p_1, p_2, p_3\} = \{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\}; \\ \text{with} \qquad p_j = 0, \quad j \notin \{0, 1, 2, 3\}, \end{cases}$$

so that the conditions (1.2.5), (1.4.1) and (1.2.8) hold with $\nu = 3$. Once again we deduce from Theorems 1.2.2 and 1.4.1 that there exists a unique $\phi \in \mathscr{C}_0$ such that properties (2.1.1), (1.2.15) and (1.4.5) are satisfied. Here the matrix

$$\mathscr{P} = \begin{bmatrix} p_1 & p_0 \\ p_3 & p_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix},$$

from which we verify that $\lambda = 1$ is a simple eigenvalue of $\mathscr{P}$. Hence we may calculate the one-dimensional eigenvalue space corresponding to eigenvalue $\lambda = 1$, and use the normalization condition $\sum_j \phi(j) = 1$, to obtain the values

$$\phi(1) = \frac{1}{2}, \qquad \phi(2) = \frac{1}{2}. \tag{2.1.7}$$

A graphical illustration of (2.1.7) is shown in Figure 2.1.2.

■

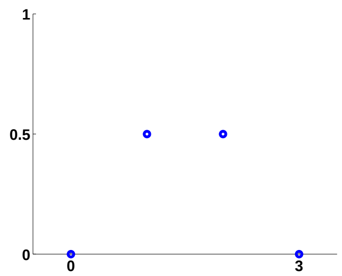


Figure 2.1.2: *Refinable function values at integers given as in Example 2.1.2.*

Example 2.1.3

In this example, suppose $\{p_j\}$ is given by

$$\begin{cases} \{p_0, p_1, p_2, p_3, p_4, p_5\} = \{\frac{3}{16}, \frac{7}{16}, \frac{6}{16}, \frac{6}{16}, \frac{7}{16}, \frac{3}{16}\}; \\ \\ \text{with} \quad p_j = 0, \quad j \notin \{0,1,2,3,4,5\}, \end{cases}$$

so that the conditions (1.2.5), (1.4.1) and (1.2.8) hold with $\nu = 5$. As in the Example 2.1.1 and 2.1.2, we deduce from Theorems 1.2.2 and 1.4.1 that there exists a unique $\phi \in \mathscr{C}_0$ such that properties (2.1.1), (1.2.15) and (1.4.5) are satisfied. Here the matrix

$$\mathscr{P} = \begin{bmatrix} \frac{7}{16} & \frac{3}{16} & 0 & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{7}{16} & \frac{3}{8} & \frac{3}{8} \\ 0 & 0 & \frac{3}{16} & \frac{9}{16} \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 7 & 3 & 0 & 0 \\ 6 & 6 & 7 & 3 \\ 3 & 7 & 6 & 6 \\ 0 & 0 & 3 & 7 \end{bmatrix}.$$

After verifying that $\lambda = 1$ is a simple eigenvalue of $\mathscr{P}$, we may calculate the one-dimensional eigenvalue space corresponding to eigenvalue $\lambda = 1$, and use the normalization condition $\sum_j \phi(j) = 1$, to obtain the values

$$\phi(1) = \frac{1}{8}, \qquad \phi(2) = \frac{3}{8}, \qquad \phi(3) = \frac{3}{8}, \qquad \phi(4) = \frac{1}{8}. \tag{2.1.8}$$

A graphical illustration of (2.1.8) is shown in Figure 2.1.3. ■

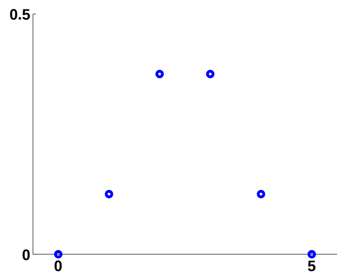


Figure 2.1.3: *Refinable function values at integers given as in Example 2.1.3.*

Example 2.1.4

Finally, suppose $\{p_j\} \in \ell_0$ is given by

$$\begin{cases} \{p_0, p_1, p_2, p_3\} = \left\{\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\right\}; \\ \\ \text{with} \quad p_j = 0, \quad j \notin \{0,1,2,3\}. \end{cases} \tag{2.1.9}$$

Observe that in this case, the conditions (1.2.8) and (1.2.5) stated in Theorem 1.4.1 hold with $\nu = 3$, but that (1.4.1) does not hold, since $p_3 < 0$. It was shown by Daubechies (1988, 1992) by means of Fourier analysis that there does indeed exist a unique corresponding refinable function $\phi = \phi^D \in \mathscr{C}_0$ associated with the mask coefficients (2.1.9), called the Daubechies refinable function, satisfying (2.1.1) and (1.2.15). In this case

$$\mathscr{P} = \begin{bmatrix} p_1 & p_0 \\ p_3 & p_2 \end{bmatrix} = \begin{bmatrix} \frac{3+\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} \\ \frac{1-\sqrt{3}}{4} & \frac{3-\sqrt{3}}{4} \end{bmatrix}.$$

As in the previous examples, after verifying that $\lambda = 1$ is a simple eigenvalue of $\mathscr{P}$, we may calculate the one-dimensional eigenvalue space corresponding to eigenvalue $\lambda = 1$, and use the normalization condition $\sum_j \phi(j) = 1$, to obtain

$$\phi^D(1) = \frac{1+\sqrt{3}}{2}, \qquad \phi^D(2) = \frac{1-\sqrt{3}}{2}. \tag{2.1.10}$$

A graphical illustration of (2.1.10) is shown in Figure 2.1.4. ■

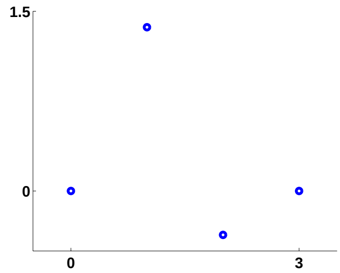


Figure 2.1.4: *Refinable function values at integers given as in Example 2.1.4.*

2.2 From $\{\phi(j)\}$ to $\{p_j\}$: A necessary condition

It was shown in Section 2.1 that, for a given sequence $\{p_j\} \in \ell_0$ such that supp $\{p_j\} = [0,\nu]|_{\mathbb{Z}}$ with $\nu \geq 2$, if there exists a refinable function ϕ with refinement sequence $\{p_j\}$, then the sequence $\{\phi(j) : j = 1,\ldots,\nu-1\}$ may be obtained as an eigenvector corresponding to the eigenvalue $\lambda = 1$ of the matrix $\mathscr{P}$ in (2.1.5), under the assumption also that $\lambda = 1$ is a simple eigenvalue of $\mathscr{P}$. The principle aim of this thesis is to investigate the following inverse problem:

For a given sequence $\{y_j : j = 1,\ldots,\nu-1\} \subset \mathbb{R}$, where $\nu \geq 2$, investigate the existence and construction of a sequence $\{p_j\} \in \ell_0$ satisfying the sum-rule condition (1.2.8), and with supp $\{p_j\} = [0,\nu]|_{\mathbb{Z}}$, for which there exists a function $\phi \in \mathscr{C}_0$ such that ϕ is a refinable function with refinement sequence $\{p_j\}$, that is,

$$\phi(x) = \sum_j p_j \phi(2x - j), \quad x \in \mathbb{R}, \tag{2.2.1}$$

with, moreover,

$$\phi(j) = y_j, \quad j = 1,\ldots,\nu-1. \tag{2.2.2}$$

We proceed in this section to derive a necessary condition on a refinement sequence $\{p_j\}$ for its corresponding refinable function ϕ to satisfy the condition (2.2.2). To this end, for a refinable function ϕ as in Theorem 1.2.1, we define the polynomial

$$\Phi(z) := \sum_j \phi(j+1)z^j = \sum_{j=0}^{\nu-2} \phi(j+1)z^j. \tag{2.2.3}$$

Also, we shall find it convenient to define, for a polynomial R defined by

$$R(z) := \sum_j r_j z^j,$$

the *even* part $R^{(e)}$ and *odd* part $R^{(o)}$ respectively by

$$R^{(e)}(z) := \sum_j r_{2j} z^{2j}, \qquad \text{and} \qquad R^{(0)}(z) := \sum_j r_{2j-1} z^{2j-1}. \tag{2.2.4}$$

Observe that then

$$\left.\begin{aligned} R(z) &= R^{(e)}(z) + R^{(o)}(z), \\ R(-z) &= R^{(e)}(z) - R^{(o)}(z), \end{aligned}\right\} \tag{2.2.5}$$

and thus,

$$\left.\begin{aligned} R^{(e)}(z) &= \frac{R(z)+R(-z)}{2}, \\ R^{(o)}(z) &= \frac{R(z)-R(-z)}{2}. \end{aligned}\right\} \tag{2.2.6}$$

We shall rely on the following fundamental polynomial identity satisfied by the polynomials P and Φ in respectively, (1.2.7) and (2.2.3).

Theorem 2.2.1 *Let ϕ be a refinable function with refinement sequence $\{p_j\}$ satisfying the support property* (1.2.5) *for an integer $\nu \geq 2$. Then the two polynomials P and Φ, as defined by, respectively,* (1.2.7) *and* (2.2.3)*, satisfy the polynomial identity*

$$P(z)\Phi(z) - P(-z)\Phi(-z) = z\Phi\big(z^2\big), \quad z \in \mathbb{C}. \tag{2.2.7}$$

Proof. First, observe that (2.2.1) implies

$$\phi(k) = \sum_j p_j \phi(2k-j) = \sum_j p_{2k-j}\phi(j), \quad k \in \mathbb{Z}. \tag{2.2.8}$$

Now observe that (2.2.8) holds if and only if, for any $z \in \mathbb{C}$, we have

$$\begin{aligned}
\sum_k \phi(k) z^{2k} &= \sum_k \left[\sum_j p_{2k-j}\phi(j)\right] z^{2k} \\
&= \sum_k \left[\sum_j p_{2k-2j}\phi(2j) + \sum_j p_{2k-2j-1}\phi(2j+1)\right] z^{2k} \\
&= \sum_j \left[\sum_k p_{2k-2j} z^{2k-2j}\right] \phi(2j) z^{2j} + \sum_j \left[\sum_k p_{2k-2j-1} z^{2k-2j-1}\right] \phi(2j+1) z^{2j+1} \\
&= z\left[\sum_k p_{2k} z^{2k}\right]\left[\sum_j \phi(2j) z^{2j-1}\right] + z\left[\sum_k p_{2k-1} z^{2k-1}\right]\left[\sum_j \phi(2j+1) z^{2j}\right]
\end{aligned}$$

and thus, from (1.2.7), (2.2.3), together with (2.2.4) and (2.2.6),

$$\begin{aligned}
z\Phi\left(z^2\right) &= z\sum_k \phi(k+1)\left(z^2\right)^k \\
&= z\sum_k \phi(k+1) z^{2k} \\
&= z\sum_k \phi(k) z^{2k-2} \\
&= z^{-1}\sum_k \phi(k) z^{2k} \\
&= \left[\sum_k p_{2k} z^{2k}\right]\left[\sum_j \phi(2j) z^{2j-1}\right] + \left[\sum_k p_{2k-1} z^{2k-1}\right]\left[\sum_j \phi(2j+1) z^{2j}\right] \\
&= 2\left(\frac{P(z)+P(-z)}{2}\right)\left(\frac{\Phi(z)-\Phi(-z)}{2}\right) + 2\left(\frac{P(z)-P(-z)}{2}\right)\left(\frac{\Phi(z)+\Phi(-z)}{2}\right) \\
&= P(z)\Phi(z) - P(-z)\Phi(-z),
\end{aligned}$$

and thereby proving the desired polynomial identity (2.2.7). ■

The following necessary condition on a refinement sequence $\{p_j\}$ for its corresponding refinable function ϕ to possess prescribed values at the integers can now be deduced from Theorem 2.2.1.

Theorem 2.2.2 *Let ϕ be a refinable function with refinement sequence $\{p_j\}$ satisfying the support property* (1.2.5) *for an integer $\nu \geq 2$, and suppose ϕ has prescribed values $\{y_1, \ldots, y_{\nu-1}\}$ at the integers $\{1, \ldots, \nu-1\}$, as specified in* (2.2.2). *Then the corresponding mask symbol P, as defined by* (1.2.7), *satisfies the polynomial identity*

$$Y(z)P(z) - Y(-z)P(-z) = zY\left(z^2\right), \quad z \in \mathbb{C}, \tag{2.2.9}$$

where, with the definition

$$y_j := 0, \quad j \notin \{1, \ldots, \nu-1\}, \tag{2.2.10}$$

Y is the polynomial given by

$$Y(z) := \sum_j y_{j+1} z^j = \sum_{j=0}^{\nu-2} y_{j+1} z^j. \tag{2.2.11}$$

Proof. Since (2.2.2) holds, it follows from (2.2.3) and (2.2.11) that $\Phi = Y$, which can now be substituted into the identity (2.2.7) of Theorem 2.2.1 to immediately yield the desired result (2.2.9). ■

2.3 Subdivision convergence as a sufficient condition

As one of the main results of this thesis, we proceed to show that, subject to the additional condition of subdivision convergence, together with a further natural constraint on the polynomial Y, a polynomial solution P of the identity (2.2.9) yields, by means of (1.2.7), a refinement sequence $\{p_j\}$ with corresponding refinable function ϕ satisfying the condition (2.2.2). Our work therefore extends the paper De Villiers, Micchelli and Sauer (2000), where the necessary condition (2.2.9) of Theorem 2.2.2 was directly used to obtain P for specific choices of Y, without investigating sufficient conditions on P for ϕ to have prescribed values at the integers as in (2.2.2). Our result is as follows.

Theorem 2.3.1 *Let $\{p_j\} \in \ell_0$ be a sequence with support interval* (1.2.5) *for an integer $\nu \geq 2$, and such that the subdivision operator $S_{\boldsymbol{p}}$ provides a convergent subdivision scheme. Suppose, moreover, that the corresponding mask symbol P, as defined by* (1.2.7)*, satisfies the polynomial identity* (2.2.9)*, with Y denoting any polynomial in $\pi_{\nu-2}$ such that*

$$Y(1) = 1. \tag{2.3.1}$$

Then the corresponding limit (scaling function) ϕ is a refinable function with the values $\{y_1, \ldots, y_{\nu-1}\}$ at the integers $\{1, \ldots, \nu-1\}$, as in (2.2.2)*, where*

$$\sum_{j=0}^{\nu-2} y_{j+1} z^j := Y(z). \tag{2.3.2}$$

Proof. The fact that ϕ is a refinable function with refinement sequence $\{p_j\}$ has already been stated in Theorem 1.3.1 (b)(i).

Next, we apply (2.3.2) and (1.2.7) to deduce from (2.2.9) that, for any $z \in \mathbb{C}$,

$$\begin{aligned}
\sum_j y_{j+1} z^{2j+1} &= \frac{1}{2}\left[\sum_k y_{k+1} z^k\right]\left[\sum_j p_j z^j\right] - \frac{1}{2}\left[\sum_k (-1)^k y_{k+1} z^k\right]\left[\sum_j (-1)^j p_j z^j\right] \\
&= \frac{1}{2}\sum_k y_{k+1}\left[\sum_j p_j z^{k+j}\right] - \frac{1}{2}\sum_k (-1)^k y_{k+1}\left[\sum_j (-1)^j p_j z^{k+j}\right] \\
&= \frac{1}{2}\sum_k y_{k+1}\left[\sum_j p_{j-k} z^j\right] - \frac{1}{2}\sum_k y_{k+1}\left[\sum_j (-1)^j p_{j-k} z^j\right] \\
&= \frac{1}{2}\left\{\sum_k y_{k+1}\left[\sum_j p_{2j-k} z^{2j}\right] + \sum_k y_{k+1}\left[\sum_j p_{2j+1-k} z^{2j+1}\right]\right\} \\
&\quad -\frac{1}{2}\left\{\sum_k y_{k+1}\left[\sum_j p_{2j-k} z^{2j}\right] - \sum_k y_{k+1}\left[\sum_j p_{2j+1-k} z^{2j+1}\right]\right\} \\
&= \sum_k y_{k+1}\left[\sum_j p_{2j+1-k} z^{2j+1}\right] \\
&= \sum_j \left[\sum_k p_{2j+1-k} y_{k+1}\right] z^{2j+1},
\end{aligned}$$

and thus, for $j \in \mathbb{Z}$,

$$y_{j+1} = \sum_k p_{2j+1-k} y_{k+1} = \sum_k p_{2j+2-k} y_k,$$

or equivalently,

$$y_j = \sum_k p_{2j-k} y_k = \sum_k p_k y_{2j-k}. \tag{2.3.3}$$

It follows from (2.3.3) and (1.3.2) that

$$\begin{aligned}
y_j = \sum_k p_k \left[\sum_\ell p_\ell y_{4j-2k-\ell}\right] &= \sum_k p_k \left[\sum_\ell p_{\ell-2k} y_{4j-\ell}\right] \\
&= \sum_\ell \left[\sum_k p_{\ell-2k} p_k y_{4j-\ell}\right] \\
&= \sum_\ell (S_{\boldsymbol{p}}\boldsymbol{p})_\ell y_{4j-\ell}.
\end{aligned} \tag{2.3.4}$$

Now observe from (1.3.2) and (1.3.7) that

$$(S_{\boldsymbol{p}}\boldsymbol{\delta})_j = \sum_k p_{j-2k}\delta_k = p_j, \quad j \in \mathbb{Z},$$

that is,

$$S_{\boldsymbol{p}}\boldsymbol{\delta} = \boldsymbol{p}. \tag{2.3.5}$$

With the sequence $\{p_j^{[r]}\}$ defined, for any $r \in \mathbb{N}$, as in (1.3.6), and (1.3.7), it follows from (2.3.3), (2.3.4), (2.3.5), together with the definition (1.3.6), that

$$y_j = \sum_k p_k^{[1]} y_{2j-k}, \quad j \in \mathbb{Z}; \tag{2.3.6}$$

$$y_j = \sum_k p_k^{[2]} y_{2^2 j-k}, \quad j \in \mathbb{Z}. \tag{2.3.7}$$

By applying (2.3.7), (2.3.3), (1.3.2) and (2.3.5), we obtain, for $j \in \mathbb{Z}$, and with the notation $\boldsymbol{p}^{[r]} = \{p_j^{[r]}\}$, $r = 1,2,\ldots,$

$$\begin{aligned} y_j &= \sum_k p_k^{[2]} \left[\sum_\ell p_\ell y_{2^3 j-2k-\ell} \right] \\ &= \sum_k p_k^{[2]} \left[\sum_\ell p_{\ell-2k} y_{2^3 j-\ell} \right] \\ &= \sum_\ell \left[\sum_k p_{\ell-2k} p_k^{[2]} \right] y_{2^3 j-\ell} \\ &= \sum_\ell \left(S_{\boldsymbol{p}} \boldsymbol{p}^{[2]} \right)_\ell y_{2^3 j-\ell} \\ &= \sum_\ell p_\ell^{[3]} y_{2^3 j-\ell}. \end{aligned} \tag{2.3.8}$$

By continuing similarly, we prove that (2.3.6), (2.3.7) and (2.3.8) generalizes, for $r \in \mathbb{N}$ and $j \in \mathbb{Z}$, to

$$y_j = \sum_k p_k^{[r]} y_{2^r j-k} = \sum_k p_{2^r j-k}^{[r]} y_k,$$

that is,

$$y_j = \sum_k p_{2^r j-k}^{[r]} y_k, \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}. \tag{2.3.9}$$

Next, by using the definition (2.2.10), we deduce from (2.3.2) and (2.3.1) that

$$\sum_j y_j = \sum_j y_{j+1} = \sum_{j=0}^{\nu-2} y_{j+1} = Y(1) = 1. \tag{2.3.10}$$

It then follows from (2.3.9), (2.3.10) and (2.2.10) that, for any $j \in \{1,\ldots,\nu-1\}$,

$$y_j - \phi(j) = \sum_k \left[p_{2^r j-k}^{[r]} - \phi(j) \right] y_k = \sum_{k=1}^{\nu-1} \left[p_{2^r j-k}^{[r]} - \phi(j) \right] y_k, \quad r \in \mathbb{N},$$

and thus

$$\left|\phi(j) - y_j\right| \leq \sum_{k=1}^{\nu-1} \left|\phi(j) - p^{[r]}_{2^r j-k}\right| \left|y_k\right|, \quad r \in \mathbb{N}. \tag{2.3.11}$$

Since the sequence $\{p_j\}$ provides a convergent subdivision scheme, we know from (1.3.5) that

$$\sup_{\ell} \left|\phi\left(\frac{\ell}{2^r}\right) - p^{[r]}_{\ell}\right| \to 0, \quad r \to \infty. \tag{2.3.12}$$

Let $j \in \{1,\ldots,\nu-1\}$ and $k \in \{1,\ldots,\nu-1\}$ be fixed. It then follows from (2.3.12) that

$$\begin{aligned} \left|\phi\left(j - \frac{k}{2^r}\right) - p^{[r]}_{2^r j-k}\right| &= \left|\phi\left(\frac{2^r j - k}{2^r}\right) - p^{[r]}_{2^r j-k}\right| \\ &\leq \sup_{\ell} \left|\phi\left(\frac{\ell}{2^r}\right) - p^{[r]}_{\ell}\right| \to 0, \quad r \to \infty, \end{aligned} \tag{2.3.13}$$

whereas the continuity at $x = j$ of the function ϕ implies

$$\left|\phi\left(j - \frac{k}{2^r}\right) - \phi(j)\right| \to 0, \quad r \to \infty. \tag{2.3.14}$$

By applying (2.3.11), (2.3.13) and (2.3.14), we deduce that, for any $j \in \{1,\ldots,\nu-1\}$,

$$\begin{aligned} \left|\phi(j) - y_j\right| &\leq \sum_{k=1}^{\nu-1} \left\{\left|\phi(j) - \phi\left(j - \frac{k}{2^r}\right)\right| + \left|\phi\left(j - \frac{k}{2^r}\right) - p^{[r]}_{2^r j-k}\right|\right\} |y_k| \\ &\leq \max_{1\leq k\leq \nu-1} |y_k| \left[\sum_{k=1}^{\nu-1} \left|\phi(j) - \phi\left(j - \frac{k}{2^r}\right)\right| + \sum_{k=1}^{\nu-1} \left|\phi\left(j - \frac{k}{2^r}\right) - p^{[r]}_{2^r j-k}\right|\right] \\ &\to \max_{1\leq k\leq \nu-1} |y_k| [0+0] = 0, \quad r \to \infty, \end{aligned}$$

which completes our proof of the desired result (2.2.2). ■

By combining Theorem 1.4.1 and Theorem 2.3.1, and recalling also Theorem 1.2.1, Theorem 1.3.1 (a), (b), as well as Theorem 1.2.2, we immediately obtain the following result.

Theorem 2.3.2 *For an integer $\nu \geq 2$, let $\{p_j\} \in \ell_0$ denote a sequence satisfying the finite support property* (1.2.5)*, the sum-rule condition* (1.2.8)*, and the positivity condition* (1.4.1)*. Suppose, moreover, that the corresponding mask symbol P, as defined by* (1.2.7)*, satisfies the polynomial identity* (2.2.9)*, with Y denoting any polynomial in $\pi_{\nu-2}$ such that* (2.3.1) *holds. Then there exists precisely one function $\phi \in \mathscr{C}_0$ such that ϕ is refinable with refinement sequence $\{p_j\}$, with also*

$$supp^c \phi = [0, \nu]; \tag{2.3.15}$$

$$\sum_j \phi(x-j) = \sum_j \phi(j) = \int_{-\infty}^{\infty} \phi(s)\,ds = 1, \quad x \in \mathbb{R}; \tag{2.3.16}$$

$$\phi(x) > 0, \quad x \in (0,\nu), \tag{2.3.17}$$

and ϕ *has the values* $\{y_1,\ldots,y_{\nu-1}\}$ *at the integers* $\{1,\ldots,\nu-1\}$, *that is,*

$$\phi(j) = y_j, \quad j = 1,\ldots,\nu-1, \tag{2.3.18}$$

where the sequence $\{y_j : j = 1,\ldots,\nu-1\}$ *is defined by* (2.3.2). *Also, the subdivision operator* $S_{\boldsymbol{p}}$ *provides a convergent subdivision scheme with limit (scaling) function* ϕ.

We proceed in Chapter 3 to establish, for polynomials Y in Theorem 2.3.2 satisfying certain further conditions, a constructive existence theory for a one-parameter family of mask symbols P which solve the polynomial identity (2.2.9).

Chapter 3

A one-parameter family of mask symbols

In this chapter, we develop a constructive method for polynomial solution $P \in \pi_\nu$ of the identity (2.2.9), for a given polynomial $Y \in \pi_{\nu-2}$ satisfying certain condition.

3.1 A general polynomial identity

First, we prove, in Theorem 3.1.1 below, a constructive existence result, on which we shall rely in later chapters, for a general class of polynomial identities which allows (2.2.9) as special case, and which extends the result of Chui and de Villiers (2010, Theorem 7.1.1), in the sense that the right hand side of the identity (3.1.6) below is more general than the one considered in Chui and de Villiers (2010). Our constructive proof follows the general structure of the one used to prove Chui and de Villiers (2010, Theorem 7.1.1).

First, we introduce the following concepts.

Definition 3.1.1 *For a polynomial f, if $f(z_0) = f(-z_0) = 0$ for some $z_0 \in \mathbb{C}\backslash\{0\}$, then z_0 and $-z_0$ are called* ***symmetric zeros*** *of f.*

Also, for any function f defined on $\mathbb{C}$, we adopt the notation

$$f_-(z) := f(-z), \quad z \in \mathbb{C}.$$

Observe that if f is a polynomial, then f_- is also a polynomial, with $\deg(f_-) = \deg(f)$. In the following gcd stands for greatest common divisor, and we denote by $\lfloor x \rfloor$ the largest integer $\leq x$.

Theorem 3.1.1 *For polynomials G and F such that* $\deg(G) = d \geq 2$; $G(0) \neq 0$; G *has no symmetric zeros;* $F \in \pi_{2d-1}$, *and F is an odd polynomial, that is,* $F_- = -F$, *let* $\{U, V\}$ *and* $\{Q, R\}$, *with* $R \in \pi_{d-1}$, *denote the polynomial pairs obtained from, respectively, the Euclidean algorithm result*

$$G(z)U(z) + G(-z)V(z) = 1, \quad z \in \mathbb{C}, \tag{3.1.1}$$

and the polynomial division result

$$VF = QG + R. \tag{3.1.2}$$

Then:

(a) The polynomial

$$\tilde{H} := -R_- \in \pi_{d-1}, \tag{3.1.3}$$

is not the zero polynomial, and satisfies the identity

$$G\tilde{H} - G_-\tilde{H}_- = F. \tag{3.1.4}$$

(b) If $F \in \pi_{2d-3}$, *then* $\tilde{H} \in \pi_{d-2}$.

(c) If H^* *is any polynomial satisfying*

$$GH^* - G_-H^*_- = F, \tag{3.1.5}$$

then the general polynomial solution H of the identity

$$GH - G_-H_- = F \tag{3.1.6}$$

is given by

$$H(z) = H^*(z) + J\left(z^2\right)G(-z), \tag{3.1.7}$$

with J denoting an arbitrary polynomial.

(d) The polynomial $H = \tilde{H}$, *as given in* (3.1.3), *is the only solution in* π_{d-1} *of the identity* (3.1.6).

Proof. (a) Since G has no symmetric zeros, and $G(0) \neq 0$, it follows that $\gcd\{G, G_-\} = 1$, and thus there exist polynomials U and V, as obtained by means of the Euclidean algorithm, such that (3.1.1) is satisfied. It follows from (3.1.1) that

$$GUF + G_-VF = F. \tag{3.1.8}$$

Let $\{Q, R\}$ denote the unique polynomial pair, with

$$R \in \pi_{d-1}, \tag{3.1.9}$$

as obtained by means of polynomial division, such that (3.1.2) is satisfied. It follows from (3.1.8) and (3.1.2) that

$$G(UF + QG_-) + G_-R = F,$$

and thus

$$G\tilde{H} + G_-R = F, \tag{3.1.10}$$

where

$$\tilde{H} := UF + QG_- . \tag{3.1.11}$$

Since $F_- = -F$, it follows that F is not the zero polynomial, so that we may deduce from (3.1.10) that, if $\tilde{H}$ and R are both not the zero polynomial, then

$$\begin{aligned} d + \deg\left(\tilde{H}\right) = \deg\left(G\tilde{H}\right) &= \deg\left(F - G_-R\right) \\ &\leq \max\left\{\deg(F), \deg(G_-R)\right\} \\ &\leq \max\left\{2d-1, \deg(GR)\right\} \\ &= \max\left\{2d-1, d+\deg(R)\right\} \leq 2d-1, \end{aligned}$$

by virtue of (3.1.9), from which it follows that

$$\deg\left(\tilde{H}\right) \leq d-1,$$

and thus

$$\tilde{H} \in \pi_{d-1}, \tag{3.1.12}$$

whereas, if $\tilde{H}$ is not the zero polynomial and R is the zero polynomial, then (3.1.10) gives $G\tilde{H} = F$, and (3.1.12) similarly follows. Since also (3.1.12) is trivially true if $\tilde{H}$ is the zero polynomial, we deduce that (3.1.12) holds in all cases. Note from (3.1.10) that

$$G_-\tilde{H}_- + GR_- = F_- = -F,$$

that is,

$$GR_{\_} + G_{\_}\tilde{H}_{\_} = -F. \tag{3.1.13}$$

By adding the equations (3.1.10) and (3.1.13), we obtain

$$G\big(\tilde{H} + R_{\_}\big) = -G_{\_}\big(\tilde{H}_{\_} + R\big). \tag{3.1.14}$$

Since, moreover, $\gcd\{G, G_{\_}\} = 1$, it follows from (3.1.14) that

$$\tilde{H} + R_{\_} = KG_{\_} \tag{3.1.15}$$

for some polynomial K. But deg $(G_{\_})$ = deg $(G) = d$, whereas (3.1.9) and (3.1.12) yield $\tilde{H} + R_{\_} \in \pi_{d-1}$, so that we may deduce from (3.1.15) that $K = 0$, the zero polynomial, and thus, from (3.1.15),

$$\tilde{H} = -R_{\_}, \tag{3.1.16}$$

or equivalently,

$$R = -\tilde{H}_{\_}. \tag{3.1.17}$$

Hence we may substitute (3.1.17) into (3.1.10) to deduce that (3.1.4) does indeed hold, with (3.1.3) following from (3.1.16) and (3.1.9). Note from (3.1.4), together with the fact that F is not the zero polynomial, as follows from $F_{\_} = -F$, that $\tilde{H}$ is not the zero polynomial.

(b) Suppose $F \in \pi_{2d-3} \subset \pi_{2d-1}$. Since G is a polynomial, with deg $(G) = d$, we may write

$$G(z) = \sum_{j=0}^{d} g_j z^j, \quad z \in \mathbb{C}, \tag{3.1.18}$$

with

$$g_d \neq 0. \tag{3.1.19}$$

Also, according to (3.1.12), we have

$$\tilde{H}(z) = \sum_{j=0}^{d-1} \tilde{h}_j z^j, \quad z \in \mathbb{C}. \tag{3.1.20}$$

By substituting (3.1.18) and (3.1.20) into (3.1.4), we obtain

$$2g_d\tilde{h}_{d-1}z^{2d-1} + Q(z) = F(z), \quad z \in \mathbb{C}, \tag{3.1.21}$$

where $Q \in \pi_{2d-3}$. Since $F \in \pi_{2d-3}$, we deduce from (3.1.21) that $g_d \tilde{h}_{d-1} = 0$, and thus, from (3.1.19), $\tilde{h}_{d-1} = 0$, which, together with (3.1.20), yields the required result $\tilde{H} \in \pi_{d-2}$.

(c) Suppose H^* is any polynomial satisfying the identity (3.1.5), and let H denote any polynomial solution of the identity (3.1.6). By subtracting (3.1.4) from (3.1.6), we obtain

$$G\big(H - H^*\big) = G\_\big(H\_ - H^*\_\big), \tag{3.1.22}$$

and thus, since $\gcd\{G, G\_\} = 1$, we obtain

$$H - H^* = J^* G\_\,, \tag{3.1.23}$$

for some polynomial J^*. By substituting (3.1.23) into (3.1.22), we deduce that

$$G J^* G\_ = G\_\, J^*\_\, G,$$

which yields

$$J^*\_ = J^*, \tag{3.1.24}$$

according to which

$$J^*(z) = J\big(z^2\big), \tag{3.1.25}$$

for some polynomial J, which, together with (3.1.23), shows that H is given as (3.1.7). Hence we have shown that, if H is a polynomial solution of the identity (3.1.6), then H must be given by (3.1.7), for some polynomial J.

Conversely, if H is given by (3.1.7) for an arbitrary polynomial J, then, for any $z \in \mathbb{C}$,

$$\begin{aligned} G(z)H(z) - G(-z)H(-z) &= G(z)\big[H^*(z) + J\big(z^2\big)G(-z)\big] - G(-z)\big[H^*(-z) + J\big(z^2\big)G(z)\big] \\ &= G(z)H^*(z) - G(-z)H^*(-z) + J\big(z^2\big)\big[G(z)G(-z) - G(-z)G(z)\big] \\ &= F(z) + J\big(z^2\big)(0) = F(z), \end{aligned}$$

from (3.1.5), and it follows that H is indeed a polynomial solution of the identity (3.1.6).

(d) Let H denote any solution in π_{d-1} of polynomial identity (3.1.6). It follows from (a) and (c) that

$$H(z) = \tilde{H}(z) + J\big(z^2\big)G(-z), \quad z \in \mathbb{C}, \tag{3.1.26}$$

for some polynomial J, and thus

$$H(z) - \tilde{H}(z) = J(z^2)G(-z), \quad z \in \mathbb{C}. \tag{3.1.27}$$

Since $H \in \pi_{d-1}$, and (3.1.3) holds, we have $H - \tilde{H} \in \pi_{d-1}$. Also, deg $(G_) = $ deg $(G) = d$. It therefore follows from (3.1.27) that J is the zero polynomial, and thus polynomial $H = \tilde{H}$, which completes our proof. ∎

In Chui and de Villiers (2010, Theorem 7.1.1) the special case $F(z) = z^{-2\lfloor d/2 \rfloor - 1}$ of Theorem 3.1.1, was proved.

3.2 The mask polynomial identity

Based on Theorem 2.3.1 and Theorem 1.3.1 (a), together with the equivalent mask symbol formulation (1.2.9) of the sum-rule condition (1.2.8), we proceed to find, by applying Theorem 3.1.1, the general solution $P \in \pi_\nu$ of

$$\left.\begin{aligned} Y(z)P(z) - Y(-z)P(-z) = zY(z^2), \quad z \in \mathbb{C}; \\ P(1) = 1 \quad ; \quad P(-1) = 0, \end{aligned}\right\} \tag{3.2.1}$$

where Y is a given polynomial in $\pi_{\nu-2}$, and satisfying certain conditions, as formulated in the following result.

Theorem 3.2.1 *For a polynomial $Y \in \pi_{\nu-2}$, with $\nu \geq 2$, satisfying the conditions*

$$\deg(Y) = \nu - 2; \tag{3.2.2}$$

$$Y(0) \neq 0 \quad ; \quad Y(1) = 1, \tag{3.2.3}$$

and where Y has no symmetric zeros, let $\{U, V\}$ and $\{Q, R\}$, with $R \in \pi_{\nu-2}$, be the polynomial pairs obtained from, respectively, the Euclidean algorithm result

$$\left[\left(\frac{1+z}{2}\right)Y(z)\right]U(z) + \left[\left(\frac{1-z}{2}\right)Y(-z)\right]V(z) = 1, \quad z \in \mathbb{C}, \tag{3.2.4}$$

and the polynomial division result

$$zY(z^2) = Q(z)\left[\left(\frac{1+z}{2}\right)Y(z)\right] + R(z), \quad z \in \mathbb{C}. \tag{3.2.5}$$

Then:

(a) The polynomial solutions P in π_ν of (3.2.1) *are given by the one-parameter family*

$$P(z) = P(t|z) := P_0(z) + t\left(1-z^2\right)Y(-z), \quad t \in \mathbb{R}, \tag{3.2.6}$$

where P_0 is the non-trivial polynomial in $\pi_{\nu-1}$ defined by

$$P_0(z) := -\left(\frac{1+z}{2}\right)R(-z). \tag{3.2.7}$$

(b) If $P^* \in \pi_\nu$ satisfies

$$\left.\begin{aligned} Y(z)P^*(z) - Y(-z)P^*(-z) = zY\left(z^2\right), \quad z \in \mathbb{C}; \\ P^*(1) = 1 \quad ; \quad P^*(-1) = 0, \end{aligned}\right\} \tag{3.2.8}$$

then the polynomial solutions P in π_ν of (3.2.1) *are given by the one-parameter family*

$$P(z) = P(t|z) := P^*(z) + t\left(1-z^2\right)Y(-z), \qquad t \in \mathbb{R}. \tag{3.2.9}$$

Proof. (a) Let the polynomials G and F be defined by

$$G(z) := \left(\frac{1+z}{2}\right)Y(z); \tag{3.2.10}$$

$$F(z) := zY\left(z^2\right). \tag{3.2.11}$$

Observe from (3.2.10) and (3.2.2) that

$$\deg(G) = \nu - 1. \tag{3.2.12}$$

Also, note from (3.2.10) and (3.2.3) that

$$G(0) \neq 0 \quad ; \quad G(1) = 1. \tag{3.2.13}$$

Next, suppose G has a symmetric zero, that is, there exists a point $z_0 \in \mathbb{C}\setminus\{0\}$ such that

$$G(z_0) = G(-z_0) = 0, \tag{3.2.14}$$

and thus, from (3.2.10),

$$(1+z_0)Y(z_0) = (1-z_0)Y(-z_0) = 0, \tag{3.2.15}$$

according to which either $z_0 = -1$ or $Y(z_0) = 0$. If $z_0 = -1$, the second equation in (3.2.15) gives $Y(1) = 0$, which contradicts (3.2.3). If $Y(z_0) = 0$, then, since the polynomial Y has no symmetric zeros, we have $Y(-z_0) \neq 0$, so that the second equation in (3.2.15) gives $z_0 = 1$, and thus $Y(1) = 0$, which again contradict (3.2.3). Hence (3.2.15) is not satisfied by any $z_0 \in \mathbb{C}\setminus\{0\}$, and it follows that G has no symmetric zeros.

Next, we note from (3.2.11), (3.2.2) and (3.2.12) that

$$\deg(F) = 2\nu - 3 = 2\deg(G) - 1,$$

whereas (3.2.11) also gives

$$F(-z) = -zY\left(z^2\right) = -F(z), \quad z \in \mathbb{C},$$

that is, $F_- = -F$. Hence we have now shown that the polynomials G and F, as given by (3.2.10) and (3.2.11), respectively, satisfy the conditions of Theorem 3.1.1 with $d = \nu - 1$.

We may therefore apply Theorem 3.1.1(a), together with (3.2.10), (3.2.11), and (3.2.12), to deduce that the polynomial solutions P in π_ν of the identity in the first line of (3.2.1) is given by the one-parameter family as described in (3.2.6), (3.2.7), (3.2.5), (3.2.4).

Observe from (3.2.6) and (3.2.7) that

$$P(t|-1) = 0, \quad t \in \mathbb{R}. \tag{3.2.16}$$

Also, since $P = P(t|\cdot)$ satisfies the identity in the first line of (3.2.1), that is

$$Y(z)P(t|z) - Y(-z)P(t|-z) = zY\left(z^2\right), \quad z \in \mathbb{C}, \quad t \in \mathbb{R}, \tag{3.2.17}$$

we may now set $z = 1$ in (3.2.17), and apply (3.2.16) and $Y(1) = 1$, from (3.2.3), to obtain

$$P(t|1) = 1, \quad t \in \mathbb{R}. \tag{3.2.18}$$

It follows from (3.2.16) and (3.2.18) that the one-parameter family of polynomials $P \in \pi_\nu$, as given by (3.2.6), also satisfies the second line of (3.2.1).

(b) Let $P^* \in \pi_\nu$ satisfy (3.2.8). It follows from the second line of (3.2.8) that there exists a polynomial P^{**} such that

$$P^*(z) = \left(\frac{1+z}{2}\right)P^{**}(z), \tag{3.2.19}$$

with also

$$P^{**}(1) = 1. \tag{3.2.20}$$

Hence, by using (3.2.19) and the first line of (3.2.8), it follows that

$$\left[\left(\frac{1+z}{2}\right)Y(z)\right]P^{**}(z) - \left[\left(\frac{1-z}{2}\right)Y(-z)\right]P^{**}(-z) = zY\left(z^2\right), \quad z \in \mathbb{C}. \tag{3.2.21}$$

Analogously to the proof in (a), we now apply Theorem 3.1.1 (c), with the polynomials G and F defined as in (3.2.10) and (3.2.11), respectively, to deduce that the polynomial solutions P in π_ν of (3.2.1) are given by the one-parameter family $P = P(t|\cdot)$, as formulated in (3.2.9). ■

By combining Theorems 2.3.1 and 3.2.1, we immediately deduce the following result.

Theorem 3.2.2 *For a mask symbol $P = P(t|\cdot) \in \pi_\nu$ as in either* (3.2.6) *or* (3.2.9), *respectively, of Theorem 3.2.1 (a) and (b), suppose there exists a set $I \subset \mathbb{R}$ such that the sequence $\boldsymbol{p}(t) = \{p_j(t)\} \in \ell_0$ defined by*

$$\frac{1}{2}\sum_j p_j(t)z^j := P(t|z) \tag{3.2.22}$$

satisfies

$$supp\{p_j(t)\} = [0,\nu]|_{\mathbb{Z}}, \quad t \in I, \tag{3.2.23}$$

and the subdivision operator $S_{\boldsymbol{p}(t)}$ provides, for each $t \in I$, a convergent subdivision scheme. Then, for $t \in I$, the corresponding limit (scaling) function $\phi(t|\cdot)$ is a refinable function with refinement sequence $\{p_j(t)\}$, that is,

$$\phi(t|x) = \sum_j p_j(t)\phi(t|2x-j), \quad x \in \mathbb{R}, \quad t \in I, \tag{3.2.24}$$

and

$$\phi(t|j) = y_j, \quad j = 1,\ldots,\nu-1, \quad t \in I, \tag{3.2.25}$$

where the sequence $\{y_j : j = 1,\ldots,\nu-1\}$ is defined by

$$\sum_{j=0}^{\nu-2} y_{j+1}z^j := Y(z). \tag{3.2.26}$$

Similarly, by applying Theorems 2.3.2 and 3.2.1, we obtain the following.

Theorem 3.2.3 *For a mask symbol* $P = P(t|\cdot) \in \pi_\nu$ *as in either* (3.2.6) *or* (3.2.9), *respectively, of Theorem 3.2.1 (a) and (b), suppose there exists a set* $I \subset \mathbb{R}$ *such that the sequence* $\boldsymbol{p}(t) = \{p_j(t)\} \in \ell_0$ *defined by* (3.2.22) *satisfies the positivity condition*

$$p_j(t) > 0, \quad j = 0, \ldots, \nu, \quad t \in I. \tag{3.2.27}$$

Then, for each $t \in I$, *there exists precisely one function* $\phi(t|\cdot)$ *such that* $\phi(t|\cdot)$ *is refinable with refinement sequence* $\{p_j(t)\}$, *as in* (3.2.24), *with also*

$$\mathrm{supp}^c \phi(t|\cdot) = [0, \nu], \quad t \in I; \tag{3.2.28}$$

$$\sum_j \phi(t|x - j) = \sum_j \phi(t|j) = \int_{-\infty}^{\infty} \phi(t|s)ds = 1, \quad x \in \mathbb{R}, \quad t \in I; \tag{3.2.29}$$

$$\phi(t|x) > 0, \quad x \in (0, \nu), \quad t \in I. \tag{3.2.30}$$

Moreover, for each $t \in I$, $\phi(t|\cdot)$ *has the values* $\{y_1, \ldots, y_{\nu-1}\}$ *at the integers* $\{1, \ldots, \nu - 1\}$, *as in* (3.2.25). *Also, for each* $t \in I$, *the the subdivision operator* $S_{\boldsymbol{p}(t)}$ *provides a convergent subdivision scheme, with limit function* $\phi(t|\cdot)$.

After recalling from Theorem 1.4.2 that the regularity of a refinable function ϕ increases with the integer value m for which the corresponding refinement mask symbol P contains the factor $\left(\frac{1+z}{2}\right)^m$, we proceed to prove that the one-parameter family of the polynomials $P(t|\cdot)$ of Theorem 3.2.1 contains precisely one mask symbol $P(t_0|\cdot)$ for which $m \geq 2$.

Theorem 3.2.4 *Let* $P(t|\cdot)$ *denote a mask symbol as in either* (3.2.6) *or* (3.2.9) *of Theorem 3.2.1. Then* $P(t|\cdot)$ *contains the factor* $\left(\frac{1+z}{2}\right)^2$ *if and only if*

(a) for the case (3.2.6),

$$t = t_0 := \frac{1}{4}R(1), \tag{3.2.31}$$

with R *denoting the remainder polynomial as obtained from* (3.2.5) *and* (3.2.4);

(b) for the case (3.2.9),

$$t = t_0 := -\frac{1}{2}\left(P^*\right)'(-1), \tag{3.2.32}$$

with P^* *denoting any solution in* π_ν *of* (3.2.8).

Proof. (a) First, observe from (3.2.6) and (3.2.7) that

$$P(t|-1) = 0, \quad t \in \mathbb{R}. \tag{3.2.33}$$

Next, we differentiate the polynomial $P(t|\cdot)$, as given in (3.2.6), (3.2.7), to obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}P(t|z) = -\frac{1}{2}R(-z) + \left(\frac{1+z}{2}\right)R'(-z) - 2tzY(-z) - t\left(1-z^2\right)Y'(-z),$$

which, together with $Y(1) = 1$, as in (3.2.3), yields

$$\frac{\mathrm{d}}{\mathrm{d}z}P(t|-1) = -\frac{1}{2}R(1) + 2t,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}z}P(t|-1) = 0 \Leftrightarrow t = \frac{1}{4}R(1). \tag{3.2.34}$$

It follows from (3.2.33) and (3.2.34) that the polynomial $P(t|\cdot)$ contains the factor $\left(\frac{1+z}{2}\right)^2$ if and only if $t = t_0$, as given in (3.2.31).

(b) First, note from (3.2.9), together with $P^*(-1) = 0$, as in the second line of (3.2.8), that (3.2.33) is satisfied. By differentiating (3.2.9) with respect to z, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}P(t|z) = \left(P^*\right)'(z) - 2tzY(-z) - t\left(1-z^2\right)Y'(-z),$$

which, together with $Y(1) = 1$, yield

$$\frac{\mathrm{d}}{\mathrm{d}z}P(t|-1) = \left(P^*\right)'(-1) + 2t,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}z}P(t|-1) = 0 \Leftrightarrow t = -\frac{1}{2}\left(P^*\right)'(-1). \tag{3.2.35}$$

It follows from (3.2.33) and (3.2.35) that the polynomial $P(t|\cdot)$ contains the factor $\left(\frac{1+z}{2}\right)^2$ if and only if $t = t_0$, as given in (3.2.32). ∎

3.3 An example

As an application of Theorems 3.2.1 and 3.2.3, we choose the polynomial Y as

$$Y(z) := \frac{1}{5}\left(1 + 3z + z^2\right), \tag{3.3.1}$$

from which we verify that Y satisfies the condition (3.2.2) and (3.2.3), with $\nu = 4$, and Y has no symmetric zeros. Let the polynomial L be defined by

$$L(z) := \left(\frac{1+z}{2}\right)Y(z) = \frac{1}{10}\left(1 + 4z + 4z^2 + z^3\right). \tag{3.3.2}$$

- According to (3.2.4) and (3.3.2), our first step is to apply the Euclidean algorithm to find the polynomial pair $\{U, V\}$ such that

$$L(z)U(z) + L(-z)V(z) = 1, \quad z \in \mathbb{C}, \tag{3.3.3}$$

as follows:

$$L(z) = (-1)L(-z) + \left(\frac{1}{5} + \frac{4}{5}z^2\right);$$

$$L(-z) = \left(\frac{1}{2} - \frac{1}{8}z\right)\left(\frac{1}{5} + \frac{4}{5}z^2\right) - \left(\frac{3}{8}z\right);$$

$$\left(\frac{1}{5} + \frac{4}{5}z^2\right) = \left(-\frac{32}{15}z\right)\left(-\frac{3}{8}z\right) + \frac{1}{5};$$

and thus, by back-substitution,

$$\begin{aligned}
\frac{1}{5} &= \left(\frac{1}{5} + \frac{4}{5}z^2\right) - \left(-\frac{32}{15}z\right)\left(-\frac{3}{8}z\right) \\
&= \left(\frac{1}{5} + \frac{4}{5}z^2\right) - \left(-\frac{32}{15}z\right)\left[L(-z) - \left(\frac{1}{2} - \frac{1}{8}z\right)\left(\frac{1}{5} + \frac{4}{5}z^2\right)\right] \\
&= \{L(z) + L(-z)\} - \left(-\frac{32}{15}z\right)\left[L(-z) - \left(\frac{1}{2} - \frac{1}{8}z\right)\{L(z) + L(-z)\}\right] \\
&= L(z)\left[1 + \left(-\frac{32}{15}z\right)\left(\frac{1}{2} - \frac{1}{8}z\right)\right] + L(-z)\left[1 - \left(-\frac{32}{15}z\right) + \left(-\frac{32}{15}z\right)\left(\frac{1}{2} - \frac{1}{8}z\right)\right] \\
&= L(z)\left[\frac{1}{15}(15 - 16z + 4z^2)\right] + L(-z)\left[\frac{1}{15}(15 + 16z + 4z^2)\right],
\end{aligned}$$

from which it follows that the polynomials

$$\left.\begin{aligned}
U(z) &:= \frac{1}{3}(15 - 16z + 4z^2); \\
V(z) &:= \frac{1}{3}(15 + 16z + 4z^2);
\end{aligned}\right\} \tag{3.3.4}$$

satisfy the identity (3.3.3).

- According to (3.2.5), our next step is to apply polynomial division to obtain the (unique) polynomial pair $\{Q, R\}$, with $R \in \pi_2$, such that

$$zV(z)Y\left(z^2\right) = Q(z)L(z) + R(z), \quad z \in \mathbb{C}. \tag{3.3.5}$$

We find that

$$zV(z)Y\left(z^2\right) = \frac{1}{15}\left(15z+16z^2+49z^3+48z^4+27z^5+48z^6+4z^7\right) = Q(z)L(z)+R(z),$$

where

$$\left.\begin{aligned} Q(z) &:= \frac{1}{3}\left(10+22z+8z^2\right); \\ R(z) &:= \frac{1}{3}\left(-1-z-3z^2\right). \end{aligned}\right\} \tag{3.3.6}$$

- According to (3.2.7), the polynomial $P_0 \in \pi_3$ is given by

$$P_0(z) = -\left(\frac{1+z}{2}\right)R(-z) = \frac{1}{6}\left(1+2z^2+3z^3\right). \tag{3.3.7}$$

- By applying (3.2.6) in Theorem 3.2.1(a), we deduce from (3.3.7) and (3.3.1) that the polynomial solutions P in π_4 of (3.2.1) are given, for any $t \in \mathbb{R}$, by

$$\begin{aligned} P(z) = P(t|z) &= \frac{1}{6}\left(1+2z^2+3z^3\right)+t\left(1-z^2\right)\frac{1}{5}\left(1-3z+z^2\right) \\ &= \left(\frac{1+z}{2}\right)\left(\frac{(5+6t)-(5+24t)z+(15+24t)z^2+6tz^3}{15}\right) \\ &= \left(\frac{1}{6}+\frac{t}{5}\right)-\frac{3t}{5}z+\frac{1}{3}z^2+\left(\frac{1}{2}+\frac{3t}{5}\right)z^3-\frac{t}{5}z^4. \end{aligned} \tag{3.3.8}$$

By using the definition (3.2.22), we deduce that the corresponding one-parameter refinement sequence $\boldsymbol{p}(t) = \{p_j(t)\} \in \ell_0$ is given by

$$\left\{\begin{aligned} &\{p_0(t), p_1(t), p_2(t), p_3(t), p_4(t)\} = \left\{\tfrac{1}{3}+\tfrac{2t}{5}, -\tfrac{6t}{5}, \tfrac{2}{3}, 1+\tfrac{6t}{5}, -\tfrac{2t}{5}\right\}; \\ &\text{with} \quad p_j(t) = 0, \quad j \notin \{0,1,2,3,4\}. \end{aligned}\right. \tag{3.3.9}$$

It follows from (3.3.9) that

$$p_j(t) > 0, \quad j = 0,\ldots,4 \Longleftrightarrow t \in \left(-\tfrac{5}{6}, 0\right). \tag{3.3.10}$$

By applying Theorem 3.2.3, we deduce from (3.3.10) that, for any parameter value $t \in \left(-\frac{5}{6}, 0\right)$, there exists a unique function $\phi(t|\cdot) \in \mathscr{C}_0$ satisfying the properties

$$\operatorname{supp}^c \phi(t|\cdot) = [0,4]; \tag{3.3.11}$$

$$\phi(t|x) = \sum_j p_j(t)\phi(t|2x-j), \quad x \in \mathbb{R}; \tag{3.3.12}$$

$$\sum_j \phi(t|x-j) = \sum_j \phi(t|j) = \int_{-\infty}^{\infty} \phi(t|s)\mathrm{d}s = 1, \quad x \in \mathbb{R}; \tag{3.3.13}$$

$$\phi(t|x) > 0, \quad x \in (0,4); \tag{3.3.14}$$

with also, from (3.2.25), (3.2.26) and (3.3.1),

$$\phi(t|1) = \frac{1}{5} \quad ; \quad \phi(t|2) = \frac{3}{5} \quad ; \quad \phi(t|3) = \frac{1}{5}. \tag{3.3.15}$$

Also, the subdivision operator $S_{\boldsymbol{p}(t)}$ provides a convergent subdivision scheme for $t \in \left(-\frac{5}{6}, 0\right)$.

Next, in order to apply Theorem 3.2.4(a), we see from the second line of (3.3.6) that $R(1) = -\frac{5}{3}$, so that (3.2.31) gives the value

$$t_0 = -\frac{5}{12}, \tag{3.3.16}$$

which, when substituted into (3.3.8), yields the mask symbol

$$P\left(-\frac{5}{12}\middle| z\right) = \frac{1}{12}\left(1+3z+4z^2+3z^3+z^4\right) = \left(\frac{1+z}{2}\right)^2 \left(\frac{1+z+z^2}{3}\right), \tag{3.3.17}$$

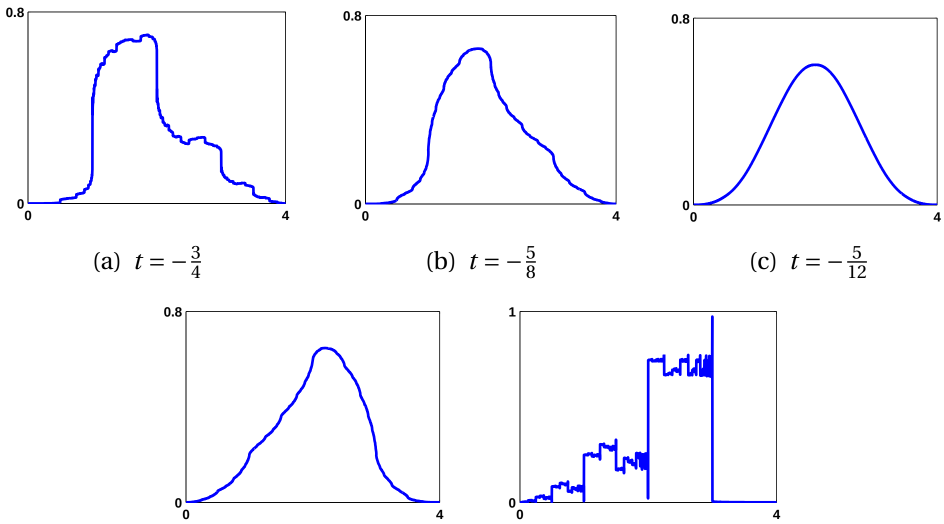
as guaranteed by Theorem 3.2.4 (a).

In Table 3.3.1, we give, for the choices $t = -\frac{3}{4}, -\frac{5}{8}, -\frac{5}{12}, -\frac{1}{4}, -\frac{1}{40} \in \left(-\frac{5}{6}, 0\right)$, the mask coefficients $\{p_j(t)\}$, as computed from (3.3.9). In Figure 3.3.1, we plot the

Table 3.3.1: *One-parameter mask coefficients* $\{p_j(t)\}$ *for different values of* $t \in \left(-\frac{5}{6}, 0\right)$.

t	$\{p_j(t) : j = 0,1,2,3,4\}$
$-\frac{3}{4}$	$\left\{\frac{1}{30}, \frac{27}{30}, \frac{20}{30}, \frac{3}{30}, \frac{9}{30}\right\}$
$-\frac{5}{8}$	$\left\{\frac{1}{12}, \frac{9}{12}, \frac{8}{12}, \frac{3}{12}, \frac{3}{12}\right\}$
$-\frac{5}{12}$	$\left\{\frac{1}{6}, \frac{3}{6}, \frac{4}{6}, \frac{3}{6}, \frac{1}{6}\right\}$
$-\frac{1}{4}$	$\left\{\frac{7}{30}, \frac{9}{30}, \frac{20}{30}, \frac{21}{30}, \frac{3}{30}\right\}$
$-\frac{1}{40}$	$\left\{\frac{97}{300}, \frac{9}{300}, \frac{200}{300}, \frac{291}{300}, \frac{3}{300}\right\}$

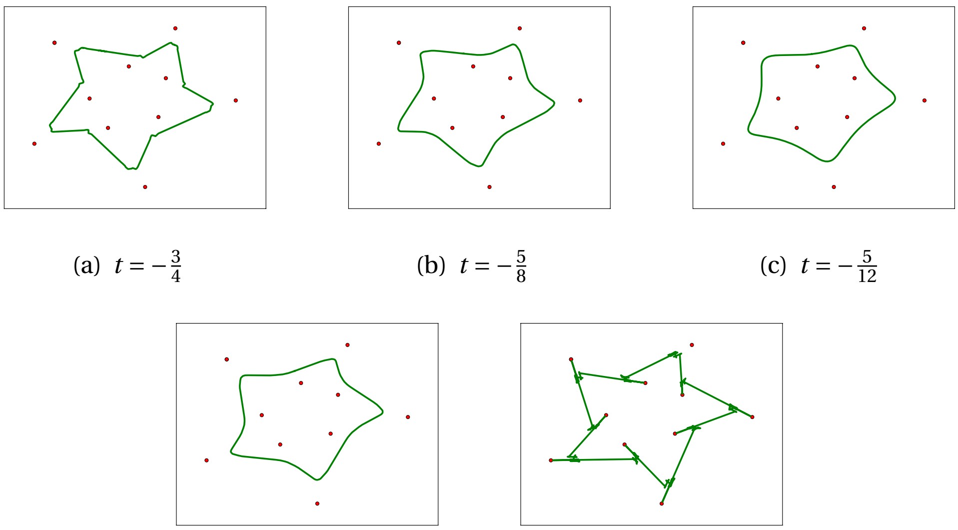
graphs of the corresponding refinable functions $\phi(t|\cdot)$ on their support interval $[0,4]$, by using Algorithm 4.3.1 in Chui and de Villiers (2010), as based on the convergent subdivision scheme implied by (1.3.5), (1.3.6), (1.3.7), whereas, in Figure 3.3.2, we render, for a given initial control point sequence, the corresponding closed subdivision limit curves from the refinement sequences $\{p_j(t)\}$ in Table 3.3.1, by using (a shifted version of) Algorithm 3.3.1(a) in Chui and de Villiers (2010), as obtained from the formulation (1.3.4) of a subdivision scheme.



(a) $t=-\frac{3}{4}$ (b) $t=-\frac{5}{8}$ (c) $t=-\frac{5}{12}$

(d) $t=-\frac{1}{4}$ (e) $t=-\frac{1}{40}$

Figure 3.3.1: *The refinable functions $\phi(t|\cdot)$ for the refinement sequences $\{p_j(t)\}$ in Table 3.3.1.*



(a) $t=-\frac{3}{4}$ (b) $t=-\frac{5}{8}$ (c) $t=-\frac{5}{12}$

(d) $t=-\frac{1}{4}$ (e) $t=-\frac{1}{40}$

Figure 3.3.2: *Closed subdivision curves obtained from the refinement sequences $\{p_j(t)\}$ in Table 3.3.1.*

Next, by applying Theorem 1.4.2, we obtain, as given in Table 3.3.2, the Hölder regularity of the refinable functions $\phi(t|\cdot)$ of Figure 3.3.1.

As expected from (3.3.17), Table 3.3.2 shows that $\phi\big(-\frac{5}{12}\big|\cdot\big)$ is the smoothest of the refinable functions considered, as graphically illustrated by Figure 3.3.1(c) and Figure 3.3.2(c), whereas, in contrast, the regularity results for $t = -\frac{3}{4}$ and $t - \frac{1}{40}$ in Table 3.3.2 are consistent with the fractal-like graphs in Figure 3.3.1(a) and (e).

■

Table 3.3.2: *The Hölder regularity of the refinable functions* $\phi(t|\cdot)$ *of Figure 3.3.1* .

	$\phi(t\|\cdot) \in \mathscr{C}_0^{k,\alpha}$	
t	k	α
$-\frac{3}{4}$	0	$\log_2\left(\frac{30}{29}\right) \approx 0.0489$
$-\frac{5}{8}$	0	$\log_2\left(\frac{12}{11}\right) \approx 0.1255$
$-\frac{5}{12}$	1	$\log_2\left(\frac{3}{2}\right) \approx 0.5850$
$-\frac{1}{4}$	0	$\log_2\left(\frac{10}{9}\right) \approx 0.1520$
$-\frac{1}{40}$	0	$\log_2\left(\frac{1000}{999}\right) \approx 0.0014$

Chapter 4

Perturbed cardinal B-splines

In this chapter, we shall apply Theorem 3.2.1(b) and Theorem 3.2.3 to establish, for $m \geq 2$, a one-parameter family $\phi_m(t|\cdot)$ of refinable functions coinciding at the integers with the cardinal B-splines N_m.

4.1 The cardinal B-splines

Cardinal B-splines have been studied extensively (see e.g., Schoenberg (1973, Lecture 2), Chui (1992, Chapter 4) and Chui and de Villiers (2010, Chapter 2)), and we proceed here to give a definition, and the properties which we shall rely on.

Definition 4.1.1 *The* ***cardinal B-splines*** $\{N_m : m = 2,3,\ldots\}$ *of order* $m \in \mathbb{N}$ *are defined recursively by*

$$N_2(x) := h(x); \quad N_m(x) := \int_0^1 N_{m-1}(x-t)\,dt, \quad x \in \mathbb{R}, \; m = 3,4,\ldots, \tag{4.1.1}$$

with h *denoting the hat function defined by* (1.2.2).

For $k \in \mathbb{Z}_+$, we define the truncated power, $(\cdot)_+^k : \mathbb{R} \to [0,\infty)$ of degree k by

$$t_+^k := \begin{cases} t^k, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{4.1.2}$$

and with the convection $0^0 = 1$. We shall write $(\cdot)_+$ for $(\cdot)_+^1$. Also, for any non-negative integer k, we define the binomial coefficient

$$\binom{k}{j} = \begin{cases} \dfrac{k!}{j!(k-j)!}, & j = 0,\ldots,k; \\ 0, & j \notin \{0,\ldots,k\}, \end{cases} \tag{4.1.3}$$

with the convention $0! := 1$.

Cardinal B-splines possess the following properties, for the proof of which we refer to Chui (1992, Theorem 4.3), (see also Chui and de Villiers (2010, Theorem 2.3.1)).

Theorem 4.1.1 *For an integer $m \geq 2$, the cardinal B-spline N_m of order $m \in \mathbb{N}$, as defined by* (4.1.1), *satisfies the following properties:*

(i) N_m *is a refinable function with refinement sequence* $\boldsymbol{p}_m = \{p_{m,j}\}$ *given by*

$$p_{m,j} = \frac{1}{2^{m-1}}\binom{m}{j}, \quad j \in \mathbb{Z}; \tag{4.1.4}$$

(ii) N_m *is compactly supported, with*

$$supp^c N_m = [0, m]; \tag{4.1.5}$$

(iii)

$$N_m(x) > 0, \quad x \in (0, m); \tag{4.1.6}$$

(iv)

$$\sum_j N_m(x-j) = 1, \quad x \in \mathbb{R}; \tag{4.1.7}$$

(v)

$$\int_{-\infty}^{\infty} N_m(x)\,dx = 1; \tag{4.1.8}$$

(vi)

$$N_m \in \mathscr{C}_0^{m-2}; \tag{4.1.9}$$

(vii)

$$N'_{m+1}(x) = N_m(x) - N_m(x-1), \quad x \in \mathbb{R}, \; \textit{for} \;\; m \geq 3; \tag{4.1.10}$$

(viii) N_m *is a symmetric function, in the sense that*

$$N_m\left(\frac{m}{2} - x\right) = N_m\left(\frac{m}{2} + x\right), \quad x \in \mathbb{R}, \tag{4.1.11}$$

or equivalently,

$$N_m(m-x) = N_m(x), \quad x \in \mathbb{R}, \tag{4.1.12}$$

(ix)

$$N_m(x) = \frac{1}{(m-1)!}\sum_{j=0}^{m}(-1)^j \binom{m}{j}(x-j)_+^{m-1}, \quad x \in \mathbb{R}; \tag{4.1.13}$$

(x) N_m *satisfies the recursive formulation*

$$N_{m+1}(x) = \frac{x}{m}N_m(x) + \frac{m+1-x}{m}N_m(x-1), \quad x \in \mathbb{R}. \tag{4.1.14}$$

Observe in particular from (4.1.13), (4.1.2), (4.1.5) and (4.1.9) that N_m is a compactly supported piecewise polynomial, with breakpoints at the integers and polynomial pieces in π_{m-1}, and with continuous derivatives up to order $m-2$.

According to (4.1.4), the definition (1.2.7) yields the cardinal B-spline mask symbol

$$P_m(z) := \left(\frac{1+z}{2}\right)^m. \tag{4.1.15}$$

It then follows from Theorem 1.2.3 that the refinement sequence $\{p_{m,j}\}$ satisfies the sum-rule of order m, and thus, in particular, the sum-rule condition (1.2.8), that is,

$$\sum_j p_{m,2j} = 1 \quad ; \quad \sum_j p_{m,2j-1} = 1. \tag{4.1.16}$$

Note also from (4.1.4) and (4.1.3) that

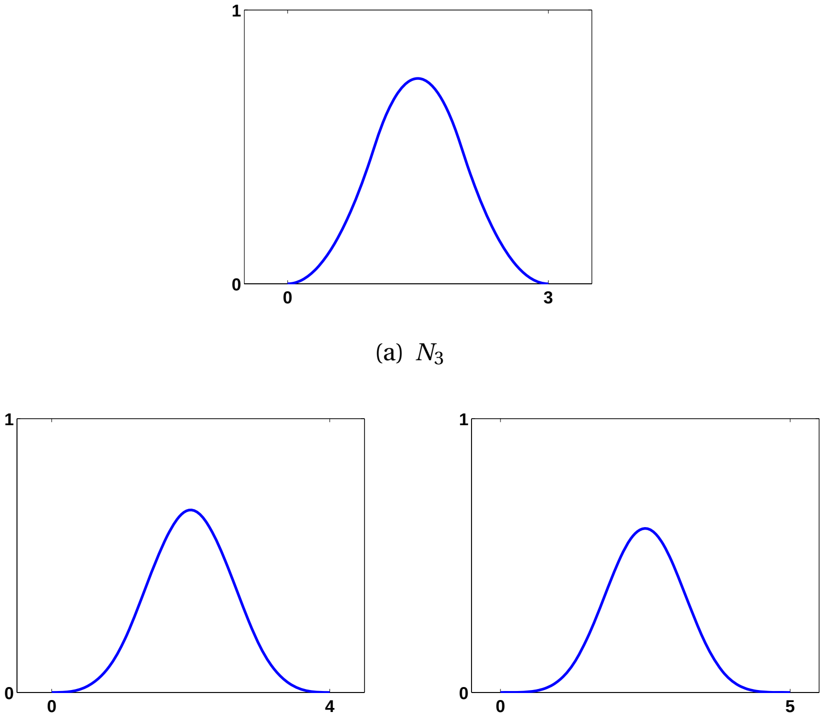
$$\text{supp}\,\{p_{m,j}\} = [0,m]|_{\mathbb{Z}}; \tag{4.1.17}$$

$$p_{m,j} > 0, \quad j = 0,\ldots,m. \tag{4.1.18}$$

Since the sequence $\boldsymbol{p}_m = \{p_{m,j}\}$ satisfies (4.1.16), (4.1.17) and (4.1.18), it follows from Theorem 1.4.1 that the subdivision operator $S_{\boldsymbol{p}_m}$ provides a convergent subdivision scheme. Hence we may apply the subdivision scheme implied by (1.3.5), (1.3.6), (1.3.7), as formulated in Chui and de Villiers (2010, Algorithm 4.3.1), to render the graphs of the cardinal B-splines N_3, N_4 and N_5, as given in Figure 4.1.1.

Definition 4.1.2 *The space of* ***cardinal splines*** *of (integer) order* $m \geq 2$ *with respect to the integer knots is defined by*

$$S_{m,\mathbb{Z}} := \left\{ f \in \mathscr{C}^{m-2}(\mathbb{R}) : f|_{[j,j+1)} = p_j \in \pi_{m-1}, \quad j \in \mathbb{Z} \right\}. \tag{4.1.19}$$



(a) N_3

(b) N_4

(c) N_5

Figure 4.1.1: *The cardinal B-splines N_m, for $m = 3,4,5$.*

From Theorem 4.1.1 (vi) and (ix), together with (4.1.2), it follows that

$$N_m(\cdot - j) \in S_{m,\mathbb{Z}}, \quad j \in \mathbb{Z}. \tag{4.1.20}$$

In fact, as proved in Micchelli (1995, Theorem 2.1)(see also Chui and de Villiers (2010, Theorem 2.5.1)), the sequence $\{N_m(\cdot - j) : j \in \mathbb{Z}\}$ is a basis for the cardinal spline space $S_{m,\mathbb{Z}}$, as follows.

Theorem 4.1.2 *For any integer $m \geq 2$, let $f \in S_{m,\mathbb{Z}}$. Then there exists a sequence $\{c_j\} \in \ell(\mathbb{Z})$, with $\{c_j\}$ uniquely determined by f, such that*

$$f(x) = \sum_j \boldsymbol{c}_j N_m(x - j), \quad x \in \mathbb{R}. \tag{4.1.21}$$

Next, we set $\phi = N_m$ in the definition (2.2.3) to define the polynomial

$$\Phi_m(z) := \sum_j N_m(j+1)z^j = \sum_{j=0}^{m-2} N_m(j+1)z^j, \tag{4.1.22}$$

which in Schoenberg (1973) was called the *Euler-Frobenius* polynomial of order $m \in \mathbb{N}$, and which, as proved in Chui (1992, Theorem 6.13), satisfies the following property.

Theorem 4.1.3 *For any integer $m \geq 2$, the zeros of Euler-Frobenius polynomial Φ_m, as define by* (4.1.22), *are all negative numbers.*

Observe from (4.1.22), (4.1.6) and (4.1.7), that, for any integer $m \geq 2$, we have

$$\deg\,(\Phi_m) = m-2; \tag{4.1.23}$$

$$\Phi_m(1) = 1 \quad ; \quad \Phi_m(0) \neq 0. \tag{4.1.24}$$

In order to obtain the polynomials Φ_m, we first observe from (4.1.13) and (4.1.2) that

$$N_m(k) = \sum_{j=0}^{k-1} (-1)^j \binom{m}{j} (k-j)^{m-1}, \quad k = 1, \ldots, m-1, \tag{4.1.25}$$

which, together with (4.1.22), yields the formula

$$\Phi_m(z) = \sum_{k=0}^{m-2} \left[\sum_{j=0}^{k} (-1)^j \binom{m}{j} (k+1-j)^{m-1} \right] z^k. \tag{4.1.26}$$

Calculating by means of (4.1.26), we obtain the polynomials

$$\left.\begin{aligned}
\Phi_2(z) &= 1, \\
\Phi_3(z) &= \tfrac{1}{2}\,(1+z)\,, \\
\Phi_4(z) &= \tfrac{1}{6}\left(1+4z+z^2\right), \\
\Phi_5(z) &= \tfrac{1}{24}\left(1+11z+11z^2+z^3\right), \\
\Phi_6(z) &= \tfrac{1}{120}\left(1+26z+66z^2+26z^3+z^4\right),
\end{aligned}\right\} z \in \mathbb{C}. \tag{4.1.27}$$

4.2 The one-parameter family

Let m denote any integer, with $m \geq 2$. In order to obtain a one-parameter family of refinable functions $\phi_m(t|\cdot)$ coinciding with the cardinal B-spline N_m at the integers, we first observe from (4.1.23) and (4.1.24) that the choice $Y = \Phi_m$ in Theorem 3.2.1 satisfies the conditions (3.2.2) and (3.2.3) in Theorem 3.2.1, with $\nu = m$. Moreover, Theorem 4.1.3 implies that Φ_m has no symmetric zeros.

Next, we deduce from Theorem 4.1.1(i), together with (4.1.15), that $\{p_j\} = \{p_{m,j}\}$ and $\phi = N_m$ satisfy the conditions of Theorem 2.2.1, with $\nu = m$, from which it then follows that the polynomial identity

$$\Phi_m(z)\left(\frac{1+z}{2}\right)^m - \Phi_m(-z)\left(\frac{1-z}{2}\right)^m = z\Phi_m\left(z^2\right), \quad z \in \mathbb{C}, \tag{4.2.1}$$

is satisfied. It follows that the polynomial

$$P^*(z) := \left(\frac{1+z}{2}\right)^m \tag{4.2.2}$$

satisfies both lines of (3.2.8) with $Y = \Phi_m$, so that we may apply Theorem 3.2.1(b), with $\nu = m$, to deduce that the polynomial solutions P in π_m of the identity

$$\Phi_m(z)P(z) - \Phi_m(-z)P(-z) = z\Phi_m\left(z^2\right), \quad z \in \mathbb{C}, \tag{4.2.3}$$

are given, as in (3.2.9) by

$$P(z) = P_m(t|z) := \left(\frac{1+z}{2}\right)^m + t\left(1-z^2\right)\Phi_m(-z), \quad t \in \mathbb{R}. \tag{4.2.4}$$

Calculating by means of (4.2.4) and (4.1.27), we obtain, for $t \in \mathbb{R}$ and $z \in \mathbb{C}$, the formulas

$$P_2(t|z) = \left(\frac{1}{4}+t\right) + \frac{z}{2} + \left(\frac{1}{4}-t\right)z^2, \tag{4.2.5}$$

$$P_3(t|z) = \left(\frac{1}{8}+t\right) + \left(\frac{3}{8}-t\right)z + \left(\frac{3}{8}-t\right)z^2 + \left(\frac{1}{8}+t\right)z^3, \tag{4.2.6}$$

$$P_4(t|z) = \left(\frac{1}{16}+t\right) + \left(\frac{1}{4}-4t\right)z + \frac{3}{8}z^2 + \left(\frac{1}{4}+4t\right)z^3 + \left(\frac{1}{16}-t\right)z^4 \tag{4.2.7}$$

$$\begin{aligned} P_5(t|z) = \left(\frac{1}{32}+t\right) + \left(\frac{5}{32}-11t\right)z + \left(\frac{5}{16}+10t\right)z^2 + \left(\frac{5}{16}+10t\right)z^3 + \left(\frac{5}{32}-11t\right)z^4 \\ + \left(\frac{1}{32}+t\right)z^5 \end{aligned} \tag{4.2.8}$$

$$\begin{aligned} P_6(t|z) = \left(\frac{1}{64}+t\right) + \left(\frac{3}{32}-26t\right)z + \left(\frac{15}{16}+65t\right)z^2 + \frac{5}{16}z^3 + \left(\frac{15}{16}-65t\right)z^4 \\ + \left(\frac{3}{32}+26t\right)z^5 + \left(\frac{1}{64}-t\right)z^6 \end{aligned} \tag{4.2.9}$$

As in (3.2.22), we now define the one-parameter family of refinement sequence $\{p_{m,j}(t)\}$ by

$$\frac{1}{2}\sum_j p_{m,j}(t)z^j := P_m(t|z), \qquad t \in \mathbb{R}, \tag{4.2.10}$$

and with the mask symbol $P_m(t|\cdot)$ given by the formula (4.2.4).

In order to apply Theorem 3.2.3, as well as the Hölder regularity result of Theorem 1.4.2, we proceed to successively consider the cases $m = 2$ and $m = 3$ in Examples 4.2.1 and 4.2.2 below.

Example 4.2.1

For $m = 2$, it follows from (4.2.5) and (4.2.10) that the sequence $\boldsymbol{p}_2(t) = \{p_{2,j}(t)\} \in \ell_0$ is given by

$$\left\{\begin{array}{l} \{p_{2,0}(t), p_{2,1}(t), p_{2,2}(t)\} = \{\tfrac{1}{2} + 2t, 1, \tfrac{1}{2} - 2t\}; \\ \\ \text{with} \quad p_{2,j}(t) = 0, \quad j \notin \{0,1,2\}, \end{array}\right. \tag{4.2.11}$$

for which it holds that

$$p_{2,j}(t) > 0, \quad j = 0,1,2 \Longleftrightarrow t \in \left(-\frac{1}{4}, \frac{1}{4}\right). \tag{4.2.12}$$

It follows from Theorem 3.2.3 that, for each $t \in \left(-\frac{1}{4}, \frac{1}{4}\right)$, the subdivision operator $S_{\boldsymbol{p}_2(t)}$ provides a convergent subdivision scheme with limit (scaling) function $\phi_2(t|\cdot)$ satisfying the properties

$$\left.\begin{array}{l} \mathrm{supp}^c \phi_2(t|\cdot) = [0,2]; \\ \phi_2(t|x) = \sum_j p_{2,j}(t)\phi_2(t|2x - j), \quad x \in \mathbb{R}; \\ \sum_j \phi_2(t|x - j) = \sum_j \phi_2(t|j) = \int_{-\infty}^{\infty} \phi_2(t|s)\mathrm{d}s = 1, \quad x \in \mathbb{R}; \\ \phi_2(t|x) > 0, \quad x \in (0,2); \\ \phi_2(t|1) = N_2(1) = 1; \end{array}\right\} t \in \left(-\frac{1}{4}, \frac{1}{4}\right).$$

For the choices $t = -\frac{1}{5}, -\frac{1}{20}, 0, \frac{1}{20}, \frac{1}{5} \in \left(-\frac{1}{4}, \frac{1}{4}\right)$, we give in Table 4.2.1 the refinement sequence $\{p_{2,j}(t)\}$, as computed from (4.2.11). Graphical illustrations, as generated by means of the same algorithms used to render the graphs and curves in Figures 3.3.1 and 3.3.2, are provided in Figures 4.2.1 and 4.2.2.

Table 4.2.1: *The refinement sequences* $\{p_{2,j}(t)\}$ *for different values of* $t \in \left(-\frac{1}{4}, \frac{1}{4}\right)$.

t	$\{p_{2,j}(t) : j = 0,1,2\}$
$-\frac{1}{5}$	$\left\{\frac{1}{10}, 1, \frac{9}{10}\right\}$
$-\frac{1}{20}$	$\left\{\frac{2}{5}, 1, \frac{3}{5}\right\}$
0	$\left\{\frac{1}{2}, 1, \frac{1}{2}\right\}$
$\frac{1}{20}$	$\left\{\frac{3}{5}, 1, \frac{2}{5}\right\}$
$\frac{1}{5}$	$\left\{\frac{9}{10}, 1, \frac{1}{10}\right\}$

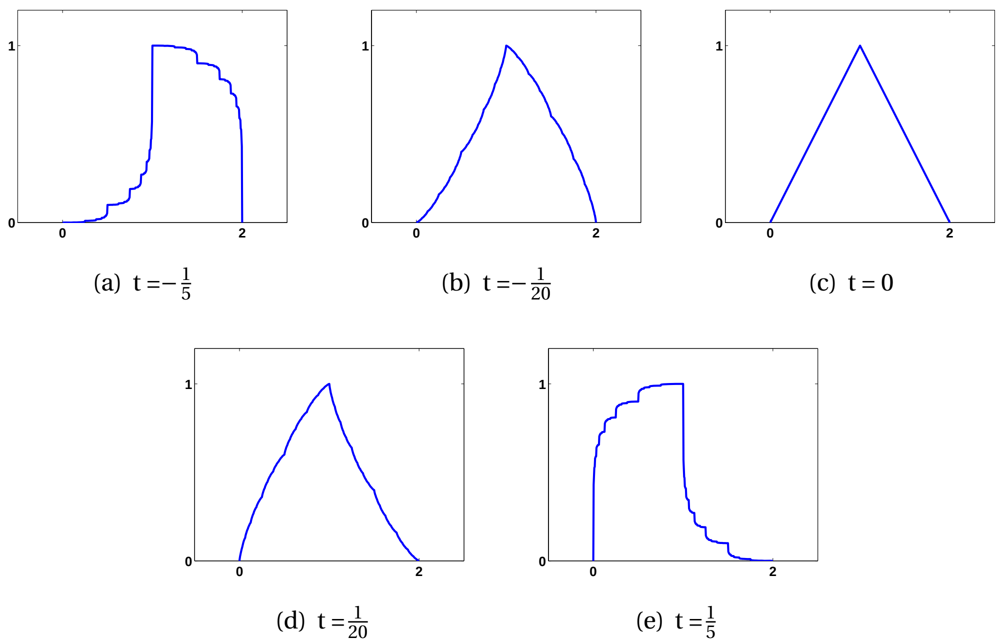
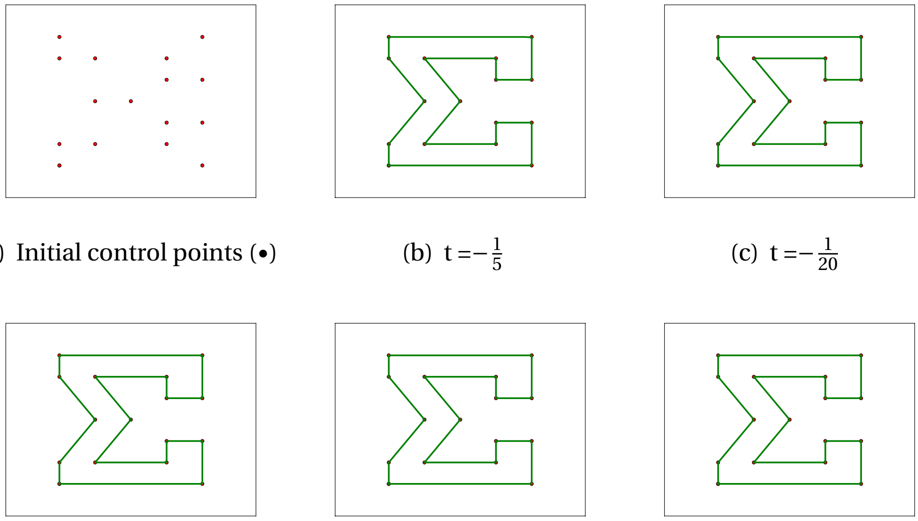


Figure 4.2.1: *The refinable functions* $\phi_2(t|\cdot)$ *for the refinement sequence* $\{p_{2,j}(t)\}$ *in Table 4.2.1.*



(a) Initial control points (•) (b) $t = -\frac{1}{5}$ (c) $t = -\frac{1}{20}$

(d) $t = 0$ (e) $t = \frac{1}{20}$ (f) $t = \frac{1}{5}$

Figure 4.2.2: *Closed subdivision curves obtained from the refinement sequences* $\{p_{2,j}(t)\}$.

Next, by applying Theorem 1.4.2, we obtain, as given in Table 4.2.2, the Hölder regularity of the refinable function $\phi_2(t|\cdot)$ of Figure 4.2.1.

Table 4.2.2: *The Hölder regularity of the refinable functions* $\phi_2(t|\cdot)$ *of Figure 4.2.1* .

	$\phi_2(t\|\cdot) \in \mathscr{C}_0^{k,\alpha}$	
t	k	α
$-\frac{1}{5}$	0	$\log_2\left(\frac{10}{9}\right) \approx 0.1520$
$-\frac{1}{20}$	0	$\log_2\left(\frac{5}{3}\right) \approx 0.7370$
0	0	1
$\frac{1}{20}$	0	$\log_2\left(\frac{5}{3}\right) \approx 0.7370$
$\frac{1}{5}$	0	$\log_2\left(\frac{10}{9}\right) \approx 0.1520$

Since $\phi_2(0|\cdot) = h$, the hat function, we note from Table 4.2.2 that $\phi_2(0|\cdot) \in \mathscr{H}^1$, so that, according to Definition 1.4.2 with $\alpha = 1$, $\phi_2(0|\cdot)$ is Lipschitz continuous on $\mathbb{R}$.

■

Example 4.2.2

For $m = 3$, it follows from (4.2.6) and (4.2.10) that the sequence $\boldsymbol{p}_3(t) = \{p_{3,j}(t)\} \in \ell_0$ is given by

$$\left\{\begin{aligned} &\{p_{3,0}(t), p_{3,1}(t), p_{3,2}(t), p_{3,3}(t)\} = \left\{\tfrac{1}{4}+2t, \tfrac{3}{4}-2t, \tfrac{3}{4}-2t, \tfrac{1}{4}+2t\right\}; \\ &\text{with} \quad p_{3,j}(t) = 0, \quad j \notin \{0,1,2,3\}, \end{aligned}\right. \tag{4.2.13}$$

for which it holds that

$$p_{3,j}(t) > 0, \quad j = 0,1,2,3 \Longleftrightarrow t \in \left(-\frac{1}{8}, \frac{3}{8}\right). \tag{4.2.14}$$

As in Example 4.2.1, it follows from Theorem 3.2.3 that, for each $t \in \left(-\frac{1}{8}, \frac{3}{8}\right)$, the subdivision operator $S_{\boldsymbol{p}_3(t)}$ provides a convergent subdivision scheme with limit (scaling) function $\phi_3(t|\cdot)$ satisfying the properties

$$\left.\begin{aligned} &\operatorname{supp}^c \phi_3(t|\cdot) = [0,3]; \\ &\phi_3(t|x) = \sum_j p_{3,j}(t)\phi_3(t|2x-j), \quad x \in \mathbb{R}; \\ &\sum_j \phi_3(t|x-j) = \sum_j \phi_3(t|j) = \int_{-\infty}^{\infty} \phi_3(t|s)\mathrm{d}s = 1, \quad x \in \mathbb{R}; \\ &\phi_3(t|x) > 0, \quad x \in (0,3). \end{aligned}\right\}$$

Also,

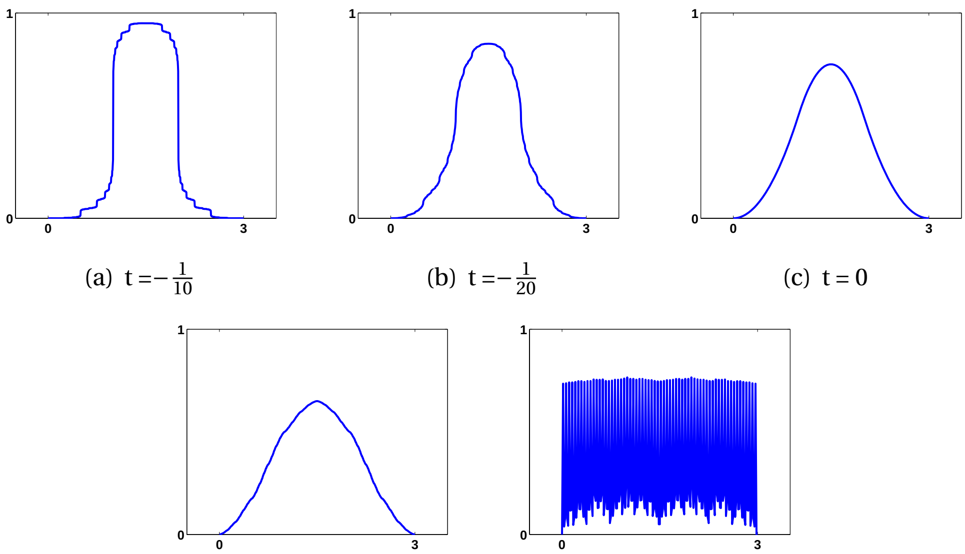
$$\phi_3(t|1) = N_3(1) = \frac{1}{2} \quad ; \quad \phi_3(t|2) = N_3(2) = \frac{1}{2}, \quad t \in \left(-\frac{1}{8}, \frac{3}{8}\right).$$

For the choices $t = -\frac{1}{10}, -\frac{1}{20}, 0, \frac{1}{20}, \frac{7}{20} \in \left(-\frac{1}{8}, \frac{3}{8}\right)$, we give in Table 4.2.3 the refinement sequence $\{p_{3,j}(t)\}$, as computed from (4.2.13).

Graphical illustrations, as generated by means of the same algorithm used to render the graphs and curves in Figures 3.3.1 and 3.3.2, are provided in Figures 4.2.3 and 4.2.4.

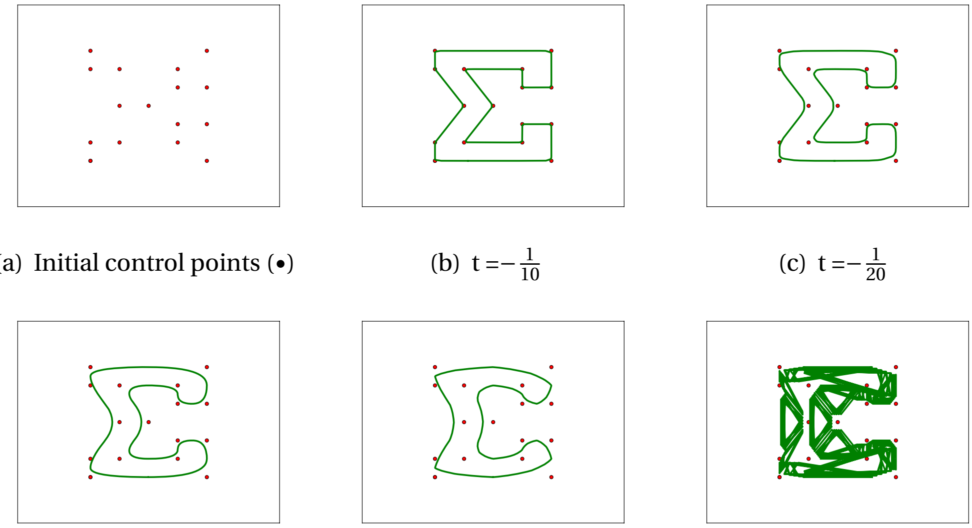
Table 4.2.3: *The refinement sequences* $\{p_{3,j}(t)\}$ *for different values of* t.

t	$\{p_{3,j}(t) : j = 0,1,2,3\}$
$-\frac{1}{10}$	$\left\{\frac{1}{20}, \frac{19}{20}, \frac{19}{20}, \frac{1}{20}\right\}$
$-\frac{1}{20}$	$\left\{\frac{3}{20}, \frac{17}{20}, \frac{17}{20}, \frac{3}{20}\right\}$
0	$\left\{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right\}$
$\frac{1}{20}$	$\left\{\frac{7}{20}, \frac{13}{20}, \frac{13}{20}, \frac{7}{20}\right\}$
$\frac{7}{20}$	$\left\{\frac{19}{20}, \frac{1}{20}, \frac{1}{20}, \frac{19}{20}\right\}$



(a) $t = -\frac{1}{10}$ (b) $t = -\frac{1}{20}$ (c) $t = 0$ (d) $t = \frac{1}{20}$ (e) $t = \frac{7}{20}$

Figure 4.2.3: *The refinable functions* $\phi_3(t|\cdot)$ *for the refinement sequences* $\{p_{3,j}(t)\}$ *in Table 4.2.3.*



(a) Initial control points (•) (b) t = $-\frac{1}{10}$ (c) t = $-\frac{1}{20}$

(d) t = 0 (e) t = $\frac{1}{20}$ (f) t = $\frac{7}{20}$

Figure 4.2.4: *Closed subdivision curves obtained from the refinement sequences* $\{p_{3,j}(t)\}$.

Next, by applying Theorem 1.4.2, we obtain, as given in Table 4.2.4, the Hölder regularity of the refinable function $\phi_3(t|\cdot)$ of Figure 4.2.3. ■

Table 4.2.4: *The Hölder regularity of the refinable functions* $\phi_3(t|\cdot)$ *of Figure 3.3.1 .*

	$\phi_3(t\|\cdot) \in \mathscr{C}_0^{k,\alpha}$	
t	k	α
$-\frac{1}{10}$	0	$\log_2\left(\frac{20}{19}\right) \approx 0.2345$
$-\frac{1}{20}$	0	$\log_2\left(\frac{20}{17}\right) \approx 0.3219$
0	1	1
$\frac{1}{20}$	0	$\log_2\left(\frac{20}{13}\right) \approx 0.6215$
$\frac{7}{20}$	0	$\log_2\left(\frac{20}{19}\right) \approx 0.0740$

Proceeding similarly for $m = 4,5$ and 6, we obtain the t-intervals yielding $p_{m,j}(t) > 0, \quad j = 0,\ldots,m$, as given in Table 4.2.5.

Table 4.2.5: *The t-intervals for which $p_{m,j}(t) > 0, \quad j = 0,\ldots,m$.*

m	t-intervals
2	$\left(-\frac{1}{4}, \frac{1}{4}\right)$
3	$\left(-\frac{1}{8}, \frac{3}{8}\right)$
4	$\left(-\frac{1}{16}, \frac{1}{16}\right)$
5	$\left(-\frac{1}{32}, \frac{5}{32}\right)$
6	$\left(-\frac{1}{64}, \frac{1}{64}\right)$

Chapter 5

Normalized binomial coefficients at the integers

In this chapter, we apply the results from Chapter 1 to 3 to investigate the existence of a refinable function ϕ with normalized binomial coefficient values at the integers.

5.1 Pascal refinement sequences

For an integer $\nu \geq 3$, our aim here is to seek a refinable function ϕ such that

$$\operatorname{supp}^c \phi = [0, \nu], \tag{5.1.1}$$

and

$$\phi(j) = \frac{1}{2^{\nu-2}} \binom{\nu-2}{j-1}, \quad j = 1, \ldots, \nu - 1. \tag{5.1.2}$$

Note from (2.2.3) that (5.1.2) implies

$$\Phi(z) = \left(\frac{1+z}{2}\right)^{\nu-2}, \tag{5.1.3}$$

and thus

$$\sum_j \phi(j) = \sum_{j=1}^{\nu-1} \phi(j) = \Phi(1) = 1, \tag{5.1.4}$$

which is consistent with Theorem 1.2.2, if the corresponding refinement sequence $\{p_j\}$ is required to satisfy the sum-rule condition (1.2.8).

Based on (5.1.3), we define the polynomial

$$Y_\nu(z) := \left(\frac{1+z}{2}\right)^{\nu-2}, \tag{5.1.5}$$

and observe that the choice $Y = Y_\nu$ then satisfies the conditions (3.2.2) and (3.2.3) in Theorem 3.2.1, and (5.1.5) shows that Y_ν has no symmetric zeros. We proceed to obtain a polynomial solution $P^* \in \pi_\nu$ of (3.2.8), by means of which (3.2.9) in Theorem 3.2.1(b) will then yield the one-parameter family $P = P(t\cdot)$ of mask symbols satisfying (3.2.1), with $Y = Y_\nu$ as in (5.1.5).

We shall rely on the polynomial sequence introduced in Theorem 5.1.1 below, in which we adopt a different approach to the one used in (De Villiers, Micchelli and Sauer, 2000, Section 3), in the following sense. Here, our starting point is the polynomial identity (5.1.6) below, for which we first prove the existence of a polynomial solution of minimum degree by means of the results from Chapter 3, after which the properties (5.1.7)-(5.1.11) of this solution then follows in a natural manner.

In contrast to our approach, the starting point in De Villiers, Micchelli and Sauer (2000), is the recursive formulation (5.1.10), (5.1.11), which is then shown to yield a polynomial sequence with properties as in (5.1.7)-(5.1.9) below, and in particular it is shown there that the polynomial identity in (5.1.6) below is then satisfied by this sequence. Our result is as follows.

Theorem 5.1.1 *For any positive integer k, there exists precisely one polynomial $\tilde{H}_k \in \pi_{2k+1}$ satisfying the identity*

$$\left(\frac{1+z}{2}\right)^{2k+2}\tilde{H}_k(z) - \left(\frac{1-z}{2}\right)^{2k+2}\tilde{H}_k(-z) = z\left(\frac{1+z^2}{2}\right)^{2k-1}, \quad z \in \mathbb{C}, \tag{5.1.6}$$

with, moreover,

$$\tilde{H}_k \in \pi_{2k-2}; \tag{5.1.7}$$

$$\tilde{H}_k(1) = 1; \tag{5.1.8}$$

and $\tilde{H}_k$ satisfies the symmetry condition

$$z^{2k-2}\tilde{H}_k\left(\frac{1}{z}\right) = \tilde{H}_k(z), \quad z \in \mathbb{C}\setminus\{0\}. \tag{5.1.9}$$

Also, the polynomial sequence $\{\tilde{H}_k : k = 1, 2, \ldots\}$ satisfies the recursive formulation

$$\tilde{H}_1(z) = 1, \quad z \in \mathbb{C}; \tag{5.1.10}$$

$$\tilde{H}_{k+1}(z) = \frac{\left(\frac{1+z^2}{2}\right)^2 \tilde{H}_k(z) - \tilde{H}_k(-1)\left(\frac{1-z}{2}\right)^{2k+2}}{\left(\frac{1+z}{2}\right)^2}, \quad k = 1, 2, \ldots. \tag{5.1.11}$$

Proof. Let $k \in \mathbb{N}$ be fixed, and define the polynomials G and F by

$$G(z) := \left(\frac{1+z}{2}\right)^{2k+2}; \tag{5.1.12}$$

$$F(z) := z\left(\frac{1+z^2}{2}\right)^{2k-1}. \tag{5.1.13}$$

Note from (5.1.12) that

$$\deg(G) = 2k+2 \geq 4; \tag{5.1.14}$$

$$G(1) = 1 \quad ; \quad G(0) \neq 0, \tag{5.1.15}$$

and G has no symmetric zeros, whereas (5.1.13) shows that F is an odd polynomial, that is $F_- = -F$, with also

$$\deg(F) = 4k-1 = 2\deg(G) - 5. \tag{5.1.16}$$

Therefore, the polynomials G and F satisfy the conditions of Theorem 3.1.1, with

$$d = 2k+2, \tag{5.1.17}$$

and thus also, from (5.1.16) and (5.1.17),

$$F \in \pi_{2d-5} \subset \pi_{2d-3} \subset \pi_{2d-1}. \tag{5.1.18}$$

Hence we may apply Theorem 3.1.1(a) and (d) to deduce that there exists precisely one polynomial $\tilde{H}_k \in \pi_{d-1} = \pi_{2k+1}$, that is, the polynomial $\tilde{H}$ in Theorem 3.1.1(a), such that the identity (5.1.6) is satisfied. Moreover, since (5.1.18) is satisfied, it follows from Theorem 3.1.1(b), with $\tilde{H} = \tilde{H}_k$, that

$$\tilde{H}_k \in \pi_{d-2} = \pi_{2k}. \tag{5.1.19}$$

Next, we replace z by $\frac{1}{z}$ in (5.1.6) to obtain

$$\left(\frac{1+\frac{1}{z}}{2}\right)^{2k+2} \tilde{H}_k\left(\frac{1}{z}\right) - \left(\frac{1-\left(\frac{1}{z}\right)}{2}\right)^{2k+2} \tilde{H}_k\left(-\frac{1}{z}\right) = \frac{1}{z}\left(\frac{1+\frac{1}{z^2}}{2}\right)^{2k-1}, \quad z \in \mathbb{C}\setminus\{0\},$$

or equivalently,

$$\left(\frac{1+z}{2}\right)^{2k+2}\left[z^{2k-2}\tilde{H}_k\left(\frac{1}{z}\right)\right]-\left(\frac{1-z}{2}\right)^{2k+2}\left[(-z)^{2k-2}\tilde{H}_k\left(-\frac{1}{z}\right)\right]=z\left(\frac{1+z^2}{2}\right)^{2k-1}, \quad z\in\mathbb{C}\setminus\{0\}, \tag{5.1.20}$$

By subtracting (5.1.6) from (5.1.20), we deduce that

$$\begin{aligned}\left(\frac{1+z}{2}\right)^{2k+2}&\left[z^{2k-2}\tilde{H}_k\left(\frac{1}{z}\right)-\tilde{H}_k(z)\right]\\ &=\left(\frac{1-z}{2}\right)^{2k+2}\left[(-z)^{2k-2}\tilde{H}_k\left(-\frac{1}{z}\right)-\tilde{H}_k(-z)\right], \quad z\in\mathbb{C}\setminus\{0\}.\end{aligned} \tag{5.1.21}$$

Since the polynomials $\left(\frac{1+z}{2}\right)^{2k+2}$ and $\left(\frac{1-z}{2}\right)^{2k+2}$ have no symmetric zeros, it follows from(5.1.21) that there exists a polynomial M such that

$$z^{2k-2}\tilde{H}_k\left(\frac{1}{z}\right)-\tilde{H}_k(z)=M(z)\left(\frac{1-z}{2}\right)^{2k+2}, \quad z\in\mathbb{C}\setminus\{0\}, \tag{5.1.22}$$

or equivalently

$$z^{2k}\tilde{H}_k\left(\frac{1}{z}\right)=z^2\tilde{H}_k(z)+z^2M(z)\left(\frac{1-z}{2}\right)^{2k+2}, \quad z\in\mathbb{C}. \tag{5.1.23}$$

According to (5.1.19), we have

$$\tilde{H}_k(z)=\sum_{j=0}^{2k}\tilde{h}_{k,j}z^j, \tag{5.1.24}$$

for some sequence $\{\tilde{h}_{k,j} : j=0,\ldots,2k\}$. By substituting (5.1.24) into (5.1.23), we obtain, for any $z\in\mathbb{C}$,

$$\begin{aligned} z^2M(z)\left(\frac{1-z}{2}\right)^{2k+2} &= \sum_{j=0}^{2k}\tilde{h}_{k,j}z^{2k-j}-\sum_{j=0}^{2k}\tilde{h}_{k,j}z^{j+2}\\ &= \sum_{j=0}^{2k}\tilde{h}_{k,2k-j}z^{j}-\sum_{j=2}^{2k+2}\tilde{h}_{k,j-2}z^{j},\end{aligned}$$

that is,

$$\begin{aligned} z^2M(z)\left(\frac{1-z}{2}\right)^{2k+2} &= \tilde{h}_{k,2k}+\tilde{h}_{k,2k-1}z+\sum_{j=2}^{2k}\left(\tilde{h}_{k,2k-j}-\tilde{h}_{k,j-2}\right)z^j\\ &\quad -\tilde{h}_{k,2k-1}z^{2k+1}-\tilde{h}_{k,2k}z^{2k+2}, \quad z\in\mathbb{C}.\end{aligned} \tag{5.1.25}$$

If M is not zero polynomial, then the left hand side of (5.1.25) is a polynomial of degree at least $2k+4$, whereas the right hand side of (5.1.25) is a polynomial of degree $\leq 2k+2$, which therefore yields a contradiction. Hence M is the zero polynomial, according to which (5.1.25) then implies

$$\tilde{h}_{k,2k} = \tilde{h}_{k,2k-1} = 0,$$

which, together with (5.1.24), yields (5.1.7). Also, by setting M equals to the zero polynomial in (5.1.22), we obtain the symmetry condition (5.1.9). After observing that (5.1.8) is obtained by setting $z = 1$ in (5.1.6), it then remains to prove the recursive formulation (5.1.10).

To this end, we first note that

$$\left(\frac{1+z}{2}\right)^4 - \left(\frac{1-z}{2}\right)^4 = z\left(\frac{1+z^2}{2}\right), \quad z \in \mathbb{C},$$

from which it follows that $\tilde{H}_1$ is indeed the polynomial given by (5.1.10). Proceeding inductively, suppose next that $k \in \mathbb{N}$ is fixed. It follows from (5.1.6) that

$$\left(\frac{1+z}{2}\right)^{2k+4} \tilde{H}_{k+1}(z) - \left(\frac{1-z}{2}\right)^{2k+4} \tilde{H}_{k+1}(-z) = z\left(\frac{1+z^2}{2}\right)^{2k+1}, \quad z \in \mathbb{C}. \quad (5.1.26)$$

Now multiply the identity (5.1.6) by $\left(\frac{1+z^2}{2}\right)^2$ to obtain, for $z \in \mathbb{C}$,

$$\left(\frac{1+z}{2}\right)^{2k+2}\left(\frac{1+z^2}{2}\right)^2 \tilde{H}_k(z) - \left(\frac{1-z}{2}\right)^{2k+2}\left(\frac{1+(-z)^2}{2}\right)^2 \tilde{H}_k(-z) = z\left(\frac{1+z^2}{2}\right)^{2k-1} \quad (5.1.27)$$

By subtracting (5.1.27) from (5.1.26), we get the identity

$$\begin{aligned} &\left(\frac{1+z}{2}\right)^{2k+2}\left[\left(\frac{1+z}{2}\right)^2 \tilde{H}_{k+1}(z) - \left(\frac{1+z^2}{2}\right)^2 \tilde{H}_k(z)\right] \\ = \; &\left(\frac{1-z}{2}\right)^{2k+2}\left[\left(\frac{1-z}{2}\right)^2 \tilde{H}_{k+1}(-z) - \left(\frac{1+(-z)^2}{2}\right)^2 \tilde{H}_k(-z)\right], \quad z \in \mathbb{C} \end{aligned} \quad (5.1.28)$$

Since the polynomials $\left(\frac{1+z}{2}\right)^{2k+2}$ and $\left(\frac{1-z}{2}\right)^{2k+2}$ have no common zeros, it follows from (5.1.28) that we must have

$$\left(\frac{1+z}{2}\right)^2 \tilde{H}_{k+1}(z) - \left(\frac{1+z^2}{2}\right)^2 \tilde{H}_k(z) = K(z)\left(\frac{1-z}{2}\right)^{2k+2}, \quad z \in \mathbb{C}, \quad (5.1.29)$$

for some polynomial K.

According to (5.1.7), the left hand side of (5.1.29) is a polynomial in π_{2k+2}, and it follows from (5.1.29) that K is a constant polynomial, that is,

$$\left(\frac{1+z}{2}\right)^2 \tilde{H}_{k+1}(z) - \left(\frac{1+z^2}{2}\right)^2 \tilde{H}_k(z) = c\left(\frac{1-z}{2}\right)^{2k+2}, \quad z \in \mathbb{C}, \tag{5.1.30}$$

for some constant c. By setting $z = -1$ in (5.1.30), we deduce that

$$c = -\tilde{H}_k(-1),$$

which can now be inserted into (5.1.30) to obtain

$$\left(\frac{1+z}{2}\right)^2 \tilde{H}_{k+1}(z) = \left(\frac{1+z^2}{2}\right)^2 \tilde{H}_k(z) - \tilde{H}_k(-1)\left(\frac{1-z}{2}\right)^{2k+2}, \quad z \in \mathbb{C}, \tag{5.1.31}$$

from which the desired result (5.1.11) then immediately follows. ■

By using Theorem 5.1.1, we can now prove the following result.

Theorem 5.1.2 *For any positive integer $\nu \geq 3$, let the polynomial $P_\nu^* \in \pi_\nu$ be defined by*

$$P_\nu^*(z) := \begin{cases} \left(\frac{1+z}{2}\right)^3 \tilde{H}_k(z), & \text{if } \nu = 2k+1; & \text{(5.1.32a)} \\ \left(\frac{1+z}{2}\right)^2 \left(\frac{1+z^2}{2}\right) \tilde{H}_k(z), & \text{if } \nu = 2k+2, & \text{(5.1.32b)} \end{cases}$$

where the polynomial sequence $\{\tilde{H}_k : k = 1, 2, \ldots\}$ is defined as in Theorem 5.1.1. Then

$$P_\nu^* \in \pi_\nu \tag{5.1.33}$$

and P_ν^* satisfies

$$\left.\begin{aligned} &\left(\frac{1+z}{2}\right)^{\nu-2} P_\nu^*(z) - \left(\frac{1-z}{2}\right)^{\nu-2} P_\nu^*(-z) = z\left(\frac{1+z^2}{2}\right)^{\nu-2}, \quad z \in \mathbb{C}; \\ &P_\nu^*(1) = 1 \quad ; \quad P_\nu^*(-1) = 0, \end{aligned}\right\} \tag{5.1.34}$$

Proof. Suppose first $\nu = 2k+1$ for an integer $k \in \mathbb{N}$. It follows from (5.1.32a), together with (5.1.6), that, for any $z \in \mathbb{C}$,

$$\left(\frac{1+z}{2}\right)^{\nu-2} P_\nu^*(z) - \left(\frac{1-z}{2}\right)^{\nu-2} P_\nu^*(-z) = \left(\frac{1+z}{2}\right)^{2k+2} \tilde{H}_k(z) - \left(\frac{1-z}{2}\right)^{2k+2} \tilde{H}_k(-z)$$

$$= \quad z\left(\frac{1+z^2}{2}\right)^{2k-1} = z\left(\frac{1+z^2}{2}\right)^{\nu-2},$$

which yields the first line of (5.1.34) for $\nu = 2k+1$.

Similarly, if $\nu = 2k+2$ for an integer $k \in \mathbb{N}$, (5.1.32b), together with (5.1.6), yields, for any $z \in \mathbb{C}$,

$$\left(\frac{1+z}{2}\right)^{\nu-2} P_\nu^*(z) - \left(\frac{1-z}{2}\right)^{\nu-2} P_\nu^*(-z)$$
$$= \left(\frac{1+z^2}{2}\right)\left[\left(\frac{1+z}{2}\right)^{2k+2} \tilde{H}_k(z) - \left(\frac{1-z}{2}\right)^{2k+2} \tilde{H}_k(-z)\right]$$
$$= z\left(\frac{1+z^2}{2}\right)^{2k} = z\left(\frac{1+z^2}{2}\right)^{\nu-2},$$

which yields the first line of (5.1.34) for $\nu = 2k+2$.

Finally, note that the second line of (5.1.34) follows immediately from (5.1.32a), (5.1.32b) and (5.1.8). ■

As already noted before Theorem 5.1.1, we may now apply (3.2.9) in Theorem 3.2.1(b), with $Y = Y_\nu$, as given by (5.1.5), and $P^* = P_\nu^*$, as given in Theorem 5.1.2, since (5.1.34) shows that (3.2.8) is satisfied with $P^* = P_\nu^*$, to immediately obtain the following result.

Theorem 5.1.3 *For any positive integer* $\nu \geq 3$, *the polynomial solutions* P *in* π_ν *of*

$$\left.\begin{aligned} \left(\frac{1+z}{2}\right)^{\nu-2} P(z) - \left(\frac{1-z}{2}\right)^{\nu-2} P(-z) = z\left(\frac{1+z^2}{2}\right)^{\nu-2}, \qquad z \in \mathbb{C}, \\ P(1) = 1 \quad ; \quad P(-1) = 0. \end{aligned}\right\} \tag{5.1.35}$$

are given by the one-parameter family

$$P(z) = P_\nu(t|z) := P_\nu^*(z) + t\left(1-z^2\right)\left(\frac{1-z}{2}\right)^{\nu-2}, \quad t \in \mathbb{R}, \tag{5.1.36}$$

with $P_\nu^* \in \pi_\nu$ *given as in Theorem 5.1.2.*

By using the recursive formulation (5.1.10), (5.1.11), we obtain the following formulas for the polynomials $\{\tilde{H}_k : k = 2,3,4\}$ of Theorem 5.1.1:

$$\tilde{H}_2(z) \quad = \quad \frac{1}{4}\left(3 - 2z + 3z^2\right); \tag{5.1.37}$$

$$\tilde{H}_3(z) \quad = \quad \frac{1}{8}\left(5 - 8z + 14z^2 - 8z^3 + 5z^4\right); \tag{5.1.38}$$

$$\tilde{H}_4(z) = \frac{1}{64}\left(35-94z+205z^2-228z^3+205z^4-94z^5+35z^6\right). \quad (5.1.39)$$

Next, we use (5.1.32) and (5.1.37)-(5.1.39) to obtain the polynomials $\{P_\nu^* : \nu = 3,\ldots,10\}$ of Theorem 5.1.2:

$$P_3^*(z) = \frac{1}{8}\left(1+3z+3z^2+z^3\right); \quad (5.1.40)$$

$$P_4^*(z) = \frac{1}{8}\left(1+2z+2z^2+2z^3+z^4\right); \quad (5.1.41)$$

$$P_5^*(z) = \frac{1}{32}\left(3+7z+6z^2+6z^3+7z^4+3z^5\right); \quad (5.1.42)$$

$$P_6^*(z) = \frac{1}{32}\left(3+4z+5z^2+8z^3+5z^4+4z^5+3z^6\right); \quad (5.1.43)$$

$$P_7^*(z) = \frac{1}{64}\left(5+7z+5z^2+15z^3+15z^4+5z^5+7z^6+5z^7\right); \quad (5.1.44)$$

$$P_8^*(z) = \frac{1}{64}\left(5+2z+8z^2+14z^3+6z^4+14z^5+8z^6+2z^7+5z^8\right); \quad (5.1.45)$$

$$P_9^*(z) = \frac{1}{512}\Big(35+11z+28z^2+140z^3+42z^4+42z^5+140z^6+28z^7 +11z^8+35z^9\Big); \quad (5.1.46)$$

$$P_{10}^*(z) = \frac{1}{512}\Big(35-24z+87z^2+64z^3+6z^4+176z^5+6z^6+64z^7+87z^8 -24z^9+35z^{10}\Big). \quad (5.1.47)$$

Note from (5.1.32) that

$$\left(P_\nu^*\right)'(-1) = 0, \quad \nu \geq 3. \quad (5.1.48)$$

It follows from (5.1.48) and Theorem 3.2.4(b) that, in the one-parameter family formulated by (5.1.36), $P_\nu(0|\cdot) = P_\nu^*$ is the only polynomial containing the factor $\left(\frac{1+z}{2}\right)^2$. We shall call P_ν^* the *Pascal mask symbol* of order ν, whereas the sequence $\{p_{\nu,j}^*\} \in \ell_0$ defined by

$$\frac{1}{2}\sum_j p_{\nu,j}^* z^j := P_\nu^*(z), \quad (5.1.49)$$

will be called the *Pascal refinement sequence*. Note from (5.1.40)-(5.1.47) that

$$\operatorname{supp}\{p_{\nu,j}^*\} = [0,\nu]\big|_{\mathbb{Z}}, \quad \nu = 3,\ldots,10. \quad (5.1.50)$$

Based on (5.1.49) and (5.1.40)-(5.1.47), we obtain, as given in Table 5.1.1, the Pascal refinement sequences $\{p^*_{\nu,j}\}$, $\nu = 3,\dots,10$.

Table 5.1.1: *The Pascal refinement sequences* $\{p^*_{\nu,j}\}$, $\nu = 3,\dots,10$.

ν	supp $\{p^*_{\nu,j}\}$	$\{p^*_{\nu,j} : j = 0,\dots,\nu\}$
3	$[0,3]\|_{\mathbb{Z}}$	$\frac{1}{4}\{1,3,3,1\}$
4	$[0,4]\|_{\mathbb{Z}}$	$\frac{1}{4}\{1,2,2,2,1\}$
5	$[0,5]\|_{\mathbb{Z}}$	$\frac{1}{16}\{3,7,6,6,7,3\}$
6	$[0,6]\|_{\mathbb{Z}}$	$\frac{1}{16}\{3,4,5,8,5,4,3\}$
7	$[0,7]\|_{\mathbb{Z}}$	$\frac{1}{32}\{5,7,5,15,15,5,7,5\}$
8	$[0,8]\|_{\mathbb{Z}}$	$\frac{1}{32}\{5,2,8,14,6,14,8,2,5\}$
9	$[0,9]\|_{\mathbb{Z}}$	$\frac{1}{256}\{35,11,28,140,42,42,140,28,11,35\}$
10	$[0,10]\|_{\mathbb{Z}}$	$\frac{1}{256}\{35,-24,87,64,6,176,6,64,87,-24,35\}$

Similarly, we define, for $t \in \mathbb{R}\backslash\{0\}$, the perturbed Pascal refinement sequence $\{p_{\nu,j}(t)\}$ by

$$\frac{1}{2}\sum_j p_{\nu,j}(t) z^j := P_\nu(t|z). \tag{5.1.51}$$

By using (5.1.37) and (5.1.40)-(5.1.47), we obtain following formulas for the polynomials $P_\nu(t|\cdot) : \nu = 3,\dots,10$ of Theorem 5.1.2:

$$P_3(t|z) = \left(\frac{1}{8}+t\right) + \left(\frac{3}{8}-t\right)z + \left(\frac{3}{8}-t\right)z^2 + \left(\frac{1}{8}-t\right)z^3; \tag{5.1.52}$$

$$P_4(t|z) = \left(\frac{1}{8}+t\right) + \left(\frac{1}{4}-2t\right)z + \frac{1}{4}z^2 + \left(\frac{1}{4}+2t\right)z^3 + \left(\frac{1}{8}-t\right)z^4; \tag{5.1.53}$$

$$P_5(t|z) = \left(\frac{3}{32}+t\right) + \left(\frac{7}{32}-3t\right)z + \left(\frac{3}{16}+2t\right)z^2 + \left(\frac{3}{16}+2t\right)z^3 + \left(\frac{7}{32}-3t\right)z^4$$

$$+\left(\frac{3}{32}+t\right)z^5; \tag{5.1.54}$$

$$P_6(t|z)=\left(\frac{3}{32}+t\right)+\left(\frac{1}{8}-4t\right)z+\left(\frac{5}{32}+5t\right)z^2+\frac{1}{4}z^3+\left(\frac{5}{32}-5t\right)z^4+\left(\frac{1}{8}+4t\right)z^5$$
$$+\left(\frac{3}{32}-t\right)z^6; \tag{5.1.55}$$

$$P_7(t|z)=\left(\frac{5}{64}+t\right)+\left(\frac{7}{64}-5t\right)z+\left(\frac{5}{64}+9t\right)z^2+\left(\frac{15}{64}-5t\right)z^3+\left(\frac{15}{64}-5t\right)z^4$$
$$+\left(\frac{5}{64}+9t\right)z^5+\left(\frac{7}{64}-5t\right)z^6+\left(\frac{5}{64}+t\right)z^7; \tag{5.1.56}$$

$$P_8(t|z)=\left(\frac{5}{64}+t\right)+\left(\frac{1}{32}-6t\right)z+\left(\frac{1}{8}+14t\right)z^2+\left(\frac{7}{32}-14t\right)z^3+\frac{3}{32}z^4+\left(\frac{7}{32}+14t\right)z^5$$
$$+\left(\frac{1}{8}-14t\right)z^6+\left(\frac{1}{32}+6t\right)z^7+\left(\frac{5}{64}+t\right)z^8; \tag{5.1.57}$$

$$P_9(t|z)=\left(\frac{35}{512}+t\right)+\left(\frac{11}{512}-7t\right)z+\left(\frac{7}{128}+20t\right)z^2+\left(\frac{35}{128}-28t\right)z^3+\left(\frac{21}{256}+14t\right)z^4$$
$$+\left(\frac{21}{256}+14t\right)z^5+\left(\frac{35}{128}-28t\right)z^6+\left(\frac{7}{128}+20t\right)z^7+\left(\frac{11}{512}-7t\right)z^8+\left(\frac{35}{512}+t\right)z^9; \tag{5.1.58}$$

$$P_{10}(t|z)=\left(\frac{35}{512}+t\right)-\left(\frac{3}{64}+8t\right)z+\left(\frac{87}{512}+27t\right)z^2+\left(\frac{1}{8}-48t\right)z^3+\left(\frac{3}{256}+42t\right)z^4$$
$$+\frac{11}{32}z^5+\left(\frac{3}{256}-42t\right)z^6+\left(\frac{1}{8}+48t\right)z^7+\left(\frac{87}{512}-27t\right)z^8+\left(-\frac{3}{64}+8t\right)z^9+\left(\frac{35}{512}-t\right)z^{10}. \tag{5.1.59}$$

The corresponding perturbed Pascal refinement sequence $\{p_{\nu,j}(t)\}$, $\nu = 3,\ldots,10$, as obtained from (5.1.51), together with (5.1.52)-(5.1.59), are given in Table 5.1.1.

Table 5.1.2: *The perturbed Pascal refinement sequences* $\{p_{\nu,j}(t)\}$, $\nu = 3,\dots,10$.

ν	$\{p^*_{\nu,j}(t) : j = 0,\dots,\nu\}$
3	$\frac{1}{4}\{1+8t, 3-8t, 3-8t, 1+8t\}$
4	$\frac{1}{4}\{1+8t, 2-16t, 2, 2+16t, 1-8t\}$
5	$\frac{1}{16}\{3+32t, 7-96t, 6+64t, 6+64t, 7-96t, 3+32t\}$
6	$\frac{1}{16}\{3+32t, 4-32t, 5+160t, 8, 5+160t, 4-32t, 3+32t\}$
7	$\frac{1}{32}\{5+64t, 7-160t, 5+288t, 15-480t, 15-480t, 5+288t, 7-160t, 5+64t\}$
8	$\frac{1}{32}\{5+64t, 2-192t, 8+448t, 14-448t, 6, 14+448t, 8-448t, 2+192t, 5+64t\}$
9	$\frac{1}{256}\{35+512t, 11-1792t, 28+5120t, 140-7168t, 42+3584t, 42+3584t, 140-7168t, 28+5120t, 11-1792t, 35+512t\}$
10	$\frac{1}{256}\{35+512t, -24-2048t, 87+6912t, 64-12288t, 6+10752t, 48, 6-10752t, 64+12288t, 87-6912t, -24+2048t, 35-512t\}$

5.2 Pascal refinable functions

First, observe from Table 5.1.1 that, for $\nu \in \{3,\ldots,9\}$, the Pascal refinement sequences $\{p^*_{\nu,j}\}$ satisfies

$$p^*_{\nu,j} > 0, \quad j = 0,\ldots,9, \tag{5.2.1}$$

with also

$$\text{supp}\,\{p^*_{\nu,j}\} = [0,\nu]|_{\mathbb{Z}}, \tag{5.2.2}$$

and

$$\sum_j p^*_{\nu,2j} = 1 \quad ; \quad \sum_j p^*_{\nu,2j-1} = 1, \tag{5.2.3}$$

It follows from (5.2.1), (5.2.2) and (5.2.3) that we may apply Theorem 2.3.2 to immediately deduce the following result.

Theorem 5.2.1 *For any integer ν such that $3 \le \nu \le 9$, denote by $\boldsymbol{p}^*_\nu = \{p^*_{\nu,j}\}$ the Pascal refinement sequence, as given in Table 5.1.1. Then there exists precisely one function $\phi^*_\nu \in \mathscr{C}_0$ such that ϕ^*_ν is refinable with refinement sequence $\{p^*_{\nu,j}\}$. Moreover, ϕ^*_ν satisfies*

$$supp^c \phi^*_\nu = [0,\nu]; \tag{5.2.4}$$

$$\sum_j \phi^*_\nu(x-j) = \sum_j \phi^*_\nu(j) = \int_{-\infty}^{\infty} \phi^*_\nu(s)ds = 1, \quad x \in \mathbb{R}; \tag{5.2.5}$$

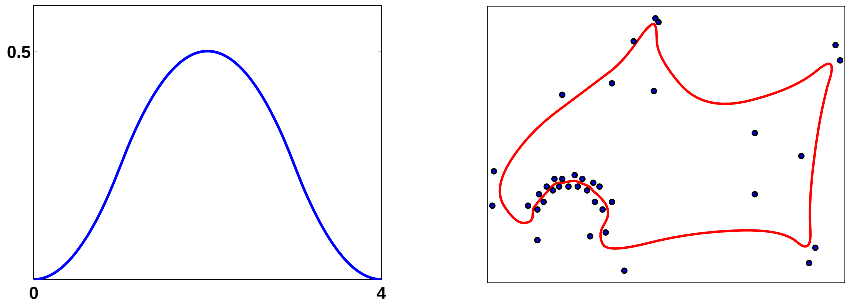
$$\phi^*_\nu(x) > 0, \quad x \in (0,\nu); \tag{5.2.6}$$

and ϕ^*_ν has normalized binomial coefficient values at the integers, that is,

$$\phi^*_\nu(j) = \frac{1}{2^{\nu-2}}\binom{\nu-2}{j-1}, \quad j = 1,\ldots,\nu-1. \tag{5.2.7}$$

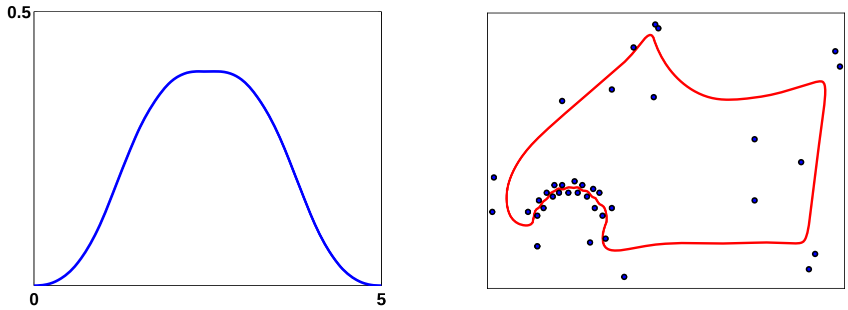
Also, the subdivision operator $S_{\boldsymbol{p}^*}$ provides a convergent subdivision scheme with limit (scaling) function ϕ^*_ν.

We shall call a refinable function ϕ^*_ν as in Theorem 5.2.1 a *Pascal refinable function*. After noting from (4.1.27) and (5.1.3) that the case $\nu = 3$ of Theorem 5.2.1 yields $\phi^*_3 = N_3$, the quadratic cardinal B-spline case, we proceed to consider the cases $\nu = 4,\ldots,9$ of Theorem 5.2.1. First, in Figures 5.2.1–5.2.6, we display, as obtained from Algorithms 4.3.1 and 3.3.1 of Chui and de Villiers (2010), the graphs of the Pascal refinable function ϕ^*_ν, $\nu = 4,\ldots 9$, as well as a corresponding closed subdivision curves.



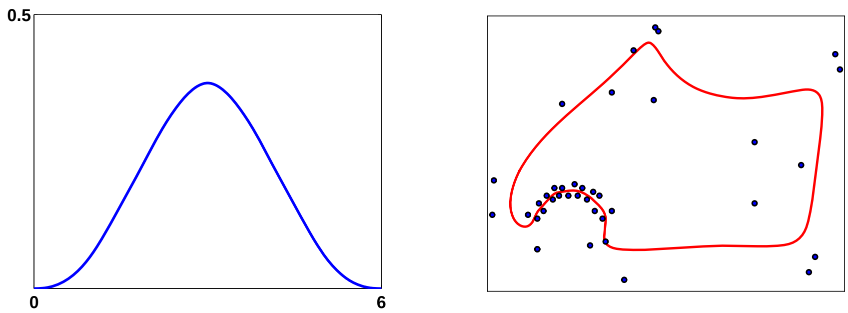
(a) ϕ_4^* (b) Initial control points (•) and subdivision curve (-)

Figure 5.2.1: *The Pascal refinable function* ϕ_4^* *and the closed subdivision curve obtained from the Pascal refinement sequence* $\{p_{4,j}^*\}$.



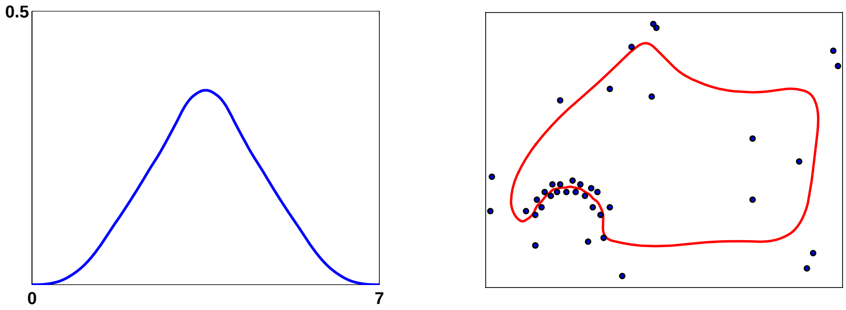
(a) ϕ_5^* (b) Initial control points(•) and subdivision curve (-)

Figure 5.2.2: *The Pascal refinable function* ϕ_5^* *and the closed subdivision curve obtained from the Pascal refinement sequence* $\{p_{5,j}^*\}$.



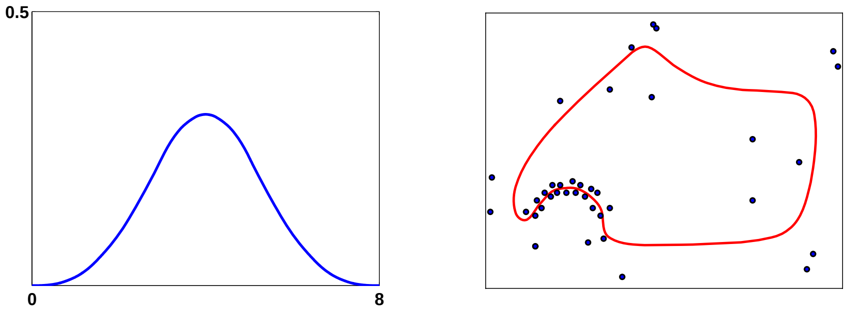
(a) ϕ_6^* (b) Initial control points(•) and subdivision curve(-)

Figure 5.2.3: *The Pascal refinable function* ϕ_6^* *and the closed subdivision curve obtained from the Pascal refinement sequence* $\{p_{6,j}^*\}$.



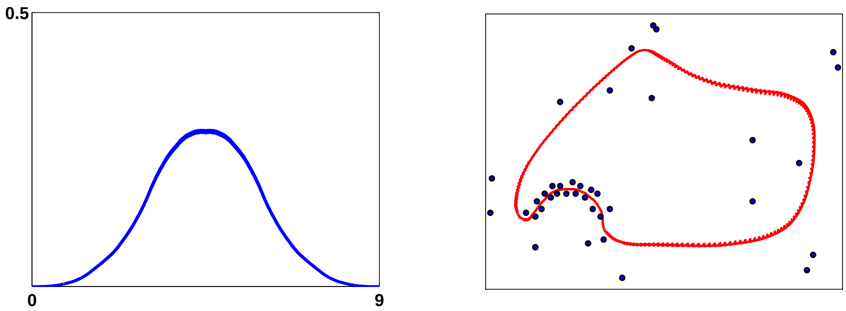
(a) ϕ_7^* (b) Initial control points(•) and subdivision curve (-)

Figure 5.2.4: *The Pascal refinable function* ϕ_7^* *and the closed subdivision curve obtained from the Pascal refinement sequence* $\{p_{7,j}^*\}$.



(a) ϕ_8^* (b) Initial control points(•) and subdivision curve (-)

Figure 5.2.5: *The Pascal refinable function* ϕ_8^* *and the closed subdivision curve obtained from the Pascal refinement sequence* $\{p_{8,j}^*\}$.



(a) ϕ_9^* (b) Initial control points(•) and subdivision curve (-)

Figure 5.2.6: *The Pascal refinable function* ϕ_9^* *and the closed subdivision curve obtained from the Pascal refinement sequence* $\{p_{9,j}^*\}$.

Next, we use Theorem 1.4.2 to obtain the Hölder regularity results of the Pascal refinable functions ϕ_ν^*, $\nu = 4,\ldots,9$, as given in Table 5.2.1. Note that for the

Table 5.2.1: *The Hölder regularity of the refinable functions ϕ_ν^* of Figures 5.2.1-Figure:5.2.6.*

	$\phi_\nu^* \in \mathscr{C}_0^{k,\alpha}$	
ν	k	α
4	1	$\log_2(2) = 1$
5	2	$\log_2\left(\frac{4}{3}\right) \approx 0.4150$
6	1	$\log_2\left(\frac{8}{7}\right) \approx 0.1926$
7	1	$\log_2\left(\frac{8}{7}\right) \approx 0.1926$
8	0	$\log_2\left(\frac{8}{7}\right) \approx 0.0875$
9	0	$\log_2\left(\frac{16}{15}\right) \approx 0.0931$

cases $\nu = 7,8,9$, we cannot apply Theorem 1.4.2 to get the Hölder regularity results of the Pascal refinable functions ϕ_ν^*, $\nu = 7,8,9$ in Figures 5.2.4–5.2.6.

In the next two examples, we consider the cases $\nu = 4$ and $\nu = 5$ of the perturbed Pascal refinement sequences $\{p_{\nu,j}(t)\}$, as given in Table 5.1.2.

Example 5.2.1

For $\nu = 4$ and $t \in \mathbb{R}\backslash\{0\}$, the perturbed Pascal mask sequence $\boldsymbol{p}(t) = \{p_{4,j}(t)\} \in \ell_0$ is given according to Table 5.1.2 by

$$\begin{cases} \{p_{4,0}(t), p_{4,1}(t), p_{4,2}(t), p_{4,3}(t), p_{4,4}(t)\} = \left\{\frac{1}{4}+2t, \frac{1}{2}-4t, \frac{1}{2}, \frac{1}{2}+4t, \frac{1}{4}-2t\right\}, \\ \text{with} \quad p_{4,j}(t) = 0, \quad j \notin \{0,1,2,3,4\}. \end{cases} \tag{5.2.8}$$

Since the refinement sequence in (5.2.8) satisfies

$$p_{4,j}(t) > 0 \Longleftrightarrow t \in \left(-\frac{1}{8}, \frac{1}{8}\right),$$

we may apply Theorem 3.2.3 to deduce that, for any $t \in \left(-\frac{1}{8}, \frac{1}{8}\right)$, there exists precisely one function $\phi_4(t|\cdot) \in \mathscr{C}_0$, called a perturbed Pascal refinable function, such that $\phi_4(t|\cdot)$ is refinable with refinement sequence $\{p_{4,j}(t)\}$, with also

$$\begin{aligned}
&\operatorname{supp}^c \phi_4(t|\cdot) = [0,4], \quad t \in \left(-\frac{1}{8}, \frac{1}{8}\right);\\
&\sum_j \phi_4(t|x-j) = \sum_j \phi_4(t|j) = \int_{-\infty}^{\infty} \phi_4(t|s)ds = 1, \quad x \in \mathbb{R};\\
&\phi_4(t|x) > 0, \quad x \in (0,4), \quad t \in \left(-\frac{1}{8}, \frac{1}{8}\right);
\end{aligned}$$

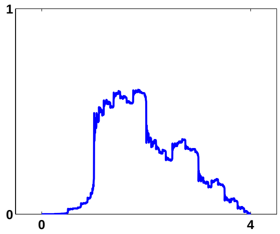
Moreover, for any $t \in \left(-\frac{1}{8}, \frac{1}{8}\right)$, the subdivision operator $S_{\boldsymbol{p}_4(t)}$ provides a convergent subdivision scheme and

$$\phi_4(t|1) = \frac{1}{4} \quad ; \quad \phi_4(t|2) = \frac{1}{2} \quad ; \quad \phi_4(t|3) = \frac{1}{4}, \quad t \in \left(-\frac{1}{8}, \frac{1}{8}\right).$$

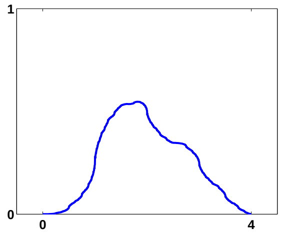
For the choices $t = -\frac{1}{10}, -\frac{3}{50}, \frac{3}{50}, \frac{1}{10} \in \left(-\frac{1}{8}, \frac{1}{8}\right)$, the corresponding perturbed Pascal refinement sequences, as computed from (5.2.8), are given in Table 5.2.2. Graphical illustrations are provided in Figures 5.2.7 and 5.2.8.

Table 5.2.2: *The perturbed Pascal refinement sequences* $\{p_{4,j}(t)\}$ *for different values of* $t \in \left(-\frac{1}{8}, \frac{1}{8}\right)$.

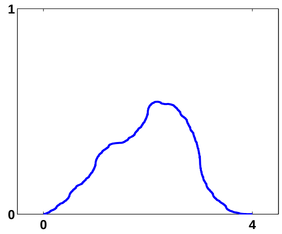
t	$\{p_{4,j}(t) : j = 0,1,2,3,4\}$
$-\frac{1}{10}$	$\left\{\frac{1}{20}, \frac{18}{20}, \frac{10}{20}, \frac{2}{20}, \frac{9}{20}\right\}$
$-\frac{3}{50}$	$\left\{\frac{13}{100}, \frac{74}{100}, \frac{50}{100}, \frac{13}{100}, \frac{37}{100}\right\}$
$\frac{3}{50}$	$\left\{\frac{37}{100}, \frac{13}{100}, \frac{50}{100}, \frac{74}{100}, \frac{13}{100}\right\}$
$\frac{1}{10}$	$\left\{\frac{1}{20}, \frac{2}{20}, \frac{10}{20}, \frac{18}{20}, \frac{1}{20}\right\}$



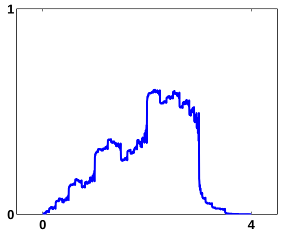
(a) $t=-\frac{1}{10}$



(b) $t=-\frac{3}{50}$

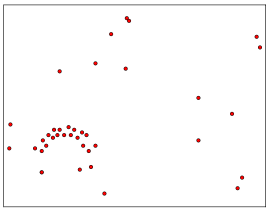


(c) $t=\frac{3}{50}$

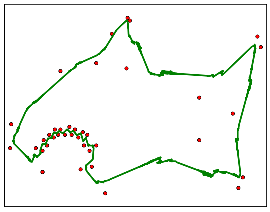


(d) $t=\frac{1}{10}$

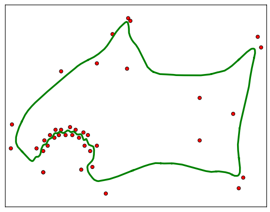
Figure 5.2.7: *The perturbed Pascal refinable function $\phi_4(t|\cdot)$ of Example 5.2.1.*



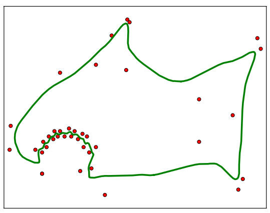
(a) Initial control points (•)



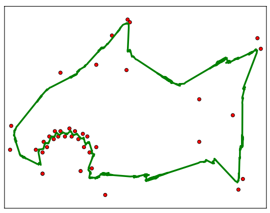
(b) $t=-\frac{1}{10}$



(c) $t=-\frac{3}{50}$



(d) $t=\frac{3}{50}$



(e) $t=\frac{1}{10}$

Figure 5.2.8: *Closed subdivision curves obtained from the refinement sequences in Table 5.2.2.*

By applying Theorem 1.4.2, we obtain, as given in Table 5.2.3, the Hölder regularity of the refinable function $\phi_4(t|\cdot)$ of Figure 5.2.7. ■

Table 5.2.3: *The Hölder regularity of the perturbed Pascal refinable functions $\phi_4(t|\cdot)$ of Figure 5.2.7 .*

	$\phi_4(t\|\cdot) \in \mathscr{C}_0^{k,\alpha}$	
t	k	α
$-\frac{1}{10}$	0	$\log_2\left(\frac{20}{19}\right) \approx 0.0740$
$-\frac{3}{50}$	0	$\log_2\left(\frac{100}{87}\right) \approx 0.2009$
$\frac{3}{50}$	0	$\log_2\left(\frac{100}{87}\right) \approx 0.2009$
$\frac{1}{10}$	0	$\log_2\left(\frac{20}{19}\right) \approx 0.0740$

Example 5.2.2

For $\nu = 5$ and $t \in \mathbb{R}\setminus\{0\}$, the perturbed Pascal mask sequence $\boldsymbol{p}(t) = \{p_{5,j}(t)\} \in \ell_0$ is given according to Table 5.1.2 by

$$\left\{\begin{array}{l} \{p_{5,0}(t), p_{5,1}(t), p_{5,2}(t), p_{5,3}(t), p_{5,4}(t), p_{5,5}(t)\} \\ = \left\{\frac{3}{16}+2t, \frac{7}{16}-6t, \frac{1}{2}, \frac{3}{8}+4t, \frac{3}{8}+4t, \frac{7}{16}-6t, \frac{3}{16}-2t\right\}; \\ \\ \text{with} \quad p_{5,j}(t) = 0, \quad j \notin \{0,1,2,3,4,5\}. \end{array}\right. \tag{5.2.9}$$

Since the refinement sequence in (5.2.8) satisfies

$$p_{5,j}(t) > 0 \Longleftrightarrow t \in \left(-\frac{3}{32}, \frac{7}{96}\right),$$

we may apply Theorem 3.2.3 to deduce that, for any $t \in \left(-\frac{3}{32}, \frac{7}{96}\right)$, there exists precisely one function $\phi_5(t|\cdot) \in \mathscr{C}_0$, called a perturbed Pascal refinable function, such that is refinable with refinement sequence $\{p_{5,j}(t)\}$, with also

$$\operatorname{supp}^c \phi_5(t|\cdot) = [0,5], \quad t \in \left(-\frac{3}{32}, \frac{7}{96}\right);$$
$$\sum_j \phi_5(t|x-j) = \sum_j \phi_5(t|j) = \int_{-\infty}^{\infty} \phi_5(t|s)\mathrm{d}s = 1, \quad x \in \mathbb{R};$$
$$\phi_5(t|x) > 0, \quad x \in (0,5), \quad t \in \left(-\frac{3}{32}, \frac{7}{96}\right);$$

Moreover, for any $t \in \left(-\frac{3}{32}, \frac{7}{96}\right)$, the subdivision operator $S_{\boldsymbol{p}_5(t)}$ provides a convergent subdivision scheme and

$$\phi_5(t|1) = \phi_5(t|4) = \frac{1}{8} \quad ; \quad \phi_5(t|2) = \phi_5(t|3 = \frac{3}{8}, \quad t \in \left(-\frac{3}{16}, \frac{7}{48}\right).$$

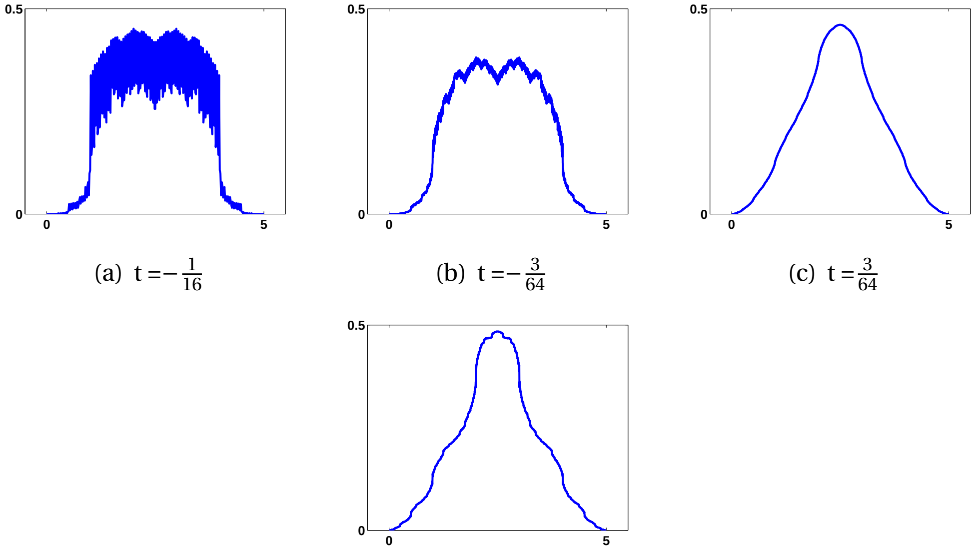
For the choices $t = -\frac{1}{16}, -\frac{3}{64}, \frac{3}{64}, \frac{1}{16} \in \left(-\frac{3}{16}, \frac{7}{96}\right)$, the corresponding perturbed Pascal refinement sequences, as computed from (5.2.9), are given in Table 5.2.4.

Table 5.2.4: *The perturbed Pascal refinement sequences* $\{p_{5,j}(t)\}$ *for different values of* $t \in \left(-\frac{3}{16}, \frac{7}{96}\right)$.

t	$\{p_{5,j}(t) : j = 0,1,2,3,4,5\}$
$-\frac{1}{16}$	$\left\{\frac{1}{16}, \frac{13}{16}, \frac{2}{16}, \frac{2}{16}, \frac{13}{16}, \frac{1}{16}\right\}$
$-\frac{3}{64}$	$\left\{\frac{3}{32}, \frac{23}{32}, \frac{6}{32}, \frac{6}{32}, \frac{23}{32}, \frac{3}{32}\right\}$
$\frac{3}{64}$	$\left\{\frac{9}{32}, \frac{5}{32}, \frac{18}{32}, \frac{18}{32}, \frac{5}{32}, \frac{9}{32}\right\}$
$\frac{1}{16}$	$\left\{\frac{5}{16}, \frac{1}{16}, \frac{10}{16}, \frac{10}{16}, \frac{1}{16}, \frac{5}{16}\right\}$

■

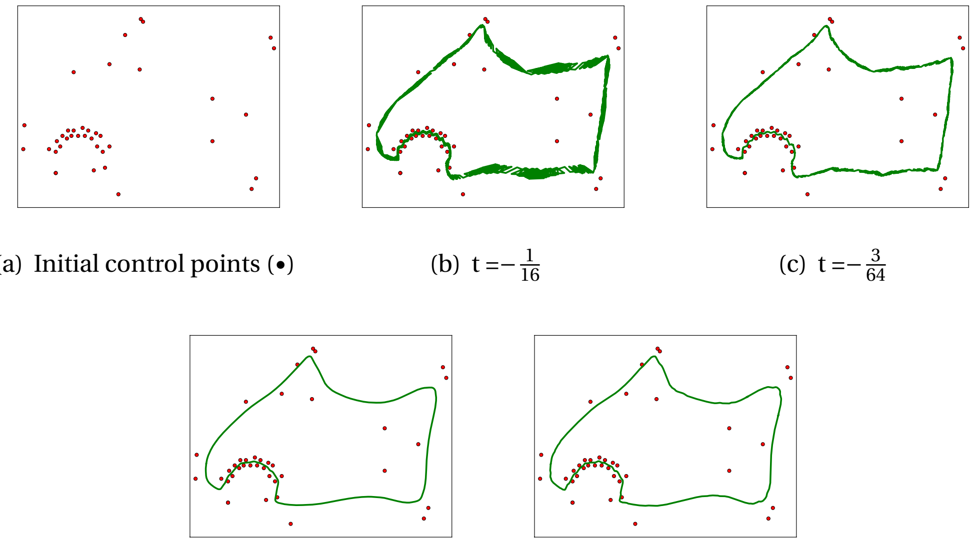
Graphical illustrations are provided in Figures 5.2.9 and 5.2.10.



(a) $t = -\frac{1}{16}$ (b) $t = -\frac{3}{64}$ (c) $t = \frac{3}{64}$

(d) $t = \frac{1}{16}$

Figure 5.2.9: *The perturbed Pascal refinable function $\phi_5(t|\cdot)$ of Example 5.2.2.*



(a) Initial control points (•) (b) $t = -\frac{1}{16}$ (c) $t = -\frac{3}{64}$

(d) $t = \frac{3}{64}$ (e) $t = \frac{1}{16}$

Figure 5.2.10: *Closed subdivision curves obtained from the refinement sequences in Table 5.2.4.*

We may apply Theorem 1.4.2, to obtain, as given in Table 5.2.5, the Hölder regularity of the refinable function $\phi_5(t|\cdot)$ of Figure 5.2.9.

Table 5.2.5: *The Hölder regularity of the perturbed Pascal refinable functions $\phi_5(t|\cdot)$ of Figure 5.2.9 .*

	$\phi_5(t\|\cdot) \in \mathscr{C}_0^{k,\alpha}$	
t	k	α
$-\frac{1}{16}$	0	$\log_2\left(\frac{16}{15}\right) \approx 0.0931$
$-\frac{3}{64}$	0	$\log_2\left(\frac{32}{29}\right) \approx 0.2295$
$\frac{3}{64}$	0	$\log_2\left(\frac{32}{27}\right) \approx 0.3326$
$\frac{1}{16}$	0	$\log_2\left(\frac{16}{15}\right) \approx 0.0931$

Proceeding similarly for $\nu = 6,7,8,9$, we obtain the t-intervals yielding $p_{\nu,j}(t) > 0, \quad j = 0,\ldots,\nu$ as given in Table 5.2.6.

Table 5.2.6: *The t-intervals for which $p_{\nu,j}(t) > 0, \quad j = 0,\ldots,\nu$.*

ν	t-intervals
4	$\left(-\frac{1}{8}, \frac{1}{8}\right)$
5	$\left(-\frac{3}{32}, \frac{7}{96}\right)$
6	$\left(-\frac{3}{32}, \frac{3}{32}\right)$
7	$\left(-\frac{5}{64}, \frac{3}{64}\right)$
8	$\left(-\frac{5}{64}, \frac{5}{54}\right)$
9	$\left(-\frac{35}{512}, \frac{5}{1024}\right)$

Chapter 6

Local interpolation with polynomial exactness

In this chapter, we show how the values at the integers of a refinable function can be used to construct a local interpolation operator with polynomial exactness order given by the order of sum-rule condition satisfied by the corresponding refinement sequence.

6.1 The fundamental interpolatory function

For a given function $\phi \in \mathscr{C}_0$, with

$$\operatorname{supp}^c \phi = [0, \nu], \tag{6.1.1}$$

for an integer $\nu \geq 2$, and a sequence $\{v_j\} \in \ell_0$, let the function $v \in \mathscr{C}_0$ be defined by

$$v(x) := \sum_j v_j \phi(x - j), \quad x \in \mathbb{R}. \tag{6.1.2}$$

We proceed to investigate the existence of a sequence $\{v_j\} \in \ell_0$ such that the function v in (6.1.2) is a compactly supported fundamental interpolatory function, in the sense that

$$v(j) = \delta_j, \quad j \in \mathbb{Z}. \tag{6.1.3}$$

First, note from (6.1.2) that (6.1.3) is satisfied if and only if, for any $k \in \mathbb{Z}$, the sequence $\{v_j\}$ satisfies

$$\delta_k = v(k) = \sum_j v_j \phi(k - j) = \sum_k \phi(j) v_{k-j},$$

that is,

$$\sum_j \phi(j) v_{k-j} = \delta_k, \quad k \in \mathbb{Z}. \tag{6.1.4}$$

Let the polynomial Φ be defined as in (2.2.3), that is,

$$\Phi(z) := \sum_j \phi(j+1) z^j = \sum_{j=0}^{\nu-2} \phi(j+1) z^j, \tag{6.1.5}$$

from (6.1.1), and for any sequence $\{v_j\} \in \ell_0$, let V be a Laurent polynomial defined by

$$V(z) := \sum_j v_j z^j. \tag{6.1.6}$$

It follows from (6.1.5) and (6.1.6) that a sequence $\{v_j\} \in \ell_0$ satisfies the condition (6.1.4) if and only if, for any $z \in \mathbb{C} \backslash \{0\}$,

$$\begin{aligned} 1 = \sum_k \delta_k z^k &= \sum_k \left[\sum_j \phi(j) v_{k-j} \right] z^k \\ &= \sum_j \phi(j) \left[\sum_k v_{k-j} z^{k-j} \right] z^j \\ &= \left[\sum_k v_k z^k \right] \left[\sum_j \phi(j) z^j \right] \\ &= V(z)\Phi(z), \end{aligned}$$

that is,

$$\Phi(z)V(z) = 1, \quad z \in \mathbb{C} \backslash \{0\}. \tag{6.1.7}$$

Now observe that (6.1.7) can only be satisfied if both Φ and V are monomials, so that, from (6.1.1), (6.1.2), (6.1.5) and (6.1.6), we must have $\nu = 2$; $\Phi(z) = 1, \ z \in \mathbb{Z}$; $V(z) = 1, \ z \in \mathbb{Z}$ and $\phi = v = h$, the hat function given by (1.2.2).

In order to build a fundamental interpolatory function in $\mathscr{C}_0$ from ϕ, we proceed to investigate the existence of a sequence $\{w_j\} \in \ell_0$ such that the function

$$w(x) := \sum_j w_j \phi(2x - j), \quad x \in \mathbb{R}, \tag{6.1.8}$$

satisfies the fundamental interpolatory property

$$w(j) = \delta_j, \quad j \in \mathbb{Z}. \tag{6.1.9}$$

To this end, we first use (6.1.9) and (6.1.8) to deduce that the condition (6.1.9) is satisfied if and only if, the sequence $\{w_j\}$ satisfies, for any $k \in \mathbb{Z}$,

$$\delta_k = w(k) = \sum_j w_j \phi(2k-j) = \sum_k w_{2k-j}\phi(j),$$

that is,

$$\sum_j \phi(j) w_{2k-j} = \delta_k, \quad k \in \mathbb{Z}. \tag{6.1.10}$$

For any sequence $\{w_j\} \in \ell_0$, let the Laurent polynomial W be defined by

$$W(z) := \frac{1}{2}\sum_j w_j z^j. \tag{6.1.11}$$

By using the definitions (6.1.5) and (6.1.11), as well as (2.2.4) and (2.2.6), we deduce that the condition (6.1.10) holds if and only if, for any $z \in \mathbb{C}\setminus\{0\}$,

$$\begin{aligned}
1 &= \sum_k \delta_k z^{2k} \\
&= \sum_k \left[\sum_j \phi(2j) w_{2k-2j} + \sum_j \phi(2j+1) w_{2k-2j-1}\right] z^{2k} \\
&= \sum_j \left[\sum_k w_{2k-2j} z^{2k-2j}\right] \phi(2j) z^{2j} + \sum_j \left[\sum_k w_{2k-2j-1} z^{2k-2j-1}\right] \phi(2j+1) z^{2j+1} \\
&= z\left[\sum_j \phi(2j) z^{2j-1}\right]\left[\sum_k w_{2k} z^{2k}\right] + z\left[\sum_j \phi(2j+1) z^{2j}\right]\left[\sum_k w_{2k+1} z^{2k+1}\right] \\
&= z\left[\Phi^{(e)}(z) W^{(e)}(z) + \Phi^{(o)}(-z) W^{(o)}(-z)\right] \\
&= 2z\left[\frac{\Phi(z)-\Phi(-z)}{2}\,\frac{W(z)+W(-z)}{2} + \frac{\Phi(z)+\Phi(-z)}{2}\,\frac{W(z)-W(-z)}{2}\right] \\
&= z\left[\Phi(z)W(z) - \Phi(-z)W(-z)\right],
\end{aligned}$$

according to which the following result has now been established.

Theorem 6.1.1 *Let* ϕ *denote any function in* $\mathscr{C}_0$ *satisfying* (6.1.1) *for an integer* $\nu \geq 2$. *Then* $\{w_j\} \in \ell_0$ *is a sequence such that the function* $w \in \mathscr{C}_0$ *defined by* (6.1.8) *satisfies the fundamental interpolatory condition* (6.1.9) *if and only if the Laurent polynomial* W *defined by* (6.1.11) *satisfies the identity*

$$\Phi(z)W(z) - \Phi(-z)W(-z) = z^{-1}, \quad z \in \mathbb{C}\setminus\{0\}, \tag{6.1.12}$$

with Φ *denoting the polynomial given by* (6.1.5).

Hence, in principle, for any function $\phi \in \mathscr{C}_0$, whether refinable or not, a Laurent polynomial solution W of the identity (6.1.12), as obtained for example by means of the methods of Chapter 3, yields, through the definitions (6.1.11) and (6.1.8), a function $w \in \mathscr{C}_0$ satisfying the interpolatory condition (6.1.9).

Definition 6.1.1 *For any function $\phi \in \mathscr{C}_0$, the linear space sequence $\{S_\phi^r : r \in \mathbb{Z}\}$ is defined by*

$$S_\phi^r := \left\{ \sum_j c_j \phi(2^r \cdot - j) : \{c_j\} \in \ell(\mathbb{Z}) \right\}, \quad r \in \mathbb{Z}. \tag{6.1.13}$$

We shall write S_ϕ for S_ϕ^0.

The following result then holds.

Theorem 6.1.2 *Let $\phi \in \mathscr{C}_0$ be such that* (6.1.1) *holds for an integer $\nu \geq 2$. Then, for any sequence $\{w_j\} \in \ell_0$ such that its symbol W as in* (6.1.11) *satisfies the identity* (6.1.12) *in Theorem 6.1.1, and any $r \in \mathbb{Z}$, the linear operator*

$$\mathscr{L}_r = \mathscr{L}_r^\phi : \mathscr{C}(\mathbb{R}) \to S_\phi^{r+1}, \tag{6.1.14}$$

as defined for any $f \in \mathscr{C}(\mathbb{R})$ by

$$\mathscr{L}_r f := \sum_j f\left(\frac{j}{2^r}\right) w(2^r \cdot - j), \tag{6.1.15}$$

where $w \in \mathscr{C}_0$ is defined by (6.1.8)*, is a local interpolation operator, with*

$$(\mathscr{L}_r f)\left(\frac{k}{2^r}\right) = f\left(\frac{k}{2^r}\right), \quad k \in \mathbb{Z}, \quad f \in \mathscr{C}(\mathbb{R}). \tag{6.1.16}$$

Moreover, $\mathscr{L}_r$ satisfies, for any $f \in \mathscr{C}(\mathbb{R})$ and $r \in \mathbb{Z}$, the formulation

$$\mathscr{L}_r f = \sum_j \left[\sum_k w_{j-2k} f\left(\frac{k}{2^r}\right) \right] \phi(2^{r+1} \cdot - j). \tag{6.1.17}$$

Proof. Since the Laurent polynomial W satisfies the identity (6.1.12), it follows from Theorem 6.1.1 that the function $w \in \mathscr{C}_0$ defined by (6.1.8) satisfies the fundamental interpolatory property (6.1.9). Hence we may apply (6.1.15) and (6.1.9) to deduce that, for any $r \in \mathbb{Z}$ and $k \in \mathbb{Z}$, we have

$$(\mathscr{L}_r f)\left(\frac{k}{2^r}\right) = \sum_j f\left(\frac{j}{2^r}\right) w(k-j) = \sum_j f\left(\frac{j}{2^r}\right) \delta_{k-j} = f\left(\frac{k}{2^r}\right),$$

which proves (6.1.16)

To prove the formulation (6.1.17) of $\mathscr{L}_r$, we use (6.1.15) and (6.1.8) to obtain, for any $f \in \mathscr{C}(\mathbb{R})$ and $r \in \mathbb{Z}$,

$$\begin{aligned}
\mathscr{L}_r f &= \sum_j f\left(\frac{j}{2^r}\right) \sum_k w_k \phi(2^{r+1} \cdot -2j - k) \\
&= \sum_j f\left(\frac{j}{2^r}\right) \sum_k w_{k-2j} \phi(2^{r+1} \cdot -k) \\
&= \sum_k \left[\sum_j w_{k-2j} f\left(\frac{j}{2^r}\right) \right] \phi(2^{r+1} \cdot -k),
\end{aligned}$$

which proves the required formulation (6.1.17). ■

Hence we have shown in Theorems 6.1.1 and 6.1.2 how the values at the integers of any given function $\phi \in \mathscr{C}_0$ can be used to generate a sequence $\{\mathscr{L}_r : r \in \mathbb{Z}\}$ of local interpolation operators satisfying (6.1.14) and (6.1.16). If, moreover, ϕ is a scaling (and therefore refinable) function, such an operator sequence is particularly useful for an efficient corresponding wavelet decomposition process, where, as the initial step, it is required to map a given signal $f \in \mathscr{C}(\mathbb{R})$ into the space S_ϕ^r, for a sufficiently large value of r.

In the next Section 6.2, we proceed to show that, for a refinable function ϕ with refinement sequence $\{p_j\}$, and with the polynomial Φ in (6.1.5) satisfying certain conditions, the sequence $\{w_j\}$ in Theorem 6.1.2 can be chosen in such a way that the local interpolation operator sequence $\{\mathscr{L}_r : j \in \mathbb{Z}\}$, as given by (6.1.17), has polynomial exactness order equal to the order of the sum-rule condition satisfied by $\{p_j\}$.

6.2 Polynomial exactness

Let ϕ denote a refinable function with refinement sequence $\{p_j\}$ such that supp $\{p_j\} = [0, \nu]|_{\mathbb{Z}}$ for an integer $\nu \geq 2$, and satisfying the sum rule of order $m \leq \nu$. It follows from Theorem 1.2.1 that ϕ satisfies the support property (6.1.1). Also, Theorem 1.3.2 (c) gives the polynomial containment result

$$\pi_{m-1} \subset S_\phi, \tag{6.2.1}$$

that is, for any integer $\ell \in \{0, \ldots, m-1\}$, there exist a coefficient sequence $\{\alpha_{\ell,j} : j \in \mathbb{Z}\} \in \ell(\mathbb{Z})$ such that

$$x^\ell = \sum_j \alpha_{\ell,j} \phi(x - \ell), \quad x \in \mathbb{R}. \tag{6.2.2}$$

For any fixed $r \in \mathbb{Z}$, it follows from (6.2.2) that

$$x^\ell = \frac{1}{2^{r\ell}} \sum_j \alpha_{\ell,j} \phi(2^r x - \ell), \quad x \in \mathbb{R},$$

from which we deduce that (6.2.1) extends to

$$\pi_{m-1} \subset S_\phi^r, \quad r \in \mathbb{Z}. \tag{6.2.3}$$

We proceed to investigate the question of whether a sequence $\{w_j\} \in \ell_0$ as in Theorem 6.1.2, which yields the local interpolation operator sequence $\{\mathscr{L}_r : \mathscr{C}(\mathbb{R}) \to S_\phi^{r+1} : r \in \mathbb{Z}\}$ given by (6.1.17), can be chosen in such a way that the polynomial exactness condition

$$\mathscr{L}_r f = f, \quad f \in \pi_{m-1}, \quad r \in \mathbb{Z}, \tag{6.2.4}$$

is obtained.

To this end, we define the positive integers

$$\mu := \left\lfloor \frac{m+\nu}{2} \right\rfloor - 1; \tag{6.2.5}$$

$$\sigma := \begin{cases} \mu, & \text{if } m+\nu \text{ is even;} \\ \mu+1, & \text{if } m+\nu \text{ is odd,} \end{cases} \tag{6.2.6}$$

and we set

$$w_j := 0, \quad j \notin \{-2\mu, \ldots, 2\sigma - \nu\}, \tag{6.2.7}$$

which, together with (6.1.8), gives

$$w(x) = \sum_{j=-2\mu}^{2\sigma-\nu} w_j \phi(2x - j). \tag{6.2.8}$$

It follows from (6.2.8) and (6.1.1) that

$$\operatorname{supp}^c w \subset [-\mu, \sigma]. \tag{6.2.9}$$

Also, note from (6.1.11) and (6.2.7) that

$$W(z) = \frac{1}{2} \sum_j w_j z^j = \frac{1}{2} \sum_{j=-2\mu}^{2\sigma-\nu} w_j z^j. \tag{6.2.10}$$

It follows from (6.2.10) that, if we define

$$X(z) := z^{2\mu} W(z), \tag{6.2.11}$$

then X is a polynomial, with

$$X \in \pi_{2\mu+2\sigma-\nu}. \tag{6.2.12}$$

We proceed to prove the following sufficient condition on the polynomial X for the local interpolation operator $\mathscr{L}_r$ in (6.1.17) of Theorem 6.1.2 to possess the polynomial exactness property (6.2.4).

Theorem 6.2.1 *Let ϕ be a refinable function with refinement sequence $\{p_j\}$ such that supp $\{p_j\} = [0,\nu]|_{\mathbb{Z}}$ for an integer $\nu \geq 2$, and where $\{p_j\}$ satisfies the sum rule of order $m \leq \nu$. Suppose furthermore that $\{w_j\} \in \ell_0$ is a sequence such that the function w defined by* (6.1.8) *satisfies the fundamental interpolatory condition* (6.1.9)*, and such that the polynomial X defined by* (6.2.11)*,* (6.2.10)*, with the positive integers μ and σ given as in* (6.2.5)*,* (6.2.6)*, satisfies*

$$X(z) = \left(\frac{1+z}{2}\right)^m \tilde{X}(z), \tag{6.2.13}$$

for some polynomial $\tilde{X}$. Then the local interpolation operator $\{\mathscr{L}_r : r \in \mathbb{Z}\}$, as defined by (6.1.15) *in Theorem 6.1.2, satisfies the polynomial exactness property* (6.2.4)*. Moreover, the locality of the operator $\mathscr{L}_r$ is, for each $r \in \mathbb{Z}$, specified by* (6.1.15) *and* (6.2.9).

Proof. Our first step is to prove that

$$\sum_j f(j)w(x-j) = f(x), \quad x \in \mathbb{R}, \quad f \in \pi_{m-1}. \tag{6.2.14}$$

To this end, we let $f \in \pi_{m-1}$, and use (6.1.8), as well as the refinement equation (2.2.1), to obtain, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_j f(j)w(x-j) &= \sum_j f(j)\sum_k w_k\phi(2x-2j-k) \\
&= \sum_j f(j)\sum_k w_{k-2j}\phi(2x-k) \\
&= \sum_k \left[\sum_j w_{k-2j} f(j)\right]\phi(2x-k).
\end{aligned} \tag{6.2.15}$$

Observe next from (6.2.11) and (6.2.10) that, with the sequence $\{x_j\} \in \ell_0$ defined by

$$\frac{1}{2}\sum_j x_j z^j := X(z), \tag{6.2.16}$$

we have

$$\sum_j x_j z^j = \sum_j w_j z^{j+2\mu} = \sum_j w_{j-2\mu} z^j, \quad z \in \mathbb{C},$$

and thus

$$x_j = w_{j-2\mu}, \quad j \in \mathbb{Z},$$

or equivalently,

$$w_j = x_{j+2\mu}, \quad j \in \mathbb{Z},$$

which, together with (6.2.15), yields

$$\sum_j f(j)w(x-j) = \sum_k \left[\sum_j x_{k+2\mu-2j} f(j)\right] \phi(2x-k), \quad x \in \mathbb{R}. \tag{6.2.17}$$

Now define

$$\tilde{f}(x) := f(x+\mu), \tag{6.2.18}$$

so that $f \in \pi_{m-1}$ implies $\tilde{f} \in \pi_{m-1}$. It follows from (6.2.18) that, for any $k \in \mathbb{Z}$,

$$\sum_j x_{k+2\mu-2j} f(j) = \sum_j x_{k-2j} f(j+\mu) = \sum_j x_{k-2j} \tilde{f}(j) = \left(S_{\boldsymbol{x}} \tilde{\boldsymbol{c}}\right)_k, \tag{6.2.19}$$

where $S_{\boldsymbol{x}} : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ denotes the subdivision operator defined by

$$\left(S_{\boldsymbol{x}} \boldsymbol{c}\right)_j := \sum_k x_{j-2k} \boldsymbol{c}_k, \quad j \in \mathbb{Z}, \quad \boldsymbol{c} = \{\boldsymbol{c}_j\} \in \ell(\mathbb{Z}), \tag{6.2.20}$$

and where

$$\tilde{\boldsymbol{c}} = \left\{\tilde{\boldsymbol{c}}_j\right\} := \left\{\tilde{f}(j)\right\}, \tag{6.2.21}$$

according to which, since $\tilde{f} \in \pi_{m-1}$, we have

$$\tilde{\boldsymbol{c}} \in \pi_{m-1}^d. \tag{6.2.22}$$

Since $\{p_j\}$ satisfies the sum rule of order m, we may apply Theorem 1.2.2, together with (6.1.5), to obtain

$$\Phi(1) = \sum_j \phi(j) = 1. \tag{6.2.23}$$

Also, (6.2.11) and (6.2.13) yield

$$W(-1) = X(-1) = 0. \tag{6.2.24}$$

By recalling from Theorem 6.1.1 that the function w satisfies the condition (6.1.9) if and only if the polynomial W satisfies the identity (6.1.12) in Theorem 6.1.1, we may now set $z = 1$ in (6.1.12), and use (6.2.23) and (6.2.24), to deduce that

$$W(1) = 1. \tag{6.2.25}$$

It then follows from (6.2.11),

$$X(1) = 1, \tag{6.2.26}$$

and thus also, from (6.2.13), that

$$\tilde{X}(1) = 1. \tag{6.2.27}$$

It follows from Theorem 1.2.3, together with (6.2.16), (6.2.13) and (6.2.27), that the sequence $\{x_j\} \in \ell_0$ satisfies the sum-rule condition of order at least m, and thus, from Theorem 1.3.2(a), since (6.2.22) holds, we deduce that there exists a polynomial $g \in \pi_{m-1}$ such that

$$(S_{\boldsymbol{x}}\tilde{\boldsymbol{c}})_k = g(k), \quad k \in \mathbb{Z}. \tag{6.2.28}$$

By applying (6.2.17), (6.2.19) and (6.2.28), we obtain

$$\sum_j f(j)w(x-j) = \sum_k g(k)\phi(2x-k), \quad x \in \mathbb{R}. \tag{6.2.29}$$

Since $g \in \pi_{m-1}$, with also $\{p_j\}$ satisfying the sum-rule condition of order m, we next apply the identity (1.3.14) in Theorem 1.3.2 (b), as well as the fact that, from Theorem 1.2.1, supp $\{p_j\} = [0,\nu]|_{\mathbb{Z}}$ implies (6.1.1), to deduce that, for any $x \in \mathbb{R}$,

$$\sum_k g(k)\phi(2x-k) = \sum_k \phi(k)g(2x-k) = \sum_{k=1}^{\nu-1} \phi(k)g(2x-k). \tag{6.2.30}$$

Hence if we define

$$\tilde{g}(x) := \sum_{k=1}^{\nu-1} \phi(k)g(2x-k), \tag{6.2.31}$$

it follows from $g \in \pi_{m-1}$ that $\tilde{g} \in \pi_{m-1}$. Moreover, (6.2.29), (6.2.30) and (6.2.31) yield

$$\sum_j f(j)w(x-j) = \tilde{g}(x), \quad x \in \mathbb{R}. \tag{6.2.32}$$

Next, we apply (6.1.9) to deduce from (6.2.32) that

$$\tilde{g}(k) = \sum_j f(j)w(k-j) = f(k), \quad k \in \mathbb{Z},$$

and thus

$$\left(\tilde{g} - f\right)(k) = 0, \quad k \in \mathbb{Z}. \tag{6.2.33}$$

But $f, \tilde{g} \in \pi_{m-1}$ implies $\tilde{g} - f \in \pi_{m-1}$, so that (6.2.33), together with the fact that only the zero polynomial in π_{m-1} has infinitely many zeros, yield

$$\tilde{g}(x) = f(x), \quad x \in \mathbb{R}. \tag{6.2.34}$$

The desired result (6.2.14) is now an immediate consequence of (6.2.32) and (6.2.34).

Let $f \in \pi_{m-1}$ and $r \in \mathbb{Z}$. With the definition

$$\tilde{f}(x) := f\left(\frac{x}{2^r}\right), \quad x \in \mathbb{R}, \tag{6.2.35}$$

according to which

$$f(x) = \tilde{f}\left(2^r x\right), \quad x \in \mathbb{R}, \tag{6.2.36}$$

we deduce from $f \in \pi_{m-1}$ that $\tilde{f} \in \pi_{m-1}$. Hence we may now apply the definition (6.1.15) of the local interpolation operator $\mathscr{L}_r$, as well as (6.2.15), to obtain, for $x \in \mathbb{R}$,

$$(\mathscr{L}_r f)(x) = \sum_j \tilde{f}(j)w\left(2^r x - j\right) = \tilde{f}\left(2^r x\right) = f(x),$$

by virtue of (6.2.36), and thereby completing our proof of the polynomial exactness property (6.2.4).

Finally, observe that, for each $r \in \mathbb{Z}$, the locality of the operator $\mathscr{L}_r$ is made precise by (6.1.15) and (6.2.9). ■

Based on Theorems 6.1.1, 6.1.2 and 6.2.1, together with (6.2.10), and the fact that (6.2.11) and (6.2.13) imply

$$W(z) = z^{-2\mu} X(z) = z^{-2\mu} \left(\frac{1+z}{2}\right)^m \tilde{X}(z), \tag{6.2.37}$$

we proceed to investigate the existence of a polynomial $\tilde{X}$, where, from (6.2.10), (6.2.37), (6.2.5) and (6.2.6), we have

$$\tilde{X} \in \pi_{2\mu+2\sigma-m-\nu} = \pi_{m+\nu-4}, \tag{6.2.38}$$

such that the identity (6.1.12) is satisfied with W given as in (6.2.37), that is, we shall seek a polynomial $\tilde{X} \in \pi_{m+\nu-4}$ satisfying the identity

$$\left[\left(\frac{1+z}{2}\right)^m \Phi(z)\right]\tilde{X}(z) - \left[\left(\frac{1-z}{2}\right)^m \Phi(-z)\right]\tilde{X}(-z) = z^{2\mu-1}, \quad z \in \mathbb{C}. \tag{6.2.39}$$

To this end, we define the polynomials

$$G(z) := \left(\frac{1+z}{2}\right)^m \Phi(z); \tag{6.2.40}$$

$$F(z) := z^{2\mu-1}, \tag{6.2.41}$$

and assume that the refinable function ϕ is such that the polynomial Φ in (6.1.5) satisfies the properties

$$\deg\,(\Phi) = \nu - 2; \tag{6.2.42}$$

$$\Phi(0) \neq 0, \tag{6.2.43}$$

and Φ has symmetric zeros. It then follows from (6.2.40) and(6.2.42) that

$$\deg\,(G) = m + \nu - 2 \geq 2, \tag{6.2.44}$$

since $m \in \mathbb{N}$ and $\nu \geq 3$, whereas, from (6.2.40), (6.2.23) and (6.2.43),

$$G(1) = 1 \quad ; \quad G(0) \neq 0. \tag{6.2.45}$$

Moreover, since Φ has no symmetric zeros, it follows, as in the argument following (3.2.14), that G has no symmetric zeros.

Next, we note from (6.2.41) that $F\_ = -F$, whereas, from (6.2.41) and (6.2.5),

$$\deg\,(F) = 2\mu - 1 = 2\left\lfloor \frac{m+\nu}{2} \right\rfloor - 3 = \begin{cases} m+\nu-3, & \text{if } m+\nu \text{ is even;} \\ m+\nu-4, & \text{if } m+\nu \text{ is odd,} \end{cases} \tag{6.2.46}$$

and thus, by using also (6.2.44), and the fact that $m+\nu \geq 4$, we obtain

$$\deg\,(F) \leq 2\deg\,(G) - 3. \tag{6.2.47}$$

We have therefore shown that the polynomials G and F, as given by (6.2.40) and (6.2.41), satisfy the conditions of Theorem 3.1.1 with $d = m+\nu-2$. Hence we may apply Theorem 3.1.1(a),(b),(c), to deduce the existence of a least-degree polynomial solution $\tilde{X} \in \pi_{d-2} = \pi_{m+\nu-4}$ of the identity (6.2.39), and thereby yielding the following main result of this chapter.

Theorem 6.2.2 *Let ϕ be a refinable function with refinement sequence $\{p_j\}$, with supp $\{p_j\} = [0,\nu]|_{\mathbb{Z}}$ for an integer $\nu \geq 3$, and satisfying the sum rule of order $m \leq \nu$. Suppose, moreover, that the polynomial $\Phi \in \pi_{\nu-2}$, as given by* (6.1.5), *and where* (6.2.23) *holds, satisfies the properties* (6.2.42), (6.2.43), *and Φ has no symmetric zeros. Also, for positive integers μ and σ as in* (6.2.5), (6.2.6), *let the Laurent polynomial W be given by* (6.2.37), *where the polynomial $\tilde{X}$ is the unique solution in $\pi_{m+\nu-3}$ of the polynomial identity* (6.2.39), *as described in Theorem 3.1.1, with G and F given by* (6.2.40), (6.2.41), *and $d = m+\nu-2$, and where, moreover,*

$$\tilde{X} \in \pi_{m+\nu-4}. \tag{6.2.48}$$

Then the coefficient sequence $\{w_j\} \in \ell_0$ defined by

$$\frac{1}{2}\sum_j w_j z^j := W(z) \tag{6.2.49}$$

yields a compactly supported continuous function w as in (6.1.8), *and where w satisfies the properties* (6.2.9), (6.1.9) *and* (6.2.14). *Furthermore, for any $r \in \mathbb{Z}$, the operator $\mathscr{L}_r$, as in* (6.1.14) *and* (6.1.17), *is local in the sense of* (6.1.15) *and* (6.2.9), *is interpolatory as in* (6.1.16), *and has polynomial exactness of order m as in* (6.2.4).

An important implication of Theorem 6.2.2 is the fact that, if the values ϕ at the integers are prescribed in such a manner that the corresponding polynomial Φ, as given by (6.1.5), yields an identity (6.2.39) for which the polynomial solution $\tilde{X} \in \pi_{m+\nu-4}$ is easily obtainable, we would thereby have established an efficient construction procedure for the corresponding local interpolation operator sequence $\{\mathscr{L}_r : r \in \mathbb{Z}\}$ as in Theorem 6.2.2. We proceed to provide examples in Section 6.3.

6.3 Examples

In this section, we provide three examples as applications of Theorem 6.2.2.

Example 6.3.1

Let $\phi = \phi(-\frac{5}{12}|\cdot)$ denote the refinable function established by means of (3.3.16) and (3.3.17) in Section 3.3, for which $\text{supp}^c\phi = [0,4]$, and with values at integers given, according to (3.3.15), by

$$\phi(1) = \frac{1}{5} \quad ; \quad \phi(2) = \frac{3}{5} \quad ; \quad \phi(3) = \frac{1}{5}, \tag{6.3.1}$$

and thus, from (6.1.5), with $\nu = 4$, and corresponding also to (3.3.1),

$$\Phi(z) = \frac{1}{5}\left(1 + 3z + z^2\right). \tag{6.3.2}$$

Also, note from (3.3.17) that we have here $m = 2$ so that (6.2.5), (6.2.6) give $\mu = \sigma = 2$. Hence the identity (6.2.39) is here given by

$$\left[\left(\frac{1+z}{2}\right)^2\left(\frac{1+3z+z^2}{5}\right)\right]\tilde{X}(z) - \left[\left(\frac{1-z}{2}\right)^2\left(\frac{1-3z+z^2}{5}\right)\right]\tilde{X}(-z) = z^3, \;\; z \in \mathbb{C}. \tag{6.3.3}$$

By applying Theorem 3.1.1(a), with the polynomials G and F defined, according to (6.2.40), (6.3.2) and (6.2.41), by

$$G(z) = \left(\frac{1+z}{2}\right)^2\left(\frac{(1+3z+z^2)}{5}\right) \quad ; \quad F(z) = z^3, \tag{6.3.4}$$

we now calculate the unique solution $\tilde{X}$ in π_2 of the identity (6.3.3) to be given by

$$\tilde{X}(z) = \frac{1}{3}\left(-1 + 5z - z^2\right). \tag{6.3.5}$$

Note furthermore from (6.2.10) that

$$W(z) = \frac{1}{2}\sum_j w_j z^j = \frac{1}{2}\sum_{j=-4}^{0} w_j z^j. \tag{6.3.6}$$

Also, (6.2.37) gives

$$W(z) = \frac{1}{z^4}\left(\frac{1+z}{2}\right)^2\tilde{X}(z), \tag{6.3.7}$$

into which we may now insert (6.3.5) to obtain

$$W(z) := \frac{1}{12z^4}\left(-1 + 3z + 8z^2 + 3z^3 - z^4\right), \tag{6.3.8}$$

and thus, from (6.3.6),

$$\{w_{-4}, w_{-3}, w_{-2}, w_{-1}, w_0\} = \left\{-\frac{1}{6}, \frac{3}{6}, \frac{8}{6}, \frac{3}{6}, -\frac{1}{6}\right\}. \tag{6.3.9}$$

It follows from (6.1.8), (6.2.7) and (6.3.9) that

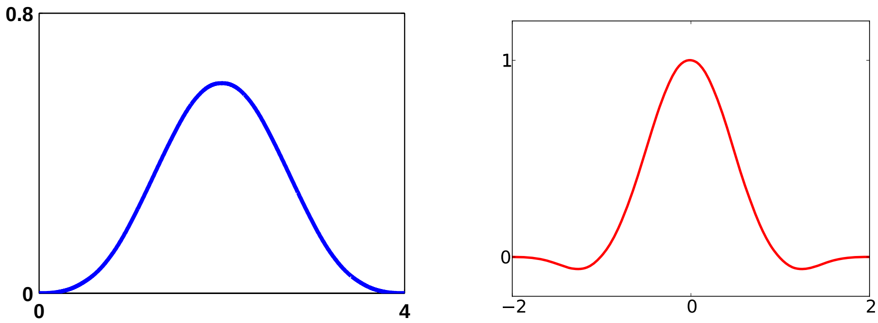
$$w(x) = -\frac{1}{6}\phi(2x+4) + \frac{1}{2}\phi(2x+3) + \frac{4}{3}\phi(2x+2) + \frac{1}{2}\phi(2x+1) - \frac{1}{6}\phi(2x),$$

with also, from (6.2.9),

$$\text{supp}^c w \subset [-2, 2]. \tag{6.3.10}$$

Moreover, the local interpolation operator sequence $\{\mathscr{L}_r : r \in \mathbb{Z}\}$ can now be constructed by means (6.1.17) and (6.3.9). The graphs of ϕ and w, as generated by means of respectively, Algorithms 4.3.1 and 4.4.2 in Chui and de Villiers (2010), are displayed in Figure 6.3.1.

■



(a) ϕ (b) w

Figure 6.3.1: *The refinable function ϕ and fundamental interpolatory function w of Example 6.3.1.*

Example 6.3.2

As in Chapter 4, we let $\phi = N_m$, the cardinal B-spline of order $m \geq 2$, so that, from (4.1.15), we have here $\nu = m$, whereas (6.2.5) and (6.2.6) yield $\mu = \sigma = m-1$, and thus, also from (6.2.10),

$$W(z) = \frac{1}{2}\sum_j w_j z^j = \frac{1}{2}\sum_{j=-2m+2}^{m-2} w_j z^j. \tag{6.3.11}$$

Moreover, the polynomial identity (6.2.39) is given here by

$$\left[\left(\frac{1+z}{2}\right)^m \Phi_m(z)\right]\tilde{X}(z)-\left[\left(\frac{1-z}{2}\right)^m \Phi_m(-z)\right]\tilde{X}(-z)=z^{2m-3},\quad z\in\mathbb{C},\tag{6.3.12}$$

with Φ_m denoting the Euler-Frobenius polynomial defined by (4.1.22).

We shall consider the specifically the case $m=3$, for which, according to the second line of (4.1.27), the identity (6.3.12) is given by

$$\left(\frac{1+z}{2}\right)^4 \tilde{X}(z)-\left(\frac{1-z}{2}\right)^4 \tilde{X}(-z)=z^3,\quad z\in\mathbb{C}.\tag{6.3.13}$$

To solve the identity (6.3.13), as well as an analogous identities in Example 6.3.3 below, we shall rely on the following result from Chui and de Villiers (2010, Theorem 7.2.1).

Theorem 6.3.1 *For any positive integer n, the unique solution $H=\tilde{H}_n$ in the polynomial class π_{2n-1} of the polynomial identity*

$$\left(\frac{1+z}{2}\right)^{2n} H_{2n}(z)-\left(\frac{1-z}{2}\right)^{2n} H_{2n}(-z)=z^{2n-1},\quad z\in\mathbb{C},\tag{6.3.14}$$

is given explicitly by the formula

$$H_{2n}(z):=z^{n-1}\sum_{j=0}^{n-1}\binom{n+j-1}{j}\left[\frac{1}{2}\left(1-\frac{z+z^{-1}}{2}\right)\right]^j,\tag{6.3.15}$$

so that, moreover,

$$\deg\,(H_{2n})=2n-2.\tag{6.3.16}$$

By applying Theorem 6.3.1 with $n=2$, we calculate that the unique solution $\tilde{X}=H_4\in\pi_2$ of the polynomial identity (6.3.13) is given by

$$\tilde{X}(z):=\frac{1}{2}\left(-1+4z-z^2\right);\tag{6.3.17}$$

and thus, from (6.2.37),

$$W(z)=\frac{1}{2}z^{-4}\left(\frac{1+z}{2}\right)^3\left(-1+4z-z^2\right),$$

yielding the formula

$$W(z)=\frac{1}{16z^4}\left(-1+z+8z^2+8z^3+z^4-z^5\right).\tag{6.3.18}$$

It follows from (6.3.18) and (6.2.10) that

$$\{w_{-4}, w_{-3}, w_{-2}, w_{-1}, w_0, w_1\} = \left\{-\frac{1}{8}, \frac{1}{8}, 1, 1, \frac{1}{8}, -\frac{1}{8}\right\}. \tag{6.3.19}$$

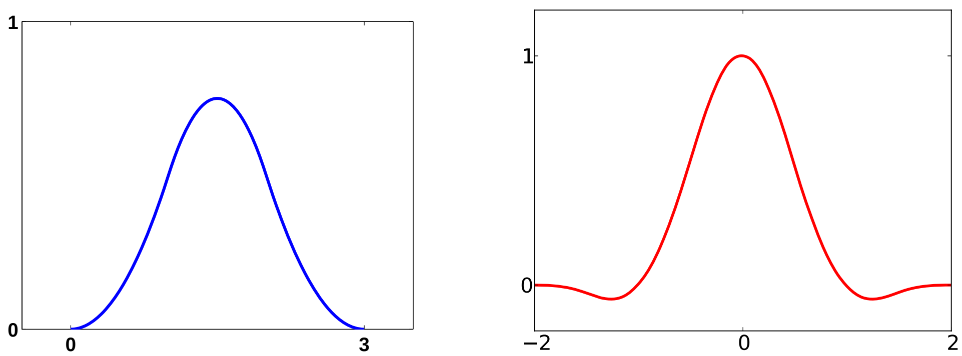
Also, (6.1.8), (6.2.7) and (6.3.19) give

$$w(x) = -\frac{1}{8}N_3(2x+4) + \frac{1}{8}N_3(2x+3) + N_3(2x+2) + N_3(2x+1) + \frac{1}{8}N_3(2x) - \frac{1}{8}N_3(2x-1), \tag{6.3.20}$$

with also, from (6.2.9), as well as (4.1.5) and (4.1.6),

$$\operatorname{supp}^c w = [-2, 2]. \tag{6.3.21}$$

The graphs of N_3 and w, as generated by means of, respectively, Algorithm 4.3.1 and 4.4.2 in Chui and de Villiers (2010), are displayed in Figure 6.3.2. ■



(a) N_3 (b) w

Figure 6.3.2: *The quadratic B-spline N_3 and fundamental interpolatory function w of Example 6.3.2.*

Moreover, the local interpolation operator sequence $\{\mathscr{L}_r : r \in \mathbb{Z}\}$ can now be constructed by means (6.1.17) and (6.3.19). This quadratic spline operator sequence is the special (uniform knot) case of a class of local spline interpolation operators, as constructed by means an alternative method, and extensively analysed further in the papers De Villiers and Rohwer (1987, 1991); De Villiers (1993*a*,*b*); De Villiers and Rohwer (1994).

For $m \geq 4$, the local interpolation sequence $\{\mathscr{L}_r : r \in \mathbb{Z}\}$ can be obtained by applying Theorem 3.1.1(a), together with (6.2.37), (6.2.10) and (6.1.17), and thereby establishing a convenient construction method for the uniform knot case of the local spline interpolation operators studied in Dahmen, Goodman and Micchelli (1988) and Chui and De Villiers (1996).

■

Example 6.3.3

For any integer ν, with $3 \le \nu \le 9$, let $\phi = \phi_\nu^*$, the Pascal refinable function of Theorem 5.2.1. According to the formulation (5.1.32) in Theorem 5.1.2, together with formulas (5.1.10), (5.1.37), (5.1.38) and (5.1.39), which shows that $\tilde{H}_1(-1) \neq 0$, $k = 1,2,3,4$, as well as Theorem 1.2.3, the corresponding mask sequence $\{p_{\nu,j}^*\}$ satisfies the sum-rule condition of order

$$m = \begin{cases} 3, & \text{if } \nu \text{ is odd;} \\ \\ 2, & \text{if } \nu \text{ is even.} \end{cases} \tag{6.3.22}$$

It follows from (6.3.22) and (6.2.5) that

$$\mu = \left\lfloor \frac{\nu+1}{2} \right\rfloor. \tag{6.3.23}$$

Hence, from (6.3.22) and (6.3.23), together with (5.1.5), the identity (6.2.39) is given by either:

(a) if ν is odd,

$$\left(\frac{1+z}{2}\right)^{\nu+1} \tilde{X}(z) - \left(\frac{1-z}{2}\right)^{\nu+1} \tilde{X}(-z) = z^\nu, \quad z \in \mathbb{C}; \tag{6.3.24}$$

(b) if ν is even,

$$\left(\frac{1+z}{2}\right)^{\nu} \tilde{X}(z) - \left(\frac{1-z}{2}\right)^{\nu} \tilde{X}(-z) = z^{\nu-1}, \quad z \in \mathbb{C}. \tag{6.3.25}$$

By applying Theorem 6.3.1, we deduce that the polynomial $\tilde{X}$ of Theorem 6.2.2 is given here explicitly by

(a) if ν is odd,

$$\tilde{X}(z) = z^{\frac{\nu-1}{2}} \sum_{j=0}^{\frac{1}{2}(\nu-1)} \binom{\frac{1}{2}(\nu-1)+j-1}{j} \left[\frac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j; \tag{6.3.26}$$

(b) if ν is even,

$$\tilde{X}(z) = z^{\frac{\nu}{2}-1} \sum_{j=0}^{\frac{\nu}{2}-1} \binom{\frac{1}{2}\nu + j}{j} \left[\frac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j. \tag{6.3.27}$$

Since (6.2.37), (6.3.22) and (6.3.23) imply

(a) if ν is odd,

$$W(z) = \frac{1}{z^{\nu+1}} \left(\frac{1+z}{2}\right)^3 \tilde{X}(z); \tag{6.3.28}$$

(b) if ν is even,

$$W(z) = \frac{1}{z^{\nu}} \left(\frac{1+z}{2}\right)^2 \tilde{X}(z), \tag{6.3.29}$$

we may now substitute (6.3.26), (6.3.27) into (6.3.28), (6.3.29) to obtain the formulas

(a) if ν is odd,

$$W(z) = \frac{1}{z^{\frac{1}{2}(\nu+3)}} \left(\frac{1+z}{2}\right)^3 \sum_{j=0}^{\frac{1}{2}(\nu-1)} \binom{\frac{1}{2}(\nu-1) + j - 1}{j} \left[\frac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j; \tag{6.3.30}$$

(b) if ν is even,

$$W(z) = \frac{1}{z^{\frac{1}{2}\nu+1}} \left(\frac{1+z}{2}\right)^2 \sum_{j=0}^{\frac{\nu}{2}-1} \binom{\frac{1}{2}\nu + j}{j} \left[\frac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j. \tag{6.3.31}$$

First, consider the case $\nu = 4$, for which (6.3.31) gives

$$\begin{aligned} W(z) &= \frac{1}{z^3} \left(\frac{1+z}{2}\right)^2 \sum_{j=0}^{1} \binom{2+j}{j} \left[\frac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j \\ &= \frac{1}{8}\left(-z^{-4} + 2z^{-3} + 6z^{-2} + 2z^{-1} - 1\right). \end{aligned} \tag{6.3.32}$$

It follows from (6.3.32) and (6.2.10) that

$$\{w_{-4}, w_{-3}, w_{-2}, w_{-1}, w_0\} = \left\{-\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, -\frac{1}{4}\right\}.$$

Next, for $\nu = 5$, we apply (6.3.30) to obtain

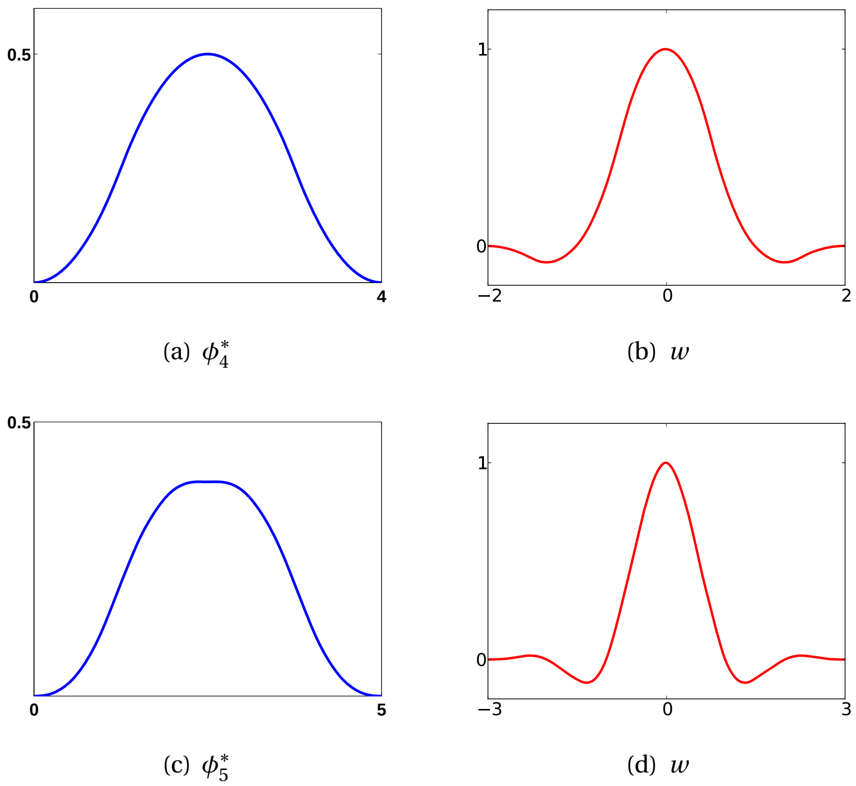
$$\begin{aligned} W(z) &= \frac{1}{z^4}\left(\frac{1+z}{2}\right)^3 \sum_{j=0}^{2} \binom{2+j}{j} \left[\frac{1}{2}\left(1-\frac{z+z^{-1}}{2}\right)\right]^j \\ &= \frac{1}{64}\left(3z^{-6}-9z^{-5}-7z^{-4}+45z^{-3}+45z^{-2}-7z^{-1}-9+3z\right) \quad (6.3.33) \end{aligned}$$

It follows from (6.3.33) and (6.2.10) that

$$\{w_{-6}, w_{-6}, w_{-4}, w_{-3}, w_{-2}, w_{-1}, w_0, w_1\} = \left\{\frac{3}{32}, -\frac{9}{32}, -\frac{7}{32}, \frac{45}{32}, \frac{45}{32}, -\frac{7}{4}, -\frac{9}{32}, \frac{3}{32}\right\}.$$

For $\nu = 4$ and $\nu = 5$, the graphs of ϕ^*_ν and w, as generated by means of, respectively, Algorithms 4.3.1 and 4.4.2 in Chui and de Villiers (2010), are displayed in Figure 6.3.3.

■



(a) ϕ^*_4 (b) w

(c) ϕ^*_5 (d) w

Figure 6.3.3: *The refinable functions ϕ^*_ν, for $\nu = 4,5$, and corresponding fundamental interpolation functions w of Example 6.3.3.*

Chapter 7

Minimally supported synthesis wavelets

In this chapter, we apply the synthesis wavelet construction methods introduced in Chui and de Villiers (2010, Chapter 9) to (formally) construct minimally supported synthesis wavelets corresponding to the scaling functions established in Section 3.3 and Chapter 5, as could be useful in wavelet subdivision methods for the efficient rendering of curves and surfaces.

7.1 Wavelet decomposition

Following Chui and de Villiers (2010, Chapter 9), we show in this section how scaling functions can be used as building blocks to generate their corresponding minimally-supported synthesis wavelets.

Let $\phi \in \mathscr{C}_0$ denote a refinable function with refinement sequence $\boldsymbol{p} = \{p_j\}$ such that

$$\operatorname{supp}\{p_j\} = [0,\nu]|_{\mathbb{Z}}, \tag{7.1.1}$$

for an integer $\nu \geq 2$, and where $\{p_j\}$ satisfies the sum-rule condition (1.2.8).

Definition 7.1.1 *For a scaling function ϕ and a sequence $\boldsymbol{q} := \{q_j\} \in \ell_0$, the function $\psi := \psi_\phi$ is defined by*

$$\psi(x) := \sum_j q_j \phi(2x - j), \quad x \in \mathbb{R}. \tag{7.1.2}$$

Definition 7.1.2 *For a refinable function ϕ and a function ψ, as in Definition 7.1.1, we define the (vector) spaces*

$$\left.\begin{aligned} S_\phi^r &:= \left\{ \sum_j c_j \phi(2^r \cdot - j) : \{c_j\} \in \ell(\mathbb{Z}) \right\}, \\ W_\phi^r &:= \left\{ \sum_j d_j \psi(2^r \cdot - j) : \{d_j\} \in \ell(\mathbb{Z}) \right\}, \end{aligned}\right\} r \in \mathbb{Z}. \tag{7.1.3}$$

The following result then holds.

Theorem 7.1.1 *Let ϕ denote a refinable function with refinement sequence $\{p_j\}$. Then*

(a)

$$S_\phi^r \subset S_\phi^{r+1}, \quad r \in \mathbb{Z}, \tag{7.1.4}$$

where, for any sequence $\{c_j\} \in \ell(\mathbb{Z})$ and $r \in \mathbb{Z}$,

$$\sum_j c_j \phi(2^r x - j) = \sum_j \left[\sum_k p_{j-2k} c_k \right] \phi(2^{r+1} x - j), \quad x \in \mathbb{R}; \tag{7.1.5}$$

(b) for any sequence $\{q_j\} \in \ell_0$,

$$W_\phi^r \subset S_\phi^{r+1}, \quad r \in \mathbb{Z}, \tag{7.1.6}$$

where, for any sequence $\{d_j\} \in \ell(\mathbb{Z})$ and $r \in \mathbb{Z}$,

$$\sum_j d_j \psi(2^r x - j) = \sum_j \left[\sum_k q_{j-2k} d_k \right] \phi(2^{r+1} x - j), \quad x \in \mathbb{R}. \tag{7.1.7}$$

The concept of space decomposition is introduced as follows.

Definition 7.1.3 *For vector spaces U, V and W, with $V \subset U$ and $W \subset U$, the space decomposition notation*

$$U = V \oplus W$$

means that, for any $u \in U$, there exist elements $v \in V$ and $w \in W$ such that $u = v + w$, and with the pair $\{v, w\}$ uniquely determined by u.

Based on the inclusions (7.1.4) and (7.1.6), the concept of a synthesis wavelet is now introduced as follows.

Definition 7.1.4 *In Definitions 7.1.1 and 7.1.3, if the refinable function ϕ and the sequence $\{q_j\} \in \ell_0$ are such that the space decomposition result*

$$S_\phi^{r+1} = S_\phi^r \oplus W_\phi^r, \quad r \in \mathbb{Z}, \tag{7.1.8}$$

is satisfied, then ψ is called a synthesis wavelet.

In order to construct such synthesis wavelets, we state the following two results for the proof of which we refer to Chui and de Villiers (2010, Theorem 9.1.2).

Theorem 7.1.2 *Let ϕ be a scaling function with refinement sequence $\{p_j\}$ and corresponding two-scale Laurent polynomial P, given by*

$$P(z) := \frac{1}{2}\sum_j p_j z^j, \quad z \in \mathbb{C}\backslash\{0\}, \tag{7.1.9}$$

and such that ϕ possesses linearly independent integer shifts on $\mathbb{R}$. Also, let $\boldsymbol{q} = \{q_j\}$, $\boldsymbol{a} = \{a_j\}$ and $\boldsymbol{b} = \{b_j\}$ denote three sequences in ℓ_0, with respective corresponding Laurent polynomial symbols

$$\left.\begin{array}{c} Q(z) := \dfrac{1}{2}\displaystyle\sum_j q_j z^j, \\ \\ A(z) := \displaystyle\sum_j a_j z^j, \quad B(z) := \displaystyle\sum_j b_j z^j, \end{array}\right\} z \in \mathbb{C}\backslash\{0\}, \tag{7.1.10}$$

and let the function ψ be defined by (7.1.2)*. Then the relation*

$$\phi(2x - j) = \sum_k a_{2k-j}\phi(x-k) + \sum_k b_{2k-j}\psi(x-k), \quad x \in \mathbb{R}, \ j \in \mathbb{Z}, \tag{7.1.11}$$

holds if and only if the Laurent polynomials P, Q, A and B satisfy the identities:

$$\left.\begin{array}{rcl} P(z)A(z) + P(-z)A(-z) & = & 1, \\ \\ Q(z)A(z) + Q(-z)A(-z) & = & 0, \\ \\ P(z)B(z) + P(-z)B(-z) & = & 0, \\ \\ Q(z)B(z) + Q(-z)B(-z) & = & 1, \end{array}\right\} z \in \mathbb{C} \setminus \{0\}. \tag{7.1.12}$$

Theorem 7.1.3 *In Theorem 7.1.2, suppose that one of the two conditions* (7.1.11) *or* (7.1.12) *is satisfied. Then:*

(a) for any $\{c_j\} \in \ell(\mathbb{Z})$ *and* $r \in \mathbb{Z}$*,*

$$\sum_j c_j \phi(2^{r+1}x - j) = \sum_j \left[\sum_k a_{2j-k} c_k\right] \phi(2^r x - j) + \sum_j \left[\sum_k b_{2j-k} c_k\right] \psi(2^r x - j), \quad x \in \mathbb{R}; \tag{7.1.13}$$

(b) for $r \in \mathbb{Z}$*,*

(i) any sequence $\{c_j\} \in \ell(\mathbb{Z})$ *such that*

$$\sum_j c_j \phi(2^{r+1} \cdot - j) \in S_\phi^r \tag{7.1.14}$$

satisfies

$$\sum_k b_{2j-k} c_k = 0, \quad j \in \mathbb{Z}; \tag{7.1.15}$$

(ii) any sequence $\{c_j\} \in \ell(\mathbb{Z})$ *such that*

$$\sum_j c_j \phi(2^{r+1} \cdot - j) \in W_\phi^r \tag{7.1.16}$$

satisfies

$$\sum_k a_{2j-k} c_k = 0, \quad j \in \mathbb{Z}; \tag{7.1.17}$$

(c)

$$S_\phi^r \cap W_\phi^r = \{0\}, \quad r \in \mathbb{Z}; \tag{7.1.18}$$

(d) the space decomposition (7.1.8) *is satisfied for every* $r \in \mathbb{Z}$*, that is,* ψ *is a synthesis wavelet.*

As was done in Chui and de Villiers (2010, pp. 354-360), we proceed to construct, for a given mask symbol P satisfying certain conditions, a synthesis wavelet ψ as in Theorem 7.1.3, by constructing Laurent polynomials Q, A and B that satisfy the system (7.1.12), such that the corresponding coefficient sequences $\{q_j\}, \{a_j\}$ and $\{b_j\}$ in (7.1.10) have minimum support.

Suppose that ϕ is a refinable function with refinement sequence $\{p_j\}$, where supp $\{p_j\} = [0, \nu]|_{\mathbb{Z}}$ for an integer $\nu \geq 2$, and with $\{p_j\}$ satisfying the sum-rule condition (1.2.8), according to which the corresponding mask symbol P, as defined by (1.2.7), satisfies the conditions (1.2.9). Suppose moreover, that P has no symmetric zeros. Then the polynomial G defined by

$$G(z) := P(z), \quad z \in \mathbb{C}, \tag{7.1.19}$$

satisfies,

$$\deg(G) = \nu \geq 2; \tag{7.1.20}$$

$$G(0) \neq 0 \quad ; \quad G(1) = 1. \tag{7.1.21}$$

and G has no symmetric zeros. Also, define the polynomial F by

$$F(z) := z^{2\lfloor \nu/2 \rfloor + 1}, \tag{7.1.22}$$

from which we deduce that $F_{-} = -F$, that is, F is an odd polynomial, and $F \in \pi_{\nu-1} \subset \pi_{2\nu-3}$, since $\nu \geq 2$. Hence, we may apply Theorem 3.1.1 (a), (b) and (d) to deduce that there exists precisely one polynomial $\tilde{H}_{\boldsymbol{p}}$ in $\pi_{\nu-2}$ with explicit construction as described in Theorem 3.1.1, satisfying the identity

$$G(z)\tilde{H}_{\boldsymbol{p}}(z) - G(-z)\tilde{H}_{\boldsymbol{p}}(-z) = F(z), \quad z \in \mathbb{C}, \tag{7.1.23}$$

with moreover,

$$\tilde{H}_{\boldsymbol{p}} \in \pi_{\nu-2}, \tag{7.1.24}$$

according to which $\tilde{H}_{\boldsymbol{p}}$ is the least-degree polynomial satisfying the identity (7.1.23). Also, by setting $z = 1$ in (7.1.23), and using (1.2.9), we obtain

$$\tilde{H}_{\boldsymbol{p}}(1) = 1. \tag{7.1.25}$$

It follows from (7.1.23) that the Laurent polynomial $A_{\boldsymbol{p}}$ defined by

$$A_{\boldsymbol{p}}(z) := z^{-2\lfloor \nu/2 \rfloor + 1}\tilde{H}_{\boldsymbol{p}}(z), \quad z \in \mathbb{C}\setminus\{0\}, \tag{7.1.26}$$

then satisfies the identitity

$$P(z)A_{\boldsymbol{p}}(z) + P(-z)A_{\boldsymbol{p}}(-z) = 1, \quad z \in \mathbb{C}\setminus\{0\}, \tag{7.1.27}$$

with also, from (7.1.25) and (7.1.26),

$$A_{\boldsymbol{p}}(1) = 1. \tag{7.1.28}$$

Next, for the Laurent polynomials $Q_{\boldsymbol{p}}$ and $B_{\boldsymbol{p}}$ defined by

$$Q_{\boldsymbol{p}}(z) := \tilde{H}_{\boldsymbol{p}}(-z), \quad z \in \mathbb{C}\setminus\{0\}; \tag{7.1.29}$$

$$B_{\boldsymbol{p}}(z) := -z^{-2\lfloor\frac{\nu}{2}\rfloor+1} P(-z), \quad z \in \mathbb{C}\setminus\{0\}, \tag{7.1.30}$$

we deduce from (7.1.26) and (7.1.23) that

$$\left.\begin{array}{rcl} Q_{\boldsymbol{p}}(z)A_{\boldsymbol{p}}(z) + Q_{\boldsymbol{p}}(-z)A_{\boldsymbol{p}}(-z) & = & 0, \\ P(z)B_{\boldsymbol{p}}(z) + P(-z)B_{\boldsymbol{p}}(-z) & = & 0, \\ Q_{\boldsymbol{p}}B_{\boldsymbol{p}}(z) + Q_{\boldsymbol{p}}(-z)B_{\boldsymbol{p}}(-z) & = & 1, \end{array}\right\} z \in \mathbb{C}\setminus\{0\}. \tag{7.1.31}$$

Also, observe from (7.1.29) and (7.1.25) that

$$Q_{\boldsymbol{p}}(-1) = 1, \tag{7.1.32}$$

whereas (7.1.30) and (1.2.9) yield

$$B_{\boldsymbol{p}}(-1) = 1; \quad B_{\boldsymbol{p}}(1) = 0. \tag{7.1.33}$$

Moreover, let the sequences $\{q_{\boldsymbol{p},j}\}$, $\{a_{\boldsymbol{p},j}\}$, $\{b_{\boldsymbol{p},j}\} \in \ell_0$ be defined by

$$\left.\begin{array}{c} \frac{1}{2}\sum_j q_j z^j := Q_{\boldsymbol{p}}(z), \\ \sum_j a_j z^j := A_{\boldsymbol{p}}(z), \quad \sum_j b_j z^j := B_{\boldsymbol{p}}(z), \end{array}\right\} z \in \mathbb{C}\setminus\{0\}. \tag{7.1.34}$$

According to (7.1.27) and (7.1.31), the system (7.1.12) is solved by the Laurent polynomials $Q := Q_{\boldsymbol{p}}$, $A := A_{\boldsymbol{p}}$ and $B := B_{\boldsymbol{p}}$. Hence, we may apply Theorems 7.1.2 and 7.1.3, and with $\tilde{H} = \tilde{H}_{\boldsymbol{p}}$ as in Theorem 3.1.1 with G and F defined by (7.1.19), (7.1.22), to deduce the following result.

Theorem 7.1.4 *Let ϕ denote a refinable function with refinement sequence $\{p_j\}$ such that supp $\{p_j\} = [0,\nu]|_{\mathbb{Z}}$ for an integer $\nu \geq 2$, and such that $\{p_j\}$ satisfies the sum-rule condition* (1.2.8). *Also, suppose that ϕ has linearly independent integer shifts on $\mathbb{R}$, and the Laurent polynomial symbol P defined by* (7.1.9) *has no symmetric zeros in $\mathbb{C}\setminus\{0\}$. Then the function*

$$\psi_{\boldsymbol{p}}(x) := \sum_j q_{\boldsymbol{p},j}\phi(2x-j), \quad x \in \mathbb{R}, \tag{7.1.35}$$

with $\boldsymbol{q_p} = \{q_{\boldsymbol{p},j}\} \in \ell_0$ *defined by* (7.1.34) *and* (7.1.29), *with* $\tilde{H}_{\boldsymbol{p}} \in \pi_{\nu-2}$ *denoting the minimum-degree solution of the polynomial identity* (7.1.23), *as obtained from Theorem 3.1.1 with G and F given by* (7.1.19), (7.1.22), *is a minimally supported synthesis wavelet, with corresponding decomposition relation given by* (7.1.13), *with* $\{a_j\} = \{a_{\boldsymbol{p},j}\}$ *and* $\{b_j\} = \{b_{\boldsymbol{p},j}\}$, *as defined by* (7.1.34), (7.1.26), (7.1.30).

The following support properties with respect to Theorem 7.1.4 are now immediate consequences of (7.1.29), (7.1.30), (7.1.26), (7.1.24), and (7.1.35), together with the fact that supp $\{p_j\} = [0,\nu]|_{\mathbb{Z}}$ implies $\text{supp}^c\phi = [0,\nu]$, from Theorem 1.2.1.

Theorem 7.1.5 *In Theorem 7.1.4, the sequences* $\{q_{\boldsymbol{p},j}\}, \{a_{\boldsymbol{p},j}\}, \{b_{\boldsymbol{p},j}\}$, *and the synthesis wavelet* $\psi_{\boldsymbol{p}}$, *have the following support properties:*

(a)

$$\textit{supp}\,\{q_{\boldsymbol{p},j}\} \subset [0,\nu-2]|_{\mathbb{Z}}\,; \tag{7.1.36}$$

(b)

$$\textit{supp}\,\{a_{\boldsymbol{p},j}\} \subset \begin{cases} [-\nu+1,-1]|_{\mathbb{Z}}\,, & \textit{if} \quad \nu \textit{ is even,} \\ [-\nu+2,0]|_{\mathbb{Z}}\,, & \textit{if} \quad \nu \textit{ is odd;} \end{cases} \tag{7.1.37}$$

(c)

$$\textit{supp}\,\{b_{\boldsymbol{p},j}\} = \begin{cases} [-\nu+1,1]|_{\mathbb{Z}}\,, & \textit{if} \quad \nu \textit{ is even,} \\ [-\nu+2,2]|_{\mathbb{Z}}\,, & \textit{if} \quad \nu \textit{ is odd;} \end{cases} \tag{7.1.38}$$

(d)

$$\textit{supp}^c\psi_{\boldsymbol{p}} \subset [0,\nu-1]. \tag{7.1.39}$$

We proceed in Section 7.2 to apply the synthesis wavelet construction in Theorem 7.1.4 to the scaling functions ϕ and corresponding refinement sequences $\{p_j\}$ of Section 3.1 and Chapter 5.

7.2 Examples

The following two examples are applications of the synthesis wavelet construction method of Theorem 7.1.4, subject to the assumption of integer-shift linear independence on $\mathbb{R}$ of the underlying scaling function ϕ.

Example 7.2.1

Let $\phi = \phi\left(-\frac{5}{12}|\cdot\right)$ denote the refinable function established in Section 3.3, and with, according to (3.3.17), corresponding refinement mask symbol

$$P\left(\frac{5}{12}\middle|z\right) = \left(\frac{1+z}{2}\right)^2\left(\frac{1+z+z^2}{3}\right) = \frac{1}{12}\left(1+3z+4z^2+3z^3+z^4\right), \tag{7.2.1}$$

and the refinement sequence $\{p_j\} = \{p_j\left(\frac{5}{12}\right)\}$ given, according to Table 3.3.1, by

$$\left\{\begin{array}{l} \{p_0, p_1, p_2, p_3, p_4\} = \{\frac{1}{6}, \frac{3}{6}, \frac{4}{6}, \frac{3}{6}, \frac{1}{6}\}; \\ \\ \text{with} \quad p_j = 0, \quad j \notin \{0,1,2,3,4\}, \end{array}\right. \tag{7.2.2}$$

according to which $\text{supp}\,\{p_j\} = [0,4]|_{\mathbb{Z}}$, and $\{p_j\}$ satisfies the sum rule condition (1.2.8).

It is clear from (7.2.1) that the polynomial P has no symmetric zeros. Subject to the assumption that ϕ has linearly independent integer shifts on $\mathbb{R}$, we may therefore apply Theorem 7.1.4 to first obtain, by means of method described in Theorem 3.1.1, the polynomial

$$\tilde{H}_{\boldsymbol{p}}(z) = -1 + 3z - z^2, \tag{7.2.3}$$

and then calculating from (7.1.26), (7.1.29) and (7.1.30), the Laurent polynomials

$$\left.\begin{array}{rcl} A_{\boldsymbol{p}}(z) & = & -z^{-3} + 3z^{-2} - z^{-1}; \\ \\ Q_{\boldsymbol{p}}(z) & = & -1 - 3z - z^2; \\ \\ B_{\boldsymbol{p}}(z) & = & \frac{1}{12}\left(-z^{-3} + 3z^{-2} - 4z^{-1} + 3 - z\right). \end{array}\right\} \tag{7.2.4}$$

It follows from (7.2.4), (7.1.34) that

$$\left\{\begin{array}{l} \{a_{\boldsymbol{p},-3}, a_{\boldsymbol{p},-2}, a_{\boldsymbol{p},-1}\} = \{-1,-3,-1\}; \\ \\ \text{with} \quad \text{supp}\,\{a_{\boldsymbol{p},j}\} = [-3,-1]|_{\mathbb{Z}}; \end{array}\right. \tag{7.2.5}$$

$$\begin{cases} \{q_{\boldsymbol{p},0}, q_{\boldsymbol{p},1}, q_{\boldsymbol{p},2}\} = \{-2,-6,-2\}; \\ \\ \text{with} \quad \text{supp}\,\{q_{\boldsymbol{p},j}\} = [0,2]|_{\mathbb{Z}}; \end{cases} \tag{7.2.6}$$

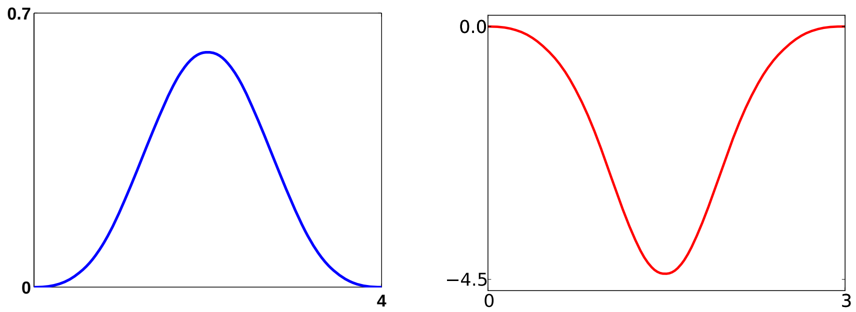
$$\begin{cases} \{b_{\boldsymbol{p},-3}, b_{\boldsymbol{p},-2}, b_{\boldsymbol{p},-1}, b_{\boldsymbol{p},0}, b_{\boldsymbol{p},1}\} = \{-\frac{1}{12}, \frac{1}{4}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{12}\}; \\ \\ \text{with} \quad \text{supp}\,\{b_{\boldsymbol{p},j}\} = [-3,1]|_{\mathbb{Z}}. \end{cases} \tag{7.2.7}$$

Moreover, from (7.2.6), (7.1.35) and (7.1.39), the corresponding minimally-supported synthesis wavelet ψ satisfies

$$\begin{cases} \psi(x) = -\phi(2x) - 3\phi(2x-1) - \phi(2x-2); \\ \\ \text{with} \quad \text{supp}^c\psi \subset [0,3]. \end{cases} \tag{7.2.8}$$

In Figure 7.2.1, we plot the graph of the scaling function ϕ on its support interval $[0,4]$, and the corresponding synthesis wavelet ψ on its support interval $[0,3]$.

■



(a) ϕ (b) ψ

Figure 7.2.1: *The scaling function ϕ and synthesis wavelet ψ of Example 7.2.1.*

Example 7.2.2

For an integer ν, with $3 \leq \nu \leq 9$, let $\phi = \phi_\nu^*$ denote the Pascal refinable function of Theorem 5.2.1. We shall assume that ϕ_ν^* has linearly independent integer shift on $\mathbb{R}$. Moreover, note from the formulation (5.1.32) in Theorem 5.1.2, together with (5.1.8), that the corresponding Pascal refinement mask symbol P_ν^* satisfies

$$P_\nu^*(1) = 1 \quad ; \quad P_\nu^*(-1) = 0, \tag{7.2.9}$$

from which we deduce that the corresponding refinement sequence $\{p_{\nu,j}^*\}$, as defined by

$$\frac{1}{2}\sum_j p_{\nu,j}^* z^j := P_\nu^*(z), \tag{7.2.10}$$

satisfies the sum-rule condition

$$\sum_j p_{\nu,2j}^* = 1 \quad ; \quad \sum_j p_{\nu,2j-1}^* = 1. \tag{7.2.11}$$

Hence, in order to apply Theorem 7.1.4, it remains to verify that the polynomial P_ν^* possesses no symmetric zeros.

To this end, we first observe from Theorem 2.2.1, together with (5.1.5), that the identity

$$\left(\frac{1+z}{2}\right)^{\nu-2} P_\nu^*(z) - \left(\frac{1-z}{2}\right)^{\nu-2} P_\nu^*(-z) = z\left(\frac{1+z^2}{2}\right)^{\nu-2}, \quad z \in \mathbb{C}, \tag{7.2.12}$$

is satisfied.

Suppose P_ν^* has a symmetric zero $z_0 \in \mathbb{C}\setminus\{0\}$, that is,

$$P_\nu^*(z_0) = P_\nu^*(-z_0) = 0. \tag{7.2.13}$$

By inserting $z = z_0$ and (7.2.13) into (7.2.12), it follows that we must have $1 + z_0^2 = 0$, and thus $z_0 \in \{-i, i\}$. According to (5.1.32) (a) and (b), together with (5.1.10) and (5.1.37)–(5.1.39), we have, for $3 \leq \nu \leq 9$, that

$$P_\nu^*(\pm i) \neq 0 \Longleftrightarrow \nu \text{ is odd},$$

and it follows that

$$P_\nu^* \text{ has no symmetric zeros for } 3 \leq \nu \leq 9 \iff \nu \text{ is odd}.$$

Hence, we may apply Theorem 7.1.4 to construct minimally-supported synthesis wavelets ψ_ν^* corresponding to Pascal scaling function ϕ_ν^*.

After observing that the case $\nu = 3$ corresponds to the minimally supported quadratic B-spline wavelet obtained in Chui and de Villiers (2010, Section 9.4), we proceed to calculate here the minimally-supported synthesis wavelet ψ_5^* corresponding to the Pascal scaling function ϕ_5^*.

As in Example 7.2.1, since P_ν^* has no symmetric zeros, and subject to the assumption that ϕ_5^* has linearly independent integer shifts on $\mathbb{R}$, we may therefore apply Theorem 7.1.4 to first obtain, by means of method described in Theorem 3.1.1, the polynomial

$$\tilde{H}_{\boldsymbol{p}_5^*}(z) = \frac{1}{8}\left(9 - 21z + 35z^2 - 15z^3\right), \tag{7.2.14}$$

and then calculate the Laurent polynomials

$$\left.\begin{aligned} A_{\boldsymbol{p}_5^*}(z) &= \tfrac{1}{8}\left(9z^{-3} - 21z^{-2} + 35z^{-1} - 15\right); \\ Q_{\boldsymbol{p}_5^*}(z) &= \tfrac{1}{8}\left(9 + 21z + 35z^2 + 15z^3\right); \\ B_{\boldsymbol{p}_5^*}(z) &= \tfrac{1}{32}\left(-3z^{-3} + 7z^{-2} - 6z^{-1} + 6 - 7z + 3z^2\right). \end{aligned}\right\} \tag{7.2.15}$$

It follows from (7.2.15), (7.1.34), and Theorem 7.1.5, that

$$\left\{\begin{aligned} &\{a_{\boldsymbol{p}_5^*,-3}, a_{\boldsymbol{p}_5^*,-2}, a_{\boldsymbol{p}_5^*,-1}, a_{\boldsymbol{p}_5^*,0}\} = \{\tfrac{9}{8}, \tfrac{-21}{8}, \tfrac{35}{8}, \tfrac{-15}{8}\}; \\ &\text{with} \quad \text{supp}\,\{a_{\boldsymbol{p}_5^*,j}\} = [-3,0]|_{\mathbb{Z}}; \end{aligned}\right. \tag{7.2.16}$$

$$\left\{\begin{aligned} &\{q_{\boldsymbol{p}_5^*,0}, q_{\boldsymbol{p}_5^*,1}, q_{\boldsymbol{p}_5^*,2}, q_{\boldsymbol{p}_5^*,3}\} = \{\tfrac{9}{8}, \tfrac{-21}{8}, \tfrac{35}{8}, \tfrac{-15}{8}\}; \\ &\text{with} \quad \text{supp}\,\{q_{\boldsymbol{p}_5^*,j}\} = [0,3]|_{\mathbb{Z}}; \end{aligned}\right. \tag{7.2.17}$$

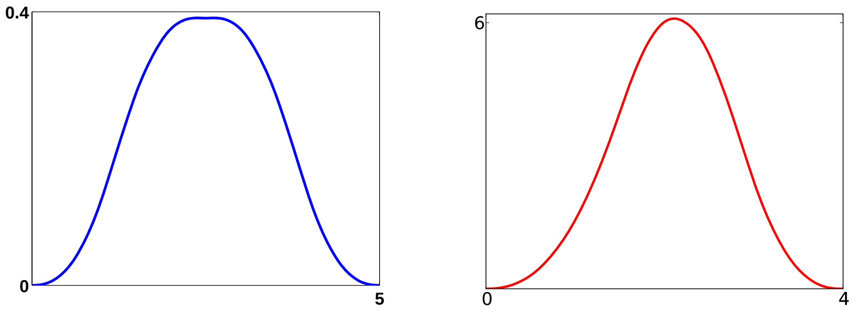
$$\left\{\begin{aligned} &\{b_{\boldsymbol{p}_5^*,-3}, b_{\boldsymbol{p}_5^*,-2}, b_{\boldsymbol{p}_5^*,-1}, b_{\boldsymbol{p}_5^*,0}, b_{\boldsymbol{p}_5^*,1}, b_{\boldsymbol{p}_5^*,2}\} = \{-\tfrac{3}{32}, \tfrac{7}{32}, -\tfrac{6}{32}, \tfrac{6}{32}, -\tfrac{7}{32}, \tfrac{3}{32}\}; \\ &\text{with} \quad \text{supp}\,\{b_{\boldsymbol{p}_5^*,j}\} = [-3,2]|_{\mathbb{Z}}. \end{aligned}\right. \tag{7.2.18}$$

Moreover, from (7.2.17), (7.1.35) and (7.1.39), the corresponding minimally-supported synthesis wavelet ψ_5^* satisfies

$$\left\{\begin{aligned} &\psi_5^*(x) = \tfrac{9}{8}\phi_5^*(2x) + \tfrac{21}{8}\phi_5^*(2x-1) + \tfrac{35}{8}\phi_5^*(2x-2) + \tfrac{15}{8}\phi_5^*(2x-3); \\ &\text{with} \quad \text{supp}^c \psi_5^* \subset [0,4]. \end{aligned}\right. \tag{7.2.19}$$

In Figure 7.2.2, we plot the graph of the scaling function ϕ_5^* on its support interval $[0,5]$, and the corresponding synthesis wavelet function ψ_5^* on its support interval $[0,4]$.

■



(a) ϕ_5^* (b) ψ_5^*

Figure 7.2.2: *The scaling function ϕ_5^* and synthesis wavelet ϕ_5^* of Example 7.2.2.*

In Example 7.2.1 and Example 7.2.2, we assumed that the scaling function ϕ has linearly independent integer shift on $\mathbb{R}$. As future research, it is intended to rigorously investigate the truth of this assumption, as well as the related issue of the robust stability on $\mathbb{R}$ of the integer shifts of ϕ and its corresponding synthesis wavelets.

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