On a unified categorical setting for homological diagram lemmas

by

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Declaration

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Abstract

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Some of the diagram lemmas of Homological Algebra, classically known for abelian categories, are not characteristic of the abelian context; this naturally leads to investigations of those non-abelian categories in which these diagram lemmas may hold. In this Thesis we attempt to bring together two different directions of such investigations; in particular, we unify the five lemma from the context of homological categories due to F. Borceux and D. Bourn, and the five lemma from the context of modular semi-exact categories in the sense of M. Grandis.

Opsomming

Op 'n verenigde kategoriese instelling vir homologiese diagram lemmata

("On a unified categorical setting for homological diagram lemmas")

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Verskeie diagram lemmata van Homologiese Algebra is aanvanklik ontwikkel in die konteks van abelse kategorieë, maar geld meer algemeen as dit behoorlik geformuleer word. Dit lei op 'n natuurlike wyse na 'n ondersoek van ander kategorieë waar hierdie lemmas ook geld. In hierdie tesis bring ons twee moontlike rigtings van ondersoek saam. Dit maak dit vir ons moontlik om die vyf-lemma in die konteks van homologiese kategoieë, deur F. Borceux en D. Bourn, en vyf-lemma in die konteks van semi-eksakte kategorieë, in die sin van M. Grandis, te verenig.

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Contents

Introduction

Abelian categories are a special type of categories, which in some sense resemble the category of abelian groups, and which have been used for an axiomatic study of Homological Algebra (see e.g. [Mac48]). In particular, various diagram lemmas of Homological Algebra can be extended from categories of modules over a ring, to arbitrary abelian categories. However, most of these diagram lemmas, if suitably formulated, also hold true in more general non-abelian contexts. One example of such a context is given by the class of homological categories in the sense of F. Borceux and D. Bourn [BB04]. A homological category is defined as a pointed regular protomodular category in the sense of D. Bourn [Bou91], and among its examples are not only all abelian categories but also categories such as the category of groups and the category of non-unitary rings. In this Thesis we attempt to replace regularity with an additional structure on the category (namely, a "cover relation" in the sense of Z. Janelidze [Jan09]) and reobtain the five lemma for homological categories in this more general context. As a result of this we obtain a general form of the five lemma, which also includes as its another special case the five lemma in the context of modular semi-exact categories in the sense of M. Grandis [Gra92]. Our five lemma is actually formulated in the same style as suggested in [Gra92], where assumptions on the category are replaced with assumptions on arrows involved in the lemma.

The main new results of this Thesis have been reported in [Mic], as well as in talks given by the author at several occasions in the Mathematics Division of Department of Mathematical Sciences of Stellenbosch University.

Chapter 1

The classical context: abelian categories

1.1 Ab-categories

Definition 1.1.1 *An* **Ab**-*category is a category* $\mathbb{C}$*, where for every two objects* X, Y *in* $\mathbb{C}$*, a binary operation (written as* $+$*) is given on* $\hom(X,Y)$*, in such a way that:*

1. *The set* $\hom(X,Y)$ *together with* $+$ *forms an additive abelian group;*
2. *If* $f, g \in \hom(X,Y)$ *and* $h \in \hom(Y,Z)$*, then* $h(f+g) = hf + hg$*, that is, composition distributes over addition from the left;*
3. *If* $f, g \in \hom(X,Y)$ *and* $h \in \hom(Z,X)$*, then* $(f+g)h = fh + gh$*, that is, composition distributes over addition from the right.*

The category **Grp** of groups is not an **Ab**-category; however, the category **Ab** of abelian groups is an **Ab**-category, and more generally, for any ring R, the category R-**Mod** of R-modules is an **Ab**-category, where for any two module homomorphisms $f, g : X \to Y$, the sum $f + g$ is defined component-wise:

$$(f+g)(x) = f(x) + g(x).$$

If $\mathbb{C}$ is an **Ab**-category, then its dual category $\mathbb{C}^{\mathrm{op}}$ is also an **Ab**-category with the abelian group structure on each hom-set $\hom(X,Y)$ in $\mathbb{C}^{\mathrm{op}}$ being defined as the same abelian group structure on the hom-set $\hom(Y,X)$ in $\mathbb{C}$. This fact can be used in showing that in an **Ab**-category $\mathbb{C}$, an object Z is terminal if and only if it is initial; such an object is usually called a *zero* object, and the category $\mathbb{C}$ is said to be *pointed* if it has a zero object. For any two objects X, Y in an **Ab**-category $\mathbb{C}$, a *zero morphism* $0_{X,Y} : X \to Y$ is a morphism which factors through the zero object Z; that is, it is the composite fg, where g is the unique morphism $X \to Z$ and f is the unique morphism $Z \to Y$.

Notice that conditions (1), (2) and (3) above imply that the zero morphisms of an **Ab**-category $\mathbb{C}$ act as identities with respect to the binary operation $+$; that is, for any morphism $h : X \to Y$ in $\mathbb{C}$,

$$h + 0_{X,Y} = h = 0_{X,Y} + h.$$

Henceforth we shall simply write 0 for a zero morphism, and assume that its 'direction' is understood. Note, in passing, that zero morphisms are useful in characterizing monomorphisms and epimorphisms in **Ab**-categories:

Proposition 1.1.1 *Let $f : X \to Y$ be a morphism in an **Ab**-category $\mathbb{C}$. Then*

1. *f is a monomorphism if and only if for every morphism $m : S \to X$, we have*
$$fm = 0_{S,Y} \;\Rightarrow\; m = 0_{S,X};$$
2. *f is an epimorphim if and only if for every morphism $e : Y \to Z$, we have*
$$ef = 0_{X,Z} \;\Rightarrow\; e = 0_{Y,Z}.$$

Proof. Since these two statements are dual to each other, it suffices to prove one of them; let us prove the first one.

Suppose first that f is a monomorphism. Consider a morphism $m : S \to X$ such that $fm = 0$. Then $fm = f0$, whence $m = 0$. Conversely, suppose that the given condition holds. To show that f is a monomorphism, let $m_1, m_2 : S \to X$ be two morphisms such that $fm_1 = fm_2$. Then $f(m_1 - m_2) = fm_1 - fm_2 = 0$. By the assumption on f, this means that $m_1 - m_2 = 0$, and so $m_1 = m_2$, proving that f is a monomorphism. $\square$

Next, we investigate products and coproducts in an **Ab**-category $\mathbb{C}$.

Definition 1.1.2 *A biproduct diagram for two objects X, Y in an **Ab**-category $\mathbb{C}$ is a diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "i_1"'] & X\Box Y \arrow[l, shift left, "p_1"'] \arrow[r, shift left, "p_2"] & Y \arrow[l, shift left, "i_2"] \end{tikzcd} \tag{1.1.1}$$

where the morphisms p_1, p_2, i_1, i_2 satisfy the identities

$$p_1 \circ i_1 = 1_X, \quad p_2 \circ i_2 = 1_Y, \quad i_1 \circ p_1 + i_2 \circ p_2 = 1_{X\Box Y}. \tag{1.1.2}$$

In an **Ab**-category $\mathbb{C}$, products and coproducts coincide, as given by the following

Proposition 1.1.2 *Let (1.1.1) be a diagram in an **Ab**-category $\mathbb{C}$. Then the following are equivalent:*

1. *$(i_1, i_2, X\square Y, p_1, p_2)$ is a biproduct of X and Y;*

2. *$(X\square Y, p_1, p_2)$ is a product of X and Y, and for $j,k = 1,2$, $p_k i_j = \delta_{jk}$, where $\delta_{jk} = \begin{cases} 1, \text{ if } j = k \\ 0, \text{ otherwise;} \end{cases}$*

3. *$(i_1, i_2, X\square Y)$ is a coproduct of X and Y, and for $j,k = 1,2$, $p_k i_j = \delta_{jk}$.*

Proof. Since the notion of a biproduct is self-dual and since (2) and (3) are dual to each other, it is sufficient to prove that (1) and (2) are equivalent. To this end, first suppose that $X\square Y$ is a biproduct of X and Y. Observe that

$$\begin{aligned} p_1 \circ i_2 &= p_1 \circ 1_{X\square Y} \circ i_2 \\ &= p_1 \circ (i_1 \circ p_1 + i_2 \circ p_2) \circ i_2 \\ &= (p_1 \circ i_1 \circ p_1 + p_1 \circ i_2 \circ p_2) \circ i_2 \\ &= 1_X \circ p_1 \circ i_2 + p_1 \circ i_2 \circ 1_Y \\ &= p_1 \circ i_2 + p_1 \circ i_2, \end{aligned}$$

from which we get $p_1 \circ i_2 = 0$. Similarly, $p_2 \circ i_1 = 0$. We now check that the diagram

$$X \xleftarrow{p_1} X\square Y \xrightarrow{p_2} Y \tag{1.1.3}$$

is a product diagram. Consider any diagram

$$X \xleftarrow{f_1} C \xrightarrow{f_2} Y$$

Consider the morphism $h : C \to X\square Y$,

$$h = i_1 \circ f_1 + i_2 \circ f_2.$$

Then

$$\begin{aligned} p_1 \circ h &= p_1 \circ (i_1 \circ f_1 + i_2 \circ f_2) \\ &= p_1 \circ i_1 \circ f_1 + p_1 \circ i_2 \circ f_2 \\ &= 1_X \circ f_1 + 0 \circ f_2 \\ &= 1_X \circ f_1 + 0 \\ &= f_1. \end{aligned}$$

Similarly, $p_2 \circ h = f_2$. Suppose there is another morphism $g : C \to X\square Y$ such that $p_1 \circ g = f_1$ and $p_2 \circ g = f_2$. Then

$$\begin{aligned} g &= 1_{X\square Y} \circ g \\ &= (i_1 \circ p_1 + i_2 \circ p_2) \circ g \\ &= i_1 \circ p_1 \circ g + i_2 \circ p_2 \circ g \\ &= i_1 \circ f_1 + i_2 \circ f_2 \\ &= h \end{aligned}$$

This shows that diagram (1.1.3) is indeed a product diagram. Conversely, suppose that (1.1.3) is a product diagram, with $p_k \circ i_j = \delta_{jk}$. We show that the diagram (1.1.1) with $i_1 = (1_X, 0)$ and $i_2 = (0, 1_Y)$ is a biproduct diagram. Since

$$p_1 \circ (i_1 \circ p_1 + i_2 \circ p_2) = p_1 \circ i_1 \circ p_1 + p_1 \circ i_2 \circ p_2 = p_1$$

and

$$p_2 \circ (i_1 \circ p_1 + i_2 \circ p_2) = p_2 \circ i_1 \circ p_1 + p_2 \circ i_2 \circ p_2 = p_2,$$

the universal property of the product gives $i_1 \circ p_1 + i_2 \circ p_2 = 1_{X \Box Y}$, which proves that $(i_1, i_2, X \Box Y, p_1, p_2)$ is a biproduct of X and Y. $\Box$

In what follows, the brackets $\langle , \rangle$ will be used to denote the canonical morphism going into a product, while the brackets $[,]$ will be used to denote the canonical morphism going out from a coproduct.

Lemma 1.1.1 *Let $(i_1, i_2, X_1 \Box X_2, p_1, p_2)$ and $(l_1, l_2, Y_1 \Box Y_2, \pi_1, \pi_2)$ be biproducts in an* **Ab***-category $\mathbb{C}$, and let $f : X_1 \to Y_1, h : X_1 \to Y_2, g : X_2 \to Y_1$ and $k : X_2 \to Y_2$ be morphisms in $\mathbb{C}$. Then the morphisms*

$$x = [\langle f, h \rangle, \langle g, k \rangle] : X_1 \Box X_2 \to Y_1 \Box Y_2$$

and

$$y = \langle [f, g], [h, k] \rangle : X_1 \Box X_2 \to Y_1 \Box Y_2$$

are the same.

Proof. Since $X_1 \Box X_2$ is a coproduct of X_1 and X_2, and $Y_1 \Box Y_2$ is a product of Y_1 and Y_2, we can form the following product/coproduct diagram:

$$\begin{tikzcd}
Y_1 & Y_1 \times Y_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y_2 \\
X_1 \arrow[r, "i_1"'] \arrow[dr, "\langle f,h \rangle"'] & X_1 \sqcup X_2 \arrow[ul, "{[f,g]}"] \arrow[ur, "{[h,k]}"'] \arrow[u, "y"'] \arrow[d, "x"] & X_2 \arrow[l, "i_2"] \arrow[dl, "\langle g,k \rangle"] \\
& Y_1 \times Y_2 &
\end{tikzcd}$$

Observe that $x \circ i_1, y \circ i_1$ and $\langle f, h \rangle$ are all morphisms from the object X_1 into the product $Y_1 \times Y_2$; similarly, $x \circ i_2, y \circ i_2$ and $\langle g, k \rangle$ are all morphisms from the object X_2 into the product $Y_1 \times Y_2$. Composing with the product projections π_1, π_2 shows that $\pi_1 \circ (x \circ i_1) = f$ and $\pi_1 \circ (y \circ i_1) = f$; $\pi_2 \circ (x \circ i_1) = h$ and $\pi_2 \circ (y \circ i_1) = h$. By the universal property of the product, this means that $x \circ i_1 = y \circ i_1$. In the same way, $x \circ i_2 = y \circ i_2$. Therefore, by the universal property of the coproduct, we conclude that $x = y$, as desired. $\Box$

Definition 1.1.3 *A category $\mathbb{C}$ is said to be additive if it is an* **Ab***-category with finite products.*

1.2 Kernels and cokernels

Definition 1.2.1 *In a category $\mathbb{C}$, an equalizer of a parallel pair $f, g : X \to Y$ of morphisms is a morphism $e : E \to X$ such that $fe = ge$, and for any other morphism $p : P \to X$ with $fp = gp$, there exists a unique morphism $u : P \to E$ such that $p = eu$.*

The dual notion of an equalizer is the notion of a *coequalizer*.

If $\mathbb{C} =$ **Set**, **Grp**,R-**Mod**, **Top** and if $f, g : X \to Y$ are morphisms in $\mathbb{C}$, then if E denotes the set $\{x \in X \mid f(x) = g(x)\}$ considered as a subset (resp. subgroup, submodule, subspace) of X and if $e : E \to X$ is the inclusion map, then (E, e) is an equalizer of f and g.

Proposition 1.2.1 *If (E, e) is an equalizer of $f, g : X \to Y$, then (E, e) is a subobject of X. Moreover, any two equalizers of $f, g : X \to Y$ are isomorphic subobjects of X.*

Proof. Let $e : E \to X$ be an equalizer of $f, g : X \to Y$. Because of the universal mapping property defining an equalizer, every equalizer is a monomorphism; thus, (E, e) is a subobject of X. For the same reason, any two equalizers of a parallel pair $f, g : X \to Y$ of morphisms are isomorphic. □

Definition 1.2.2 *A morphism $e : E \to X$ in a category $\mathbb{C}$ is called a regular monomorphism if it is an equalizer of some parallel pair $f, g : X \to Y$ of morphisms in $\mathbb{C}$. In this case, (E, e) is called a regular subobject of X.*

The corresponding dual notions are *regular epimorphism* and *regular quotient object*.

Proposition 1.2.2 *For a morphism $f : X \to Y$ in a category $\mathbb{C}$, the following are equivalent:*

1. *f is an isomorphism;*
2. *f is a regular monomorphism and an epimorphism;*
3. *f is a regular epimorphism and a monomorphism.*

Proof. It suffices to prove the equivalence of (1) and (2), since the notion of an isomorphism is self-dual, and (2) and (3) are dual to each other. Suppose first that $f : X \to Y$ is an isomorphism. Then, clearly, it is an epimorphism. We now show that f is also a regular monomorphism. Indeed, f is an equalizer of any pair of identical morphisms (with domain Y), say $1_Y, 1_Y : Y \to Y$, since trivially $1_Y f = 1_Y f$, and for any other morphism $h : H \to Y$ (trivially) satisfying $1_Y h = 1_Y h$, the unique morphism $u : H \to X$ is given by $u = f^{-1}h$. Conversely, suppose that f is both a regular monomorphism and an epimorphism. If f equalizes some parallel pair $p, q : Y \to Z$ of morphisms, then $pf = qf$. Since f is an epimorphism, this gives $p = q$, whence f is an isomorphism. $\square$

Definition 1.2.3 *Let $\mathbb{C}$ be a pointed category.*

1. *A* kernel *of a morphism $f : X \to Y$ in $\mathbb{C}$ is the equalizer $k : K \to X$ of the pair $f, 0 : X \to Y$.*
2. *The category $\mathbb{C}$ is said to have kernels provided that a kernel exists for each morphism f in $\mathbb{C}$.*
3. *A morphism $k : K \to X$ is said to be a normal monomorphism if it is a kernel of some morphism $f : X \to Y$.*

The corresponding dual notions are *cokernel* and *normal epimorphism.*

Proposition 1.2.3 *In an* **Ab***-category $\mathbb{C}$, an equalizer of a parallel pair $f, g : X \to Y$ of morphisms is the same as the kernel of the morphism $f - g : X \to Y$. Thus, in an* **Ab***-category, regular monomorphisms are the same as normal monomorphisms.*

Proof. This follows from the fact that in an **Ab**-category we have: $(f - g)e = 0$ if and only if $fe = ge$. $\square$

Proposition 1.2.4 *Let $\mathbb{C}$ be a pointed category.*

1. *If $k : K \to X$ is a kernel of some morphism and the cokernel of k exists, then k will be a kernel of its cokernel.*
2. *If $c : Y \to Z$ is a cokernel of some morphism and the kernel of c exists, then c will be a cokernel of its kernel.*
3. *Every monomorphism in $\mathbb{C}$ has a kernel, and every epimorphism in $\mathbb{C}$ has a cokernel.*

Proof. We prove (1) and the first part of (3) ((2) and the second part of (3) follow dually).

(1) Suppose that $k : K \to X$ is a kernel of $f : X \to Y$, and let $c : X \to C$ be a cokernel of k. Since $ck = 0$ and $fk = 0$, the universal property of the cokernel gives a unique morphism $u : C \to Y$ such that $f = uc$:

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "f"] \arrow[dr, "c"'] & Y \\ L \arrow[u, "v"] \arrow[ur, "l"'] & & C \arrow[u, "u"'] \end{tikzcd}$$

We already had $ck = 0$. To show that k is indeed a kernel of c, suppose that there is another morphism $l : L \to X$ such that $cl = 0$. Then $fl = (uc)l =$

$u(cl) = 0$, and so by the universal property of the kernel, there is a unique morphism $v : L \to K$ such that $l = kv$.

(3) If $f : X \to Y$ is a monomorphism, then its kernel is the (zero) morphism from the zero object to X. □

We trivially obtain the following

Corollary 1.2.1 *Suppose that a pointed category $\mathbb{C}$ has kernels and cokernels. Then*

1. *A morphism f is a normal monomorphism if and only if f is the kernel of its cokernel.*

2. *A morphism f is a normal epimorphism if and only if f is the cokernel of its kernel.*

We write $\ker(f)$ for the kernel of a morphism f, and $\mathrm{coker}(f)$ for the cokernel of f, whenever they exist.

Lemma 1.2.1 *Let $\mathbb{C}$ be a pointed category that has kernels and cokernels. Then*

1. *Every morphism $f : X \to Y$ in $\mathbb{C}$ has a canonical factorization $f = mq$, with $m = \ker(\mathrm{coker}(f))$;*

2. *if also $f = m'q'$, where m' is a normal monomorphism, then there exists a unique morphism t such that $m = m't$ and $q' = tq$;*

3. *if $\mathbb{C}$ has equalizers and every monomorphism in $\mathbb{C}$ is normal, then q is an epimorphism.*

Proof. (1) Let $f : X \to Y$ be any morphism in $\mathbb{C}$. Consider the following set-up for kernels:

$$\begin{tikzcd}
& Q \arrow[d, "m=\ker(\mathrm{coker}(f))"] & \\
X \arrow[ur, "q"] \arrow[r, "f"'] & Y \arrow[r, "\mathrm{coker}(f)"'] & C
\end{tikzcd}$$

Since $\mathrm{coker}(f) \circ \ker(\mathrm{coker}(f)) = 0$ and $\mathrm{coker}(f) \circ f = 0$, the universal property of the kernel gives a unique morphism $q : X \to Q$ such that $f = mq$.

(2) Suppose now that we also have $f = m'q'$, where m' is a kernel (i.e. a normal monomorphism). Then $m' = \ker(s')$ where $s' = \mathrm{coker}(m')$. Also, set $s = \mathrm{coker}(m)$ (noting that $s = \mathrm{coker}(m) = \mathrm{coker}(f)$) and consider the following set-up for kernels and cokernels:

$$\begin{tikzcd}
X \arrow[r, "q"] \arrow[d, "q'"'] & Q \arrow[d, "m"] & \\
Q' \arrow[r, "m'"'] & Y \arrow[r, "s'"'] \arrow[d, "s"'] & Z' \\
& Z \arrow[ur, "v"'] &
\end{tikzcd}$$

Since $s'm' = 0$, we have that $s'f = s'm'q' = 0$, and by the cokernel property, $s' = vs$ for some unique morphism v. Furthermore, $s'm = vsm = 0$, so m factors (uniquely) through m'; put $m = m't$. Since m' is a monomorphism and $m'q' = mq = m'tq$, we get $q' = tq$, as desired.

(3) Suppose that the given conditions are satisfied. To prove that q is an epimorphism, let $a, b : Q \to W$ be morphisms in $\mathbb{C}$ such that $aq = bq$. If $e : E \to Q$ is an equalizer of the pair a, b, then, as shown below

$$\begin{tikzcd}
X \arrow[r, "q"] \arrow[d, "x"'] & Q \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & W \\
E, \arrow[ur, "e"'] & &
\end{tikzcd}$$

q factors through e as $q = ex$ for some unique morphism x, and so $f = mq = m(ex) = (me)x$. Since e is an equalizer, it is a monomorphism; therefore, me is a monomorphism and by assumption, it is a normal monomorphism. From (2) above, there is a morphism t such that $m = met$ (and $x = tq$), and this gives $1_Q = et$; the latter meaning that e has a right inverse, and so e must be an isomorphism. Since $ae = be$, we get $a = b$, which proves that q is an epimorphism. $\square$

1.3 Diagram chasing and the five lemma in an abelian category

Definition 1.3.1 *An abelian category is an additive category $\mathbb{C}$ satisfying the following conditions:*

1. *Every morphism in $\mathbb{C}$ has a kernel and a cokernel.*
2. *Every monomorphism is a kernel and every epimorphism is a cokernel.*

For any ring R, the category R-**Mod** of R-modules is an abelian category. Note that if $\mathbb{C}$ is an abelian category, then so is its dual category $\mathbb{C}^{\mathrm{op}}$. For an abelian category we have the following extended version of Proposition 1.1.1:

Proposition 1.3.1 *In an abelian category $\mathbb{C}$, a morphism $f : X \to Y$ is*

1. *a monomorphism if and only if* $\ker(f) = 0$*;*

2. *an epimorphism if and only if* $\operatorname{coker}(f) = 0$*;*

3. *an isomorphism if and only if* $\ker(f) = 0$ *and* $\operatorname{coker}(f) = 0$.

Proposition 1.3.2 *In an abelian category* $\mathbb{C}$*, every morphism* f *has a canonical factorization* $f = me$*, where* $m = \ker(\operatorname{coker}(f))$ *and* $e = \operatorname{coker}(\ker(f))$*; moreover, these factorizations are functorial.*

Proof. The factorization $f = me$ is obtained as in Lemma 1.2.1. Since any abelian category has equalizers and every monomorphism is normal, Lemma 1.2.1(3) shows that e is an epimorphism. Then, it is easy to see that $\ker(e) = \ker(f)$, and since e is a normal epimorphism, we have $e = \operatorname{coker}(\ker(e)) = \operatorname{coker}(\ker(f))$. Thus, we obtain the desired factorization.

To show that these factorizations are functorial, consider an (epimorphism, monomorphism)-factorization $f' = m'e'$ of another morphism $f' : X' \to Y'$ and a commutative rectangle of solid arrows

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "g"'] & E \arrow[r, "m"] \arrow[d, dashed] & Y \arrow[d, "h"'] \\ X' \arrow[r, "e'"'] & E' \arrow[r, "m'"'] & Y'; \end{tikzcd}$$

we show that there is a unique morphism $u : E \to E'$ such that the new diagram (the two squares so formed) commutes. Let $k = \ker(f) = \ker(e)$. Then $m'e'gk = hmek = 0$, and so $e'gk = 0$. Since $e = \operatorname{coker}(k)$ (because $k = \ker(e)$), there is a unique morphism $u : E \to E'$ such that $e'g = ue$, by the universal property of the cokernel. Moreover, $m'ue = m'e'g = hme$, whence $m'u = hm$ because e is an epimorphism. Thus, the two new squares commute and we have the desired functoriality of the factorizations. $\square$

Given a morphism $f : X \to Y$ in an abelian category $\mathbb{C}$, we set

$$m = \operatorname{im}(f), \ \ e = \operatorname{coim}(f),$$

where $f = me$ is the canonical factorization of f. Note that m is a subobject of Y and e is a quotient object of X.

Definition 1.3.2 *In an abelian category* $\mathbb{C}$*, a diagram*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{1.3.1}$$

is said to be exact (at Y*) provided* $\operatorname{im}(f) \approx \ker(g)$ *(that is,* $\operatorname{im}(f)$ *and* $\ker(g)$ *are isomorphic as subobjects of* Y*).*

Proposition 1.3.3 *The following are equivalent for a diagram (1.3.1) in an abelian category* $\mathbb{C}$*:*

1. *$\langle f, g\rangle$ is an exact sequence;*
2. $\operatorname{coker}(f) \approx \operatorname{coim}(g)$*;*
3. $gf = 0$ *and* $\operatorname{coker}(f) \circ \ker(g) = 0$.

Proof. (1⇒2) Suppose that $\langle f, g\rangle$ is an exact sequence, then by definition, $\ker(\operatorname{coker}(f)) = \operatorname{im}(f) \approx \ker(g)$; therefore,

$$\operatorname{coker}(f) \approx \operatorname{coker}(\ker(\operatorname{coker}(f))) \approx \operatorname{coker}(\ker(g)) = \operatorname{coim}(g).$$

(2⇒3) Using the canonical factorization of g, we have

$$gf = (\operatorname{im}(g) \circ \operatorname{coim}(g)) \circ f = \operatorname{im}(g) \circ (\operatorname{coim}(g) \circ f)$$

$$= \operatorname{im}(g) \circ (\operatorname{coker}(f) \circ f) = \operatorname{im}(g) \circ 0 = 0$$

and

$$\operatorname{coker}(f) \circ \ker(g) = \operatorname{coim}(g) \circ \ker(g) = \operatorname{coker}(\ker(g)) \circ \ker(g) = 0.$$

(3⇒1) Using the canonical factorization of f and the fact that $gf = 0$, we have

$$0 = gf = g \circ (\operatorname{im}(f) \circ \operatorname{coim}(f)) = (g \circ \operatorname{im}(f)) \circ \operatorname{coim}(f),$$

whence $g \circ \operatorname{im}(f) = 0$ because $\operatorname{coim}(f)$ is an epimorphism. Thus, by the kernel property, there is a morphism h such that $\operatorname{im}(f) = \ker(g) \circ h$, meaning that we have inclusion of subobjects $\operatorname{im}(f) \preceq \ker(g)$. Also, since $\operatorname{coker}(f) \circ \ker(g) = 0$, there is a morphism k such that $\ker(g) = \ker(\operatorname{coker}(f)) \circ k = \operatorname{im}(f) \circ k$; that is, $\ker(g) \preceq \operatorname{im}(f)$. Thus, $\ker(g) \approx \operatorname{im}(f)$, and so $\langle f, g\rangle$ is an exact sequence. □

If we adjoin a zero morphism to the *left*, *right*, or *both* sides of an exact sequence (1.3.1) as follows

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0,$$

we obtain a *short left exact* sequence, a *short right exact* sequence, or (simply) a *short exact* sequence, respectively, if the new sequence is exact at X and Y, or exact at Y and Z, or exact at X, Y and Z, respectively. If this sequence is short exact, then by Proposition 1.3.1, $0 = \ker(f)$ implies that f is a monomorphism; also, $\operatorname{coker}(g) = 0$ and so g is an epimorphism. Since f is a normal monomorphism, $f \approx \ker(\operatorname{coker}(f)) = \operatorname{im}(f)$, whence $f = \ker(g)$, by exactness at Y; similarly, $g = \operatorname{coker}(f)$. Conversely, if $f = \ker(g)$ and $g = \operatorname{coker}(f)$, then the above sequence will be short exact.

Proposition 1.3.4 *Consider a pullback square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "a"'] & Y' \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in an abelian category $\mathbb{C}$.

1. *If* f *is an epimorphism, then so if* f';
2. *If* $k = \ker(f)$, *then* (k *factors as*) $k = ak'$ *for a* k' *which turns out to be the kernel of* f'.

Proof. (1) Recall that the pullback X' can be constructed canonically from products and equalizers. Take $X \square Y'$ with projections p_1, p_2, and take the equalizer e of the parallel pair fp_1, bp_2 of morphisms in $\mathbb{C}$, where, as usual, $e = \ker(fp_1 - bp_2)$. Together with $a = p_1 e$ and $f' = p_2 e$, we have constructed the pullback. Observe that $fp_1 - bp_2$ is an epimorphism. Indeed, if $h(fp_1 - bp_2) = 0$, then

$$0 = 0i_1 = h(fp_1 - bp_2)i_1 = hfp_1 i_1 = hf,$$

using the usual identities satisfied by the injections i_1, i_2 and the projections p_1, p_2 of the biproduct. Since f is an epimorphism, we obtain $h = 0$, so that $fp_1 - bp_2$ is also an epimorphism as claimed; moreover, $fp_1 - bp_2 = \operatorname{coker}(e)$, because every epimorphism in $\mathbb{C}$ is normal. To show that f' is an epimorphism, suppose that $sf' = 0$ for some s. This means that $s(p_2 e) = (sp_2)e = 0$, and so sp_2 factors uniquely through $fp_1 - bp_2$ as $sp_2 = t(fp_1 - bp_2)$, by the universal property of the cokernel. Using the identity $p_2 i_1 = 0$ and composing both sides with s, we have that

$$0 = s0 = s(p_2 i_1) = (sp_2)i_1 = t(fp_1 - bp_2)i_1 = tfp_1 i_1 = sf,$$

whence $s = 0$ because f is an epimorphism. This proves that f' is an epimorphism.

(2) Suppose that $k = \ker(f)$ and consider the morphisms $k : K \to X$ and $0 : K \to Y'$ which satisfy $fk = 0 = b0$. By the pullback property, there is a unique morphism $k' : K \to X'$ for which $ak' = k$ and $f'k' = 0$. To see that k' is indeed the kernel of f', suppose that we also have $f't = 0$ for some morphism $t : T \to X'$. Then $fat = bf't = 0$, and so at factors through k as $at = ku$, since $k = \ker(f)$. This gives $at = ku = (ak')u = a(k'u)$; since we also have $f't = 0 = f'k'u$ (and hence $b0 = 0 = f(at) = f(k'u)$), the uniqueness required in the pullback forces $t = k'u$. This proves that $k' = \ker(f')$, as desired. □

As stated in [Mac98], the above proposition is useful in making diagram chases in any abelian category $\mathbb{C}$, using 'members' (in $\mathbb{C}$) instead of elements as in familiar categories like **Ab** or R-**Mod**. For each object X in $\mathbb{C}$, define a *member* x of X to be a morphism with codomain X. When x is a member of X, we write $x \in_m X$ as in [Mac98]. If $x, y \in_m X$, define $x \equiv y$ to mean that there are suitable epimorphisms u, v such that $xu = yv$. This relation is clearly reflexive and symmetric; to show that it is transitive, one uses the above proposition. Thus, since $\equiv$ is an equivalence relation, one takes a member of X to be a $\equiv$-equivalence class of morphisms with codomain X. The proof of the following theorem, which can be found in [Mac98], will be omitted here:

Theorem 1.3.1 *(Elementary rules for chasing diagrams). For the members in any abelian category* $\mathbb{C}$

1. *$f : X \to Y$ is a monomorphism if and only if for all $a \in_m X$, $fa \equiv 0$ implies $a \equiv 0$;*
2. *$f : X \to Y$ is a monomorphism if and only if for all $a, a' \in_m X$, $fa \equiv fa'$ implies $a \equiv a'$;*
3. *$g : Y \to Z$ is an epimorphism if and only if for each $d \in_m Z$ there exists some $c \in_m Y$ with $gc \equiv d$;*
4. *$h : Z \to W$ is a zero morphism if and only if for all $e \in_m Z$, $he \equiv 0$;*
5. *A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact at Y if and only if $gf = 0$ and to every $b \in_m Y$ with $gb \equiv 0$ there exists $a \in_m X$ such that $fa \equiv b$;*
6. *(Subtraction) Given $g : Y \to Z$ and $a, b \in_m Y$ with $ga \equiv gb$, there is a member $c \in_m Y$ such that $gc \equiv 0$; moreover, any $h : Y \to H$ with $ha \equiv 0$ has $hb \equiv hc$ and any $k : Y \to K$ with $kb \equiv 0$ has $ka \equiv -kc$.*

We now use the above elementary rules in proving the following basic diagram lemma of homological algebra.

Theorem 1.3.2 *(The five lemma). In an abelian category* $\mathbb{C}$*, consider a commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "a_1"] \arrow[d, "a"'] & B_1 \arrow[r, "b_1"] \arrow[d, "b"'] & C_1 \arrow[r, "c_1"] \arrow[d, "c"'] & D_1 \arrow[r, "d_1"] \arrow[d, "d"'] & E_1 \arrow[d, "e"'] \\
A_2 \arrow[r, "a_2"'] & B_2 \arrow[r, "b_2"'] & C_2 \arrow[r, "c_2"'] & D_2 \arrow[r, "d_2"'] & E_2
\end{tikzcd} \tag{1.3.2}$$

with exact rows. If a, b, d, e are isomorphisms, then c is an isomorphism.

Proof. In an abelian category, a morphism is an isomorphism if and only if it is both a monomorphism and an epimorphism. To show that c is an isomorphism, it suffices, by duality, to show that c is an epimorphism. To this end, consider a member $z \in_m C_2$. Since $c_2 z \in_m D_2$ and d is an epimorphism, there exists a member $y \in_m D_1$ such that $dy \equiv c_2 z$. Then $0 = d_2 c_2 z \equiv d_2 dy = ed_1 y$, and $d_1 y \equiv 0$ because e is a monomorphism. By exactness at D_1, there is a member $x \in_m C_1$ for which $c_1 x \equiv y$, and so $c_2 cx = dc_1 x \equiv dy \equiv c_2 z$. By the subtraction rule, there is a member $w \in_m C_2$ satisfying $c_2 w \equiv 0$; moreover w is given explicitly as $w = zq - cxp$ for some suitable epimorphisms p, q. By exactness at C_2, there is a member $v \in_m B_2$ such that $b_2 v \equiv w$; and, since b is an epimorphism, there is a member $u \in_m B_1$ with $bu \equiv v$. Then $cb_1 u = b_2 bu \equiv b_2 v \equiv w$. Since $cb_1 u \equiv w$ means that there are suitable epimorphisms r, s for which $(cb_1 u)r = ws$, we get that $c(b_1 ur) = ws = (zq - cxp)s = z(qs) - c(xps)$, whence $c(b_1 ur + xps) = z(qs)$. Since qs is an epimorphism, we have that $c(b_1 ur + xps) \equiv z$. This proves that c is an epimorphism. $\square$

As a corollary of the above theorem, we get:

Proposition 1.3.5 *(The short five lemma). In any abelian category $\mathbb{C}$, consider a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & B_1 \arrow[r, "b_1"] \arrow[d, "b"'] & C_1 \arrow[r, "c_1"] \arrow[d, "c"'] & D_1 \arrow[r] \arrow[d, "d"'] & 0 \\ 0 \arrow[r] & B_2 \arrow[r, "b_2"'] & C_2 \arrow[r, "c_2"'] & D_2 \arrow[r] & 0 \end{tikzcd} \tag{1.3.3}$$

with short exact rows. If b, d are isomorphisms, then c is an isomorphism.

Notice that in the proof of the five lemma above, we use more than just the elementary rules of diagram chasing presented in Theorem 1.3.1. In fact, we seem to use quite strongly the additive structure of a category. In [Mac98] this is avoided by giving a dual argument, for which the elementary rules are enough. In the general context where we are going to obtain the five lemma, we no longer have duality and so we will have to rely on the proof of Theorem 1.3.1 given above, but at the same time extend it beyond the additive context. As mentioned in the Introduction, two such separate extensions are known, and ours, obtained in Chapter 3, is more general and unifies them.

Chapter 2

Towards the general context

2.1 Regular categories

In a category $\mathbb{C}$, the *kernel pair* of a morphism $f : X \to Y$ is a pair (p_1, p_2) of morphisms arising in the pullback

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

of f along itself.

Proposition 2.1.1 *The following are equivalent for a morphism $f : X \to Y$ in a category $\mathbb{C}$:*

1. *f is a monomorphism.*
2. *$(1_X, 1_X)$ is a kernel pair of f.*
3. *The kernel pair (p_1, p_2) of f exists and is such that $p_1 = p_2$.*

In some sense, we have the following analogue of Proposition 1.2.4:

Proposition 2.1.2 *In a category $\mathbb{C}$,*

1. *if a coequalizer has a kernel pair, it is the coequalizer of its kernel pair;*
2. *if a kernel pair has a coequalizer, it is the kernel pair of its coequalizer.*

Proof. (1) Let $h, k : X \to Y$ and $c : Y \to Z$ be morphisms in $\mathbb{C}$; suppose that c is the coequalizer of the pair h, k as displayed below:

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & Y \arrow[r, "c"] & Z \end{tikzcd}$$

If $u, v : W \to Y$ is the kernel pair of c, we will show that c is indeed the coequalizer of u, v. First, $cu = cv$. Since we also have $ch = ck$, and the diagram

$$\begin{tikzcd} W \arrow[r, "v"] \arrow[d, "u"'] & Y \arrow[d, "c"] \\ Y \arrow[r, "c"'] & Z \end{tikzcd} \tag{2.1.1}$$

is a pullback, we get a unique morphism $s : X \to W$ such that $h = us$ and $k = vs$. If some other morphism $c' : Y \to Z'$ also satisfies $c'u = c'v$, then $c'h = c'(us) = (c'u)s = (c'v)s = c'(vs) = c'k$. Thus, since c is the coequalizer of h, k, there is a unique morphism $q : Z \to Z'$ such that $c' = qc$, proving that c is the coequalizer of u, v.

(2) Let $u, v : W \to Y$ be the kernel pair of $c' : Y \to Z'$, and suppose that $c : Y \to Z$ is the coequalizer of u, v; we show that u, v is again the kernel pair of c. First, $cu = cv$. Since we also have that $c'u = c'v$, the universal property of the coequalizer gives a unique morphism $q : Z \to Z'$ such that $c' = qc$. Suppose now that there are morphisms $a, b : V \to Y$ for which $ca = cb$, then $c'a = (qc)a = q(ca) = q(cb) = (qc)b = c'b$; thus, there is a unique morphism $s : V \to W$ such that $a = us$ and $b = vs$, because u, v is the kernel pair of c'. Therefore, the diagram (2.1.1) is a pullback, and so u, v is the kernel pair of c.
□

Definition 2.1.1 *A category $\mathbb{C}$ is said to be regular if it satisfies the following conditions:*

1. *Every morphism in $\mathbb{C}$ has a kernel pair.*
2. *Every kernel pair has a coequalizer.*
3. *The pullback of a regular epimorphism along any morphism exists and is again a regular epimorphism.*

In view of Propositions 1.2.3 and 1.3.4, any abelian category is a regular category. Below we recall some basic properties of regular categories. These properties can be used to conveniently extend the notion of membership from abelian to regular categories. However, instead of doing this, we will follow a slightly different approach where instead of working with members as equivalence classes of morphisms we work directly with morphisms and we replace the equivalence relation with a preorder relation.

Lemma 2.1.1 *In a regular category $\mathbb{C}$, let $f : X \to Y$ be a regular epimorphism and let $g : Y \to Z$ be an arbitrary morphism. Then the 'factorization'*

$$f \times_Z f : X \times_Z X \to Y \times_Z Y$$

exists and is an epimorphism.

Proof. First take the kernel pair of g, which (exists and) is the pullback in the lower right corner of the diagram below:

$$\begin{tikzcd}
X \times_Z X \arrow[r, "j"] \arrow[d, "i"'] & Y \times_Z X \arrow[r, "h"] \arrow[d, "e"'] & X \arrow[d, "f"] \\
X \times_Z Y \arrow[r, "d"] \arrow[d, "c"'] & Y \times_Z Y \arrow[r, "b"] \arrow[d, "a"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Since f is a regular epimorphism, its pullback along a exists and is again a regular epimorphism, that is, d is a regular epimorphism. Similarly, in the other pullbacks involved in the above diagram, e, i, j are regular epimorphisms. Therefore, $f \times_Z f = di = ej$ is an epimorphism, being a composite of two (regular) epimorphisms. □

Proposition 2.1.3 *In a regular category $\mathbb{C}$, every morphism factors as a regular epimorphism followed by a monomorphism and this factorization is unique up to isomorphism.*

Proof. Suppose that $f : X \to Y$ is a morphism in a regular category $\mathbb{C}$. Take the kernel pair (u, v) of f and the coequalizer e of this kernel pair. Since $fu = fv$, there is, as shown below

$$\begin{tikzcd}
P \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, "q"'] & X \arrow[r, "f"] \arrow[d, "e"'] & Y \\
R \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & I, \arrow[ur, "m"'] &
\end{tikzcd}$$

a unique morphism m such that $f = me$, by the universal property of the coequalizer. By our choice, e is a regular epimorphism; it then remains to show that m is a monomorphism. Let (h, k) be the kernel pair of m. Since $eu = ev$, we have that $m(eu) = m(ev)$, giving a unique morphism $q : P \to R$ such that $eu = hq$ and $ev = kq$, by the pullback property (of the kernel pair (h, k) of m). As

$$P = X \times_Y X, \;\; R = I \times_Y I, \;\; q = e \times_Y e,$$

application of Lemma 2.1.1 to the regular epimorphism e and the morphism m shows that q is an epimorphism. Thus, $hq = eu = ev = kq$ implies that $h = k$. By Proposition 2.1.1 we conclude that m is a monomorphism. This shows that every morphism f in a regular category is factorizable in the form $f = me$, where m is a monomorphism and e is a regular epimorphism.

To prove that this factorization is unique, suppose that there is another factorization $f = m'e'$ with m' a monomorphism and e' a regular epimorphism. Consider the diagram

$$\begin{tikzcd} W \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & X \arrow[r, "e"] \arrow[d, equal] & I \arrow[r, "m"] & Y \arrow[d, equal] \\ W' \arrow[r, shift left, "h'"] \arrow[r, shift right, "k'"'] & X \arrow[r, "e'"] & I' \arrow[r, "m'"] & Y \end{tikzcd}$$

where e is the coequalizer of $h, k : W \to X$ and e' is the coequalizer of $h', k' : W' \to X$. Since $m'e'h = fh = fk = m'e'k$ and m' is a monomorphism, we have that $e'h = e'k$. Thus, there is a unique morphism $s : I \to I'$ such that $e' = se$, by the universal property of the coequalizer e. Moreover, $m'se = m'e' = f = me$ and so $m's = m$. As $meh' = m'e'h' = m'e'k' = mek'$, we have that $eh' = ek'$, whence there is a unique morphism $t : I' \to I$ such that $e = te'$, again by the universal property of the coequalizer e'. Therefore, $1_I e = e = te' = t(se) = (ts)e$ implies that $ts = 1_I$; similarly, $st = 1_{I'}$. This proves that s is an isomorphism and so we have the required uniqueness of the factorization. □

One of the consequences of Proposition 2.1.3 is that in a regular category, the class of regular epimorphisms is closed under composition.

In the canonical factorization $f = me$ of any morphism $f : X \to Y$ in a regular category $\mathbb{C}$, the monomorphism part m is usually called the *(regular) image* of f, denoted by $\mathrm{Im}(f)$. If $g : Z \to Y$ is another morphism, let us set $f \leqq g$ if (and only if) $\mathrm{Im}(f)$ factors through $\mathrm{Im}(g)$.

Proposition 2.1.4 *If $f : X \to Y$ and $g : Z \to Y$ are two morphisms in a regular category $\mathbb{C}$, then the following are equivalent:*

1. *$f \leqq g$.*
2. *There exists a commutative square*

$$\begin{tikzcd} W \arrow[r, "z"] \arrow[d, "x"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where x is a regular epimorphism.

Proof. (1⇒2) If $f \leqq g$, then $\mathrm{Im}(f)$ factors through $\mathrm{Im}(g)$. Write $m_f = m_g s$, where $f = m_f e_f$ and $g = m_g e_g$ are the canonical factorizations of f and g, respectively. The required square is obtained by 'pasting' the squares

$$\begin{tikzcd}
W \arrow[r, "e'_f"] \arrow[d, "e''_g"'] & V \arrow[r, "s'"] \arrow[d, "e'_g"] & Z \arrow[d, "e_g"] \\
X \arrow[r, "e_f"'] \arrow[d, "1_X"'] & I_f \arrow[r, "s"] \arrow[d, "m_f"] & I_g \arrow[d, "m_g"] \\
X \arrow[r, "f"'] & Y \arrow[r, "1_Y"'] & Y
\end{tikzcd}$$

where the two upper squares are pullbacks. Since they are pullbacks, e'_g and e''_g are regular epimorphisms, because e_g is; similarly, e'_f is a regular epimorphism, because e_f is. Accordingly, we take $x = 1_X e''_g = e''_g$ and $z = s'e'_f$, to get the form of the required square.

(2⇒1) Suppose that we have such a commutative square with x being a regular epimorphism. Since $fx = (m_f e_f)x = m_f(e_f x)$ and $e_f x$ is a regular epimorphism (being a composite of regular epimorphisms), it follows that $\mathrm{Im}(f) = \mathrm{Im}(fx) = \mathrm{Im}(gz)$; therefore, $\mathrm{Im}(f)$ factors through $\mathrm{Im}(g)$, proving that $f \leqq g$. □

Corollary 2.1.1 *Let $f : X \to Y$ and $g : Z \to Y$ be two morphisms in a regular category $\mathbb{C}$ such that $g \leqq f$. Then there exists a morphism $e : W \to X$ such that $fe \leqq g$ and $g \leqq fe$.*

If we now replace $\leqq$ with the 'factors through' relation $\prec$, then the above proposition has the following analogue.

Proposition 2.1.5 *Let $f : X \to Y$ and $g : Z \to Y$ be morphisms in any category $\mathbb{C}$. Then the following are equivalent:*

1. *$f \prec g$.*
2. *There exists a commutative square*

$$\begin{tikzcd}
W \arrow[r, "z"] \arrow[d, "x"'] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

where x is a split epimorphism.

Proof. If f factors through g, set $f = gs$ for some $s : X \to Z$. We obtain (trivially) a commutative square with $x = 1_X$ as the required split epimorphism. Conversely, if we have such a commutative square with x being a split epimorphism, then f factors through g because $fx = gz$ yields $f = g(zx')$, where $x' : X \to W$ is the right inverse of x. □

Definition 2.1.2 *A cover relation on a category $\mathbb{C}$ is a binary relation $\sqsubset$ on the class of morphisms of $\mathbb{C}$, which is defined only for those pairs of morphisms that have the same codomain, and which has the following two properties:*

1. *Left preservation: for any two morphisms f and g having the same codomain, $f \sqsubset g$ implies $hf \sqsubset hg$, for any morphism h composable with f and g;*
2. *Right preservation: for any two morphisms f and g having the same codomain, $f \sqsubset g$ implies $fe \sqsubset g$, for any e composable with f.*
3. *Reflexivity: $f \sqsubset f$ for every morphism f.*
4. *Transitivity: if $f \sqsubset g$ and $g \sqsubset h$ then $f \sqsubset h$, for every three morphisms f, g, h.*

Remark 2.1.1 *The properties of reflexivity and transitivity are not required in the definition of a cover relation given in [Jan09]. We have included these properties in the definition since in this Thesis we always work with reflexive and transitive cover relations.*

The relation $\leqq$ in a regular category is an example of a cover relation. Recall that in concrete regular categories like **Ab**, **Grp**, or **Set**$_\star$, we have $f \leqq g$ if and only if $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$.

Another example of a cover relation is the usual 'factors through' relation $\prec$ which we also encountered earlier. In fact, it is the 'least' cover relation, that is, for any other cover relation $\sqsubset$, we always have:

$$f \prec g \Rightarrow f \sqsubset g.$$

Many properties of morphisms in a regular category can be expressed in terms of the cover relation $\leqq$ alone, which then suggests a natural generalization to the context of an arbitrary cover relation:

Definition 2.1.3 *Let $\sqsubset$ be a binary relation on the class of morphisms of a category $\mathbb{C}$. A morphism $f : X \to Y$ is said to be*

- *a* $\sqsubset$-covering *when for any morphism $g : Z \to Y$, we have $g \sqsubset f$;*
- *a* $\sqsubset$-null morphism *when for any other morphism $g : Z \to Y$, we have $f \sqsubset g$;*
- *a* $\sqsubset$-embedding *when for two morphisms $h : H \to X$ and $k : K \to X$ such that $fh \sqsubset fk$ we have $h \sqsubset k$; in the case when we require this for a $\sqsubset$-null morphism k, f is said to be a* weak $\sqsubset$-embedding*;*
- $\sqsubset$-full *when for any morphism $g : Z \to Y$ such that $g \sqsubset f$, there is a morphism $e : W \to X$ such that $g \sqsubset fe$ and $fe \sqsubset g$ (this will be written as $g \approx_\sqsubset fe$ for short).*

Sometimes we will drop the prefix $\sqsubset$ *in the above terms, provided this will cause no confusion.*

Remark 2.1.2 *For a cover relation* $\sqsubset$*, identity morphisms are always* $\sqsubset$*-coverings. In fact, cover relations can be characterized as those transitive relations which have the left preservation property and for which identity morphisms are coverings.*

For the canonical cover relation $\leqq$ of a pointed regular category, coverings are precisely the regular epimorphisms, null morphisms are the zero morphisms, weak embeddings are morphisms with trivial kernel, and any morphism is full. By Proposition 1.3.1, in an abelian category weak embeddings are the same as monomorphisms, and so, in view of Proposition 1.2.2, in an abelian category isomorphisms are the same as coverings that are at the same time weak embeddings. The generalization of the five lemma in the context of a category equipped with a cover relation, which is obtained in this Thesis, will be formulated in terms of weak embeddings and coverings, instead of isomorphisms.

2.2 $\sqsubset$-categories

If a cover relation $\sqsubset$ is specified in a category $\mathbb{C}$, we shall write the category as a pair $(\mathbb{C}, \sqsubset)$, and call the pair a $\sqsubset$-*category* if the following two axioms are satisfied for the cover relation $\sqsubset$ specified in $\mathbb{C}$:

(C_0) For any object X in $\mathbb{C}$ there exists a null morphism with codomain X.

(C_1) For any null morphism f in $\mathbb{C}$, the composite gf is a null morphism for any morphism g composable with f.

Note that under (C_0), axiom (C_1) is equivalent to the following:

(C_1') For any two morphisms f, g (having the same codomain), there exists a morphism h such that $fh \sqsubset g$.

Example 2.2.1 *Consider the ordered set* $(\mathbb{Z}, \leqslant)$*, where* $\mathbb{Z}$ *is the set of integers, and* $\leqslant$ *is the usual order relation on* $\mathbb{Z}$*. Consider the pair* $(\mathbb{C}, \prec)$*, where* $\mathbb{C}$ *is the ordered set* $(\mathbb{Z}, \leqslant)$ *regarded as a category (and* $\prec$ *is the usual 'factors through' relation on the category* $\mathbb{C}$*). Then there are no null morphisms in* $\mathbb{C}$*. Hence (*C_1*) is trivially satisfied but (*C_0*) is not satisfied.*

Proposition 2.2.1 *For any category* $\mathbb{C}$*, if the pair* $(\mathbb{C}, \prec)$ *satisfies (*C_0*), then it also satisfies (*C_1*).*

Proof. Let $\mathbb{C}$ be a category such that the pair $(\mathbb{C}, \prec)$ satisfies (C_0). In $\mathbb{C}$, consider a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

where f is a null morphism. We show that gf is also a null morphism. By axiom (C_0), there exists a null morphism $z : C \to Z$. Then $z = gh$ for some morphism $h : C \to Y$. Since f is a null morphism we have $f \prec h$, which implies $gf \prec gh = z$. This shows that gf is a null morphism. $\square$

Proposition 2.2.2 *Consider a pair* $(\mathbb{C}, \sqsubset)$ *where* $\mathbb{C}$ *has an initial object.*

1. *Any morphism in* $\mathbb{C}$ *which factors through the initial object is a* $\sqsubset$*-null morphism.*
2. $(\mathbb{C}, \sqsubset)$ *satisfies both* (C_0) *and* (C_1)*.*

Proof. (1) This is obvious in the case when $\sqsubset$ is the 'factors through' relation $\prec$. Since $\prec$ is a subrelation of any cover relation $\sqsubset$, null morphisms for it are at the same time $\sqsubset$-null morphisms.

(2) Let I be an initial object. Then for any object X the unique morphism $u : I \to X$ is a null morphism by the above, and so we have (C_0). To get (C'_1), take $h = u$ when X is the domain of f. $\square$

The above proposition provides us with ample examples of $\sqsubset$-categories; in particular, we see that regular categories having an initial object are $\sqsubset$-categories, and in particular, so are the abelian categories.

Proposition 2.2.3 *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two morphisms in a* $\sqsubset$*-category.*

1. *If* f *and* g *are null morphisms, coverings, or (weak) embeddings, then* gf *is a null morphism, a covering, or a (weak) embedding, respectively;*
2. *If* gf *is a covering, then so is* g*;*
3. *If* gf *is a (weak) embedding, then so is* f*;*
4. *Suppose that* f *is a covering. Then* gf *is null if and only if* g *is null;*
5. *Suppose that* f *is a covering. Then* gf *being full implies that* g *is full;*
6. *Suppose that* g *is an embedding. Then* gf *is full if both* f *and* g *are full;*
7. *Suppose that* g *is an embedding. Then* gf *being full implies that* f *is full.*

Proof. All statements in the proposition can be proved by simple straightforward arguments. As an illustration, we present the proof of (6). To this end, suppose that f is full and g is a full embedding. Let $h : H \to Z$ be a morphism such that $h \sqsubset gf$. Then $h \sqsubset g$, and since g is full, we get some morphism $k : K \to Y$ such that $h \sqsubset gk$ and $gk \sqsubset h$. Since g is an embedding, the relations $gk \sqsubset h \sqsubset gf$ imply $k \sqsubset f$, whence also $k \approx fl$ for some morphism $l : L \to X$, because f is full. Thus, $h \approx gk \approx g(fl) = (gf)l$, which proves that gf is full. □

2.3 Exact and homological sequences

Notation 2.3.1 *In a $\sqsubset$-category $\mathbb{C}$, the class of null morphisms (with respect to the cover relation $\sqsubset$) will be denoted by $\mathrm{N}_{\sqsubset}$ (or simply by N). Note that the assignment $\sqsubset \mapsto \mathrm{N}_{\sqsubset}$ is functorial, that is, if a cover relation $\sqsubset$ on $\mathbb{C}$ is contained in another cover relation $\sqsubset'$ on $\mathbb{C}$, then $\mathrm{N}_{\sqsubset} \subseteq \mathrm{N}_{\sqsubset'}$.*

Definition 2.3.1 *In a $\sqsubset$-category $\mathbb{C}$, a diagram*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{2.3.1}$$

is said to be exact (at Y) if the following condition holds: for any morphism $h : W \to Y$ in the category, $gh \in \mathrm{N}_{\sqsubset}$ if and only if $h \sqsubset f$.

Lemma 2.3.1 *In a $\sqsubset$-category $\mathbb{C}$, consider a diagram*

$$K \xrightarrow{c} X \xrightarrow{f} Y \xrightarrow{e} C$$

where f is a null morphism. Then

1. *c is a covering if and only if the sequence is exact at X.*
2. *e is a weak embedding if and only if the sequence is exact at Y.*

Proof. (1) The sequence is exact at Y if and only if for any morphism $h : W \to X$ we have: $fh \in \mathrm{N}$ if and only if $h \sqsubset c$. Since f is a null morphism, we always have $fh \in \mathrm{N}_{\sqsubset}$, and so exactness at Y states that for any h we have $h \sqsubset c$, i.e. c is a covering.

(2) Suppose that e is a weak embedding. If $s : V \to Y$ is a morphism such that $es \in \mathrm{N}$, then $s \in \mathrm{N}$, whence $s \sqsubset f$. On the other hand, if $s \sqsubset f \in \mathrm{N}$, then $es \in \mathrm{N}$, in view of (C_1). This proves that the sequence is exact at Y. Conversely, suppose that the sequence is exact at Y. Then e is clearly a weak embedding because for any morphism $s : V \to Y$, $es \in \mathrm{N}$ implies that $s \sqsubset f \in \mathrm{N}$. □

Definition 2.3.2 *In a $\sqsubset$-category $\mathbb{C}$, a diagram (2.3.1) is said to be homological (at Y) if for any morphism $u : U \to Y$ such that $gu \sqsubset gf$, we have $u \sqsubset f$. It is said to be weakly homological (at Y) if for any morphism u such that $gu \in \mathrm{N}_{\sqsubset}$, we have $u \sqsubset f$. An exact/(weakly) homological sequence is defined as a diagram*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n+1}} X_{n+2}$$

such that

$$X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1}$$

is exact/(weakly) homological for each $i \in \{1, ..., n+1\}$.

It is easy to see that if a diagram (2.3.1) is either exact or homological, then it is weakly homological.

Proposition 2.3.1 *For any diagram (2.3.1) in a $\sqsubset$-category $\mathbb{C}$, the following are equivalent:*

1. *(2.3.1) is exact.*
2. *(2.3.1) is homological and $gf \in \mathrm{N}_{\sqsubset}$.*
3. *(2.3.1) is weakly homological and $gf \in \mathrm{N}_{\sqsubset}$.*

Proof. (1⇒2) Suppose (2.3.1) is exact. Since $f \sqsubset f$, we have $gf \in \mathrm{N}_{\sqsubset}$. To prove that (2.3.1) is homological, consider a morphism $u : U \to Y$ such that $gu \sqsubset gf$. Then $gu \in \mathrm{N}_{\sqsubset}$, and by exactness of the diagram, we have that $u \sqsubset f$.

(2⇒3) is trivial.

(3⇒1) If $h \sqsubset f$ then $gh \sqsubset gf$ and hence $gh \in \mathrm{N}_{\sqsubset}$. □

Proposition 2.3.2 *In a $\sqsubset$-category $\mathbb{C}$, a morphism $f : X \to Y$ is an embedding (resp. a weak embedding) if and only if the sequence*

$$W \xrightarrow{e} X \xrightarrow{f} Y$$

is homological (resp. weakly homological) for every morphism $e : W \to X$.

Proof. If a given morphism $f : X \to Y$ is an embedding, it is easy to see that the above sequence is homological for every morphism $e : W \to X$. Conversely, suppose that the sequence is homological for every morphism $e : W \to X$. To show that f is an embedding, let $u : U \to X$ and $v : V \to X$ be morphisms such that $fu \sqsubset fv$; we claim that $u \sqsubset v$. But this is immediate since the sequence

$$V \xrightarrow{v} X \xrightarrow{f} Y$$

is homological. A similar argument works for the case when f is a weak embedding. □

Definition 2.3.3 *In a $\sqsubset$-category $\mathbb{C}$, a morphism $f : X \to Y$ is said to be a homological morphism if every weakly homological sequence*

$$W \xrightarrow{e} X \xrightarrow{f} Y$$

is homological.

From Proposition 2.3.2 we get:

Corollary 2.3.1 *In a $\sqsubset$-category $\mathbb{C}$, a morphism $f : X \to Y$ is an embedding if and only if it is both a homological morphism and a weak embedding.*

Example 2.3.1 *Every morphism in* **Grp** *is homological. To see this, let $f : X \to Y$ be any group homomorphism. If the sequence*

$$W \xrightarrow{e} X \xrightarrow{f} Y$$

is weakly homological (that is, for all $x \in X$, $f(x) = 0$ implies that there is some $w \in W$ such that $e(w) = x$), then it is also homological: let $r, s \in X$ such that $f(r) = f(s)$ and let $r = e(a)$ for some $a \in W$. Then it is easy to see that $s \in \mathrm{Im}(e)$ also, because $f(s - r) = 0$ implies (by weak homologicity) that there is some $b \in W$ such that $e(b) = s - r$. This says that the above sequence is homological, and hence that f is a homological morphism.

Let us now add the following to the list of the composition properties of some special morphisms in a $\sqsubset$-category $\mathbb{C}$.

Proposition 2.3.3 *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms in a $\sqsubset$-category $\mathbb{C}$. If g is an embedding, then gf is a homological morphism if and only if f is a homological morphism.*

Proof. Let g be an embedding. We first show that gf is a homological morphism if f is a homological morphism. To this end, consider a weakly homological sequence

$$W \xrightarrow{e} X \xrightarrow{gf} Z$$

and suppose that for some morphism $s : V \to X$, we have that $(gf)s \sqsubset (gf)e$; we must show that $s \sqsubset e$. First, we get that $fs \sqsubset fe$, because g is an embedding. Moreover, the sequence

$$W \xrightarrow{e} X \xrightarrow{f} Y$$

is also a weakly homological sequence, because the previous one is. Therefore, since f is a homological morphism, we get that the above sequence is actually a homological sequence, so that $fs \sqsubset fe$ yields $s \sqsubset e$, as desired. Thus, gf is a homological morphism. Conversely, suppose that gf is a homological morphism. To show that f is also a homological morphism, suppose that the above sequence is a weakly homological sequence; we show that it is a homological sequence. So, let $s : V \to X$ be a morphism such that $fs \sqsubset fe$. Note that since g is a weak embedding, we have that gf is again a weak embedding, so that the previous sequence is weakly homological, and moreover it is homological because gf is a homological morphism. Then $fs \sqsubset fe$ gives $(gf)s = g(fs) \sqsubset g(fe) = (gf)e$, which in turn gives $s \sqsubset e$, as desired. This proves that f is a homological morphism. □

Proposition 2.3.4 *In a $\sqsubset$-category $\mathbb{C}$, let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a homological sequence where f and g are full. Then gf is also full.

Proof. Let s be a morphism with codomain Z such that $s \sqsubset gf$. Then $s \sqsubset gf \sqsubset g$, and since g is full, there is some morphism t such that $s \approx gt$. Thus, $gt \sqsubset s \sqsubset gf$; this gives $gt \sqsubset gf$, and we can then homologicity to deduce that $t \sqsubset f$. Since f is full, there is some morphism u such that $t \approx fu$. By left preservation, $gt \approx g(fu) = (gf)u$, whence $s \approx (gf)u$. This shows that gf is also full. □

2.4 Subtractivity and 3×3 lemmas

Definition 2.4.1 *In a $\sqsubset$-category $\mathbb{C}$, a* span

$$\begin{array}{ccc} S & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \\ X, & & \end{array} \tag{2.4.1}$$

(denoted by $[f, g]$) is said to be subtractive if for any two morphisms $a : A \to S$, $b : B \to S$ in $\mathbb{C}$ such that $fa \sqsubset fb$ and $gb \in \mathrm{N}_{\sqsubset}$, there exists a morphism $c : C \to S$ such that $ga \sqsubset gc$ and $fc \in \mathrm{N}_{\sqsubset}$.

In a $\sqsubset$-category $\mathbb{C}$, by a 3×3 diagram we a mean a commutative diagram of the form

$$\begin{tikzcd} & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \\ \bullet \arrow[r] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"'] & B_1 \arrow[r, "v_1"] \arrow[d, "g_1"'] & C_1 \arrow[r] \arrow[d, "h_1"] & \bullet \\ \bullet \arrow[r] & A_2 \arrow[r, "u_2"] \arrow[d, "f_2"'] & B_2 \arrow[r, "v_2"] \arrow[d, "g_2"'] & C_2 \arrow[r] \arrow[d, "h_2"] & \bullet \\ \bullet \arrow[r] & A_3 \arrow[r, "u_3"'] \arrow[d] & B_3 \arrow[r, "v_3"'] \arrow[d] & C_3 \arrow[r] \arrow[d] & \bullet \\ & \bullet & \bullet & \bullet & \end{tikzcd} \tag{2.4.2}$$

where all the columns are exact sequences, and arrows whose domain or codomain is represented by a bullet are null morphisms.

Theorem 2.4.1 (Upper 3×3 lemma in a $\sqsubset$-category) *Consider a 3×3 diagram (2.4.2) in a $\sqsubset$-category $\mathbb{C}$, where g_1, h_1 are homological morphisms, u_2, v_2 are full, and the span $[g_2, v_2]$ is subtractive. If the middle and bottom rows are exact, then the top row is also exact.*

Proof. Exactness of the top row at A_1 follows from Lemma 2.3.1(2) and Proposition 2.2.3(1,3). Next, we prove exactness at B_1. We have: $h_1 v_1 u_1 = v_2 g_1 u_1 = v_2 u_2 f_1 \in \mathrm{N}$. Since h_1 is a weak embedding (by exactness of the third column at C_1), we get $v_1 u_1 \in \mathrm{N}$. Now suppose $v_1 y \in \mathrm{N}$ for some y; then $v_2 g_1 y = h_1 v_1 y \in \mathrm{N}$, whence $g_1 y \sqsubset u_2$, by exactness of the middle row at B_2. Since u_2 is full, there is a morphism $z : Z \to A_2$ such that $g_1 y \approx u_2 z$. Then $u_3 f_2 z = g_2 u_2 z \approx g_2 g_1 y \in \mathrm{N}$, and $f_2 z \in \mathrm{N}$ because the bottom row is exact at A_3. Furthermore, by exactness of the first column at A_2, we have that $z \sqsubset f_1$, whence $g_1 y \sqsubset u_2 z \sqsubset u_2 f_1 = g_1 u_1$. Since g_1 is an embedding (as it is both a weak embedding and a homological morphism — see Corollary 2.3.1), $y \sqsubset u_1$, as desired. This proves exactness at B_1. Finally, we prove exactness at C_1. This time we use Lemma 2.3.1(1) and show that v_1 is a covering. For any $s : S \to C_1$, we have that $h_1 s \sqsubset v_2$, by exactness of the middle row at C_2. Since v_2 is full, there is a morphism $r : R \to B_2$ such that $h_1 s \approx v_2 r$. Then $v_3 g_2 r = h_2 v_2 r \approx h_2 h_1 s \in \mathrm{N}$, whence $g_2 r \sqsubset u_3 \sqsubset u_3 f_2 = g_2 u_2$, by exactness of the bottom row at B_3 and because f_2 is a covering (by exactness of the first column at A_3). As $v_2 u_2 \in \mathrm{N}$, we use subtractivity of the span $[g_2, v_2]$ to get a morphism $q : Q \to B_2$ such that $v_2 r \sqsubset v_2 q$ and $g_2 q \in \mathrm{N}$; the latter meaning that $q \sqsubset g_1$. Thus, $h_1 s \sqsubset v_2 r \sqsubset v_2 q \sqsubset v_2 g_1 = h_1 v_1$, and $s \sqsubset v_1$ because h_1 is an embedding (again being a weak embedding and a homological morphism). This proves that v_1 is a covering, so that the top row is exact at C_1. $\square$

Theorem 2.4.2 (Lower 3×3 lemma in a $\sqsubset$-category) *In a $\sqsubset$-category $\mathbb{C}$, consider a commutative diagram (2.4.2) in which u_2 is a homological morphism, g_1, g_2, f_2 are full, and the span $[g_2, v_2]$ is subtractive. If the top and middle rows are exact, then the bottom row is also exact.*

Proof. We prove exactness at A_3, B_3, C_3. Exactness at A_3 can be accomplished by showing that u_3 is a weak embedding. Thus, let $s : S \to A_2$ be any morphism such that $u_3 s \in \mathrm{N}$. Since f_2 is a full covering, we have that $s \approx f_2 t$ for some morphism $t : T \to A_2$. Then $g_2 u_2 t = u_3 f_2 t \approx u_3 s \in \mathrm{N}$, and so $u_2 t \sqsubset g_1$, by the exactness of the second column at B_2. Since g_1 is full, there is a morphism $r : R \to B_1$ such that $u_2 t \approx g_1 r$. Then $h_1 v_1 r = v_2 g_1 r \approx v_2 u_2 t \in \mathrm{N}$ implies that $v_1 r \in \mathrm{N}$, by exactness of the first column at C_1; also, $r \sqsubset u_1$ because the top row is exact at B_1. By left preservation, we have that $u_2 t \sqsubset g_1 r \sqsubset g_1 u_1 = u_2 f_1$, and $t \sqsubset f_1$ because u_2 is an embedding. Therefore, $s \approx f_2 t \sqsubset f_2 f_1 \in \mathrm{N}$, proving that u_3 is a weak embedding, and hence that the bottom row is exact at A_3. Next, we prove exactness at B_3. To this end, first consider a morphism $x : X \to B_3$ such that $x \sqsubset u_3$. Then $v_3 x \sqsubset v_3 u_3 \sqsubset v_3 u_3 f_2 = v_3 g_2 u_2 = h_2 v_2 u_2 \in \mathrm{N}$. Conversely, suppose that $v_3 x \in N$; we must show that $x \sqsubset u_3$. As g_2 is a covering, one has $x \sqsubset g_2$, and for some morphism $y : Y \to B_2$, one has that $x \approx g_2 y$. Proceeding in the usual way, $h_2 v_2 y = v_3 g_2 y \approx v_3 x \in \mathrm{N}$ implies that $v_2 y \sqsubset h_1 \sqsubset h_1 v_1 = v_2 g_1$. Because $g_2 g_1 \in \mathrm{N}$, we can use the subtractivity of the span $[g_2, v_2]$ at this stage. This ensures that there is a morphism $z : Z \to B_2$ such that $g_2 y \sqsubset g_2 z$ and $v_2 z \in \mathrm{N}$; the latter meaning that $z \sqsubset u_2$. Hence, $x \sqsubset g_2 y \sqsubset g_2 z \sqsubset g_2 u_2 = u_3 f_2 \sqsubset u_3$, as desired. This proves that the bottom row is exact at B_3. Finally, exactness at C_3 follows from Lemma 2.3.1(1) and Proposition 2.2.3(1,2). □

A *normal category* in the sense of [Jan10] is a pointed regular category (having finite limits) where every regular epimorphism is a normal epimorphism. Like any regular category, a normal category has a canonical cover relation $\leqq$, which makes it into a $\sqsubset$-category. Then, the notion of an exact sequence in a normal category becomes the 'usual one' (see e.g. [BB04]). Further, subtractivity of a span becomes precisely the subtractivity in the sense of [Jan10], and hence the upper and lower 3×3 lemmas obtained in Section 2.4 above become precisely the ones obtained in [Jan10] (in fact, this is also true more generally for pointed regular categories). Note that subtractivity of spans can be a seen as a weakened version of 'rule of subtraction' (see Theorem 1.3.1(6)) in an abelian category.

Remark 2.4.1 *For pointed regular categories with finite limits, if every morphism is homological then every span is subtractive (see [Jan10]). Our context of $\sqsubset$-categories seems to be far too general to reproduce this fact.*

Chapter 3

The generalized five lemma

3.1 Five lemma in a $\sqsubset$-category

Lemma 3.1.1 *In a $\sqsubset$-category $\mathbb{C}$, consider a commutative diagram with exact rows*

$$\begin{tikzcd} C_1 \arrow[r, "c_1"] \arrow[d, "c"'] & D_1 \arrow[r, "d_1"] \arrow[d, "d"'] & E_1 \arrow[d, "e"'] \\ C_2 \arrow[r, "c_2"'] & D_2 \arrow[r, "d_2"'] & E_2 \end{tikzcd}$$

where the sequence $\langle c, c_2\rangle$ is homological, d is a full covering, and e is a weak embedding. Then c is a covering.

Proof. To show that c is a covering, consider a morphism $s : S \to C_2$. Since d is a covering, we have $c_2 s \sqsubset d$, and since d is also full, there is some morphism $t : T \to D_1$ such that $c_2 s \approx dt$. By left preservation, $ed_1 t = d_2 dt \approx d_2 c_2 s \in \mathrm{N}$. Since e is a weak embedding, this gives $d_1 t \in \mathrm{N}$, and hence $t \sqsubset c_1$, by exactness of the top row. Again, by left preservation, $c_2 s \approx dt \sqsubset dc_1 = c_2 c$, so that $c_2 s \sqsubset c_2 c$. Homologicity of the sequence $\langle c, c_2\rangle$ gives $s \sqsubset c$, proving that c is a covering. $\square$

Lemma 3.1.2 *In a $\sqsubset$-category $\mathbb{C}$, consider a commutative diagram with exact rows*

$$\begin{tikzcd} A_1 \arrow[r, "a_1"] \arrow[d, "a"'] & B_1 \arrow[r, "b_1"] \arrow[d, "b"'] & C_1 \arrow[r, "c_1"] \arrow[d, "c"'] & D_1 \arrow[d, "d"'] \\ A_2 \arrow[r, "a_2"'] & B_2 \arrow[r, "b_2"'] & C_2 \arrow[r, "c_2"'] & D_2 \end{tikzcd}$$

where a is a covering, the sequence $\langle a_1, b\rangle$ is homological, b_1 is full, and d is a weak embedding. Then c is a weak embedding.

Proof. Let $x : X \to C_1$ be a morphism such that $cx \in \mathrm{N}$. We claim $x \in \mathrm{N}$. Since $cx \in \mathrm{N}$, we have $dc_1 x = c_2 cx \in \mathrm{N}$. Then $c_1 x \in \mathrm{N}$, since d is a weak embedding. By exactness of the top row at C_1, this gives $x \sqsubset b_1$, and since b_1 is full, there is a morphism $y : Y \to B_1$ such that $x \approx b_1 y$. Thus, $cx \approx cb_1 y = b_2 by$, and so $b_2 by \in \mathrm{N}$. By exactness of the bottom row at B_2, and since a is a covering, we get $by \sqsubset a_2 \approx a_2 a = ba_1$. This gives $y \sqsubset a_1$, in view of homologicity of the sequence $\langle a_1, b \rangle$. Therefore, $x \approx b_1 y \sqsubset b_1 a_1 \in \mathrm{N}$, as desired. $\square$

Theorem 3.1.1 (Five lemma in a $\sqsubset$-category) *In a $\sqsubset$-category $\mathbb{C}$, consider a commutative diagram with exact rows*

$$\begin{tikzcd} A_1 \arrow[r, "a_1"] \arrow[d, "a"'] & B_1 \arrow[r, "b_1"] \arrow[d, "b"'] & C_1 \arrow[r, "c_1"] \arrow[d, "c"'] & D_1 \arrow[r, "d_1"] \arrow[d, "d"'] & E_1 \arrow[d, "e"'] \\ A_2 \arrow[r, "a_2"'] & B_2 \arrow[r, "b_2"'] & C_2 \arrow[r, "c_2"'] & D_2 \arrow[r, "d_2"'] & E_2 \end{tikzcd} \tag{3.1.1}$$

1. *Suppose that d is full and c_2 is homological. If further b, d are coverings and e is a weak embedding, then c is a covering.*

2. *Suppose that b_1 is full and b is homological. If further b, d are weak embeddings and a is a covering, then c is a weak embedding.*

Proof. (1) We first observe that the sequence $\langle c, c_2 \rangle$ is weakly homological. Indeed, if there is some morphism $r : R \to C_2$ such that $c_2 r \in \mathrm{N}$, then $r \sqsubset b_2$, by exactness of the bottom row at C_2. Thus, since b is a covering, $b_2 \approx b_2 b$, whence $r \sqsubset b_2 b = cb_1 \sqsubset c$. So the sequence $\langle c, c_2 \rangle$ is weakly homological, as claimed. Furthermore, because c_2 is a homological morphism, the sequence $\langle c, c_2 \rangle$ is actually a homological sequence. That c is a covering now follows from Lemma 3.1.1.

(2) Since b is a weak embedding, the sequence $\langle a_1, b \rangle$ is weakly homological by Proposition 2.3.2. Since b is a homological morphism, the sequence $\langle a_1, b \rangle$ is actually a homological sequence. Therefore, from Lemma 3.1.2, c is a weak embedding. $\square$

Theorem 3.1.2 (Short five lemma in a $\sqsubset$-category) *In a $\sqsubset$-category $\mathbb{C}$, consider a commutative diagram with exact rows*

$$\begin{tikzcd} \bullet \arrow[r] & B_1 \arrow[r, "b_1"] \arrow[d, "b"'] & C_1 \arrow[r, "c_1"] \arrow[d, "c"'] & D_1 \arrow[r] \arrow[d, "d"'] & \bullet \\ \bullet \arrow[r] & B_2 \arrow[r, "b_2"] & C_2 \arrow[r, "c_2"] & D_2 \arrow[r] & \bullet \end{tikzcd} \tag{3.1.2}$$

where bullets represent arbitrary objects, while morphisms whose domain or codomain is a bullet, are null morphisms.

1. *Suppose c_2 is homological. If b and d are coverings, then c is a covering.*

2. *Suppose b_1 is full. If b and d are weak embeddings, then c is a weak embedding.*

Proof. (a) Exactness of the first row of the above diagram forces c_1 to be a covering, so that for any morphism $y : Y \to C_2$, we have $c_2 y \sqsubset d \sqsubset dc_1 = c_2 c$. That $y \sqsubset c$ follows from the fact that the sequence $\langle c, c_2 \rangle$ is homological, which in turn follows from the fact that c_2 is a homological morphism and the sequence $\langle c, c_2 \rangle$ is weakly homological. The latter can be proved in a similar way as in the proof of Theorem 3.1.1(a).

(b) Let $x : X \to C_1$ be a morphism such that $cx \in \mathrm{N}$. Then $dc_1 x = c_2 cx \in \mathrm{N}$, whence $c_1 x \in \mathrm{N}$, since d is a weak embedding. Thus, $x \sqsubset b_1$, and so $x \approx b_1 s$ for some morphism $s : S \to B_1$. By left preservation, $b_2 bs = cb_1 s \approx cx$, whence $bs \in \mathrm{N}$ (by exactness of the bottom row at B_2). Since b is a weak embedding, we have $s \in \mathrm{N}$. Therefore, $x \in \mathrm{N}$, proving that c is a weak embedding. □

3.2 Application to Borceux-Bourn homological categories

In algebra, homologicity of a homomorphism $f : X \to Y$ can be seen as a condition on the kernel congruence of f, stating that if the subalgebra of X contains the equivalence class of 0 then it must be a union of equivalence classes. In universal algebra, this condition is called 0-coherence, and as shown in [JU11], 0-coherence holds in a pointed category with finite limits if and only if the category is protomodular in the sense of [Bou91]. For pointed regular categories with finite limits, we get: every morphism is homological if and only if the category is homological in the sense of [BB04]. Since any homological category is normal, in a homological category the generalized five lemma and the generalized short five lemma obtained in Section 3.1 become the 'usual' five lemma and the 'usual' short five lemma (see e.g. [BB04], [Jan10]). In particular, since any abelian category is homological, the classical (short) five lemma for abelian categories which was recalled in Chapter 1 can be seen as a corollary of the generalized (short) five obtained in this chapter.

3.3 Application to Grandis semi-exact categories

We recall the following from [Gra92]:

Definition 3.3.1 *A semi-exact category $\mathbb{A} = (\mathbb{A}, N)$ is a pair satisfying the following axioms:*

1. *$\mathbb{A}$ is a category and N is a closed ideal of $\mathbb{A}$, i.e. N is a class of morphisms in $\mathbb{A}$ such that if in a composite fgh we have $g \in N$, then $fgh \in N$, and moreover, every morphism in N factors through an object whose identity morphism is in N.*

2. *Every morphism $f : X \to Y$ in $\mathbb{A}$ has a kernel* $\ker(f)$ *and a cokernel* $\mathrm{coker}(f)$ *(which are defined with respect to the class N).*

Given a semi-exact category $\mathbb{A}$, each object X in $\mathbb{A}$ has a lattice $\mathrm{Nsb}(X)$ of normal subobjects, and a lattice $\mathrm{Nqt}(X)$ of normal quotient objects. Moreover, each morphism $f : X \to Y$ has direct and inverse images for normal subobjects:

- $f_\star : \mathrm{Nsb}(X) \to \mathrm{Nsb}(Y),\ x \mapsto \mathrm{nim}(fx) = \ker(\mathrm{coker}(fx))$;
- $f^\star : \mathrm{Nsb}(Y) \to \mathrm{Nsb}(X),\ y \mapsto \ker(\mathrm{coker}(y)f)$.

A morphism $f : X \to Y$ is said to be *left-modular*, or *right-modular*, or *modular* if the associated map $\mathrm{Nsb}(f) : \mathrm{Nsb}(X) \to \mathrm{Nsb}(Y)$ satisfies the first, or second, or both of the conditions below:

- $f^\star f_\star x = x \vee f^\star 0,\ x \in \mathrm{Nsb}(X)$;
- $f_\star f^\star y = y \wedge f_\star 1,\ y \in \mathrm{Nsb}(Y)$.

A semiexact category $\mathbb{A}$ is *modular* if for any object X, the lattice $\mathrm{Nsb}(X)$ is modular, and any morphism is a modular morphism.

Theorem 3.3.1 *For a semiexact category $\mathbb{A} = (\mathbb{A}, N)$, its class M of kernels (with respect to N) has and is stable under all pullbacks. Further, the relation $\sqsubset$ defined as follows is a cover relation: $f \sqsubset g$ if and only if* $\mathrm{nim}(f) \prec \mathrm{nim}(g)$. *For this cover relation N is the class of null morphisms, and moreover, this cover relation makes $\mathbb{A}$ into a $\sqsubset$-category.*

Proof. Let M be the class of kernels in $\mathbb{A}$. We will show that for any $m \in M$, the pullback of m along any morphism f in $\mathbb{A}$ exists, and is again in M. Indeed, since m is a kernel of some morphism, we have $m = \ker(\mathrm{coker}(m))$. Now, consider the following commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "\ker(\mathrm{coker}(m)f)"'] & Y' \arrow[d, "m=\ker(\mathrm{coker}(m))"] \\
X \arrow[r, "f"] \arrow[dr, "\mathrm{coker}(m)f"'] & Y \arrow[d, "\mathrm{coker}(m)"] \\
 & C,
\end{tikzcd}$$

where the morphism f' arises by the universal property of the kernel m. If there are morphisms $r : W \to X$ and $s : W \to Y'$ such that $fr = ms$, then

$$\mathrm{coker}(m)fr = \mathrm{coker}(m)ms = (\mathrm{coker}(m)m)s \in N,$$

so that there is a (unique) morphism $u : W \to X'$ such that

$$r = \ker(\operatorname{coker}(m)f)u.$$

Then it follows that $mf'u = ms$, and so $f'u = s$, since m is a monomorphism. Hence, the above diagram is indeed a pullback diagram, and the construction shows that the pullback of m along f is again a kernel.

For morphisms f and g with the same codomain, we define $\sqsubset^M$ as in [Jan09]: set $f \sqsubset^M g$ if and only if for all $m \in M$, $g \prec m$ implies $f \prec m$. It is easy to see that $\sqsubset^M$ is reflexive, transitive, and has right preservation property. To show that the left preservation property holds, suppose that $f \sqsubset^M g$ and consider the following display:

$$\begin{tikzcd}
& Z \arrow[d, "g"] & \\
X \arrow[r, "f"] & Y \arrow[dr, "h"] & \\
& & W
\end{tikzcd}$$

Let $hg \prec m$ for some $m \in M$. Then $hg = mx$ for some morphism x in $\mathbb{A}$. Since M is pullback-stable, we can take the pullback of m along h to get a morphism $m' \in M$ and another morphism h' in $\mathbb{A}$ such that $hm' = mh'$. By the universal property of the pullback, there is a unique morphism u such that $g = m'u$, that is, such that $g \prec m'$. Since $f \sqsubset^M g$, this implies that $f \prec m'$, and hence that $hf \prec hm' = mh'$, whence $hf \prec m$. Therefore, $hf \sqsubset^M hg$, showing that $\sqsubset^M$ satisfies the left preservation property, and so $\sqsubset^M$ is a cover relation. It can be easily shown that $f \sqsubset^M g$ in fact coincides with the relation defined in the theorem.

Next, we show that N is the class of null morphisms for this cover relation, i.e. $N = N_{\sqsubset}$. Let $f : X \to Y$ be a morphism from the class N. We show that $\operatorname{nim}(f) \prec \operatorname{nim}(g)$, for any morphism $g : Z \to Y$ in $\mathbb{A}$. Consider the following display:

$$\begin{tikzcd}
X'' \arrow[dr, "\mathrm{nim}(f)"] & & Z'' \arrow[dl, "\mathrm{nim}(g)"'] \\
X \arrow[r, "f"] & Y \arrow[dl, "\mathrm{coker}(f)"'] \arrow[dr, "\mathrm{coker}(g)"] & Z \arrow[l, "g"'] \\
X' & & Z'
\end{tikzcd}$$

Since $\operatorname{coker}(g)f \in N$, it follows, by the universal property of the cokernel, that there is a morphism $x' : X' \to Z'$ such that $\operatorname{coker}(g) = x'\operatorname{coker}(f)$. Moreover, $\operatorname{coker}(g)\operatorname{nim}(f) = x'\operatorname{coker}(f)\operatorname{nim}(f) \in N$; hence, by the universal property of the kernel, there is a morphism $x'' : X'' \to Z''$ such that $\operatorname{nim}(f) = \operatorname{nim}(g)x''$, as desired. Conversely, suppose that $f \in \mathrm{N}_{\sqsubset}$, then $\mathrm{nim}(f) \prec \mathrm{nim}(g)$, for any morphism g with the same codomain as f. In particular, taking $g = 1_Y$, we have that $f \prec \mathrm{nim}(f) \prec \mathrm{nim}(\ker(1_Y)) = \ker(1_Y) \in N$. Therefore $f \in N$.

The fact that the pair $(\mathbb{A}, \sqsubset)$ is a $\sqsubset$-category now follows trivially from the definition of a semi-exact category, which completes the proof. $\square$

The above theorem shows that we could apply our five lemma to a semi-exact category. This would give precisely the five lemma obtained in [Gra92]. To show this we only have to confirm that all properties used in the formulation of our five lemma match with those in the formulation of the five lemma in [Gra92]. It is not difficult to verify that this is indeed so. In particular, homologicity of a morphism is precisely left-modularity and fullness is precisely right-modularity.

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