Statistical Inference for Inequality Measures Based on Semi-Parametric Estimators

by

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Abstract

Measures of inequality, also used as measures of concentration or diversity, are very popular in economics and especially in measuring the inequality in income or wealth within a population and between populations. However, they have applications in many other fields, e.g. in ecology, linguistics, sociology, demography, epidemiology and information science.

A large number of measures have been proposed to measure inequality. Examples include the Gini index, the generalized entropy, the Atkinson and the quintile share ratio measures. Inequality measures are inherently dependent on the tails of the population (underlying distribution) and therefore their estimators are typically sensitive to data from these tails (nonrobust). For example, income distributions often exhibit a long tail to the right, leading to the frequent occurrence of large values in samples. Since the usual estimators are based on the empirical distribution function, they are usually nonrobust to such large values. Furthermore, heavy-tailed distributions often occur in real life data sets, remedial action therefore needs to be taken in such cases.

The remedial action can be either a trimming of the extreme data or a modification of the (traditional) estimator to make it more robust to extreme observations. In this thesis we follow the second option, modifying the traditional empirical distribution function as estimator to make it more robust. Using results from extreme value theory, we develop more reliable distribution estimators in a semi-parametric setting. These new estimators of the distribution then form the basis for more robust estimators of the measures of inequality. These estimators are developed for the four most popular classes of measures, viz. Gini, generalized entropy, Atkinson and quintile share ratio. Properties of such estimators are studied especially via simulation. Using limiting distribution theory and the bootstrap methodology, approximate confidence intervals were derived. Through the various simulation studies, the proposed estimators are compared to the standard ones in terms of mean squared error, relative impact of contamination, confidence interval length and coverage probability. In these studies the semi-parametric methods show a clear improvement over the standard ones. The theoretical properties of the quintile share ratio have not been studied much. Consequently, we also derive its influence function as well as the limiting normal distribution of its nonparametric estimator. These results have not previously been published.

In order to illustrate the methods developed, we apply them to a number of real life data sets. Using such data sets, we show how the methods can be used in practice for inference. In order to choose between the candidate parametric distributions, use is made of a measure of sample representativeness from the literature. These illustrations show that the proposed methods can be used to reach satisfactory conclusions in real life problems.

Opsomming

Maatstawwe van ongelykheid, wat ook gebruik word as maatstawwe van konsentrasie of diversiteit, is baie populêr in ekonomie en veral vir die kwantifisering van ongelykheid in inkomste of welvaart binne ’n populasie en tussen populasies. Hulle het egter ook toepassings in baie ander dissiplines, byvoorbeeld ekologie, linguistiek, sosiologie, demografie, epidemiologie en inligtingskunde.

Daar bestaan reeds verskeie maatstawwe vir die meet van ongelykheid. Voorbeelde sluit in die Gini indeks, die veralgemeende entropie maatstaf, die Atkinson maatstaf en die kwintiel aandeel verhouding. Maatstawwe van ongelykheid is inherent afhanklik van die sterte van die populasie (onderliggende verdeling) en beramers daarvoor is tipies dus sensitief vir data uit sodanige sterte (nierobuust). Inkomste verdelings het byvoorbeeld dikwels lang regtersterte, wat kan lei tot die voorkoms van groot waardes in steekproewe. Die tradisionele beramers is gebaseer op die empiriese verdelingsfunksie, en hulle is gewoonlik dus nierobuust teenoor sodanige groot waardes nie. Aangesien swaarstert verdelings dikwels voorkom in werklike data, moet regstellings gemaak word in sulke gevalle.

Hierdie regstellings kan bestaan uit of die afknip van ekstreme data of die aanpassing van tradisionele beramers om hulle meer robuust te maak teen ekstreme waardes. In hierdie tesis word die tweede opsie gevolg deurdat die tradisionele empiriese verdelingsfunksie as beramer aangepas word om dit meer robuust te maak. Deur gebruik te maak van resultate van ekstreemwaardeteorie, word meer betroubare beramers vir verdelings ontwikkel in ’n semi-parametriese opset. Hierdie nuwe beramers van die verdeling vorm dan die basis vir meer robuuste beramers van maatstawwe van ongelykheid. Hierdie beramers word ontwikkel vir die vier mees populêre klasse van maatstawwe, naamlik Gini, veralgemeende entropie, Atkinson en kwintiel aandeel verhouding. Eienskappe van hierdie beramers word bestudeer, veral met behulp van simulasie studies. Benaderde vertrouensintervalle word ontwikkel deur gebruik te maak van limietverdelingsteorie en die skoenlus metodologie. Die voorgestelde beramers word vergelyk met tradisionele beramers deur middel van verskeie simulasie studies. Die vergelyking word gedoen in terme van gemiddelde kwadraat fout, relatiewe impak van kontaminasie, vertrouensinterval lengte en oordekkingswaarskynlikheid. In hierdie studies toon die semi-parametriese metodes ’n duidelike verbetering teenoor die tradisionele metodes. Die kwintiel aandeel verhouding se teoretiese eienskappe het nog nie veel aandag in die literatuur geniet nie. Gevolglik lei ons die invloedfunksie asook die asimptotiese verdeling van die nie-parametriese beramer daarvoor af.

Ten einde die metodes wat ontwikkel is te illustreer, word dit toegepas op ’n aantal werklike datastelle. Hierdie toepassings toon hoe die metodes gebruik kan word vir inferensie in die praktyk. ’n Metode in die literatuur vir steekproefverteenwoordiging word voorgestel en gebruik om ’n keuse tussen die kandidaat parametriese verdelings te maak. Hierdie voorbeelde toon dat die voorgestelde metodes met vrug gebruik kan word om bevredigende gevolgtrekkings in die praktyk te maak.

Résumé

Les mesures d'inégalité, également utilisées comme mesures de concentration ou de diversité, sont très populaires dans les sciences économiques, particulièrement en mesurant l'inégalité dans le revenu ou la richesse dans une population et entre des populations. Cependant, elles ont des applications dans beaucoup d'autres domaines, par exemple en écologie, linguistique, sociologie, démographie, épidémiologie et science de l'information.

Un grand nombre de mesures a été proposé pour mesurer l'inégalité. Les exemples incluent l'index de Gini, l'entropie généralisée, la mesure d'Atkinson et le rapport des quintiles du revenu. Les mesures d'inégalité dépendent des queues de la population (distribution fondamentale) et ainsi leurs estimateurs sont en général sensibles aux données de ces queues (non-robustes). Par exemple, les fonctions de répartition du revenu présentent souvent une longue queue vers la droite, conduisant à l'apparition fréquente de grandes valeurs dans les échantillons. Puisque les estimateurs habituels sont basés sur la distribution empirique, ils sont habituellement non-robustes à de telles grandes valeurs. En outre, les distributions à queue lourde se produisent souvent dans des données de la vie réelle, d'où la nécessité de prendre des mesures correctives dans de tels cas.

L'action corrective peut consister en une coupure délibérée des données extrêmes ou en une modification de l'estimateur (traditionnel) pour le rendre plus robuste aux observations extrêmes. Dans cette thèse nous suivons la deuxième option, modifiant la fonction empirique traditionnelle comme estimateur pour la rendre plus robuste. Utilisant des résultats de la théorie des valeurs extrêmes, nous développons des estimateurs plus fiables de la fonction de répartition dans un cadre semi-paramétrique. Ces nouveaux estimateurs de la fonction de répartition constituent alors la base pour des estimateurs plus robustes des mesures d'inégalité. Ces estimateurs sont développés pour les quatre classes les plus populaires des mesures d'inégalité, à savoir Gini, entropie généralisée, Atkinson et rapport des quintiles du revenu. Des propriétés de tels estimateurs sont étudiées par l'intermédiaire de simulations. En utilisant la loi limite et le bootstrap, des intervalles de confiance ont été construits. Par l'intermédiaire de simulations, les estimateurs proposés sont comparés à ceux habituels en termes d'erreur quadratique moyenne, effet relatif de contamination, longueur moyenne d'intervalle de confiance et probabilité de couverture. Dans ces études les méthodes semi-paramétriques montrent une amélioration claire par rapport aux méthodes habituelles. Les propriétés théoriques du rapport des quintiles n'ont pas été beaucoup étudiées dans la littérature. Par conséquent, nous dérivons également sa fonction d'influence aussi bien que la loi normale limite de son estimateur non paramétrique. Ces résultats n'ont pas été précédemment publiés.

Afin d'illustrer les méthodes développées, nous les appliquons à un certain nombre de données réelles. En utilisant de telles données, nous montrons comment ces méthodes peuvent être appliquées dans la pratique pour l'inférence statistique. Afin de choisir entre les distributions paramétriques candidates, nous utilisons une mesure de représentativité de l'échantillon proposée dans la littérature. Ces illustrations prouvent que les méthodes proposées sont fiables et peuvent être utilisées pour tirer des conclusions satisfaisantes sur des problèmes de la vie réelle.

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"Jehovah is my Shepherd. I shall lack nothing" ***Psalm 23:1***

Contents

Abbreviations

$ABias$	Asymptotic Bias
$ACIL$	Average Confidence Interval Length
$AMSE$	Asymptotic Mean Squared Error
$AVar$	Asymptotic Variance
$A\nu$	Atkinson measure with parameter ν
BCa	Bootstrap Calibrated and Accelerated
$BCAI$	BCa interval
$BPGPDI$	Bootstrap percentile interval when using GPD in the tail
BPI	Bootstrap percentile interval
$BPPI$	Bootstrap percentile interval when using Pa in the tail
$BTGPDI$	Bootstrap t interval when using GPD in the tail
BTI	Bootstrap t interval
$BTPI$	Bootstrap t interval when using Pa in the tail
$Bu1$	Burr distribution with $\alpha = 2$, $\tau = 0.83$ and $\lambda = 1$
$Bu2$	Burr distribution with $\alpha = 1$, $\tau = 1.4$ and $\lambda = 1$
$Bu3$	Burr distribution with $\alpha = 0.5$, $\tau = 4$ and $\lambda = 1$
CI	Confidence Interval
CIl	Confidence Interval length
CIS	Confidence Interval shape
CP	Coverage Probability
EVI	Extreme Value Index
EVT	Extreme Value Theory
$Fr1$	Frechet distribution with $\alpha = 2$
$Fr2$	Frechet distribution with $\alpha = 1.7$
GE	Generalized Entropy
GEV	Generalized Extreme Value
$GE\nu$	Generalized Entropy measure with parameter ν
GPD	Generalized Pareto Distribution
IF	Influence Function
$i.i.d.$	Independent and identically distributed
ISE	Integrated Squared Error
$LNCP$	Lower Non-Coverage Probability

MAD	Median Absolute Deviation
$MCap$	Market Capitalization
MEF	Mean Excess Function
MEP	Mean Excess Plot
MLD	Mean Logarithmic Deviation
MLE	Maximum Likelihood Estimator
MSE	Mean Squared Error
NP	Nonparametric
Pa	Pareto distribution
PDC	Partial Density Component
PPD	Perturbed Pareto Distribution
$PTBCAI$	Power Transformed bootstrap calibrated and accelerated interval
$PTBPI$	Power Transformed bootstrap percentile interval
QSR	Quintile Share Ratio
RIC	Relative Impact of Contamination
RIF	Relative Influence Function
SE	Standard Error
SNI	Standard Normal interval
SP	Semi-Parametric
$SPCo$	Cowell and Flachaire semi-parametric method
$SPGPD$	Semi-parametric method with GPD in the tail
$SPPa$	Semi-parametric method with strict Pareto in the tail
$SPPPD$	Semi-parametric method with PPD in the tail
STI	Student t interval
$T2$	Student t distribution with 2 degrees of freedom
$UNCP$	Upper Non-Coverage Probability

Notation

F	Distribution function of a random variable X
f	Density function of a random variable X
$\overline{F}$	Survival function of a random variable X with distribution function F ($\overline{F} = 1 - F$)
F_n	Empirical distribution function
$H_{k,n}$	Hill estimator
Q	Quantile function
Φ	Standard normal distribution function
ϕ	Standard normal density function
μ_k	Population moment of order k
m_k	Sample moment of order k
t_n	Student t distribution with n degrees of freedom
$T_n \xrightarrow{D} T$	The random variable T_n converges in distribution to T_n
$T_n \xrightarrow{P} T$	The random variable T_n converges in probability to T_n
U	Tail quantile function defined by $U(y) = Q\left(1 - \frac{1}{y}\right),\ y > 1$
$[x]$	Largest integer less than or equal to x

List of Figures

List of Tables

Chapter 1

Introduction

Statistical inference is the process of drawing conclusions from data that are subject to random variation. More substantially, the terms statistical inference, statistical induction and inferential statistics are used to describe systems of procedures that can be used to draw conclusions from data sets arising from systems affected by random variation. See e.g. the 2008 Oxford Dictionary of Statistics. Initial requirements of such a system of procedures for inference and induction are that the system should produce reasonable answers when applied to well-defined situations and that it should be general enough to be applied across a range of situations. There are many contexts in which inference is desirable, and there are many approaches to performing inference. This study particularly addresses statistical inference for inequality measures based on semi-parametric estimators. In this chapter we briefly give some background ideas and we state the problem under consideration. We then describe the scope of the study and we give its main contributions. Finally we provide an outline of the chapters to follow.

1.1 Background and Problem Statement

Economic inequality is an important concept in any society and even more so in a developing country where often a high level of inequality exists. It is therefore essential that reliable measures of inequality be defined and their properties investigated. Such measures, also used as measures of concentration or diversity, are very popular in economics and especially in measuring the inequality in income or wealth within a population and between populations. However, they have applications in many other fields such as ecology (see e.g. Magurran [40]), linguistics (see e.g. Herdan [31]), sociology (see e.g. Allison [1]), demography (see e.g. White [55]), epidemiology (see e.g. Harper and Lynch [30]) and information science (see e.g. Rousseau [48]), just to mention a few.

Over the years a large number of these measures have been proposed. Some of the most well known ones are the Gini index, the generalized entropy, the Atkinson and the quintile share ratio measures. Recently the Laeken European Council has adopted the so called Laeken Indicators which cover four important dimensions of social inclusion (financial poverty, employment, health and education), highlighting the multidimensionality of the phenomenon of social exclusion. Some of these indicators are measures of inequality. In particular, they include the Gini index and the quintile share ratio.

Having reliable inequality measures available is an important first step. A next step is to estimate the values of these measures using samples from the appropriate populations and, in particular, to estimate the variability of these estimators and more generally, to obtain confidence intervals for these measures. Since inequality is inherently dependent on the tails of a population, estimators of inequality are typically overly sensitive to data from these tails. We note in this regard that all the well known inequality measures have unbounded influence functions. It is well known that income distributions often exhibit a long tail to the right, making estimators of inequality particularly sensitive to large values. It is thus important to study the behavior of estimators based on data from heavy-tailed distributions. Many of the traditional estimators are sensitive to such extreme data points (see e.g. Cowell and Flachaire [10]) and remedial action needs to be taken. This remedial action can be either a trimming of the extreme data or a modification of the estimator to make it more robust to extreme observations.

Cowell and Flachaire [10] (see also Cowell and Victoria-Feser [12]) have proposed a so-called semi-parametric approach to modify estimators under heavy-tailed distributions. This method estimates the left part of the distribution, where the bulk of the distribution resides, using the usual nonparametric empirical distribution function and the right (upper) part of the distribution using a Pareto distribution. The resulting estimator is therefore partly nonparametric and partly parametric, hence semi-parametric. Based on their idea, results from extreme value theory are used in this thesis to obtain more reliable distribution estimators. These new estimators of the distribution form the basis for more robust estimators of the measures of inequality.

The bootstrap has in recent years become a very powerful technique for estimating variances of complex statistics and obtaining confidence intervals based on such statistics (see e.g. Efron and Tibshirani [26]). It has also been applied successfully to estimators of some inequality measures (see e.g. Davidson and Flachaire [18]). This is an extremely useful technique that can also be applied in the semi-parametric setting.

1.2 Scope and Contributions of the Study

Given a data set, a very important issue is to determine from which distribution the data are likely to have come from. In practice many methods are based on the empirical distribution function, which can lead to misleading conclusions especially when dealing with heavy-tailed distributions, e.g. income distributions. The main objective of this thesis is to develop improved inference for inequality measures in the case of heavy-tailed distributions. We use tools from Extreme Value Theory as basis from which to propose estimators for measures of inequality in a semi-parametric setting. The main contributions of this thesis are as follows.

1. We develop new semi-parametric estimators for the underlying distribution based on results from Extreme Value Theory (EVT). This is done by fitting three different parametric distributions in the tails, namely the Generalized Pareto distribution (GPD), the strict Pareto distribution, and the Perturbed Pareto Distribution (PPD).

2. Over the years nonparametric estimators for inequality measures have been used. Using the semi-parametric estimators for the underlying distribution, we develop semi-parametric estimators for four important measures of inequality, namely the Gini index, the Generalized Entropy (GE), the Atkinson and the Quintile Share Ratio (QSR) measures.

3. The influence function of the QSR and the limiting distribution of its nonparametric estimator are not available in literature. Both of these are derived in this study.

4. Sampling distributions of the semi-parametric estimators are studied via simulation. It is shown that the sampling distributions of semi-parametric estimators are better approximated by the limiting normal distribution.

5. With an extensive simulation study we show that in terms of mean squared errors, the semi-parametric estimators show improved performance over the nonparametric estimators as well as over the proposal of Cowell and Flachaire [10].

6. With an extensive simulation, we study the sensitivity of the estimators to outliers. Using the relative impact of contamination, we show that the proposed semi-parametric estimators are less sensitive to contamination or outliers than their nonparametric counterparts.

7. We carry out an extensive simulation to study confidence intervals for the inequality measures. Various confidence intervals were considered, viz. the standard normal, the Student t and boot-

strap intervals. These confidence intervals were obtained based on both the traditional estimators as well as on the semi-parametric estimators. In the simulation we studied the performance of the intervals in terms of the average confidence interval lengths and coverage probabilities. It appeared in most cases that confidence intervals based on semi-parametric estimators outperform the methods based on traditional estimators. Given the fact that confidence intervals are obtained for complex measures, the proposed procedures do remarkably well. The performance of the bootstrap is also remarkably good, given the complexity of the statistics underlying the bootstrap procedures.

8. Illustrations are given in order to show how the methods developed can be applied to real life data sets. The usual methods as well as the semi-parametric ones are applied to three data sets, claims data from a South African short term insurer, Norwegian fire insurance data and 2005 South African income and expenditure survey data. The illustrations show that the proposed procedures do remarkably well and can be used in practice to reach satisfactory conclusions. In order to choose between the three parametric distributions, we propose that a measure of sample representativeness be used. This was applied to the three data sets to choose the appropriate parametric distribution to use in the tail estimation.

1.3 Outline of the Study

In Chapter 2 we provide a list of definitions of inequality measures and their properties, and we give an overview of the approaches to the inequality measurement. Chapter 3 is devoted to describing a number of popular heavy-tailed distributions. In that chapter we also study the sampling distributions of the nonparametric estimators of the inequality measures described in Chapter 2. In Chapter 4 we review some methods of estimating the Extreme Value Index (EVI) and choosing the threshold above which to apply the parametric distribution when estimating the underlying distribution in a semi-parametric setting. In Chapter 5 we discuss the semi-parametric estimation of measures of inequality, in particular, the Gini, the generalized entropy, the Atkinson and the quintile share ratio measures. In the same chapter we also study the sampling distributions of the semi-parametric estimators. Chapter 6 provides an overview of various methods for constructing confidence intervals. A simulation study is conducted in Chapter 7 in order to assess some properties of the estimators and their performance in terms of confidence intervals. Chapter 8 illustrates how the different techniques described can be used in practice. Chapter 9 is devoted to conclusions and to indicate some areas that can be investigated in further research. Since the simulations generated many tables and graphs, these are mostly given in the appendices in order not to clutter the main text.

Chapter 2

Literature Overview

A large body of literature is devoted to the measurement of inequality. Many papers showed how to approach the measurement of inequality statistically, by describing the use of typical sample designs. However, extreme values in the data can have a detrimental effect on the estimators of inequality measures, especially when using some of the classical methods of estimation. This problem has been tackled by some authors. In this chapter we provide a list of definitions of inequality indices and their properties, and we give an overview of the approaches to inequality measurement. We start off by introducing the Lorenz curves and the influence functions.

2.1 Lorenz Curves

Lorenz curves constitute an important tool for analyzing economic inequality. As mentioned by Schluter and Trede [50], in the case of income distribution the Lorenz curve depicts the cumulative income share of the least well-off fraction of the population.

Definition 2.1. Let $\Im$ be the set of all univariate probability distributions with support $(0,\infty)$, and let X be a random variable with probability distribution $F \in \Im$. The Lorenz curve of X is given by (see Schluter and Trede [51])

$$\{(q, C(F;q)), 0 \leq q \leq 1\}, \tag{2.1}$$

where C is the cumulative functional defined by

$$C(F;q) = \int_0^\infty x\mathbf{1}\{x \leq F^{-1}(q)\}dF(x) = \int_0^{F^{-1}(q)} xdF(x), \tag{2.2}$$

and $\mathbf{1}\{.\}$ is the indicator function.

Consider the normalized functional

$$L(F;q) := \frac{C(F;q)}{\mu}, \tag{2.3}$$

where $\mu = C(F;1)$ is the mean functional.

The graph of $C(F;q)$ versus q describes the generalized Lorenz curve (GLC), and the graph of $L(F;q)$ versus q describes the relative Lorenz curve (RLC).

A Lorenz curve shows the degree of inequality that exists in the distributions, and is often used to illustrate the extent to which income or wealth is distributed unequally in a particular society. As we will later see, many inequality measures are closely related to the Lorenz curve.

2.2 Influence Functions

In robust statistics the influence function was developed as an important measure of sensitivity of estimators to large values. See e.g. Huber and Ronchetti [33] for discussion of this.

Definition 2.2. Let Δ_z be a point mass distribution giving probability 1 to an arbitrary point $z \in (0,\infty)$. Define the mixture distribution

$$F_{\varepsilon}^{(z)}(x) = (1-\varepsilon)F(x) + \varepsilon\Delta_z(x), \text{ for } \varepsilon \in [0,1]. \tag{2.4}$$

The influence function (IF) of a functional $T(F)$ is defined as

$$IF(z;T) = \lim_{\varepsilon\to 0}\frac{T(F_{\varepsilon}^{(z)}) - T(F)}{\varepsilon} = \frac{\partial}{\partial\varepsilon}T(F_{\varepsilon}^{(z)})|_{\varepsilon=0}. \tag{2.5}$$

Remark 2.1.

1. The relative influence function (RIF) is defined as

$$RIF(T) = \frac{IF(z;T)}{T(F)}. \tag{2.6}$$

2. The functional $T(F)$ can be estimated by the plug-in estimator $T(F_n)$, where F_n is the empirical distribution function of the sample $X_1, X_2, \ldots, X_n$, defined by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{X_i \le x\}. \tag{2.7}$$

The relative impact of a contamination on the functional T is defined by

$$RIC(T) = \frac{T(F_n) - T(F_n^\star)}{T(F_n)}, \tag{2.8}$$

where $F_n^\star$ is the empirical distribution function of the contaminated sample.

See e.g. Cowell and Flachaire [10].

2.3 Inequality Measures: Definitions and Some Properties

In order to measure inequality, a number of coefficients such as the generalized entropy (GE), the Atkinson, the quintile share ratio and the Gini measures, have been introduced in the literature. In this section we describe these inequality measures and we give some of their properties.

2.3.1 Generalized Entropy Measures of Inequality

The concept of comparing distributions using information-theoretic approaches has involved using entropy-based measures which quantify the discrepancies between the probability distributions (see Cowell et al. [11]). First introduced by Shannon [53], this concept was further developed into a relative measure of entropy by Kullback and Leibler [38]. Cowell et al. [11] then showed that generalized entropy measures are obtained by a change of variables from these entropy measures.

Let Y be a random variable distributed on the nonnegative real line, and let f be its probability density function. The generalized entropy (GE) inequality measure is defined by (see e.g. Cowell and Flachaire [10], Cowell et al. [11]):

$$I_E^\alpha = \int_0^\infty \frac{1}{\alpha(\alpha-1)}\left[\left(\frac{y}{\mu_1}\right)^\alpha - 1\right] dF(y) = \frac{1}{\alpha(\alpha-1)}\left(\frac{\mu_\alpha}{\mu_1^\alpha} - 1\right), \ \alpha \neq 0, 1, \tag{2.9}$$

where

$$\mu_\alpha = \int_0^\infty y^\alpha dF(y) \text{ and } \mu_1 = E(Y).$$

The motivation of Equation (2.9) is as follows: Consider Shannon's entropy defined as the expected information (see Shannon [53])

$$H(f) := -E[\log f(Y)] = -\int_0^\infty f(y)\log f(y)dy. \tag{2.10}$$

Letting $g(f) = -\log f$, we see that g is convex with $g(1) = 0$. It also has an additive property. The latter property is not essential and the above can be generalized to functions g_α which are convex and for which $g_\alpha(1) = 0$ (see e.g. Khinchin [35]).

An important special case is given by

$$g_\alpha(f) = \frac{1}{\alpha - 1}\left[1 - f^\alpha\right],\ \alpha > 0, \alpha \neq 1. \tag{2.11}$$

From (2.11), a generalization of (2.10) is obtained as

$$H_\alpha(f) := Eg_\alpha(f(Y)) = \frac{1}{\alpha - 1}\left[1 - E\left(f(Y)^{\alpha-1}\right)\right],\ \alpha > 0, \alpha \neq 1. \tag{2.12}$$

In order to link this entropy to inequality, the transformation $s : [0,1] \to [0,1]$ given below can be used (see Cowell et al. [11]).

Define

$$s(q) := \frac{F^{-1}(q)}{\int_0^1 F^{-1}(t)dt} = \frac{y}{\mu_1}, \tag{2.13}$$

where F is the distribution function of Y such that a proportion $q = F(y)$ of the population has a value less than y, μ_1 is the mean of the distribution. See [11] for an interpretation of the transformation s.

It is clear that the function s has the same properties as a regular density function:

$$s(q) \geq 0,\ \text{for all } q \text{ and } \int_0^1 s(q)dq = 1. \tag{2.14}$$

Substituting s for f in Equation (2.12) leads to

$$H_\alpha(s) = -\alpha I_E^\alpha. \tag{2.15}$$

Thus

$$I_E^\alpha = -\alpha^{-1} H_\alpha(s). \tag{2.16}$$

As mentioned in [11], the parameter α has a natural interpretation in terms of economic welfare: for $\alpha > 0$, the measure I_E^α is "top-sensitive" in that it gives higher importance to changes in the top of the distribution.

The equivalent measure for a finite population $y_1, y_2, \ldots, y_N$, is given in Cowell [9] as

$$I_E^{\alpha} = \frac{1}{\alpha(\alpha-1)}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{y_i}{\bar{y}}\right)^{\alpha} - 1\right], \tag{2.17}$$

where

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i.$$

The influence function of I_E^{α} is given by

$$IF(z; I_E^{\alpha}) = [z^{\alpha} - \mu_{\alpha}] - \frac{\mu_{\alpha}}{(\alpha-1)\mu_1^{\alpha+1}}[z - \mu_1], \alpha \neq 0, 1, \tag{2.18}$$

see Cowell and Flachaire [10], and for any given value of α, it is unbounded:

1. If $\alpha > 1$ the IF tends to infinity when $z \to \infty$ at the rate of z^{α};
2. If $0 < \alpha < 1$ the IF tends to infinity when $z \to \infty$ at the rate of z;
3. If $\alpha < 0$ the IF tends to infinity when $z \to \infty$ at the rate of z, and when $z \to 0$ at the rate of z^{α}.

Mean Logarithmic Deviation

The mean logarithmic deviation (MLD) measure is a special case of the GE class where $\alpha = 0$. It follows directly from Equation (2.16) that

$$I_E^0 = -\int_0^{\infty} \log\left(\frac{y}{\mu_1}\right) dF(y) = \log \mu_1 - \nu, \tag{2.19}$$

where

$$\nu = \int_0^{\infty} (\log y) dF(y).$$

The influence function, given by

$$IF(z; I_E^0) = -[\log z - \nu] + \frac{1}{\mu_1}[z - \mu_1], \tag{2.20}$$

tends to infinity at the rate z when $z \to \infty$ and at the rate of $\log z$ when $z \to 0$.

The mean logarithmic deviation for a finite population $y_1, y_2, \ldots, y_N$, is given by (see World Bank Institute [56])

$$I_E^0 = \frac{1}{N}\sum_{i=1}^{N} \log\left(\frac{\bar{y}}{y_i}\right). \tag{2.21}$$

Theil Measure

The Theil measure of inequality is also a special case of the GE class where $\alpha = 1$. It follows directly from Equation (2.16) that it is given by:

$$I_E^1 = \int_0^\infty \frac{y}{\mu_1} \log\left(\frac{y}{\mu_1}\right) dF(y) = \frac{\nu}{\mu_1} - \log \mu_1, \tag{2.22}$$

where now

$$\nu = \int_0^\infty y \log y dF(y).$$

The influence function, given by

$$IF(z; I_E^1) = \frac{1}{\mu_1}[z \log z - \nu] - \frac{\nu + \mu_1}{\mu_1^2}[z - \mu_1], \tag{2.23}$$

tends to infinity at the rate of $z \log z$ when $z \to \infty$.

The Theil Coefficient for a finite population $y_1, y_2, \ldots, y_N$, is given by (see [56]):

$$I_E^1 = \frac{1}{N}\sum_{i=1}^{N} \frac{y_i}{\bar{y}} \log\left(\frac{y_i}{\bar{y}}\right). \tag{2.24}$$

2.3.2 Atkinson Class of Inequality Measures

The Atkinson measure of inequality is defined by (see Cowell and Flachaire [10]):

$$I_A^\varepsilon = 1 - \left[\int_0^\infty \left(\frac{y}{\mu}\right)^{1-\varepsilon} dF(y)\right]^{1/(1-\varepsilon)} = 1 - \frac{\mu_{1-\varepsilon}^{1/(1-\varepsilon)}}{\mu}, \ \varepsilon > 0, \varepsilon \neq 1, \tag{2.25}$$

where

$$\mu_{1-\varepsilon} = \int_0^\infty y^{1-\varepsilon} dF(y).$$

The special case where $\varepsilon = 1$ is given by

$$I_A^1 = 1 - \frac{e^{\int_0^\infty (\log y) dy}}{\mu} = 1 - e^{-I_E^0}. \tag{2.26}$$

The Atkinson measure I_A^{ε} is a nonlinear transformation of the GE measure I_E^{α}: for $\varepsilon = 1-\alpha > 0$,

$$I_A^{1-\alpha} = 1 - [(\alpha^2-\alpha)I_E^{\alpha}+1]^{1/\alpha}.$$

With that relationship the Atkinson measures basically play the same role as the generalized entropy measures. See Cowell et al. [11].

The influence function, given by (see Cowell and Flachaire [10])

$$IF(z;I_A^{\varepsilon}) = \frac{\mu_{1-\varepsilon}^{1/(1-\varepsilon)}}{(\varepsilon-1)\mu}[z^{1-\varepsilon}-\mu_{1-\varepsilon}] + \frac{\mu_{1-\varepsilon}^{1/(1-\varepsilon)}}{\mu^2}[z-\mu], \tag{2.27}$$

has the following properties:

1. If $0<\varepsilon<1$ the IF tends to infinity when $z\to\infty$ at the rate of z;
2. If $\varepsilon>1$ the IF tends to infinity when $z\to\infty$ at the rate of z, and when $z\to 0$ at the rate of $z^{1-\varepsilon}$.

For $\varepsilon = 1$, the influence function is given by

$$IF(z;I_A^1) = -\frac{e^{\int_0^{\infty}(\log y)dy}}{\mu}[\log z - \int_0^{\infty}(\log y)dy] + \frac{e^{\int_0^{\infty}(\log y)dy}}{\mu^2}[z-\mu], \tag{2.28}$$

which tends to infinity when $z\to\infty$ at the rate of z, and when $z\to 0$ at the rate of $\log z$.

The Atkinson measure of inequality for a finite population $y_1,y_2,\ldots,y_N$, is given by (see [56]):

$$I_A^{\varepsilon} = 1 - \left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{y_i}{\bar{y}}\right)^{1-\varepsilon}\right]^{1/(1-\varepsilon)}, \ \varepsilon\neq 1, \tag{2.29}$$

and

$$I_A^1 = 1 - \frac{1}{\bar{y}}\prod_{i=1}^{N}(y_i^{1/N}). \tag{2.30}$$

2.3.3 Relationship Between the Lorenz Curve and the GE Measures and Between the Lorenz Curve and Atkinson Measures

The relative Lorenz curve was defined in Equation (2.3) as

$$L(F;q) := \mu^{-1}C(F;q). \tag{2.31}$$

Consider the quantile function

$$Q(F;q) \equiv F^{-1}(q) = \inf\{x|F(x) \geq q\}. \tag{2.32}$$

Note that

$$L'(F;q) \equiv \frac{d}{dq}L(F;q) = \mu^{-1}Q(F;q). \tag{2.33}$$

Since

$$I_E^{\alpha} = [\alpha(\alpha-1)]^{-1}\int_0^{\infty}\left[\left(\frac{y}{\mu}\right)^{\alpha}-1\right]dF(y), \tag{2.34}$$

it follows that

$$I_E^{\alpha} = [\alpha(\alpha-1)]^{-1}\int_0^1\left[\left(\mu^{-1}Q(F;u)\right)^{\alpha}-1\right]du = [\alpha(\alpha-1)]^{-1}\int_0^1\left[\left(L'(F;u)\right)^{\alpha}-1\right]du. \tag{2.35}$$

Similarly, we have

$$I_A^{\varepsilon} = 1-\left[\int_0^1\left(\mu^{-1}Q(F;u)\right)^{1-\varepsilon}du\right]^{1/(1-\varepsilon)} = 1-\left[\int_0^1\left(L'(F;u)\right)^{1-\varepsilon}du\right]^{1/(1-\varepsilon)}. \tag{2.36}$$

2.3.4 Gini Coefficient

The Gini Coefficient is the most widely used measure of inequality. It is defined by (see e.g. Cowell and Flachaire [10]):

$$I_G = 1-2\int_0^1 L(F;p)dp, \tag{2.37}$$

where $L(.)$ denotes the Lorenz curve as given in Equation (2.3).

The Gini coefficient lies between 0 and 1. The value 0 corresponds to perfect equality and the value 1 corresponds to perfect inequality. In an economic situation, perfect equality means the wealth is uniformly distributed over all the individuals in the population, and perfect inequality means the entire wealth goes to only one individual.

Letting

$$A = \int_0^1 L(F;p)dp$$

and

$$B = \int_0^1 pdp = \frac{1}{2},$$

it easily follows from Equation (2.37) that

$$I_G = 2(B - A),$$

therefore, twice the area between the 45° line and the Lorenz curve, lying below the line.

From Pan American Health Organization [43], the Lorenz curve is an accumulated frequency curve that compares the distribution of a specific variable with the uniform distribution that represents equality. This equality distribution is represented by a straight line of slope one, and the greater the deviation of the Lorenz curve from this line, the greater the Gini Coefficient.

Many alternative expressions for Gini have been given in the literature. The most important ones are the following (see e.g. Davidson [17], Dorfman [21], Duclos and Araar [24]):

- $$I_G = \frac{2}{\mu}\int_0^\infty yF(y)dF(y) - 1, \tag{2.38}$$

- $$I_G = 1 - \frac{2}{\mu}\int_0^\infty y(1 - F(y))dF(y), \tag{2.39}$$

- $$I_G = \frac{1}{\mu}\int_0^\infty (2F(y) - 1)ydF(y), \tag{2.40}$$

- $$I_G = 1 - \frac{1}{\mu}\int_0^\infty (1 - F(y))^2 dy, \tag{2.41}$$

- $$I_G = \frac{1}{\mu}\int_0^\infty F(y)(1 - F(y))dy, \text{ and} \tag{2.42}$$

- $$I_G = \frac{E|X - Y|}{2\mu}, \tag{2.43}$$

where X and Y are independent, having the same distribution function F with mean μ.

Remark 2.2. These formulas for the Gini coefficient are all related to one another. These relationships will now be proved.

- Equation (2.39) is derived from Equation (2.38) as follows:

$$\begin{aligned}
I_G &= \frac{2}{\mu}\int_0^\infty yF(y)dF(y) - 1 \\
&= \frac{2}{\mu}\left[-\int_0^\infty y(1-F(y))dF(y) + \mu\right] - 1 \\
&= -\frac{2}{\mu}\int_0^\infty y(1-F(y))dF(y) + 2 - 1 \\
&= 1 - \frac{2}{\mu}\int_0^\infty y(1-F(y))dF(y).
\end{aligned}$$

- Equation (2.40) is derived from Equation (2.38) as follows:

$$\begin{aligned}
I_G &= \frac{2}{\mu}\int_0^\infty yF(y)dF(y) - 1 \\
&= \frac{1}{\mu}\left[\int_0^\infty 2yF(y)dF(y) - \mu\right] \\
&= \frac{1}{\mu}\left[\int_0^\infty 2yF(y)dF(y) - \int_0^\infty ydF(y)\right] \\
&= \frac{1}{\mu}\int_0^\infty (2F(y)-1)ydF(y).
\end{aligned}$$

- Equation (2.39) is derived from Equation (2.37) as follows:

$$\begin{aligned}
I_G &= 1 - 2\int_0^1 L(F;p)dp \\
&= 1 - \frac{2}{\mu}\int_0^1\int_0^{F^{-1}(p)} ydF(y)dp \\
&= 1 - \frac{2}{\mu}\int_0^\infty y\int_{F(y)}^1 dpdF(y) \\
&= 1 - \frac{2}{\mu}\int_0^\infty y(1-F(y))dF(y).
\end{aligned}$$

- Equation (2.41) is derived from Equation (2.39) as follows:

$$\begin{aligned}
I_G &= 1 - \frac{2}{\mu}\int_0^\infty y(1-F(y))dF(y) \\
&= 1 + \frac{2}{\mu}\int_0^\infty y(1-F(y))d(1-F(y)) \\
&= 1 - \frac{2}{\mu}\int_0^\infty (1-F(y))^2dy + \frac{2}{\mu}\int_0^\infty y(1-F(y))dF(y) \\
&\quad \text{(by integration by parts)} \\
&= 1 - \frac{2}{\mu}\int_0^\infty (1-F(y))^2dy + 1 - I_G.
\end{aligned}$$

It follows that

$$2I_G = 2 - \frac{2}{\mu}\int_0^{\infty}(1-F(y))^2 dy,$$

thus

$$I_G = 1 - \frac{1}{\mu}\int_0^{\infty}(1-F(y))^2 dy.$$

- Equation (2.42) is derived from Equation (2.41) as follows:

$$\begin{aligned}
I_G &= 1 - \frac{1}{\mu}\int_0^{\infty}(1-F(y))^2 dy \\
&= 1 - \frac{1}{\mu}\int_0^{\infty}(1-F(y))(1-F(y))dy \\
&= 1 - \frac{1}{\mu}\int_0^{\infty}(1-F(y))dy + \frac{1}{\mu}\int_0^{\infty}F(y)(1-F(y))dy \\
&= \frac{1}{\mu}\int_0^{\infty}F(y)(1-F(y))dy.
\end{aligned}$$

- Equation (2.43) is derived from Equation (2.39) as follows:

$$\begin{aligned}
E|X-Y| &= \int_0^{\infty}\int_0^{\infty}|x-y|dF(x)dF(y) \\
&= 2\int_0^{\infty}\int_0^{x}(x-y)dF(y)dF(x) \\
&= 2\int_0^{\infty}xF(x)dF(x) - 2\int_0^{\infty}\int_0^{x}ydF(y)dF(x) \\
&= -2\int_0^{\infty}x(1-F(x))dF(x) + 2\mu - 2\int_0^{\infty}y(1-F(y))dF(y) \\
&= 2\mu - 4\int_0^{\infty}y(1-F(y))dF(y) \\
&= 2\mu\left(1 - \frac{2}{\mu}\int_0^{\infty}y(1-F(y))dF(y)\right) \\
&= 2\mu I_G \text{ (using Equation (2.39))},
\end{aligned}$$

thus

$$I_G = \frac{E|X-Y|}{2\mu}.$$

In our subsequent work, use will be made of Equation (2.42) for the Gini coefficient, as it is very convenient for estimation purposes.

The influence function of I_G is given by (see e.g. Cowell and Flachaire [10])

$$IF(z;I_G) = 2\left[R(F) - C(F;F(z)) + \frac{z}{\mu}(R(F)-(1-F(z)))\right], \tag{2.44}$$

where

$$R(F) = \int_0^1 L(F;p)dp \tag{2.45}$$

and C is as given in Equation (2.2).

This influence function tends to infinity at the rate of z when $z \to \infty$.

Remark 2.3. The above formulation of the Gini coefficient has been given for the case of an infinite population. In the case of finite populations, the integrals must be replaced by the corresponding sums. For example in this case Equation (2.43) becomes

$$I_G = \frac{1}{2N^2\bar{y}} \sum_{i=1}^{N} \sum_{j=1}^{N} |y_j - y_i| \tag{2.46}$$

for a finite population $y_1, y_2, \ldots, y_N$.

2.3.5 Quintile Share Ratio Measure of Inequality

Consider a random variable Y with distribution function F and denote by Q its quantile function. For simplification purposes we will use $Q(q)$ for $Q(F;q)$ defined in Equation (2.32).

Definition 2.3. The quintile share ratio (QSR) is defined by

$$\eta = \frac{\int_{Q(0.8)}^{\infty} ydF(y)}{\int_0^{Q(0.2)} ydF(y)} = \frac{EY\mathbf{1}\{Y > Q(0.8)\}}{EY\mathbf{1}\{Y \leq Q(0.2)\}}, \tag{2.47}$$

where $\mathbf{1}\{.\}$ is an indicator function.

In the case of income, the QSR can be interpreted as the ratio of the total income received by the 20% of a country's population with highest income to that received by the 20% of the country's population with the lowest income (see Hulliger and Schoch [34]).

The QSR for a finite population $y_1, y_2, \ldots, y_N$, is given by:

$$\eta = \left[\sum_{i=[0.8N]+1}^{N} Y_{i,N} \right] / \left[\sum_{i=1}^{[0.2N]} Y_{i,N} \right], \tag{2.48}$$

where $Y_{1,N} < Y_{2,N} < \ldots < Y_{N,N}$ are the order statistics associated with the finite population and $[x]$ is the largest integer smaller than or equal to x.

Remark 2.4. The QSR forms part of the so-called Laeken indicators, the European indicators on poverty and social exclusion (see EU-SILC [28]).

The influence function of the QSR is not available yet in the literature. This is derived in the next theorem.

Theorem 2.1. The influence function of the quintile share ratio η in Equation (2.47) is given by:

$$IF(z;\eta) = \begin{cases} [-zN(F)+0.2Q(0.8)D(F)+0.8Q(0.2)N(F)]/D^2(F), \text{ if } z \leq Q(0.2), \\ [0.2Q(0.8)D(F)-0.2Q(0.2)N(F)]/D^2(F), \text{ if } Q(0.2) < z \leq Q(0.8), \\ [zD(F)-0.8Q(0.8)D(F)-0.2Q(0.2)N(F)]/D^2(F), \text{ if } z > Q(0.8), \end{cases} \tag{2.49}$$

where

$$N(F) = \int_{Q(0.8)}^{\infty} xdF(x) \tag{2.50}$$

and

$$D(F) = \int_{0}^{Q(0.2)} xdF(x). \tag{2.51}$$

Proof. Consider the mixture distribution $F_{\varepsilon,z}$ defined in Equation (2.4), and define the influence function as in Equation (2.5). Denoting by $Q_{\varepsilon}^{(z)}$ the quantile function associated with $F_{\varepsilon}^{(z)}$, we have

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_{Q_{\varepsilon}^{(z)}(0.8)}^{\infty} xdF(x) - \int_{Q(0.8)}^{\infty} xdF(x)\right] \\ = \frac{\partial}{\partial\varepsilon}\int_{Q_{\varepsilon}^{(z)}(0.8)}^{\infty} xdF(x)|_{\varepsilon=0} \\ = -Q_{\varepsilon}^{(z)}(0.8)f(Q_{\varepsilon}^{(z)}(0.8))\frac{\partial}{\partial\varepsilon}Q_{\varepsilon}^{(z)}(0.8)|_{\varepsilon=0} \\ = -Q(0.8)f(Q(0.8))IF(z;Q(0.8)). \end{aligned}$$

It follows that for $\varepsilon \to 0$,

$$\begin{aligned} \int_{Q_{\varepsilon}^{(z)}(0.8)}^{\infty} xdF(x) &= \int_{Q(0.8)}^{\infty} xdF(x) - \varepsilon Q(0.8)f(Q(0.8))IF(z;Q(0.8)) + o(\varepsilon) \\ &= N(F) - \varepsilon Q(0.8)f(Q(0.8))IF(z;Q(0.8)) + o(\varepsilon). \end{aligned}$$

On the other hand we have

$$\begin{aligned} \varepsilon\int_{Q_{\varepsilon}^{(z)}(0.8)}^{\infty} xd\Delta_z(x) &= \varepsilon z\mathbf{1}(z \geq Q_{\varepsilon}^{(z)}(0.8)) \\ &= \varepsilon z\mathbf{1}(z \geq Q(0.8))(1+o(1)). \end{aligned}$$

Therefore

$$\begin{aligned}
N(F_\varepsilon^{(z)}) &= (1-\varepsilon)\int_{Q_\varepsilon^{(z)}(0.8)}^{\infty} xdF(x) + \varepsilon\int_{Q_\varepsilon^{(z)}(0.8)}^{\infty} xd\Delta_z(x) \\
&= (1-\varepsilon)\left[N(F) - \varepsilon Q(0.8)f(Q(0.8))IF(z;Q(0.8)) + o(\varepsilon)\right] + \varepsilon\int_{Q_\varepsilon^{(z)}(0.8)}^{\infty} xd\Delta_z(x) \\
&= N(F) - \varepsilon N(F) - \varepsilon Q(0.8)f(Q(0.8))IF(z;Q(0.8)) + \varepsilon z\mathbf{1}(z \geq Q(0.8)) + o(\varepsilon).
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{N(F_\varepsilon^{(z)}) - N(F)}{\varepsilon} &= -N(F) - Q(0.8)f(Q(0.8))IF(z;Q(0.8)) + z\mathbf{1}(z \geq Q(0.8)) \\
&\equiv N'(F). \qquad (2.52)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_0^{Q_\varepsilon^{(z)}(0.2)} xdF(x) - \int_0^{Q(0.2)} xdF(x)\right] \\
&= \frac{\partial}{\partial\varepsilon}\int_0^{Q_\varepsilon^{(z)}(0.2)} xdF(x)|_{\varepsilon=0} \\
&= Q_\varepsilon^{(z)}(0.2)f(Q_\varepsilon^{(z)}(0.2))\frac{\partial}{\partial\varepsilon}Q_\varepsilon^{(z)}(0.2)|_{\varepsilon=0} \\
&= Q(0.2)f(Q(0.2))IF(z;Q(0.2)),
\end{aligned}$$

leading to

$$\begin{aligned}
\int_0^{Q_\varepsilon^{(z)}(0.2)} xdF(x) &= \int_0^{Q(0.2)} xdF(x) + \varepsilon Q(0.2)f(Q(0.2))IF(z;Q(0.2)) + o(\varepsilon) \\
&= D(F) + \varepsilon Q(0.2)f(Q(0.2))IF(z;Q(0.2)) + o(\varepsilon).
\end{aligned}$$

Furthermore we have,

$$\begin{aligned}
\varepsilon\int_0^{Q_\varepsilon^{(z)}(0.2)} xd\Delta_z(x) &= \varepsilon z\mathbf{1}(z \leq Q_\varepsilon^{(z)}(0.2)) \\
&= \varepsilon z\mathbf{1}(z \leq Q(0.2))(1+o(1)).
\end{aligned}$$

It follows that

$$
\begin{aligned}
D(F_{\varepsilon}^{(z)}) &= (1-\varepsilon)\int_{0}^{Q_{\varepsilon}^{(z)}(0.2)} x dF(x) + \varepsilon \int_{0}^{Q_{\varepsilon}^{(z)}(0.2)} x d\Delta_z(x) \\
&= (1-\varepsilon)\left[D(F) + \varepsilon Q(0.2) f(Q(0.2)) IF(z;Q(0.2)) + o(\varepsilon)\right] + \varepsilon \int_{0}^{Q_{\varepsilon}^{(z)}(0.2)} x d\Delta_z(x) \\
&= D(F) - \varepsilon D(F) + \varepsilon Q(0.2) f(Q(0.2)) IF(z;Q(0.2)) + \varepsilon z \mathbf{1}(z \leq Q(0.2)) + o(\varepsilon).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \frac{D(F_{\varepsilon}^{(z)}) - D(F)}{\varepsilon} &= -D(F) + Q(0.2) f(Q(0.2)) IF(z;Q(0.2)) + z\mathbf{1}(z \leq Q(0.2)) \\
&\equiv D'(F). \qquad (2.53)
\end{aligned}
$$

The influence function for the QSR η is then given by

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \frac{\eta_{\varepsilon} - \eta}{\varepsilon} &= \frac{\partial}{\partial \varepsilon} \eta_{\varepsilon}|_{\varepsilon=0} \\
&= \frac{\partial}{\partial \varepsilon} \frac{N(F_{\varepsilon}^{(z)})}{D(F_{\varepsilon}^{(z)})}|_{\varepsilon=0} \\
&= \frac{N'(F)D(F) - D'(F)N(F)}{D^2(F)}. \qquad (2.54)
\end{aligned}
$$

But

$$
\begin{aligned}
N'(F)D(F) - &D'(F)N(F) \\
&= D(F)\left[-N(F) - Q(0.8) f(Q(0.8)) IF(z;Q(0.8)) + z\mathbf{1}(z \geq Q(0.8))\right] \\
&\quad - N(F)\left[-D(F) + Q(0.2) f(Q(0.2)) IF(z;Q(0.2)) + z\mathbf{1}(z \leq Q(0.2))\right] \\
&= -D(F)Q(0.8) f(Q(0.8)) IF(z;Q(0.8)) + zD(F)\mathbf{1}(z \geq Q(0.8)) \\
&\quad - N(F)Q(0.2) f(Q(0.2)) IF(z;Q(0.2)) - zN(F)\mathbf{1}(z \leq Q(0.2))
\end{aligned}
$$

and

$$
\begin{aligned}
IF(z;Q(p)) &= \frac{\partial}{\partial \varepsilon} Q_{\varepsilon}^{(z)}(p)|_{\varepsilon=0} \\
&= \frac{\partial}{\partial \varepsilon} F_{\varepsilon}^{-1}(p)|_{\varepsilon=0},
\end{aligned}
$$

where

$$
F_{\varepsilon}(x) \equiv F_{\varepsilon}^{(z)}(x) = (1-\varepsilon)F(x) + \varepsilon \Delta_z(x)
$$

as in Equation (2.4).

In order to find the above derivative, we use the relationship

$$F_\varepsilon(F_\varepsilon^{-1}(p)) = p.$$

Differentiating this expression on both sides, using the rules of differentiating a composite function, gives

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}F_\varepsilon(F_\varepsilon^{-1}(p)) = 0 &\Rightarrow \frac{\partial}{\partial x}F_\varepsilon(x)|_{x=F_\varepsilon^{-1}(p)} \cdot \frac{\partial}{\partial\varepsilon}F_\varepsilon^{-1}(p) + \frac{\partial}{\partial\varepsilon}F_\varepsilon(x)|_{x=F_\varepsilon^{-1}(p)} = 0\\ &\Rightarrow \frac{\partial}{\partial\varepsilon}F_\varepsilon^{-1}(p) = -\frac{\partial}{\partial\varepsilon}F_\varepsilon(x)|_{x=F_\varepsilon^{-1}(p)} \cdot \left[\frac{\partial}{\partial x}F_\varepsilon(x)|_{x=F_\varepsilon^{-1}(p)}\right]^{-1}\\ &\Rightarrow \frac{\partial}{\partial\varepsilon}F_\varepsilon^{-1}(p) = -\frac{\partial}{\partial\varepsilon}F_\varepsilon(x)|_{x=F_\varepsilon^{-1}(p)} \cdot \left[f_\varepsilon(F_\varepsilon^{-1}(p))\right]^{-1}.\end{aligned}$$

Finally, taking the limit as $\varepsilon \to 0$ gives

$$\frac{\partial}{\partial\varepsilon}F_\varepsilon^{-1}(p)|_{\varepsilon=0} = [p - \mathbf{1}(z < Q(p))]f(Q(p))^{-1}. \tag{2.55}$$

Thus

$$IF(z; Q(p)) = \frac{1}{f(Q(p)}(p - \mathbf{1}(z < Q(p))). \tag{2.56}$$

Substituting the terms into Equation (2.54) gives directly

$$IF(z;\eta) = \begin{cases} [-zN(F) + 0.2Q(0.8)D(F) + 0.8Q(0.2)N(F)]/D^2(F), \text{ if } z \le Q(0.2),\\ [0.2Q(0.8)D(F) - 0.2Q(0.2)N(F)]/D^2(F), \text{ if } Q(0.2) < z \le Q(0.8),\\ [zD(F) - 0.8Q(0.8)D(F) - 0.2Q(0.2)N(F)]/D^2(F), \text{ if } z > Q(0.8).\end{cases} \tag{2.57}$$

This completes the proof of the theorem. □

Remark 2.5.

1. The fact that

$$IF(z;\eta) = [0.2Q(0.8)D(F) - 0.2Q(0.2)N(F)]/D^2(F) = Constant$$

   for $Q_F(0.2) < z < Q_F(0.8)$ is due to the fact that the definition of the QSR does not take into account the data values between $Q_F(0.2)$ and $Q_F(0.8)$; that is, those values do not influence the QSR measure.

2. Given the plug-in estimator $\eta(F_n)$ of $\eta(F) \equiv \eta$, we will have (under appropriate conditions) that

$$\sqrt{n}\left(\eta(F_n) - \eta(F)\right) \xrightarrow{D} N\left(0, \sigma_\eta^2\right), \tag{2.58}$$

where

$$\sigma_\eta^2 = \int_0^\infty IF^2(z;\eta)\,dF(z). \tag{2.59}$$

The variance σ_η^2 can be estimated using the plug-in method.

2.4 Current Estimation Procedures and Effects of Extreme Values

Having reliable inequality measures available is an important first step. A next step is to estimate the values of these measures using samples from the appropriate populations and, in particular, to estimate the variability of these estimators and more generally, to obtain confidence intervals for the measures. Since inequality is inherently dependent on the tails of a population, estimators of inequality are typically sensitive to data from these tails. In this section we discuss various approaches to the estimation problem.

2.4.1 Income Distribution and Inequality Measurement: The Problem of Extreme Values

Inequality measures can be very sensitive to changes in the distribution. In the case of income distribution for example, the data often has a long right tail, and this can seriously affect the estimation procedures for the measures. Thus it is appropriate to examine the behavior of inequality measures with respect to extreme values (see Cowell and Flachaire [10]). In [10] they examined statistical performance of inequality indices in the presence of extreme values in data and showed that these indices are very sensitive to the properties of the income distribution. They considered various inequality measures (Generalized entropy (GE), MLD Coefficient (GE with $\alpha = 0$), Theil Coefficient (GE with $\alpha = 1$), Atkinson, LogVar and Gini measures) in a semi-parametric, an asymptotic and a bootstrap setup. In the case of the bootstrap, two nonstandard methods were used: moon bootstrap (m out on n bootstrap) and semi-parametric bootstrap. In order to carry out their analysis, they made use of three different distributions: the Singh-Maddala, the Pareto and the lognormal distributions. All those distributions will be discussed further in this work (see Chapter 3). Amongst others, the following points were emphasized in [10].

1. The GE measures with $\alpha > 1$ are very sensitive to high incomes in the data.

2. The Gini Coefficient is less sensitive to contamination in high incomes than the GE class of measures.

3. The inequality measures computed with a semi-parametric estimation of the income distribution are much less sensitive to contamination.

4. The MLD Coefficient is more sensitive to contamination in high incomes when the underlying distribution has a heavy upper tail. Semi-parametric MLD measures are much less sensitive.

We will investigate the semi-parametric procedures further in this work (see e.g. Chapter 5).

2.4.2 Asymptotic and Bootstrap Inference for Inequality Measures

Bootstrap techniques are useful tools for estimating properties of estimators (e.g. variances, standard errors), by calculating them when sampling from a given data set, or from an approximating distribution. One standard choice for an approximating distribution is the empirical distribution of the observed data. The bootstrap method offers an ideal opportunity to perform approximate inference. Using Monte Carlo results, Davidson and Flachaire [18] noticed that bootstrapping a commonly used measure of inequality leads to inference which is not accurate even in very large samples, although inference with poverty measures is satisfactory. They found that the major cause is the extreme sensitivity of many inequality measures to the exact nature of the upper tail of the income distribution. As a solution, they proposed two nonstandard bootstrap methods: the m out of n bootstrap, which is valid in some situations where the standard bootstrap fails, and a semi-parametric bootstrap in which the upper tail is modeled parametrically.

Through their experiment, they found three reasons for the poor performance of standard bootstrap techniques:

1. Almost all indices are nonlinear functions of sample moments, thereby inducing biases and nonnormality in estimators of these indices.

2. Estimators of the covariances of the sample moments used to construct indices are often very noisy.

3. The indices are often extremely sensitive to the exact nature of the tails of the distribution.

The simulation results showed that the third cause is often the most important. In order to circumvent this problem, the following two bootstrap techniques were proposed (see [18]):

1. The m out of n bootstrap: It is valid in the case of infinite variance. It consists of drawing subsamples of size m from the original sample of size n (with $n \geq m$), without replacement. This technique is also known as the moon bootstrap and is usually thought of as useful when the standard bootstrap fails or when it is difficult to check its consistency.

2. The semi-parametric bootstrap: It consists of drawing samples from a semi-parametric estimator of the distribution, which combines a parametric estimation of the upper tail with a non-parametric estimation of the rest of the distribution.

Davidson and Flachaire [18] analyzed the Theil inequality measure and showed that asymptotic and standard bootstrap tests for it may not yield accurate inference, even if the sample size is very large. The main reason for this, they said, is the nature of the upper tail of the income distribution. Their proposed methods performed better than standard bootstrap techniques.

2.4.3 Further Developments on Measures of Inequality

Eliazar and Sokolov [27] established a Gini-based characterization of extreme value statistics, presenting a novel connection between the Gini index and extreme value statistics.

Qin et al. [45] constructed empirical likelihood confidence intervals for the Gini coefficient and showed that these perform very well for large samples. However, the method has under-coverage problems when the sample size is small or moderate. To solve that problem they proposed the bootstrap-calibrated empirical likelihood confidence intervals.

Langel and Tillé [39] proposed an improved methodology for the estimation of the QSR and the variance of the estimator in a complex sampling design framework. They also discussed the construction of confidence intervals and made a proposition to account for skewness of the sampling distribution of the QSR. Realizing that the skewness in the distribution of the estimators of the quintile share ratio makes it difficult to achieve reliable confidence intervals, the authors applied the Box-Cox transformations to reduce the problems caused by skewness. They applied their method to a real life data set and pointed out the common problem of skewness, and the sensitivity of statistics to extreme values, both of which are obstacles to reliable inference.

Chapter 3

Heavy Tailed Distributions and Sampling Distributions of the Nonparametric Estimators of Inequality Measures

A distribution function F is heavy-tailed if its tail is heavier than an exponential tail, i.e.

$$\lim_{x\to\infty}\frac{\exp(-\lambda x)}{\overline{F}(x)}=0,\ \text{for any } \lambda>0, \tag{3.1}$$

where $\overline{F}(x)=1-F(x)$. The degree of deviation can be depicted through visual inspection of an exponential quantile plot of points with coordinates

$$(-\log(\frac{j}{n+1}),X_{n-j+1,n}),\ j=1,2,\dots,n,$$

where $X_{1,n}<X_{2,n}<\dots<X_{n,n}$ denote the order statistics associated with the sample $X_1,X_2,\dots,X_n$. Modeling extreme events through heavy-tailed distributions attracts more and more attention, especially when modeling income distributions. In this chapter we describe a number of popular heavy-tailed distributions. We also study the sampling distributions of the nonparametric estimators of the inequality measures described in Chapter 2.

3.1 Heavy Tailed Distributions

3.1.1 Pareto distribution

Definition 3.1. A random variable X is said to have a Pareto distribution if its probability distribution function has the form

$$F(x;\beta) = 1 - \left(\frac{x}{x_0}\right)^{-\beta}, \; x \geq x_0, \; x_0, \beta > 0, \tag{3.2}$$

where x_0 is the scale parameter and β is the shape parameter (see Chotikapanich [7], page 120).

The moments of the Pareto distribution are given by

$$\mu_k = \frac{\beta x_0^k}{\beta - k}, \tag{3.3}$$

and are only defined for $k < \beta$.

In particular the mean and the variance are respectively given by

$$\mu = \frac{\beta x_0}{\beta - 1} \tag{3.4}$$

and

$$\sigma^2 = \left(\frac{\beta}{\beta - 2}\right)\left(\frac{x_0}{\beta - 1}\right)^2. \tag{3.5}$$

Consider an i.i.d. sample $X_1, X_2, \ldots, X_n$ from the Pareto distribution. The maximum likelihood estimator for β is given by

$$\widehat{\beta} = \left[\frac{1}{n}\sum_{i=1}^{n}(\log X_i - \log \widehat{x}_0)\right]^{-1}, \text{ where } \widehat{x}_0 = \min_{1 \leq i \leq n}\{X_i\}. \tag{3.6}$$

3.1.2 Burr Distribution

Definition 3.2. A random variable X is said to have a Burr distribution if its probability distribution function has the form (see Beirlant et al. [3], Hogg and Klugman [32])

$$F(x;\alpha,\tau,\lambda) = 1 - \left(\frac{\lambda}{\lambda + x^{\tau}}\right)^{\alpha}, \; x > 0, \tau, \lambda, \alpha > 0, \tag{3.7}$$

where λ is the scale parameter, τ and α are shape parameters.

The moments of the Burr distribution are given by

$$\mu_k = \frac{\lambda^{k/\tau}\Gamma\left(\alpha - \frac{k}{\tau}\right)\Gamma\left(1+\frac{k}{\tau}\right)}{\Gamma(\alpha)}, \tag{3.8}$$

where $\Gamma(.)$ is the Gamma function.

The maximum likelihood estimators of the parameters λ, τ, and α can only be obtained numerically as solutions to the nonlinear equations

$$\begin{cases} \frac{n}{\alpha} + n\log\lambda - \sum_{i=1}^{n}\log(\lambda + X_i^{\tau}) = 0 \\ \frac{n}{\tau} + \sum_{i=1}^{n}\log X_i - \tau(\alpha+1)\sum_{i=1}^{n}\frac{X_i^{\tau-1}}{\lambda + X_i^{\tau}} = 0 \\ \frac{n\alpha}{\lambda} - (\alpha+1)\sum_{i=1}^{n}\frac{1}{\lambda + X_i^{\tau}} = 0. \end{cases} \tag{3.9}$$

From Equation (3.8) we obtain for the method of moments estimators, the equations

$$\begin{cases} \lambda^{1/\tau}\Gamma(\alpha - \frac{1}{\tau})\Gamma(1+\frac{1}{\tau}) = \Gamma(\alpha)m_1 \\ \lambda^{2/\tau}\Gamma(\alpha - \frac{2}{\tau})\Gamma(1+\frac{2}{\tau}) = \Gamma(\alpha)m_2 \\ \lambda^{3/\tau}\Gamma(\alpha - \frac{3}{\tau})\Gamma(1+\frac{3}{\tau}) = \Gamma(\alpha)m_3, \end{cases} \tag{3.10}$$

where

$$m_i = \frac{1}{n}\sum_{j=1}^{n} X_j^i, \; i = 1,2,3$$

represent the sample moments.

These will again have to be solved numerically.

3.1.3 Singh-Maddala Distribution

Definition 3.3. A random variable X is said to have a Singh-Maddala distribution if its probability distribution function has the form (see Kleiber and Kotz [36])

$$F(x;a,b,c) = 1 - \left[1 + \left(\frac{x}{b}\right)^a\right]^{-c}, \; x > 0, \; a,b,c > 0, \tag{3.11}$$

where b is the scale parameter, a and c are shape parameters. This distribution is a special case of the Burr distribution with $\tau = a, \lambda = b^a$ and $\alpha = c$.

The moments of the Singh-Maddala distribution are defined for $-a < k < ac$ and given by

$$\mu_k = \frac{b^k \Gamma\left(1+\frac{k}{a}\right)\Gamma\left(c-\frac{k}{a}\right)}{\Gamma(c)}. \tag{3.12}$$

As in the previous case, the maximum likelihood estimators of the parameters a, b, and c can only be obtained numerically as solutions to the nonlinear equations:

$$\begin{cases} \frac{n}{a} - n\log b + \sum_{i=1}^n \log X_i - (c+1)\sum_{i=1}^n \frac{(X_i/b)^a \log(X_i/b)}{1+(X_i/b)^a} = 0 \\ n - (c+1)\sum_{i=1}^n \frac{(X_i/b)^a}{1+(X_i/b)^a} = 0 \\ n - c\sum_{i=1}^n \log(1+(X_i/b)^a) = 0. \end{cases} \tag{3.13}$$

From Equation (3.12) we obtain for the method of moments estimators, the equations

$$\begin{cases} b\Gamma(1+\frac{1}{a})\Gamma(c-\frac{1}{a}) = \Gamma(c)m_1 \\ b^2\Gamma(1+\frac{2}{a})\Gamma(c-\frac{2}{a}) = \Gamma(c)m_2 \\ b^3\Gamma(1+\frac{3}{a})\Gamma(c-\frac{3}{a}) = \Gamma(c)m_3. \end{cases} \tag{3.14}$$

These will again have to be solved numerically.

3.1.4 Lognormal Distribution

Definition 3.4. A random variable X is said to have a lognormal distribution if its probability distribution function has the form

$$F(x;\mu,\sigma) = \Phi\left(\frac{\log x - \mu}{\sigma}\right),\ x > 0, -\infty < \mu < \infty, \sigma > 0, \tag{3.15}$$

where Φ is the standard normal distribution function.

The moments of the lognormal distribution are given by

$$\mu_k = \exp\left(k\mu + \frac{1}{2}k^2\sigma^2\right). \tag{3.16}$$

Consider an i.i.d. sample $X_1, X_2, \ldots, X_n$ from the lognormal distribution. The maximum likelihood estimators for the parameters μ and σ are given by

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \log X_i \tag{3.17}$$

and

$$\widehat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (\log X_i - \widehat{\mu})^2}. \tag{3.18}$$

3.1.5 Gamma Distribution

Definition 3.5. A random variable X is said to have a gamma distribution if its probability distribution function has the form

$$F(x;\alpha,\lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^x u^{\alpha-1} \exp(-\lambda u) du, \ x > 0, \alpha, \lambda > 0, \tag{3.19}$$

where α is the shape parameter and λ is the scale parameter.

The moments of the gamma distribution are given by

$$\mu_k = \frac{1}{\lambda^k} \prod_{i=1}^{k-1} (\alpha + i). \tag{3.20}$$

The maximum likelihood estimators for the parameters can only be obtained numerically as solutions to the nonlinear equations

$$\begin{cases} n \log \lambda - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n} \log X_i = 0 \\ \frac{n\alpha}{\lambda} - \sum_{i=1}^{n} X_i = 0, \end{cases} \tag{3.21}$$

where

$$\Gamma'(x) = \frac{d\Gamma(x)}{dx}. \tag{3.22}$$

Given an i.i.d. sample $X_1, X_2, \ldots, X_n$ from the Gamma distribution, the method of moments estimators for α and λ are given by (see Hogg and Klugman [32])

$$\widehat{\alpha} = \frac{m_1^2}{m_2 - m_1^2} \tag{3.23}$$

and

$$\widehat{\lambda} = \frac{m_1}{m_2 - m_1^2}. \tag{3.24}$$

These can also be used as starting values in solving the nonlinear equations for the maximum likelihood estimators.

3.1.6 Loggamma Distribution

Definition 3.6. A random variable X is said to have a loggamma distribution if its probability density function has the form (see Kleiber and Kotz [36])

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\beta-1} \left[\log(x)\right]^{\alpha-1}, \; x \geq 1, \; \alpha,\beta > 0. \tag{3.25}$$

The moments of the loggamma distribution are defined for $k < \beta$ and given by

$$\mu_k = \left(\frac{\beta}{\beta-k}\right)^{\alpha}. \tag{3.26}$$

As before, the maximum likelihood estimators of the parameters can only be obtained numerically as solutions to the nonlinear equations

$$\begin{cases} n\log\beta - n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n}\log(\log X_i) = 0 \\ n\alpha - \beta\sum_{i=1}^{n}\log X_i = 0, \end{cases} \tag{3.27}$$

where $\Gamma'(.)$ is defined as in Equation (3.22).

From Equation (3.26) we obtain for the method of moments estimators, the equations

$$\begin{cases} \left(\frac{\beta}{\beta-1}\right)^{\alpha} = m_1 \\ \left(\frac{\beta}{\beta-2}\right)^{\alpha} = m_2. \end{cases} \tag{3.28}$$

These will again have to be solved numerically.

3.1.7 Dagum Type I Distribution

Definition 3.7. A random variable X is said to have a Dagum Type I distribution if its probability density function has the form (see Kleiber and Kotz [36])

$$f(x;a,b,c) = \frac{abc}{x^2}\left(\frac{b}{x}\right)^{a-1}\left[1+\left(\frac{b}{x}\right)^{a}\right]^{-c-1}, \; x > 0, \; a > 0, b, c \geq 0. \tag{3.29}$$

The moments of the Dagum type I distribution are defined for $-ac < k < a$ and given by

$$\mu_k = \frac{b^k \Gamma\left(1-\frac{k}{a}\right)\Gamma\left(c+\frac{k}{a}\right)}{\Gamma(c)}. \tag{3.30}$$

The maximum likelihood estimators of the parameters can only be obtained numerically as solutions to the nonlinear equations

$$\begin{cases} \frac{n}{a} + n\log b - \sum_{i=1}^{n} \log X_i - (c+1)\sum_{i=1}^{n} \frac{(b/X_i)^a \log(b/X_i)}{1+(b/X_i)^a} = 0 \\ n - (c+1)\sum_{i=1}^{n} \frac{(b/X_i)^a}{1+(b/X_i)^a} = 0 \\ n - c\sum_{i=1}^{n} \log(1+(b/X_i)^a) = 0. \end{cases} \tag{3.31}$$

From Equation (3.30) we obtain for the method of moments estimators, the equations

$$\begin{cases} b\Gamma(1-\frac{1}{a})\Gamma(c+\frac{1}{a}) = \Gamma(c)m_1 \\ b^2\Gamma(1-\frac{2}{a})\Gamma(c+\frac{2}{a}) = \Gamma(c)m_2 \\ b^3\Gamma(1-\frac{3}{a})\Gamma(c+\frac{3}{a}) = \Gamma(c)m_3. \end{cases} \tag{3.32}$$

These will again have to be solved numerically.

3.1.8 Fréchet Distribution

A random variable X is said to have a Fréchet distribution if its probability distribution function has the form (see Beirlant et al. [3])

$$F(x;\alpha) = \exp\{-x^{-\alpha}\},\ x > 0,\ \alpha > 0. \tag{3.33}$$

The moments of the Fréchet distribution are defined for $k < \alpha$ and given by

$$\mu_k = \Gamma\left(1-\frac{k}{\alpha}\right). \tag{3.34}$$

The maximum likelihood estimator of the parameter α can only be obtained numerically as solution to the nonlinear equation

$$n + \alpha\sum_{i=1}^{n}(X_i^{-\alpha}-1)\log X_i = 0. \tag{3.35}$$

The method of moments estimator can also be obtained as solution to the nonlinear equation

$$\Gamma\left(1-\frac{1}{\alpha}\right)=m_1. \tag{3.36}$$

This will again have to be solved numerically.

3.1.9 Student t Distribution

A random variable X is said to have a Student t distribution with ν degrees of freedom if its probability density function has the form (see Beirlant et al. [3])

$$f(x;\nu)=\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}},\ x\in\mathbb{R},\ \ \nu>0. \tag{3.37}$$

The moments of the Student t distribution are defined for $k<\nu$ and given by

$$\mu_k=\frac{\nu^{k/2}\Gamma(\frac{k+1}{2})\Gamma(\frac{\nu-k}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}\text{ for } k \text{ even, and } 0 \text{ for } k \text{ odd.} \tag{3.38}$$

As before, the maximum likelihood estimator of the parameter ν can only be obtained numerically as solution to the nonlinear equation

$$n\frac{\Gamma'(\frac{\nu+1}{2})}{\Gamma(\frac{\nu+1}{2})}-n\frac{\Gamma'(\frac{\nu}{2})}{\Gamma(\frac{\nu}{2})}-\sum_{i=1}^{n}\log\left(1+\frac{X_i^2}{\nu}\right)+\frac{\nu+1}{\nu}\sum_{i=1}^{n}\frac{X_i^2}{\nu+X_i^2}-\frac{n}{\nu}=0, \tag{3.39}$$

where $\Gamma'(.)$ is defined as in Equation (3.22).

Remark 3.1. The previous distributions are discussed because they are commonly referred to in the literature involving inequality measures. However, in our simulation work we will only make use of the Pareto, the Burr, the Fréchet and the Student t as the underlying distributions.

3.2 Sampling Distributions of the Nonparametric Estimators of Inequality Measures

It is common in statistics to investigate the sampling distribution of any estimator under consideration, that is, the distribution of the estimator in all possible samples of the same size drawn from the population. From the central limit theorem, the sampling distributions of many statistics are expected to be normal or nearly normal, if the sample size is large enough. However, the speed of convergence varies highly, depending on the particular statistic. In this section we investigate the sampling distributions of the nonparametric estimators of the inequality measures described in Chapter 2. Since these distributions converge very slowly, we applied a power transformation for improvement. The idea is to be able to decide when asymptotic theory may be applied.

3.2.1 Quantile Plots for the Sampling Distributions

The underlying distributions used in the simulation are given below.

1. The Pareto distribution with $x_0 = 0.1$, $\beta = 1.5$ and $\gamma = 0.67$ (we will refer to it as Pa);
2. The Burr distribution with $\alpha = 2$, $\tau = 0.83$, $\lambda = 1$ and $\gamma = 0.96$ (we will refer to it as Bu1);
3. The Burr distribution with $\alpha = 1$, $\tau = 1.4$, $\lambda = 1$ and $\gamma = 0.71$ (we will refer to it as Bu2);
4. The Burr distribution with $\alpha = 0.5$, $\tau = 4$, $\lambda = 1$ and $\gamma = 0.25$ (we will refer to it as Bu3);
5. The Fréchet distribution with $\alpha = 2$ and $\gamma = 0.50$ (we will refer to it as Fr1);
6. The Fréchet distribution with $\alpha = 1.7$ and $\gamma = 0.59$ (we will refer to it as Fr2);
7. The $|t_2|$ distribution with $\gamma = 0.50$ (we will refer to it as T2).

We confine ourselves to giving the normal Q-Q plots for the sampling distributions of the estimators of Gini coefficient (Figures 3.1 to 3.3 below). We use samples of size $n = 10000$. The results for other measures are given in Appendix B (see Figures B.1 to B.9 for the GE and Figures B.10 to B.18 for the Atkinson measures).

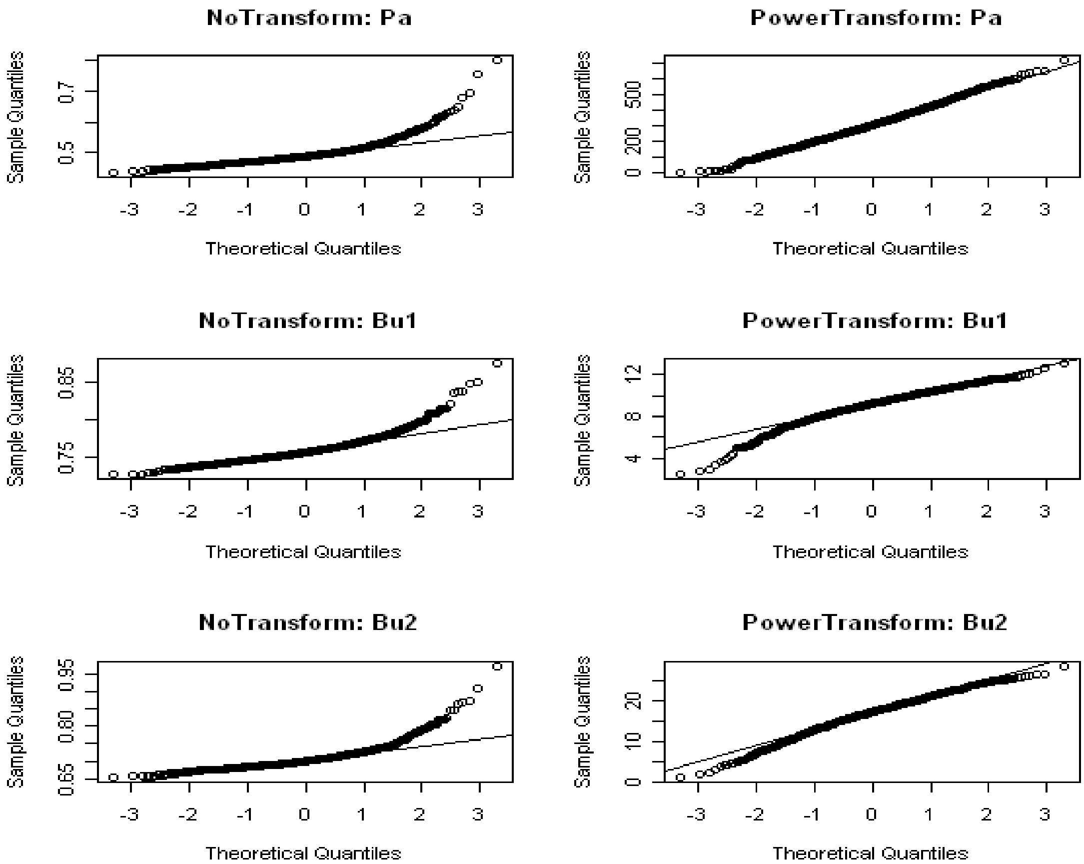

Figure 3.1: Sampling Distribution for Gini (Samples from Pa, Bu1, Bu2 Distributions)

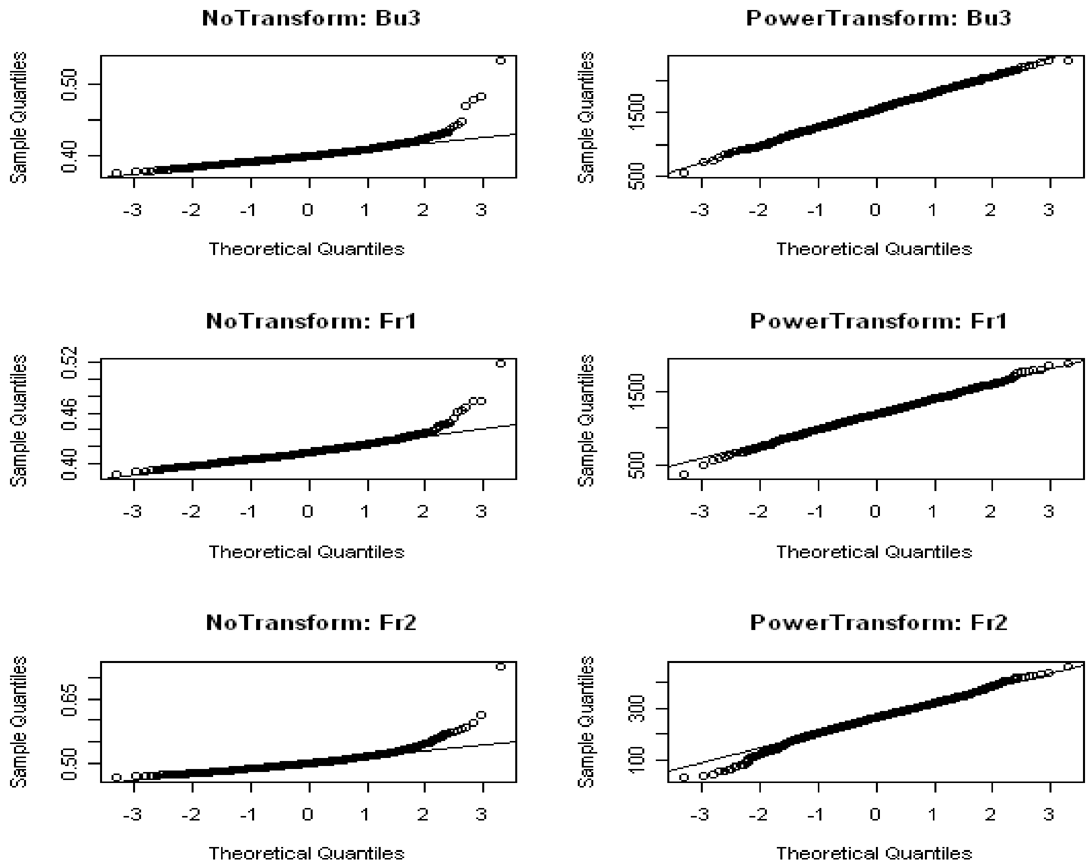

Figure 3.2: Sampling Distribution for Gini (Samples from Bu3, Fr1, Fr2 Distributions)

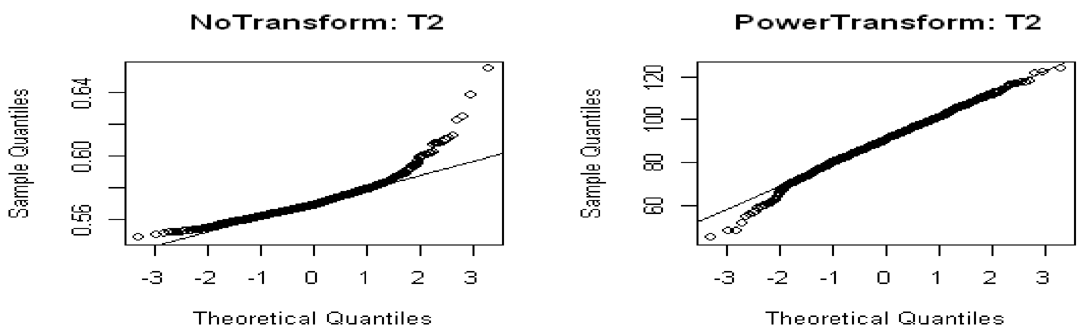

Figure 3.3: Sampling Distribution for Gini (Samples from T2 Distribution)

3.2.2 Discussion

We see from the Q-Q plots that the sampling distributions of the nonparametric estimators of the Gini (see Figures 3.1 to 3.3 above), the GE (see Figures B.1 to B.9 in Appendix B) and the Atkinson (see Figures B.10 to B.18 in Appendix B) measures of inequality converge very slowly to the normal distribution when $n \to \infty$. This slow convergence raises the need for ways in which one can improve on the nonparametric procedures. One way of doing so is to apply a power transformation. An improvement is obtained by applying such a transformation (see the right column of each graph).

For a measure $I(F)$, its nonparametric estimator is $I(F_n)$, with F_n the empirical distribution function of the sample. The sampling distribution of $T_n = I(F_n)^{-c}$ is approximately normal for the considered sample size ($n = 10000$), where c is a positive constant. In the simulation leading to the previous graphs, we considered $c = 8$ for Gini, $c = 5$ for GE0, $c = 4$ for GE1, $c = 3$ for GE1.3, $c = 6$ for A1, $c = 8$ for A1.5 and $c = 8$ for A2. To choose c in each case we ran the simulation over a range of c's and considered the value of c giving the best approximation to the normal. As the graphs show, the sampling distribution of T_n converges to the normal distribution faster than that of $I(F_n)$ as the sample size n goes to infinity.

Chapter 4

Extreme Value Index Estimation and Threshold Selection Methods

In semi-parametric modeling, the estimators of the parameters in the upper tail of the distribution, such as the extreme value index, depend on the sample fraction which is used for estimation. Therefore, a good choice of threshold is required. Too low a threshold is likely to lead to bias, and too high a threshold will generate too few excesses, leading to high variances of estimators. In order to address these problems, various methods of choosing the thresholds have been proposed. In this chapter, we review some of the methods. Prior to that, we define the Generalized Extreme Value (GEV) distribution and we describe several estimators for the extreme value index. See Berning [5] for a thorough discussion of these.

Definition 4.1. A random variable X is said to have a GEV distribution if its probability distribution function has the form

$$G_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\},\ 1+\gamma x > 0, \tag{4.1}$$

where $\gamma \in \mathbb{R}$ is called the Extreme Value Index (EVI).

Denote by U the tail quantile function associated with the extreme value distribution and assume that U satisfies the usual condition

$$\frac{U(ux)}{U(x)} = u^\gamma(1 + h_{-\beta}(u)b(x) + o(b(x))), \tag{4.2}$$

for some regular varying function b with index $-\beta$, i.e.

$$\lim_{x\to\infty} \frac{b(ux)}{b(x)} = u^{-\beta}, \forall u > 0, \tag{4.3}$$

and

$$h_{-\beta}(u) = \frac{1 - u^{-\beta}}{\beta}$$

(see Beirlant et al. [3], page 48).

4.1 Estimators for the Extreme Value Index

One of the central points of extreme value theory is the estimation of the extreme value index γ. In this section, we review several EVI estimators found in the literature. Consider an i.i.d. sample $X_1, X_2, \ldots, X_n$, from a distribution F, with the associated ordered sample $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$.

Pickands Estimator

The Pickands estimator for the EVI γ is given by (see Pickands [44])

$$\widehat{\gamma}_P = \frac{1}{\log 2} \log \left(\frac{X_{[\frac{k}{2}],n} - X_{[\frac{k}{4}],n}}{X_{k,n} - X_{[\frac{k}{2}],n}} \right), \tag{4.4}$$

where $[x]$ is the largest integer less than or equal to x, and k has the form $k = 4m, 1 \leq 4m \leq n$.

A particular characteristic of the Pickands estimator is the fact that the largest observation is not explicitly used in the estimation. This estimator is known to have poor performance, and therefore should not be included in a simulation study.

Hill estimator

The Hill estimator is defined for $\gamma > 0$ as the mean excess value of the log-transformed data points $X_{n-k+1,n}, X_{n-k+2,n}, \ldots, X_{n,n}$, and is given by (see Beirlant et al. [3], page 101)

$$H_{k,n} = \frac{1}{k} \sum_{j=1}^{k} \log X_{n-j+1,n} - \log X_{n-k,n}. \tag{4.5}$$

Different representations of $H_{k,n}$ are given in [3], one of them (using Rényi's exponential representation) being the simple average of scaled log-spacings, i.e.

$$H_{k,n} = \frac{1}{k} \sum_{j=1}^{k} Z_j, \tag{4.6}$$

with

$$Z_j = j(\log X_{n-j+1,n} - \log X_{n-j,n}).$$

The following approximation holds (see Beirlant et al. [3], page 111):

$$Z_j \stackrel{D}{\sim} \left(\gamma + \left(\frac{j}{k+1}\right)^{\beta} b\left(\frac{n+1}{k+1}\right)\right) E_j,\ j = 1,2,\dots,k, \tag{4.7}$$

and $\{E_j\}, j = 1,2,\dots,k$, are independent, exponentially distributed with mean 1. Note that in the case of the strict Pareto distribution, the transformed variables Z_j are exactly independent and exponentially distributed:

$$Z_j \stackrel{D}{=} \gamma E_j,\ j = 1,2,\dots,k. \tag{4.8}$$

The following properties of the Hill estimator are well known (see e.g. [3], pages 111-112).

1. The asymptotic bias of the Hill estimator can be approximated using the exponential representation:

$$ABias(H_{k,n}) \sim b_{n,k} \frac{1}{k} \sum_{j=1}^{k} \left(\frac{j}{k+1}\right)^{\beta} \sim \frac{b_{n,k}}{1+\beta}, \tag{4.9}$$

   where

$$b_{n,k} = b\left(\frac{n+1}{k+1}\right).$$

   Notice that the bias will be small only if $b_{n,k}$ is small, which in turn requires k to be small.

2. The asymptotic variance of the Hill estimator is given by

$$AVar(H_{n,k}) \sim \mathrm{Var}\left(\frac{\gamma}{k} \sum_{j=1}^{k} E_j\right) \sim \frac{\gamma^2}{k}. \tag{4.10}$$

   Notice in this case that the variance will be small if k is large.

3. Finally, asymptotic normality of the Hill estimator holds when $k,n \to \infty, k/n \to 0$ and $\sqrt{k} b_{n,k} \to 0$:

$$\sqrt{k}\left(\frac{H_{k,n}}{\gamma} - 1\right) \stackrel{D}{\to} N(0,1). \tag{4.11}$$

This result allows the construction of approximate confidence intervals for the extreme value index γ. At the level of significance $(1-\alpha)$, this interval is given by

$$\left(\left(1 + \frac{\Phi^{-1}(1-\alpha/2)}{\sqrt{k}}\right)^{-1} H_{k,n}, \left(1 - \frac{\Phi^{-1}(1-\alpha/2)}{\sqrt{k}}\right)^{-1} H_{k,n}\right), \tag{4.12}$$

which is an acceptable approach if the bias is not too important, that is, if $\beta \geq 1$. Typically, the condition $\sqrt{k}b_{n,k} \to 0$ severely restricts the range of k values where the confidence interval is of any practical value.

Adapted Hill Estimator

The popularity of the Hill estimator generated a tempting problem to try to extend it to the general case $\gamma \in \mathbb{R}$. Such an attempt led Beirlant et al. [4] to the so-called adapted estimator, which is applicable for any γ in the range of real numbers. This estimator is given by

$$\widehat{\gamma}_{adH} = \frac{1}{k}\sum_{i=1}^{k} \log U_i - \log U_{k+1}, \tag{4.13}$$

where

$$U_i \equiv X_{i+1,n}\left(\frac{1}{k}\sum_{j=1}^{i} \log X_{j,n} - \log X_{i+1,n}\right).$$

Moment Estimator

Dekkers et al. [20] proposed an estimator that can be considered as an adaptation of the Hill estimator, in order to obtain consistency for all $\gamma \in \mathbb{R}$. This estimator is the moment estimator, given by

$$\widehat{\gamma}_M(k) = M_1 + 1 - \frac{1}{2}\left(1 - \frac{M_1^2}{M2}\right)^{-1}, \tag{4.14}$$

where

$$M_j \equiv \frac{1}{k}\sum_{i=1}^{k} (\log X_{i,n} - \log X_{k+1,n})^j, \; j = 1, 2. \tag{4.15}$$

Q-Q Estimator

The Q-Q plot approach was proposed by Beirlant et al. [4]. According to this approach, the Hill estimator is approximately the slope of the line fitted to the upper tail of Pareto Q-Q plot. A more precise estimator under this approach, known as the Q-Q estimator, was suggested by Kratz and Resnick [37] and is given by

$$\widehat{\gamma}_{qq} = \frac{\sum_{i=1}^{k}(\log \frac{i}{k+1})\{\sum_{j=1}^{k} \log X_{j,n} - k \log X_{i,n}\}}{k\sum_{i=1}^{k}\left(\log \frac{i}{k+1}\right)^2 - \left(\sum_{i=1}^{k} \log \frac{i}{k+1}\right)^2}. \tag{4.16}$$

Moment Ratio Estimator

The moment ratio estimator was proposed by Danielsson et al. [16]. This estimator is given by

$$\widehat{\gamma}_{MR} = \frac{1}{2}\frac{M_2}{M_1}, \tag{4.17}$$

where M_1 and M_2 are defined by Equation (4.15). The authors proved that $\widehat{\gamma}_{MR}$ has lower asymptotic square bias than the Hill estimator when evaluated at the same threshold (i.e. for the same k), though the convergence rates are the same.

Peng's Estimator

An estimator related to the moment estimator $\widehat{\gamma}_M$ is Peng's estimator, suggested by Deheuvels et al. [19] and is given by

$$\widehat{\gamma}_L = \frac{M_2}{2M_1} + 1 - \frac{1}{2}\left(1 - \frac{M_1^2}{M_2}\right)^{-1}, \tag{4.18}$$

where M_1 and M_2 are defined by Equation (4.15). This estimator was developed to reduce the bias of the moment estimator.

W Estimator

Another estimator related to $\widehat{\gamma}_L$ is the W estimator given by (see [19])

$$\widehat{\gamma}_W = 1 - \frac{1}{2}\left(1 - \frac{L_1^2}{L_2}\right)^{-1}, \tag{4.19}$$

where

$$L_j \equiv \frac{1}{k}\sum_{i=1}^{k}(X_{i,n} - X_{k+1,n})^j, \; j = 1,2.$$

Remark 4.1. Deheuvels et al. [19] mentioned that $\widehat{\gamma}_L$ is consistent for any $\gamma \in \mathbb{R}$ under the usual conditions, while $\widehat{\gamma}_W$ is consistent only for $\gamma < \frac{1}{2}$.

Averaged Hill Estimator

Resnick and Stărică [46] proposed a simple averaging technique that reduces the instability of the Hill plot. This procedure consists of averaging the Hill estimator values corresponding to different values of order statistics. The corresponding estimator is given by

$$av\widehat{H}_{k,n} = \frac{1}{k-[ku]} \sum_{i=[ku]+1}^{k} \widehat{H}_{i,n},\ 0<u<1. \tag{4.20}$$

The authors proved that through averaging, the variance of the Hill estimator can be considerably reduced, and that the instability of the plot can be controlled.

Averaged Moment Estimator

Resnick and Stărică [47] applied their idea of smoothing to the more general moment estimator $\widehat{\gamma}_M$, essentially generalizing their reasoning of smoothing the Hill estimator. The proposed technique consists of averaging the moment estimator values corresponding to different numbers of order statistics. The corresponding estimator is given by

$$av\widehat{\gamma}_M(k) = \frac{1}{k-[ku]} \sum_{i=[ku]+1}^{k} \widehat{\gamma}_M(i),\ 0<u<1. \tag{4.21}$$

The authors suggested taking $u = 0.3$ or $u = 0.5$, depending on the sample size (the smaller the sample size, the larger u should be). They also showed that through averaging, the variance of the moment estimator can be considerably reduced only in the case of $\gamma < 0$. For $\gamma > 0$, the simple moment estimator turns out to be better than the averaged moment estimator.

Estimators Based on Excess Plots

The mean excess plots (MEP) proved to be useful when estimating the extreme value index. However, though the mean excess functions (MEF) theoretically estimate the EVI γ, in practice strong fluctuations of the empirical MEF and the corresponding MEP are observed, especially in the right part of the plot, since there are fewer data. In order to make the estimation procedure more robust, i.e. less sensitive to the fluctuations, the following adaptive estimators of the MEF have been considered by Beirlant et al. [4]:

1. An estimator based on the median excess plot

$$\widehat{\gamma}_{med} = \frac{1}{\log 2}(\log X_{[\frac{k}{2}]+1,n} - \log X_{k+1,n}). \tag{4.22}$$

2. An estimator based on Trimmed Mean Excess Plot

$$\widehat{\gamma}_{trim} = \frac{1}{k-[pk]} \sum_{j=[pk]+1}^{k} \log X_{j,n} - \log X_{k+1,n}, \ \gamma > 0, \ p = 0.01, 0.05, 0.10. \tag{4.23}$$

Remark 4.2. Note that not all the previous methods will be used in this thesis, but we discussed them in order to give to the reader various options when facing a problem involving the EVI estimation.

4.2 Estimation Methods for Parameters in More General Distributions

In the previous section, we describe estimators for the EVI. These estimators are only appropriate when we are dealing with the extreme value index as the GEV parameter, and so they are not relevant for other distributions. In this section we discuss methods of parameter estimation in general, given a known parametric distribution with unknown parameters.

4.2.1 Integrated Squared Error Estimation

Assuming a parametric family of distributions $\{F_{\underline{\theta}}\}$, Vandewalle et al. [54] suggested a method to find the parameter estimate $\widehat{\underline{\theta}}$ which brings the density $f_{\widehat{\underline{\theta}}}$ closest to the true unknown density f underlying the data, using an integrated squared error distance criterion. That is, $\widehat{\underline{\theta}}$ is taken as

$$\begin{aligned} \widehat{\underline{\theta}} &= \arg\min_{\underline{\theta}} \left[\int_1^\infty (f_{\underline{\theta}}(y) - f(y))^2 dy \right] \\ &= \arg\min_{\underline{\theta}} \left[\int_1^\infty f_{\underline{\theta}}^2(y)dy - 2\int_1^\infty f_{\underline{\theta}}(y) f(y) dy + \int_1^\infty f^2(y)dy \right]. \end{aligned} \tag{4.24}$$

But

$$\int_1^\infty f_{\underline{\theta}}(y) f(y) dy = E\left[f_{\underline{\theta}}(Y)\right],$$

and the minimizing value of $\underline{\theta}$ does not depend on $\int_1^\infty f^2(y)dy$. Therefore, Equation (4.24) can be rewritten as

$$\widehat{\underline{\theta}} = \arg\min_{\underline{\theta}} \left[\int_1^\infty f_{\underline{\theta}}^2(y)dy - 2E\left[f_{\underline{\theta}}(Y)\right] \right], \tag{4.25}$$

where Y is a random variable with density f.

Now consider a random sample $Y_1, Y_2, \ldots, Y_n$ on Y and estimate $E\left[f_{\underline{\theta}}(Y)\right]$ by the sample average.

This leads to the so-called integrated squared error (ISE) estimator given by

$$\widehat{\underline{\theta}}_{ISE} = \arg\min_{\underline{\theta}} \left[\int_1^\infty f_{\underline{\theta}}^2(y)dy - \frac{2}{n}\sum_{i=1}^n f_{\underline{\theta}}(Y_i) \right]. \tag{4.26}$$

For many models, $\int_1^\infty f_{\underline{\theta}}^2(y)dy$ can be found in closed form as a function of $\underline{\theta}$.

4.2.2 Partial Density Component Estimation

For the ISE estimator in the previous section, only f is supposed to be a real density function whereas $f_{\underline{\theta}}$ is not. Therefore, instead of using a complete density model $f_{\underline{\theta}}$, an incomplete mixture model $wf_{\underline{\theta}}$ where w is a positive parameter, can also be considered, yielding the so-called partial density component (PDC) estimator

$$\widehat{\underline{\theta}}_{PDC}^{w} = \arg\min_{\underline{\theta}, w} \left[w^2 \int_1^\infty f_{\underline{\theta}}^2(y)dy - \frac{2w}{n}\sum_{i=1}^n f_{\underline{\theta}}(Y_i) \right]. \tag{4.27}$$

Note that the PDC estimator is used in connection with the mixture approximation for the conditional distribution of the relative excesses over a high threshold (see [54]).

4.2.3 Minimum Power Divergence Estimation

Consider a parametric family $\{F_{\underline{\theta}}\}$, indexed by the unknown parameter $\underline{\theta} \in \Omega \subset \mathbb{R}^s$, with s a positive integer, and denote by $\{f_{\underline{\theta}}\}$ the corresponding densities with respect to Lebesgue measure. Let $\mathbb{G}$ be the class of all distributions G having density g with respect to Lebesgue measure. Define the divergence $d_\alpha(g, f)$ between the density functions g and f to be (see Basu et al. [2])

$$d_\alpha(g, f) = \int_1^\infty \{f^{1+\alpha}(z) - (1+\frac{1}{\alpha})g(z)f^\alpha(z)\}dz, \ \alpha > 0, \tag{4.28}$$

and

$$d_0(g, f) = \int_1^\infty g(z)\log(g(z)/f(z))dz. \tag{4.29}$$

Equation (4.29) is known as the Kullback-Leibler divergence.

The minimum power divergence method consists of choosing parameter values to minimize $d_\alpha(g, f_{\underline{\theta}})$. From Theorem 1 in [2], it follows that for any given α, the minimum density power divergence functional at G, defined by the requirement

$$d_\alpha(g, f_{T_\alpha(G)}) = \min_{\underline{\theta}\in\Omega} d_\alpha(g, f_{\underline{\theta}}), \tag{4.30}$$

is Fisher-consistent. Furthermore, given a random sample $X_1, X_2, \ldots, X_n$ from G, the so-called minimum power divergence (MPD) estimator is defined as

$$\widehat{\underline{\theta}}_{MPD} = \arg\min_{\underline{\theta}} \left[\int_1^\infty f_{\underline{\theta}}^{1+\alpha}(z)dz - (1+\frac{1}{\alpha})n^{-1} \sum_{i=1}^{n} f_{\underline{\theta}}^{\alpha}(X_i) \right]. \tag{4.31}$$

Although any $\alpha > 0$ can be used, Basu et al. [2] suggested the use of α values between 0 and 1 because the MPD method becomes less and less efficient as α increases.

Remark 4.3. In this work we are going to use distributions involving more parameters than just the EVI and so the previous methods are candidates for the estimation of those parameters. However, an investigation of these methods together with the maximum likelihood method showed that despite some small improvements over the MLEs, the maximum likelihood gives satisfactory results. Since the ML method is easy to handle in the simulation, we decided on using it in this work.

4.3 Threshold Selection Methods

All the estimators of the EVI described in the previous section make use of k, the number of exceedances. Suppose we want to use a subset $\{X_{n-k+1,n}, X_{n-k+2,n}, \ldots, X_{n,n}\}$ of the ordered sample. An important first step is to decide on a method for choosing k in some reasonable fashion. In this section we describe a number of different methods for doing this. The most well known method is the Hill plot, which we first discuss, and some modifications of it.

4.3.1 Hill Plot Method

Every choice of k gives a different estimator for γ and so one can plot the estimator $H_{k,n}$ versus k, yielding the so-called Hill plot:

$$\{(k, H_{k,n}) : 1 \leq k \leq n-1\} \tag{4.32}$$

(see Beirlant et al. [3]). The value of k is then taken from the values corresponding to the region where the Hill plot is roughly constant.

The problem with this method is that in practice, the Hill plot is very often far from constant, making it difficult to implement. As an alternative, Resnick and Stărică [46] proposed plotting

$$\{(\log k, H_{k,n}) : 1 \leq k \leq n-1\}. \tag{4.33}$$

This, however, does not overcome some of the problems associated with the Hill estimator.

4.3.2 AltHill Plot Method

The AltHill plot is an alternative to the Hill plot, consisting of plotting the points (see Drees et al. [22])

$$\{(\theta, H_{[n^\theta],n}),\ 0 \leq \theta \leq 1\}, \tag{4.34}$$

where $[n^\theta]$ is the largest integer less than or equal to n^θ. In this case, the choice of k is given by $\widehat{k} = [n^{\theta^*}]$, where θ^* is taken in the region where the graph is roughly constant. Although this improves on the Hill plot, it is still not satisfactory. Some alternative methods to the Hill-type, are now discussed.

4.3.3 Guillou and Hall Method

Guillou and Hall [29] proposed choosing the value $\widehat{k}$ of k as the smallest value for which

$$\sqrt{\frac{k}{12}}\frac{|\widehat{b}^+_{LS}(-1)|}{H_{k,n}} > C_{crit}, \tag{4.35}$$

where $\widehat{b}^+_{LS}$ is given by

$$\widehat{b}^+_{LS}(\widehat{\beta}) = \frac{(1+\widehat{\beta})^2}{\widehat{\beta}^2}\frac{1}{k}\sum_{j=1}^{k}\left(\left(\frac{j}{k+1}\right)^{\widehat{\beta}} - \frac{1}{1+\widehat{\beta}}\right)Z_j, \tag{4.36}$$

with

$$Z_j = (j+1)\log\frac{X_{n-j,n}H_{j,n}}{X_{n-j-1,n}H_{j+1,n}},$$

$\widehat{\beta}$ being an external estimator for β, and C_{crit} is a critical value such as 1.25 or 1.5 (see Beirlant et al. [3], page 123).

4.3.4 Minimum Mean Squared Error Method

The asymptotic mean squared error for the Hill estimator $H_{k,n}$ is given by (see Beirlant et al. [3])

$$AMSE(H_{k,n}) = \frac{\gamma^2}{k} + \left(\frac{b_{n,k}}{1+\beta}\right)^2, \tag{4.37}$$

where $b_{n,k} = b\left(\frac{n+1}{k+1}\right)$. A method to choose k is to plot the points

$$\{\left(k, \widehat{AMSE}(H_{k,n})\right), \; k = 1,2,\ldots,n-1\}, \tag{4.38}$$

where $\widehat{AMSE}(H_{k,n})$ is the estimator of $AMSE(H_{k,n})$ using the maximum likelihood estimators of the parameters, and then choosing the value $\widehat{k}$ that minimizes the plot.

4.3.5 Drees and Kaufmann Method

Drees and Kaufmann [23] proposed a sequential procedure to select the optimal sample size $\widehat{k}_{n,opt}$. This method is as follow.

1. Obtain an initial estimate $\widehat{\gamma}_0 = H_{[2\sqrt{n},n]}$ for γ.

2. For $r_n = 2.5\widehat{\gamma}_0 n^{0.25}$, compute the stopping time

$$\widehat{k}_n(r) = \min\{k \in \{1,2,\ldots,n-1\} | \max_{1\leq i\leq k} \sqrt{i}(H_{i,n} - H_{k,n}) > r_n\}. \tag{4.39}$$

3. Similarly, compute $\widehat{k}_n(r_n^{\varepsilon})$ for $\varepsilon = 0.7$.

4. With a consistent estimator $\widehat{\beta}$ for β, calculate

$$\widehat{k}_{n,opt} = \left(\frac{\widehat{k}_n(r_n^{\varepsilon})}{[\widehat{k}_n(r_n)]^{\varepsilon}}\right)^{\frac{1}{1-\varepsilon}} (1+2\widehat{\beta})^{-1/\widehat{\beta}} (2\widehat{\beta}\widehat{\gamma}_0)^{\frac{1}{1+2\widehat{\beta}}}. \tag{4.40}$$

From simulation, it was found that the method performs better if a fixed value β_0 is used for β, in particular taking $\widehat{\beta} \equiv \beta_0 = 1$.

4.3.6 Hall-Class Method

A distribution belongs to the Hall-class if its tail quantile function satisfies

$$U(x) = Cx^{\gamma}(1 + Dx^{-\beta}(1 + o(1))),\ x \to \infty, \tag{4.41}$$

for some constants $C > 0$ and $D \in \mathbb{R}$. For this class of distributions, we have in Equation (4.2)

$$b(x) = -\beta D x^{\beta}(1 + o(1)), \tag{4.42}$$

as $x \to \infty$, and so the asymptotic mean squared error of the Hill estimator is minimal for (see Beirlant et al. [3])

$$k_{n,opt} \sim (b^2(n))^{-1/(1+2\beta)} \left(\frac{\gamma^2(1+\beta)}{2\beta}\right)^{1/(1+2\beta)},\ (n \to \infty). \tag{4.43}$$

Because of the particular form of b, we obtain

$$k_{n,opt} \sim \left(\frac{nb^2}{k_0}\right)^{-1/(1+2\beta)} k_0^{2\beta/(1+2\beta)} \left(\frac{\gamma^2(1+\beta)}{2\beta}\right)^{1/(1+2\beta)}, \tag{4.44}$$

for any secondary value $k_0 \in \{1, 2, \ldots, n\}$, with $k_0 = o(n)$. After substituting consistent estimators for b_{n,k_0}, β and γ in this expression, all based on k_0, one gets an estimator $\widehat{k}_{n,opt}$. Thus for each value of k_0, we obtain an estimator $\widehat{k}_{n,opt}$ of $k_{n,opt}$, and as $k_0, n \to \infty, k_0/n \to 0$, and $\frac{\sqrt{k_0} b_{n,k_0}}{\log k_0} \to \infty$, we have

$$\frac{\widehat{k}_{n,opt}}{k_{n,opt}} \xrightarrow{P} 1. \tag{4.45}$$

In order to set up an automatic method from a practical point of view, one can use the median of the first $\left[\frac{n}{2}\right]$ k values as an overall estimator for $k_{n,opt}$, i.e.

$$\widehat{k}_{n,med} = median\{\widehat{k}_{n,opt} : k_0 = 3, 4, , \ldots, \left[\frac{n}{2}\right]\}, \tag{4.46}$$

where $\left[\frac{n}{2}\right]$ is the largest integer less than or equal to $\frac{n}{2}$.

Remark 4.4.

1. A special case of the Hall class is the Burr distribution, whose quantile function can be represented as in Equation (4.41) with

$$\gamma = (\lambda\tau)^{-1},\ C = \eta^{1/\tau},\ D = -\tau^{-1},\ \beta = \lambda^{-1}.$$

2. Other examples of distributions in the Hall class are the F distribution with ν and ϕ degrees of freedom, with distribution function given by

$$F_{\nu,\phi}(x) = 1 - \int_x^\infty \frac{\Gamma(\frac{\nu+\phi}{2})}{\Gamma(\frac{\nu}{2})\Gamma(\frac{\phi}{2})}\left(\frac{\nu}{\phi}\right)^{\nu/2} w^{\nu/2-1}\left(1+\frac{\nu}{\phi}w\right)^{-(\nu+\phi)/2} dw,\ x > 0, \nu, \phi > 0, \quad (4.47)$$

and the Student t distribution with ν degrees of freedom defined in Equation (3.37).

See Beirlant et al. [3].

4.3.7 Resampling Method

Resampling can also be used to obtain methods for choosing the threshold. One such method uses the bootstrap combined with an auxiliary statistic, the mean squared error of which converges at the same rate as the Hill estimator and which has a known asymptotic mean, independent of the parameters γ and β (see Beirlant et al. [3]). Such a statistic is

$$A_{k,n} = H_{k,n}^{(2)} - 2H_{k,n}^2, \quad (4.48)$$

with

$$H_{k,n}^{(2)} = \frac{1}{k}\sum_{j=1}^{k}(\log X_{n-j+1,n} - \log X_{n-k,n})^2. \quad (4.49)$$

Since both $H_{k,n}^{(2)}/(2H_{k,n})$ and $H_{k,n}$ are consistent estimators for γ, $A_{k,n}$ will converge to 0 for sequences of k values as $n \to \infty$. Thus

$$AMSE(A_{k,n}) = E_\infty(A_{k,n}^2), \quad (4.50)$$

and no initial parameter estimate is needed to calculate the bootstrap counterpart. Moreover, the k value that minimizes $AMSE(A_{k,n})$, denoted by $\overline{k}_{n,opt}$, is of the same order in n as $k_{n,opt}$:

$$\frac{\overline{k}_{n,opt}}{k_{n,opt}} \to \left(1+\frac{1}{\beta}\right)^{2/(1+2\beta)},\ n \to \infty \quad (4.51)$$

(see [3], page 127). Since the usual bootstrap estimator for $\overline{k}_{n,opt}$ does not converge in probability to the true value, a subsample bootstrap is used. Taking subsamples of size $n_1 = o(n^{1-\varepsilon})$ for some $0 < \varepsilon < 1$ provides a consistent estimator $\widehat{\overline{k}}_{n_1,opt}$ for $\overline{k}_{n_1,opt}$. Taking a second subsample of size

$n_2 = n_1^2/n$ then leads to the estimator

$$\widehat{k}_{n,opt} \sim \frac{\widehat{\overline{k}}^2_{n_1,opt}}{\widehat{\overline{k}}_{n_2,opt}} \left(1 + \frac{1}{\widehat{\beta}_1}\right)^{-2/(1+2\widehat{\beta})} \tag{4.52}$$

for $k_{n,opt}$, where

$$\widehat{\beta}_1 = \frac{\log \widehat{\overline{k}}_{n_1,opt}}{2\log(\widehat{\overline{k}}_{n_1,opt}/n_1)} \tag{4.53}$$

is a consistent estimator for β. The algorithm for this bootstrap procedure is summarized as follows:

1. Draw B bootstrap samples of size $n_1 \in (\sqrt{n}, n)$ from the original sample and determine the value $\widehat{\overline{k}}_{n_1,opt}$ that minimizes the mean squared error of A_{k,n_1}.

2. Repeat this for B bootstrap samples of size $n_2 = n_1^2/n$ and determine the value $\widehat{\overline{k}}_{n_2,opt}$, where the bootstrap mean squared error of A_{k,n_2} is minimal.

3. Calculate $\widehat{k}_{n,opt}$ from Equation (4.52) above.

Note that in this procedure, no preliminary parameter estimation is needed. Only the subsample size n_1 and the number B of bootstrap resamples have to be chosen. See Beirlant et al. [3].

4.3.8 Berning's Adaptive Methods

Berning [5] proposed methods of adaptive threshold selection, that is, adapting k for a given sample, based on estimators obtained from the sample. These methods consist of finding a range or region of k values where the EVI estimators are stable. The estimated value $\widehat{k}$ of k is then taken as the average of the k values corresponding to the stable region.

Let n be the size of the available sample and let $k_1 = 0.05n,\ k_2 = 0.10n,\ \ldots,\ k_{19} = 0.95n$, rounded to nearest integers. Denote by y_i the estimated EVI on the interval $i = 1, 2, \ldots, 19$. In order to quantify the stability of the estimators, Berning [5] proposed a measure of instability ν^2 given by

$$\nu^2 = \sigma^2 + b^2, \tag{4.54}$$

where σ^2 is the standard error of the y-values and b is the slope of the least squares regression line of y on the indices i. He proposed the following three methods for finding a stable region of the EVI estimates.

Fixed Region Length

This method is as follows:

1. Fix the region length to l (an integer such that $2 \leq l \leq 18$).
2. Calculate ν^2 for the region $k_1, k_2, \ldots, k_l$, for the region $k_2, k_3, \ldots, k_{l+1}$, $\ldots$, and for the region $k_{19-l+1}, k_{19-l+2}, \ldots, k_{19}$.
3. Consider as stable the region for which ν^2 is the smallest and choose k from this region.

Reducing the Region until Instability Increases

This method consists of the following steps:

1. Calculate ν^2 over $k_1, k_2, \ldots, k_{19}$ (call it ν_a^2).
2. Calculate ν^2 over $k_2, k_3, \ldots, k_{19}$, i.e. leaving out the first k value (call it ν_F^2).
3. Calculate ν^2 over $k_1, k_2, \ldots, k_{18}$, i.e. leaving out the last k value (call it ν_L^2).
4. Compare the previous values:
   - If $\nu_a^2 < \min(\nu_F^2, \nu_L^2)$, then the region $k_1, k_2, \ldots, k_{19}$ is the stable region.
   - If $\min(\nu_F^2, \nu_L^2) < \nu_a^2$, then repeat steps 1. to 3. by leaving out the first or the last k value in the region, depending on which reduction in the region decreases ν^2 to the greatest extent. In other words, leave out the first k value if $\nu_F^2 < \nu_L^2$ and leave out the last k value if $\nu_L^2 < \nu_F^2$.
5. Repeat steps 1 to 4 until neither reduction of the region decreases the instability.

Fixing the Upper Limit and Reducing the Region from the Left

This method consists of fixing the upper limit of the region, and then calculating ν^2 by leaving out the first k value in the region, then the second, and so on, until ν^2 does not decrease anymore.

Remark 4.5. The methods discussed in the previous subsections often depend on the data one has to handle. Moreover, they are more easily applicable when one restricts oneself to a given data set. In this work we simulated many data sets from various distributions and so it was extremely complex to apply any particular one to the simulated data sets. In order to circumvent this problem, we carried out a simulation to decide what threshold to choose depending on the underlying distribution and the distribution to apply in the tails. In fact, we faced two different choices of the threshold since a proportion of the data is used to estimate the parameters in the tail distribution while that distribution is applied to a different proportion. This adjusts both the variance of the parameter estimators in the tail distribution and the bias of the estimators of inequality measures.

In the simulation we considered $k_{est} = h_{est}n,\ h_{est} \in [0,1]$, to estimate the parameters in the tail distribution and we applied that distribution to $k_{SP} = h_{SP}\sqrt{n},\ h_{SP} \in [0,1]$ exceedances. Different h_{est} and h_{SP} were obtained for each underlying distribution and for each sample size.

This procedure is not the best, but as we will see further in this work, our choices of threshold gave much improved results. Further research needs to be done for better choices.

Chapter 5

Semi-Parametric Estimation of Inequality Measures

In this chapter we discuss the semi-parametric estimation of measures of inequality, in particular, the Gini, the generalized entropy, the Atkinson and the quintile share ratio measures. This estimation procedure is specifically applicable in the case of heavy-tailed distributions. Such distributions often occur in real data sets e.g. in income data which usually have a heavy right tail.

5.1 Semi-parametric Estimation of the Underlying Distribution Function

In this section we discuss the estimation procedure for the distribution function in a semi-parametric setting. Define a semi-parametric distribution function by

$$\tilde{F}(x) = \begin{cases} F(x), & x \leq x_0, \\ F(x_0) + (1 - F(x_0))F_\theta(x), & x > x_0, \end{cases} \tag{5.1}$$

for a given x_0, where F_θ is a parametric distribution satisfying the condition $F_\theta(x_0) = 0$, and F is an unknown distribution. Note that θ can be a vector parameter.

Choose $x_0 = Q(F, 1-\alpha),\ \alpha \in [0,1]$, where Q denotes the quantile function associated with F.

We then have

$$\begin{aligned} F(x_0) + (1 - F(x_0))F_\theta(x) &= 1 - \alpha + (1 - (1 - \alpha))F_\theta(x) \\ &= 1 - \alpha + \alpha F_\theta(x) \\ &= 1 - \alpha(1 - F_\theta(x)). \end{aligned}$$

It follows that

$$\tilde{F}(x) = \begin{cases} F(x), & x \le Q(F, 1-\alpha), \\ 1 - \alpha(1 - F_\theta(x)), & x > Q(F, 1-\alpha), \end{cases} \tag{5.2}$$

where F_θ satisfies the condition $F_\theta(Q(F, 1-\alpha)) = 0$.

Estimating θ by $\widehat{\theta}$, and estimating F by the empirical distribution function F_n, we estimate the underlying distribution semi-parametrically as

$$\tilde{F}_n(x) = \begin{cases} F_n(x), & x \le Q(F_n, 1-\alpha), \\ 1 - \alpha(1 - F_{\widehat{\theta}}(x)), & x > Q(F_n, 1-\alpha). \end{cases} \tag{5.3}$$

Equation (5.3) is very important as it estimates the underlying distribution. However, a choice of the parametric distribution F_θ is required in order to make the estimation process possible. We address that issue by making use of results from Extreme Value Theory (EVT). See e.g. Beirlant et al. [3] for these results.

Given a certain threshold u, we consider the conditional distribution of the exceedance of u given that u was exceeded. We consider two types of exceedances:

1. $X - u$ given $X > u$ (absolute exceedance).
2. X/u given $X > u$ (relative exceedance).

From EVT, if F belongs to the domain of attraction of the generalized extreme value distribution, the following limiting results hold when $u \to \infty$.

1. The distribution of $X - u|X > u$ converges to the generalized Pareto distribution (GPD)

$$G(x; \sigma, \gamma) = 1 - \left(1 + \frac{\gamma x}{\sigma}\right)^{-\frac{1}{\gamma}}, \; x > 0, \tag{5.4}$$

where $\sigma > 0$ is the scale parameter and $\gamma > 0$ is the EVI.

2. The distribution of $X/u|X > u$ converges to the strict Pareto distribution

$$F_P(x) = 1 - x^{-\frac{1}{\gamma}},\ x > 1,\ \gamma > 0, \tag{5.5}$$

where γ is the extreme value index.

3. A second order approximation of the distribution of the relative exceedance: the perturbed Pareto distribution.

Note that the strict Pareto is only a first order approximation. In order to better describe the departure from the strict Pareto, we can also use a second order refinement, leading to an approximation of the distribution of the relative exceedance by the mixture of two Pareto distributions, namely the so-called perturbed Pareto distribution (PPD). The motivation for this is as follows.

The definition of a Pareto-type tail is

$$1 - F(x) = x^{-1/\gamma} l_F(x), \tag{5.6}$$

with l_F a slowly varying function at infinity.

Writing

$$F_t(x) = P(X/t \leq x | X > t), \tag{5.7}$$

it follows directly that

$$1 - F_t(x) = x^{-1/\gamma} \frac{l_F(tx)}{l_F(t)} \to x^{-1/\gamma}, \text{ as } t \to \infty. \tag{5.8}$$

This gives a first order approximation.

As a second order approximation, we consider the rate of convergence of the slowly varying function l_F, that is the rate in

$$\frac{l_F(tx)}{l_F(t)} \to 1, \text{ as } t \to \infty. \tag{5.9}$$

For this, assume a function $B^{\star}(t)$ exists, $B^{\star}(t) \to 0$, as $t \to \infty$, such that

$$B^{\star}(t)^{-1} \left[\frac{l_F(tx)}{l_F(t)} - 1 \right] = \frac{l_F(tx) - l_F(t)}{B^{\star}(t) l_F(t)} \to m(x). \tag{5.10}$$

It then follows from regular EVT, that

$$m(u) = d \frac{u^{-\tau} - 1}{-\tau},\ \tau > 0, \tag{5.11}$$

where d is a constant and $B^\star$ is regularly varying at infinity with index $-\tau$, i.e.

$$\lim_{t\to\infty}\frac{B^\star(tx)}{B^\star(t)} = x^{-\tau},\ \forall x>0. \tag{5.12}$$

(See e.g. Beirlant et al. [3], p. 91.)

Substituting in (5.8) gives

$$\begin{aligned}\frac{1-F(tx)}{1-F(t)} &= x^{-1/\gamma}\left(\frac{l_F(tx)-l_F(t)}{B^\star(t)l_F(t)}B^\star(t)+1\right)\\ &= x^{-1/\gamma}\left(1+B(t)h_{-\tau}(x)+o(B(t))\right),\ \text{as } t\to\infty,\end{aligned} \tag{5.13}$$

where

$$h_{-\tau}(x) = \frac{x^{-\tau}-1}{-\tau}$$

and $B = dB^\star$ is also a regularly varying function at infinity with index $-\tau$.

Condition (5.13) can be rewritten as

$$1-F_t(x) = x^{-1/\gamma}\left[1-B(t)\tau^{-1}(x^{-\tau}-1)+o(B(t))\right],\ \text{as } t\to\infty. \tag{5.14}$$

Leaving out the error term in (5.14), one refines the original Pareto approximation to a mixture of two Pareto distributions, namely the perturbed Pareto distribution (PPD) defined by the survival function

$$\overline{G}(x;\gamma,c,\tau) = 1-G(x;\gamma,c,\tau) = (1-c)x^{-1/\gamma}+cx^{-1/\gamma-\tau}, \tag{5.15}$$

where $x>1$, $\gamma>0$, $\tau>0$ and $c\in(-1/\tau,1)$.

The idea now is to fit such a perturbed Pareto distribution to the relative exceedance, aiming for a more accurate estimation of the unknown tail. See Beirlant et al. [3] for more details.

5.2 Choices of Parametric Distribution Functions using Extreme Value Theory

In the previous section we described three different results from EVT, namely the GPD, the strict Pareto and the PPD. We will now use these to choose the parametric distribution in Equation (5.1) and to find semi-parametric estimators for the distribution function using Equation (5.3).

5.2.1 SP Estimator of the Underlying Distribution when Fitting the GPD

Given a random variable X with distribution function F, we have for a large x_0 that

$$X - x_0 | X > x_0 \overset{D}{\simeq} GPD. \tag{5.16}$$

Let G denote the distribution function of the GPD. Using Equation (5.4), we can write, for a large x_0 and $x > x_0$,

$$\begin{aligned} P[X - x_0 > x | X > x_0] &= \frac{P[X - x_0 > x]}{P[X > x_0]} \\ &= \frac{P[X > x + x_0]}{[X > x_0]} \\ &= \frac{\overline{F}(x + x_0)}{\overline{F}(x_0)} \approx \overline{G}(x; \sigma, \gamma), \end{aligned}$$

where $\overline{F}(x) = 1 - F(x)$ and $\overline{G}(x;\sigma,\gamma) = 1 - G(x;\sigma,\gamma)$. It follows that

$$\overline{F}(x + x_0) \approx \overline{F}(x_0)\overline{G}(x;\sigma,\gamma), \tag{5.17}$$

which becomes (by setting $y = x + x_0$),

$$\overline{F}(y) \approx \overline{F}(x_0)\overline{G}(y - x_0;\sigma,\gamma). \tag{5.18}$$

It follows that for a large value of x_0

$$F(x) \approx 1 - \overline{F}(x_0)\overline{G}(x - x_0;\sigma,\gamma). \tag{5.19}$$

As choice of F_θ in Equation (5.1) we can therefore take the right hand side of Equation (5.19) to get

$$\tilde{F}(x) = \begin{cases} F(x), & x \leq x_0, \\ 1 - \overline{F}(x_0)\overline{G}(x - x_0;\sigma,\gamma), & x > x_0, \end{cases} \tag{5.20}$$

for a large value of x_0.

Now estimate F by the empirical distribution function

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\{X_{i,n} \leq x\},$$
$$= \begin{cases} 0, \ x < X_{1,n}, \\ \frac{i}{n}, \ X_{i,n} \leq x < X_{i+1,n}, \\ 1, \ x \geq X_{n,n}. \end{cases} \tag{5.21}$$

and x_0 by $X_{n-k,n}$. Take k as relatively small so that the GPD is a reasonable approximation to the distribution of the exceedances $X_{n-k+1,n} - X_{n-k,n}, X_{n-k+2,n} - X_{n-k,n}, \ldots, X_{n,n} - X_{n-k,n}$. Finally, estimate σ and γ by $\widehat{\sigma}$ and $\widehat{\gamma}$ using these exceedances. A semi-parametric estimator for the underlying distribution function is then given by

$$\tilde{F}_n(x) = \begin{cases} F_n(x), & x \leq X_{n-k,n}, \\ 1 - \frac{k}{n}\overline{G}(x - X_{n-k,n}; \widehat{\sigma}, \widehat{\gamma}), & x > X_{n-k,n}. \end{cases} \tag{5.22}$$

Using the definition of the GPD as given in Equation (5.4), we get

$$\tilde{F}_n(x) = \begin{cases} F_n(x), & x \leq X_{n-k,n}, \\ 1 - \frac{k}{n}\left[1 + \frac{\widehat{\gamma}(x - X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}}, & x > X_{n-k,n}. \end{cases} \tag{5.23}$$

5.2.2 SP Estimator of the Underlying Distribution when Fitting the Strict Pareto

Instead of fitting the GPD to the absolute exceedances, we can fit the strict Pareto distribution to the relative exceedances. Let F_P denote the distribution function of the strict Pareto distribution. Using Equation (5.5), we can write for a large x_0 and $x > 1$,

$$P[\frac{X}{x_0} > x | X > x_0] = \frac{P[X > x_0 x]}{P[X > x_0]}$$
$$= \frac{\overline{F}(x_0 x)}{\overline{F}(x_0)} \approx \overline{F}_P(x; \gamma).$$

It follows that

$$\overline{F}(x_0 x) \approx \overline{F}(x_0)\overline{F}_P(x; \gamma), \tag{5.24}$$

which becomes (by setting $y = x_0 x$),

$$\overline{F}(y) \approx \overline{F}(x_0)\overline{F}_P(y/x_0;\gamma), \tag{5.25}$$

i.e.

$$F(y) \approx 1 - \overline{F}(x_0)\overline{F}_P(y/x_0;\gamma). \tag{5.26}$$

From Equation (5.26), we can write $\tilde{F}$ as

$$\tilde{F}(x) = \begin{cases} F(x), & x \leq x_0, \\ 1 - \overline{F}(x_0)\overline{F}_P(x/x_0;\gamma), & x > x_0, \end{cases} \tag{5.27}$$

for a large x_0.

Now estimate F again by the empirical distribution function F_n given in Equation (5.21) and x_0 by $X_{n-k,n}$. Take k as relatively small so that the strict Pareto is a reasonable approximation to the distribution of the relative exceedances $\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$. Finally, estimate γ by $\widehat{\gamma}$ using the exceedances. A semi-parametric estimator for the underlying distribution function in this case is then given by

$$\tilde{F}_n(x) = \begin{cases} F_n(x), & x \leq X_{n-k,n}, \\ 1 - \frac{k}{n}\overline{F}_P(x/X_{n-k,n};\widehat{\gamma}), & x > X_{n-k,n}, \end{cases} \tag{5.28}$$

which, by using the definition of the strict Pareto distribution as given in Equation (5.5), becomes

$$\tilde{F}_n(x) = \begin{cases} F_n(x), & x \leq X_{n-k,n}, \\ 1 - \frac{k}{n}X_{n-k,n}^{\frac{1}{\gamma}}x^{-\frac{1}{\gamma}}, & x > X_{n-k,n}. \end{cases} \tag{5.29}$$

5.2.3 SP Estimator of the Underlying Distribution when Fitting the PPD

In this section we approximate the distribution of the relative exceedance by the PPD given in Equation 5.15. The same procedure as in Section 5.2.2 then leads to

$$\tilde{F}(x) = \begin{cases} F(x), & x \leq x_0, \\ 1 - \overline{F}(x_0)\overline{G}(x/x_0;\gamma,c,\tau), & x > x_0, \end{cases} \tag{5.30}$$

for a large x_0. As before, estimate F by the empirical distribution function F_n given in Equation (5.21) and x_0 by $X_{n-k,n}$, and take k as relatively small so that the perturbed Pareto is a reasonable approximation to the distribution of the relative exceedances $\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$. Finally, estimate γ, c and τ respectively by $\widehat{\gamma}$, $\widehat{c}$ and $\widehat{\tau}$ using the relative exceedances. The semi-parametric estimator of the underlying distribution function is then given by

$$\tilde{F}_n(x) = \begin{cases} F_n(x), & x \leq X_{n-k,n}, \\ 1 - \frac{k}{n}\overline{G}(x/X_{n-k,n}; \widehat{\gamma}, \widehat{c}, \widehat{\tau}), & x > X_{n-k,n}, \end{cases} \tag{5.31}$$

which, by using the definition of the perturbed Pareto distribution as given in Equation (5.15), becomes

$$\tilde{F}_n(x) = \begin{cases} F_n(x), & x \leq X_{n-k,n}, \\ 1 - \frac{k}{n} X_{n-k,n}^{1/\widehat{\gamma}} \left[(1-\widehat{c})x^{-1/\widehat{\gamma}} + \widehat{c} X_{n-k,n}^{\widehat{\tau}} x^{-1/\widehat{\gamma}-\widehat{\tau}}\right], & x > X_{n-k,n}. \end{cases} \tag{5.32}$$

5.3 Semi-Parametric Estimators of Measures of Inequality

Having estimated the underlying distribution, we now use it to estimate the measures of inequality. This is done for the Gini, the generalized entropy, the Atkinson and the quintile share ratio measures.

Remark 5.1. Throughout this section the estimators for the parameters introduced via the parametric part of the distribution will not be given explicitly. The maximum likelihood estimators of these parameters will be used throughout. The distribution function F in the subsequent sections will be considered as having $\gamma > 0$.

5.3.1 Semi-Parametric Estimation of Gini Coefficient

Recall from Section 2.3.4 that, given a distribution function F of a random variable X with mean μ, the ordinary Gini coefficient can be defined as

$$I_G = \frac{1}{\mu}\int_0^\infty F(x)(1-F(x))dx. \tag{5.33}$$

For random samples from F, there exist a number of estimators for I_G, most of them involving the empirical distribution function over the whole sample. In this section, we propose an estimator for the Gini coefficient in a semi-parametric setting. Consider a random sample $X_1, X_2, \ldots, X_n$ from F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$, and suppose the threshold above which a parametric distribution is fitted is $x_0 = Q(F, 1-\alpha)$.

The Gini coefficient is then estimated semi-parametrically as

$$\widehat{I}_{SPG} = \frac{1}{\widehat{\mu}} \int_0^\infty \tilde{F}_n(x)(1 - \tilde{F}_n(x))dx, \tag{5.34}$$

where $\tilde{F}_n$ is given in Equation (5.3), and $\widehat{\mu}$ is the estimator of μ using $\tilde{F}_n$, that is

$$\widehat{\mu} = \int_0^\infty x d\tilde{F}_n(x). \tag{5.35}$$

Estimation of I_G when F_θ is Derived from a GPD

The following theorem gives a semi-parametric estimator for I_G in the case where the parametric distribution is derived from a GPD.

Theorem 5.1. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the GPD is a reasonable approximation to the distribution of the exceedances $X_{n-k+1,n} - X_{n-k,n}, X_{n-k+2,n} - X_{n-k,n}, \ldots, X_{n,n} - X_{n-k,n}$ for a given k. A semi-parametric estimator for I_G is then given by

$$\widehat{I}_{SPG} = \frac{1}{\widehat{\mu}} \sum_{i=1}^{n-k-1} \frac{i}{n}(1 - \frac{i}{n})(X_{i+1,n} - X_{i,n}) + \frac{k\widehat{\sigma}\,[2n - k - \widehat{\gamma}(n-k)]}{n^2\widehat{\mu}(1-\widehat{\gamma})(2-\widehat{\gamma})}, \tag{5.36}$$

where $\widehat{\sigma}$ and $\widehat{\gamma}$ are estimators for the unknown scale and shape parameters σ and γ of the GPD using the exceedances, with $\widehat{\gamma} < 1$, and

$$\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n-k} X_{i,n} + \frac{k}{n}\left(X_{n-k,n} + \frac{\widehat{\sigma}}{1-\widehat{\gamma}}\right) \tag{5.37}$$

is an estimator for μ.

Proof. Using the estimator in Equation (5.23) we have for $\widehat{\gamma} < 1$,

$$\begin{aligned}
\widehat{\mu} &= \int_0^\infty x d\tilde{F}_n(x) && (5.38)\\
&= \int_0^{X_{n-k,n}} x dF_n(x) + \frac{k}{n\widehat{\sigma}} \int_{X_{n-k,n}}^\infty x \left[1 + \frac{\widehat{\gamma}(x - X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx \\
&= \frac{1}{n} \sum_{i=1}^{n-k} X_{i,n} + \frac{k}{n} X_{n-k,n} + \frac{k\widehat{\sigma}}{n(1-\widehat{\gamma})} \\
&= \frac{1}{n} \sum_{i=1}^{n-k} X_{i,n} + \frac{k}{n}\left(X_{n-k,n} + \frac{\widehat{\sigma}}{1-\widehat{\gamma}}\right), && (5.39)
\end{aligned}$$

and

$$\widehat{I}_{SPG} = \frac{1}{\widehat{\mu}}\int_0^{X_{n-k,n}} F_n(x)(1-F_n(x))dx + \frac{k}{n\widehat{\mu}}\int_{X_{n-k,n}}^{\infty}\left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}}dx$$
$$-\frac{k^2}{n^2\widehat{\mu}}\int_{X_{n-k,n}}^{\infty}\left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{2}{\widehat{\gamma}}}dx. \tag{5.40}$$

Let

$$\widehat{L} = \frac{1}{\widehat{\mu}}\int_0^{X_{n-k,n}} F_n(x)(1-F_n(x))dx \tag{5.41}$$

and

$$\widehat{R} = \frac{k}{n\widehat{\mu}}\int_{X_{n-k,n}}^{\infty}\left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}}dx - \frac{k^2}{n^2\widehat{\mu}}\int_{X_{n-k,n}}^{\infty}\left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{2}{\widehat{\gamma}}}dx, \tag{5.42}$$

so that

$$\widehat{I}_{SPG} = \widehat{L}+\widehat{R}. \tag{5.43}$$

We have

$$\widehat{L} = \frac{1}{\widehat{\mu}}\sum_{i=1}^{n-k-1}(X_{i+1,n}-X_{i,n})F_n(X_{i,n})(1-F_n(X_{i,n})). \tag{5.44}$$

Since

$$F_n(X_{i,n}) = \frac{i}{n},$$

it follows that

$$\widehat{L} = \frac{1}{\widehat{\mu}}\sum_{i=1}^{n-k-1}\frac{i}{n}(1-\frac{i}{n})(X_{i+1,n}-X_{i,n}). \tag{5.45}$$

Furthermore, simple integration, for $\widehat{\gamma}<1$, leads to

$$\widehat{R} = \frac{k\widehat{\sigma}}{n\widehat{\mu}(1-\widehat{\gamma})} - \frac{k^2\widehat{\sigma}}{n^2\widehat{\mu}(2-\widehat{\gamma})},$$

and finally

$$\widehat{R} = \frac{k\widehat{\sigma}\,[2n-k-\widehat{\gamma}(n-k)]}{n^2\widehat{\mu}(1-\widehat{\gamma})(2-\widehat{\gamma})}. \tag{5.46}$$

Substituting Equations (5.45) and (5.46) in Equation (5.43), we get

$$\widehat{I}_{SPG} = \frac{1}{\widehat{\mu}}\sum_{i=1}^{n-k-1}\frac{i}{n}(1-\frac{i}{n})(X_{i+1,n}-X_{i,n}) + \frac{k\widehat{\sigma}\,[2n-k-\widehat{\gamma}(n-k)]}{n^2\widehat{\mu}(1-\widehat{\gamma})(2-\widehat{\gamma})}, \tag{5.47}$$

where

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{k}{n}\left(X_{n-k,n} + \frac{\widehat{\sigma}}{1-\widehat{\gamma}}\right).$$

This completes the proof. □

Remark 5.2. We obtained the estimator (5.47) under the assumption $\widehat{\gamma} < 1$. In the case where $\widehat{\gamma} > 1$, we have $\frac{1}{\widehat{\gamma}} < 1$ and so the integrals in Equation (5.42) lead to infinity. This yields the restriction that the semi-parametric Gini coefficient is not defined in our setting for $\widehat{\gamma} > 1$.

Estimation of I_G when F_θ is Derived from a Strict Pareto Distribution

The following theorem gives a semi-parametric estimator for I_G in the case where the parametric distribution is derived from a strict Pareto distribution.

Theorem 5.2. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the strict Pareto distribution is a reasonable approximation to the distribution of the relative exceedances

$$\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$$

for a given k. A semi-parametric estimator for I_G is given by

$$\widehat{I}_{SPG} = \frac{1}{\widehat{\mu}}\sum_{i=1}^{n-k-1} \frac{i}{n}\left(1-\frac{i}{n}\right)(X_{i+1,n} - X_{i,n}) + \frac{k\widehat{\gamma}X_{n-k,n}\left[2n-k-\widehat{\gamma}(n-k)\right]}{n^2\widehat{\mu}(1-\widehat{\gamma})(2-\widehat{\gamma})}, \tag{5.48}$$

where $\widehat{\gamma}$ is an estimator for the unknown parameter γ in the strict Pareto distribution using the relative exceedances, with $\widehat{\gamma} < 1$, and

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n(1-\widehat{\gamma})} \tag{5.49}$$

is an estimator for μ.

Proof. Using the estimator in Equation (5.29) we have

$$\begin{aligned}\int_{X_{n-k,n}}^{\infty} x\,d\tilde{F}_n(x) &= \frac{k}{n\widehat{\gamma}} X_{n-k,n}^{\frac{1}{\widehat{\gamma}}} \int_{X_{n-k,n}}^{\infty} x^{-\frac{1}{\widehat{\gamma}}}dx \\ &= \frac{kX_{n-k,n}}{n(1-\widehat{\gamma})},\end{aligned} \tag{5.50}$$

and

$$\int_{X_{n-k,n}}^{\infty} \tilde{F}_n(x)(1-\tilde{F}_n(x))dx = \frac{k}{n}\int_{X_{n-k,n}}^{\infty} X_{n-k,n}^{\frac{1}{\hat{\gamma}}} x^{-\frac{1}{\hat{\gamma}}}(1-\frac{k}{n}X_{n-k,n}^{\frac{1}{\hat{\gamma}}} x^{-\frac{1}{\hat{\gamma}}})dx$$
$$= \frac{k}{n}X_{n-k,n}^{\frac{1}{\hat{\gamma}}}\left[\int_{X_{n-k,n}}^{\infty} x^{-\frac{1}{\hat{\gamma}}}dx - \frac{k}{n}X_{n-k,n}^{\frac{1}{\hat{\gamma}}}\int_{X_{n-k,n}}^{\infty} x^{-\frac{2}{\hat{\gamma}}}dx\right]$$
$$= \frac{k}{n}X_{n-k,n}\left[\frac{\widehat{\gamma}}{1-\widehat{\gamma}} - \frac{k\widehat{\gamma}}{n(2-\widehat{\gamma})}\right]$$
$$= \frac{k\widehat{\gamma}X_{n-k,n}\left[2n-k-\widehat{\gamma}(n-k)\right]}{n^2(1-\widehat{\gamma})(2-\widehat{\gamma})}. \tag{5.51}$$

It easily follows that

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n(1-\widehat{\gamma})} \tag{5.52}$$

and

$$\widehat{I}_{SPG} = \frac{1}{\widehat{\mu}}\sum_{i=1}^{n-k-1}\frac{i}{n}(1-\frac{i}{n})(X_{i+1,n}-X_{i,n}) + \frac{k\widehat{\gamma}X_{n-k,n}\left[2n-k-\widehat{\gamma}(n-k)\right]}{n^2\widehat{\mu}(1-\widehat{\gamma})(2-\widehat{\gamma})}, \tag{5.53}$$

which completes the proof. □

Estimation of I_G when F_θ is Derived from a Perturbed Pareto Distribution

The following theorem gives a semi-parametric estimator for I_G in the case where the parametric distribution is derived from a perturbed Pareto distribution.

Theorem 5.3. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the perturbed Pareto distribution is a reasonable approximation to the distribution of the relative exceedances

$$\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$$

for a given k. A semi-parametric estimator for I_G is given by

$$\widehat{I}_{SPG} = \frac{1}{\widehat{\mu}}\sum_{i=1}^{n-k-1}\frac{i}{n}(1-\frac{i}{n})\left(X_{i+1,n}-X_{i,n}\right) + \frac{k\widehat{\gamma}X_{n-k,n}}{n\widehat{\mu}}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right]$$
$$-\frac{k^2\widehat{\gamma}X_{n-k,n}}{n^2\widehat{\mu}}\left[\frac{(1-\widehat{c})^2}{2-\widehat{\gamma}} + \frac{\widehat{c}^2}{2+\widehat{\gamma}(2\widehat{\tau}-1)} + \frac{2\widehat{c}(1-\widehat{c})}{2+\widehat{\gamma}(\widehat{\tau}-1)}\right], \tag{5.54}$$

where $\widehat{\gamma}$, $\widehat{c}$ and $\widehat{\tau}$ are estimators for the unknown parameters γ, c and τ in the PPD using the relative exceedances, with $\widehat{\gamma} < 1$, and

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right] \tag{5.55}$$

is an estimator for μ.

Proof. Using the estimator in Equation (5.32) we have

$$\int_{X_{n-k,n}}^{\infty} x d\tilde{F}_n(x) = \frac{k}{n} X_{n-k,n}^{1/\widehat{\gamma}}\left[\frac{1-\widehat{c}}{\widehat{\gamma}}\int_{X_{n-k,n}}^{\infty} x^{-1/\widehat{\gamma}}dx + \widehat{c}\left(\frac{1}{\widehat{\gamma}}+\widehat{\tau}\right) X_{n-k,n}^{\widehat{\tau}}\int_{X_{n-k,n}}^{\infty} x^{-1/\widehat{\gamma}-\widehat{\tau}}dx\right],$$

and

$$\begin{aligned}
&\int_{X_{n-k,n}}^{\infty} \tilde{F}_n(x)(1-\tilde{F}_n(x))dx \\
&\quad = \frac{k}{n} X_{n-k,n}^{1/\widehat{\gamma}}\left[(1-\widehat{c})\int_{X_{n-k,n}}^{\infty} x^{-1/\widehat{\gamma}}dx + \widehat{c}X_{n-k,n}^{\widehat{\tau}}\int_{X_{n-k,n}}^{\infty} x^{-1/\widehat{\gamma}-\widehat{\tau}}dx\right] \\
&\quad - \frac{k^2}{n^2} X_{n-k,n}^{2/\widehat{\gamma}}\left[(1-\widehat{c})^2\int_{X_{n-k,n}}^{\infty} x^{-2/\widehat{\gamma}}dx + \widehat{c}^2 X_{n-k,n}^{2\widehat{\tau}}\int_{X_{n-k,n}}^{\infty} x^{-2(1/\widehat{\gamma}+\widehat{\tau})}dx\right] \\
&\quad - \frac{k^2}{n^2} X_{n-k,n}^{2/\widehat{\gamma}}\left[2\widehat{c}(1-\widehat{c})X_{n-k,n}^{\tau}\int_{X_{n-k,n}}^{\infty} x^{-(2/\widehat{\gamma}+\widehat{\tau})}dx\right],
\end{aligned}$$

which, by simple integration, leads to

$$\int_{X_{n-k,n}}^{\infty} x d\tilde{F}_n(x) = \frac{kX_{n-k,n}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right], \tag{5.56}$$

and

$$\begin{aligned}
\int_{X_{n-k,n}}^{\infty} \tilde{F}_n(x)(1-\tilde{F}_n(x))dx &= \frac{k\widehat{\gamma}X_{n-k,n}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right] \\
&- \frac{k^2\widehat{\gamma}X_{n-k,n}}{n^2}\left[\frac{(1-\widehat{c})^2}{2-\widehat{\gamma}} + \frac{\widehat{c}^2}{2+\widehat{\gamma}(2\widehat{\tau}-1)} + \frac{2\widehat{c}(1-\widehat{c})}{2+\widehat{\gamma}(\widehat{\tau}-1)}\right].
\end{aligned} \tag{5.57}$$

It follows that

$$\begin{aligned}
\widehat{I}_{SPG} = \frac{1}{\widehat{\mu}}\sum_{i=1}^{n-k-1}\frac{i}{n}(1-\frac{i}{n})\left(X_{i+1,n}-X_{i,n}\right) &+ \frac{k\widehat{\gamma}X_{n-k,n}}{n\widehat{\mu}}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right] \\
&- \frac{k^2\widehat{\gamma}X_{n-k,n}}{n^2\widehat{\mu}}\left[\frac{(1-\widehat{c})^2}{2-\widehat{\gamma}} + \frac{\widehat{c}^2}{2+\widehat{\gamma}(2\widehat{\tau}-1)} + \frac{2\widehat{c}(1-\widehat{c})}{2+\widehat{\gamma}(\widehat{\tau}-1)}\right],
\end{aligned} \tag{5.58}$$

with
$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right]. \tag{5.59}$$

□

Remark 5.3. It is easy to show that for $c = 0$ one obtains the corresponding expressions in the case of the strict Pareto distribution.

5.3.2 Semi-Parametric Estimation of Generalized Entropy Inequality Measures

Recall from Section 2.3.1 that the generalized entropy (GE) measures are defined as follows.

$$I_E^\alpha = \frac{1}{\alpha(\alpha-1)}\left(\frac{\mu_\alpha}{\mu_1^\alpha} - 1\right),\ \alpha \neq 0, \alpha \neq 1, \tag{5.60}$$

$$I_E^0 = \log \mu_1 - \nu_1 \tag{5.61}$$

and
$$I_E^1 = \frac{\nu_2}{\mu_1} - \log \mu_1, \tag{5.62}$$

where
$$\mu_\alpha = \int_0^\infty x^\alpha dF(x), \tag{5.63}$$

$$\nu_1 = \int_0^\infty (\log x) dF(x) \tag{5.64}$$

and
$$\nu_2 = \int_0^\infty (x \log x) dF(x). \tag{5.65}$$

In this section we derive semi-parametric estimators for the GE inequality measures.

Estimation of I_E^α when F_θ is Derived from a Generalized Pareto Distribution

The following theorem gives a semi-parametric estimator for I_E^α in the case where the parametric distribution is derived from a GPD.

Theorem 5.4. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the GPD is a reasonable approximation to the distribution of the exceedances $X_{n-k+1,n} - X_{n-k,n}, X_{n-k+2,n} - X_{n-k,n}, \ldots, X_{n,n} - X_{n-k,n}$

for a given k. Denote by $\widehat{I}_\alpha, \alpha \neq 0,1$, $\widehat{I}_0$ and $\widehat{I}_1$ the numerical approximations of the integrals

$$\int_{X_{n-k,n}}^{\infty} x^{\alpha} \left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx,$$

$$\int_{X_{n-k,n}}^{\infty} (\log x) \left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx,$$

and

$$\int_{X_{n-k,n}}^{\infty} (x \log x) \left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx,$$

respectively. Estimators for the GE inequality measures are given as follows.

$$\widehat{I}^{\alpha}_{SPE} = \frac{1}{\alpha(\alpha-1)}\left(\frac{\widehat{\mu}_\alpha}{\widehat{\mu}_1^{\alpha}}-1\right), \ \alpha \neq 0, \alpha \neq 1, \tag{5.66}$$

$$\widehat{I}^{0}_{SPE} = \log \widehat{\mu}_1 - \widehat{\nu}_1 \tag{5.67}$$

and

$$\widehat{I}^{1}_{SPE} = \frac{\widehat{\nu}_2}{\widehat{\mu}_1} - \log \widehat{\mu}_1, \tag{5.68}$$

where $\widehat{\mu}_\alpha$, $\widehat{\mu}_1$, $\widehat{\nu}_1$ and $\widehat{\nu}_2$ are given by

$$\widehat{\mu}_\alpha = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^{\alpha} + \frac{k\widehat{I}_\alpha}{n\widehat{\sigma}}, \tag{5.69}$$

$$\widehat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{k}{n}\left(X_{n-k,n} + \frac{\widehat{\sigma}}{1-\widehat{\gamma}}\right), \tag{5.70}$$

$$\widehat{\nu}_1 = \frac{1}{n}\sum_{i=1}^{n-k} \log(X_{i,n}) + \frac{k\widehat{I}_0}{n\widehat{\sigma}} \tag{5.71}$$

and

$$\widehat{\nu}_2 = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}\log(X_{i,n}) + \frac{k\widehat{I}_1}{n\widehat{\sigma}}, \tag{5.72}$$

respectively, and $\widehat{\sigma}$ and $\widehat{\gamma}$ are estimators for the unknown scale and shape parameters σ and γ of the GPD using the exceedances.

Proof. Recall that in the case of the GPD, $\tilde{F}$ is estimated by

$$\tilde{F}_n(x) = \begin{cases} F_n(x), & x \leq X_{n-k,n}, \\ 1-\frac{k}{n}\left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}}, & x > X_{n-k,n}. \end{cases}$$

Therefore, we can estimate the moment μ_α by

$$\widehat{\mu}_\alpha = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^\alpha + \frac{k}{n\widehat{\sigma}}\int_{X_{n-k,n}}^{\infty} x^\alpha \left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx. \tag{5.73}$$

The integral

$$\int_{X_{n-k,n}}^{\infty} x^\alpha \left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx$$

can be approximated by numerical methods. Denoting the approximation by $\widehat{I}_\alpha$, we can estimate μ_α by

$$\widehat{\mu}_\alpha = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^\alpha + \frac{k\widehat{I}_\alpha}{n\widehat{\sigma}}. \tag{5.74}$$

On the other hand we have

$$\widehat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{k}{n}\left(X_{n-k,n} + \frac{\widehat{\sigma}}{1-\widehat{\gamma}}\right) \tag{5.75}$$

and

$$\widehat{\nu}_1 = \frac{1}{n}\sum_{i=1}^{n-k} \log(X_{i,n}) + \frac{k\widehat{I}_0}{n\widehat{\sigma}}, \tag{5.76}$$

where $\widehat{I}_0$ is an approximation of the integral

$$\int_{X_{n-k,n}}^{\infty} (\log x) \left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx,$$

using numerical methods.

Similarly, we have

$$\widehat{\nu}_2 = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}\log(X_{i,n}) + \frac{k\widehat{I}_1}{n\widehat{\sigma}}, \tag{5.77}$$

where $\widehat{I}_1$ is an approximation of the integral

$$\int_{X_{n-k,n}}^{\infty} (x\log x) \left[1+\frac{\widehat{\gamma}(x-X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx,$$

using numerical methods. The theorem result follows. □

Estimation of I_E^α when F_θ is Derived from a Strict Pareto Distribution

The following theorem gives a semi-parametric estimator for I_E^α in the case where the parametric distribution is derived from a strict Pareto distribution.

Theorem 5.5. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the strict Pareto distribution is a reasonable approximation to the distribution of the relative exceedances

$$\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$$

for a given k. Semi-parametric estimators for the GE inequality measures are given as follows.

$$\widehat{I}^{\alpha}_{SPE} = \frac{1}{\alpha(\alpha-1)}\left(\frac{\widehat{\mu}_{\alpha}}{\widehat{\mu}_1^{\alpha}} - 1\right), \ \alpha \neq 0, \alpha \neq 1, \tag{5.78}$$

$$\widehat{I}^{0}_{SPE} = \log \widehat{\mu}_1 - \widehat{\nu}_1 \tag{5.79}$$

and

$$\widehat{I}^{1}_{SPE} = \frac{\widehat{\nu}_2}{\widehat{\mu}_1} - \log \widehat{\mu}_1, \tag{5.80}$$

where $\widehat{\mu}_{\alpha}$, $\widehat{\mu}_1$, $\widehat{\nu}_1$ and $\widehat{\nu}_2$ are given by

$$\widehat{\mu}_{\alpha} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^{\alpha} + \frac{kX_{n-k,n}^{\alpha}}{n(1-\alpha\widehat{\gamma})}, \tag{5.81}$$

$$\widehat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n(1-\widehat{\gamma})}, \tag{5.82}$$

$$\widehat{\nu}_1 = \frac{1}{n}\sum_{i=1}^{n-k} \log(X_{i,n}) + \frac{k}{n}\left[\log(X_{n-k,n}) + \widehat{\gamma}\right] \tag{5.83}$$

and

$$\widehat{\nu}_2 = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}\log(X_{i,n}) + \frac{k}{n(1-\widehat{\gamma})}X_{n-k,n}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1-\widehat{\gamma}}\right], \tag{5.84}$$

respectively, and $\widehat{\gamma}$ is an estimator for the unknown shape parameter γ of the strict Pareto distribution using the exceedances.

Proof. Estimating the underlying distribution by $\tilde{F}_n$ given in Equation (5.29), we can estimate μ_{α} by

$$\widehat{\mu}_{\alpha} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^{\alpha} + \int_{X_{n-k,n}}^{\infty} x^{\alpha} d\left(1 - \frac{k}{n}X_{n-k,n}^{\frac{1}{\widehat{\gamma}}} x^{-\frac{1}{\widehat{\gamma}}}\right). \tag{5.85}$$

But

$$\begin{aligned}\int_{X_{n-k,n}}^{\infty} x^{\alpha} d\left(1-\frac{k}{n}X_{n-k,n}^{\frac{1}{\widehat{\gamma}}}x^{-\frac{1}{\widehat{\gamma}}}\right) &= \int_{X_{n-k,n}}^{\infty} \frac{k}{n}\frac{1}{\widehat{\gamma}}X_{n-k,n}^{1/\widehat{\gamma}}x^{\alpha-(1/\widehat{\gamma}+1)}dx \\ &= \frac{k}{n\widehat{\gamma}}X_{n-k,n}^{1/\widehat{\gamma}}\int_{X_{n-k,n}}^{\infty} x^{\alpha-(1/\widehat{\gamma}+1)}dx \\ &= \frac{kX_{n-k,n}^{\alpha}}{n(1-\alpha\widehat{\gamma})},\ \alpha<\frac{1}{\widehat{\gamma}}.\end{aligned}$$

It follows that

$$\widehat{\mu}_{\alpha} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^{\alpha} + \frac{kX_{n-k,n}^{\alpha}}{n(1-\alpha\widehat{\gamma})}. \tag{5.86}$$

In particular, we have

$$\widehat{\mu}_{1} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n(1-\widehat{\gamma})}. \tag{5.87}$$

On the other hand, we have

$$\widehat{\nu}_{1} = \frac{1}{n}\sum_{i=1}^{n-k} \log(X_{i,n}) + \int_{X_{n-k,n}}^{\infty}(\log x)d\left(1-\frac{k}{n}X_{n-k,n}^{\frac{1}{\widehat{\gamma}}}x^{-\frac{1}{\widehat{\gamma}}}\right) \tag{5.88}$$

and

$$\widehat{\nu}_{2} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}\log(X_{i,n}) + \frac{k}{n}\int_{X_{n-k,n}}^{\infty}(x\log x)d\left(1-\frac{k}{n}X_{n-k,n}^{\frac{1}{\widehat{\gamma}}}x^{-\frac{1}{\widehat{\gamma}}}\right). \tag{5.89}$$

But

$$\begin{aligned}\int_{X_{n-k,n}}^{\infty}(\log x)d\left(1-\frac{k}{n}X_{n-k,n}^{\frac{1}{\widehat{\gamma}}}x^{-\frac{1}{\widehat{\gamma}}}\right) &= \frac{k}{n\widehat{\gamma}}X_{n-k,n}^{1/\widehat{\gamma}}\int_{X_{n-k,n}}^{\infty}(\log x)x^{-(1/\widehat{\gamma}+1)}dx \\ &= \frac{k}{n\widehat{\gamma}}X_{n-k,n}^{1/\widehat{\gamma}}\left[\widehat{\gamma}X_{n-k,n}^{-1/\widehat{\gamma}}\log(X_{n-k,n})+\widehat{\gamma}^{2}X_{n-k,n}^{-1/\widehat{\gamma}}\right] \\ &= \frac{k}{n}\left[\log(X_{n-k,n})+\widehat{\gamma}\right],\end{aligned}$$

and

$$\begin{aligned}&\int_{X_{n-k,n}}^{\infty}(x\log x)d\left(1-\frac{k}{n}X_{n-k,n}^{\frac{1}{\widehat{\gamma}}}x^{-\frac{1}{\widehat{\gamma}}}\right) \\ &\quad= \frac{k}{n\widehat{\gamma}}X_{n-k,n}^{1/\widehat{\gamma}}\int_{X_{n-k,n}}^{\infty}(\log x)x^{-1/\widehat{\gamma}}dx \\ &\quad= \frac{k}{n\widehat{\gamma}}X_{n-k,n}^{1/\widehat{\gamma}}\left[\frac{\widehat{\gamma}}{1-\widehat{\gamma}}X_{n-k,n}^{-1/\widehat{\gamma}+1}\log(X_{n-k,n})+\frac{\widehat{\gamma}^{2}}{(1-\widehat{\gamma})^{2}}X_{n-k,n}^{-1/\widehat{\gamma}+1}\right] \\ &\quad= \frac{k}{n(1-\widehat{\gamma})}X_{n-k,n}\left[\log(X_{n-k,n})+\frac{\widehat{\gamma}}{1-\widehat{\gamma}}\right].\end{aligned}$$

It follows that

$$\widehat{\nu}_1 = \frac{1}{n}\sum_{i=1}^{n-k}\log(X_{i,n}) + \frac{k}{n}\left[\log(X_{n-k,n}) + \widehat{\gamma}\right] \tag{5.90}$$

and

$$\widehat{\nu}_2 = \frac{1}{n}\sum_{i=1}^{n-k}X_{i,n}\log(X_{i,n}) + \frac{k}{n(1-\widehat{\gamma})}X_{n-k,n}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1-\widehat{\gamma}}\right]. \tag{5.91}$$

The theorem result follows. □

Estimation of I_E^α when F_θ is Derived from a Perturbed Pareto Distribution

The following theorem gives semi-parametric estimators for I_E^α in the case where the parametric distribution is derived from a PPD.

Theorem 5.6. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the perturbed Pareto distribution (PPD) is a reasonable approximation to the distribution of the relative exceedances

$$\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$$

for a given k. Semi-parametric estimators for the GE inequality measures are given as follows.

$$\widehat{I}_{SPE}^{\alpha} = \frac{1}{\alpha(\alpha-1)}\left(\frac{\widehat{\mu}_\alpha}{\widehat{\mu}_1^\alpha} - 1\right),\ \alpha \neq 0, \alpha \neq 1, \tag{5.92}$$

$$\widehat{I}_{SPE}^{0} = \log\widehat{\mu}_1 - \widehat{\nu}_1 \tag{5.93}$$

and

$$\widehat{I}_{SPE}^{1} = \frac{\widehat{\nu}_2}{\widehat{\mu}_1} - \log\widehat{\mu}_1, \tag{5.94}$$

where $\widehat{\mu}_\alpha$, $\widehat{\mu}_1$, $\widehat{\nu}_1$ and $\widehat{\nu}_2$ are given by

$$\widehat{\mu}_\alpha = \frac{1}{n}\sum_{i=1}^{n-k}X_{i,n}^\alpha + \frac{kX_{n-k,n}^\alpha}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}\alpha} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-\alpha)}\right],\ \alpha < \frac{1}{\widehat{\gamma}}, \tag{5.95}$$

$$\widehat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n-k}X_{i,n} + \frac{kX_{n-k,n}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right], \tag{5.96}$$

$$\widehat{\nu}_1 = \frac{1}{n}\sum_{i=1}^{n-k}\log(X_{i,n}) + \frac{k(1-\widehat{c})}{n}\left[\log(X_{n-k,n}) + \widehat{\gamma}\right] + \frac{k\widehat{c}}{n}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1+\widehat{\gamma}\widehat{\tau}}\right] \tag{5.97}$$

and

$$\widehat{\nu}_2 = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}\log(X_{i,n}) + \frac{k(1-\widehat{c})X_{n-k,n}}{n(1-\widehat{\gamma})}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1-\widehat{\gamma}}\right] + \frac{k\widehat{c}(1+\widehat{\gamma}\widehat{\tau})X_{n-k,n}}{n(1-\widehat{\gamma}(1+\widehat{\tau}))}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1-\widehat{\gamma}(1+\widehat{\tau})}\right], \quad (5.98)$$

respectively, and $\widehat{\gamma}$, $\widehat{c}$ and $\widehat{\tau}$ are estimators for the unknown parameters γ, c and τ in the PPD using the relative exceedances.

Proof. Estimating the underlying distribution by $\tilde{F}_n$ given in Equation (5.32), we can estimate μ_α by

$$\widehat{\mu}_\alpha = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^{\alpha} + \frac{k(1-\widehat{c})}{n\widehat{\gamma}} X_{n-k,n}^{1/\widehat{\gamma}} \int_{X_{n-k,n}}^{\infty} x^{\alpha-(1/\widehat{\gamma}+1)}dx + \frac{k\widehat{c}}{n}\left(\frac{1}{\widehat{\gamma}}+\widehat{\tau}\right) X_{n-k,n}^{1/\widehat{\gamma}} \int_{X_{n-k,n}}^{\infty} x^{\alpha-(1/\widehat{\gamma}+\tau+1)}dx,$$

which, by simple integration leads to

$$\widehat{\mu}_\alpha = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^{\alpha} + \frac{kX_{n-k,n}^{\alpha}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}\alpha} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-\alpha)}\right], \ \alpha < \frac{1}{\widehat{\gamma}}. \quad (5.99)$$

In particular,

$$\widehat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right]. \quad (5.100)$$

On the other hand, we have

$$\widehat{\nu}_1 = \frac{1}{n}\sum_{i=1}^{n-k} \log(X_{i,n}) + \frac{k(1-\widehat{c})}{n\widehat{\gamma}} X_{n-k,n}^{1/\widehat{\gamma}} \int_{X_{n-k,n}}^{\infty} (\log x)x^{-(1/\widehat{\gamma}+1)}dx + \frac{k\widehat{c}}{n}\left(\frac{1}{\widehat{\gamma}}+\widehat{\tau}\right) X_{n-k,n}^{1/\widehat{\gamma}+\tau} \int_{X_{n-k,n}}^{\infty} (\log x)x^{-(1/\widehat{\gamma}+\tau+1)}dx.$$

But straightforward integration by parts leads to

$$\int_{X_{n-k,n}}^{\infty} (\log x)x^{-(1/\widehat{\gamma}+1)}dx = \widehat{\gamma}X_{n-k,n}^{-1/\widehat{\gamma}}\left[\log(X_{n-k,n}) + \widehat{\gamma}\right] \quad (5.101)$$

and

$$\int_{X_{n-k,n}}^{\infty} (\log x)x^{-(1/\widehat{\gamma}+\tau+1)}dx = \frac{\widehat{\gamma}X_{n-k,n}^{-(1/\widehat{\gamma}+\widehat{\tau})}}{1+\widehat{\gamma}\widehat{\tau}}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1+\widehat{\gamma}\widehat{\tau}}\right]. \quad (5.102)$$

It follows that

$$\widehat{\nu}_1 = \frac{1}{n}\sum_{i=1}^{n-k}\log(X_{i,n}) + \frac{k(1-\widehat{c})}{n}\left[\log(X_{n-k,n}) + \widehat{\gamma}\right] + \frac{k\widehat{c}}{n}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1+\widehat{\gamma}\widehat{\tau}}\right]. \tag{5.103}$$

Moreover,

$$\widehat{\nu}_2 = \frac{1}{n}\sum_{i=1}^{n-k}X_{i,n}\log(X_{i,n}) + \frac{k(1-\widehat{c})}{n\widehat{\gamma}}X_{n-k,n}^{1/\widehat{\gamma}}\int_{X_{n-k,n}}^{\infty}(\log x)x^{-1/\widehat{\gamma}}dx$$
$$+\frac{k\widehat{c}}{n}\left(\frac{1}{\widehat{\gamma}}+\widehat{\tau}\right)X_{n-k,n}^{1/\widehat{\gamma}+\tau}\int_{X_{n-k,n}}^{\infty}(\log x)x^{-(1/\widehat{\gamma}+\tau)}dx.$$

As before, easy integration by parts leads to

$$\int_{X_{n-k,n}}^{\infty}(\log x)x^{-1/\widehat{\gamma}}dx = \frac{\widehat{\gamma}X_{n-k,n}^{-1/\widehat{\gamma}+1}}{1-\widehat{\gamma}}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1-\widehat{\gamma}}\right] \tag{5.104}$$

and

$$\int_{X_{n-k,n}}^{\infty}(\log x)x^{-(1/\widehat{\gamma}+\tau)}dx = \frac{\widehat{\gamma}X_{n-k,n}^{-(1/\widehat{\gamma}+\widehat{\tau})+1}}{1-\widehat{\gamma}(1+\widehat{\tau})}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1-\widehat{\gamma}(1+\widehat{\tau})}\right]. \tag{5.105}$$

It follows that

$$\widehat{\nu}_2 = \frac{1}{n}\sum_{i=1}^{n-k}X_{i,n}\log(X_{i,n}) + \frac{k(1-\widehat{c})X_{n-k,n}}{n(1-\widehat{\gamma})}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1-\widehat{\gamma}}\right]$$
$$+\frac{k\widehat{c}(1+\widehat{\gamma}\widehat{\tau})X_{n-k,n}}{n(1-\widehat{\gamma}(1+\widehat{\tau}))}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1-\widehat{\gamma}(1+\widehat{\tau})}\right]. \tag{5.106}$$

The theorem result follows. □

5.3.3 Semi-Parametric Estimation of Atkinson Inequality Measures

Recall from Section 2.3.2 that the Atkinson measures are defined as follows.

$$I_A^{\varepsilon} = 1 - \frac{\mu_{1-\varepsilon}^{1/(1-\varepsilon)}}{\mu},\ \varepsilon > 0, \varepsilon \neq 1 \tag{5.107}$$

and

$$I_A^1 = 1 - \frac{\exp(\nu)}{\mu}, \tag{5.108}$$

where

$$\mu_{1-\varepsilon} = \int_0^{\infty} x^{1-\varepsilon}dF(x),$$

$$\mu = \int_0^\infty x dF(x)$$

and

$$\nu = \int_0^\infty (\log x) dF(x).$$

In this section we derive semi-parametric estimators for the Atkinson inequality measures.

Estimation of I_A^ε when F_θ is Derived from a Generalized Pareto Distribution

The following theorem gives semi-parametric estimators for I_A^ε in the case where the parametric distribution is derived from a GPD.

Theorem 5.7. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the GPD is a reasonable approximation to the distribution of the exceedances $X_{n-k+1,n} - X_{n-k,n}, X_{n-k+2,n} - X_{n-k,n}, \ldots, X_{n,n} - X_{n-k,n}$ for a given k. Denote by $\widehat{I_\varepsilon}$ and $\widehat{I_1}$ the numerical approximations of the integrals

$$\int_{X_{n-k,n}}^\infty x^{1-\varepsilon} \left[1 + \frac{\widehat{\gamma}(x - X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx$$

and

$$\int_{X_{n-k,n}}^\infty (\log x) \left[1 + \frac{\widehat{\gamma}(x - X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx,$$

respectively. Estimators for the Atkinson inequality measures are given as follows.

$$\widehat{I}_{SPA}^\varepsilon = 1 - \frac{\widehat{\mu}_\varepsilon^{1/(1-\varepsilon)}}{\widehat{\mu}}, \ \varepsilon > 0, \varepsilon \neq 1 \tag{5.109}$$

and

$$\widehat{I}_{SPA}^1 = 1 - \frac{\exp(\widehat{\nu})}{\widehat{\mu}}, \tag{5.110}$$

where $\widehat{\mu}_\varepsilon$, $\widehat{\mu}$ and $\widehat{\nu}$ are given by

$$\widehat{\mu}_\varepsilon = \frac{1}{n} \sum_{i=1}^{n-k} X_{i,n}^{1-\varepsilon} + \frac{k\widehat{I_\varepsilon}}{n\widehat{\sigma}}, \tag{5.111}$$

$$\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n-k} X_{i,n} + \frac{k}{n}\left(X_{n-k,n} + \frac{\widehat{\sigma}}{1-\widehat{\gamma}}\right) \tag{5.112}$$

and

$$\widehat{\nu} = \frac{1}{n} \sum_{i=1}^{n-k} \log(X_{i,n}) + \frac{k\widehat{I_1}}{n\widehat{\sigma}}, \tag{5.113}$$

respectively, and $\widehat{\sigma}$ and $\widehat{\gamma}$ are estimators for the unknown scale and shape parameters σ and γ of the

GPD using the exceedances.

Proof. The theorem easily follows by using the estimator given in Equation (5.23). □

Estimation of I_A^ε when F_θ is Derived from a Strict Pareto Distribution

The following theorem gives semi-parametric estimators for I_A^ε in the case where the parametric distribution is derived from a strict Pareto distribution.

Theorem 5.8. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the strict Pareto distribution is a reasonable approximation to the distribution of the relative exceedances $\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$ for a given k. Semi-parametric estimators for the Atkinson inequality measures are given as follows.

$$\widehat{I}_{SPA}^{\varepsilon} = 1 - \frac{\widehat{\mu}_{\varepsilon}^{1/(1-\varepsilon)}}{\widehat{\mu}}, \ \varepsilon > 0, \varepsilon \neq 1 \tag{5.114}$$

and

$$\widehat{I}_{SPA}^{1} = 1 - \frac{\exp(\widehat{\nu})}{\widehat{\mu}}, \tag{5.115}$$

where $\widehat{\mu}_\varepsilon$, $\widehat{\mu}$ and $\widehat{\nu}$ are given by

$$\widehat{\mu}_\varepsilon = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^{1-\varepsilon} + \frac{kX_{n-k,n}^{1-\varepsilon}}{n(1-(1-\varepsilon)\widehat{\gamma})}, \varepsilon > 1 - \frac{1}{\widehat{\gamma}}, \varepsilon \neq 1, \tag{5.116}$$

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n(1-\widehat{\gamma})} \tag{5.117}$$

and

$$\widehat{\nu} = \frac{1}{n}\sum_{i=1}^{n-k} \log(X_{i,n}) + \frac{k}{n}\left[\log(X_{n-k,n}) + \widehat{\gamma}\right], \tag{5.118}$$

respectively, and $\widehat{\gamma}$ is an estimator for the unknown shape parameter γ of the strict Pareto distribution using the exceedances.

Proof. As in the case of the GE measures, we make use of the estimator in Equation (5.29) to estimate the quantities μ_ε, μ_1 and ν by $\widehat{\mu}_\varepsilon$, $\widehat{\mu}$ and $\widehat{\nu}$, respectively, and the theorem result follows. □

Estimation of I_A^ε when F_θ is Derived from a Perturbed Pareto Distribution

The following theorem gives semi-parametric estimators for I_A^ε in the case where the parametric distribution is derived from a PPD.

Theorem 5.9. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the perturbed Pareto distribution (PPD) is a reasonable approximation to the distribution of the relative exceedances

$$\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$$

for a given k. Semi-parametric estimators for the Atkinson inequality measures are given as follows.

$$\widehat{I}_{SPA}^{\varepsilon} = 1 - \frac{\widehat{\mu}_\varepsilon^{1/(1-\varepsilon)}}{\widehat{\mu}}, \ \varepsilon > 0, \varepsilon \neq 1 \tag{5.119}$$

and

$$\widehat{I}_{SPA}^{1} = 1 - \frac{\exp(\widehat{\nu})}{\widehat{\mu}}, \tag{5.120}$$

where $\widehat{\mu}_\varepsilon$, $\widehat{\mu}$ and $\widehat{\nu}$ are given by

$$\widehat{\mu}_\varepsilon = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n}^{1-\varepsilon} + \frac{kX_{n-k,n}^{1-\varepsilon}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}(1-\varepsilon)} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-(1-\varepsilon))}\right], \ \varepsilon > 1 - \frac{1}{\widehat{\gamma}}, \tag{5.121}$$

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right] \tag{5.122}$$

and

$$\widehat{\nu} = \frac{1}{n}\sum_{i=1}^{n-k} \log(X_{i,n}) + \frac{k(1-\widehat{c})}{n}\left[\log(X_{n-k,n}) + \widehat{\gamma}\right] + \frac{k\widehat{c}}{n}\left[\log(X_{n-k,n}) + \frac{\widehat{\gamma}}{1+\widehat{\gamma}\widehat{\tau}}\right], \tag{5.123}$$

respectively, and $\widehat{\gamma}$, $\widehat{c}$ and $\widehat{\tau}$ are estimators for the unknown parameters γ, c and τ in the PPD using the relative exceedances.

Proof. Estimating the underlying distribution by $\tilde{F}_n$ given in Equation (5.32), we can estimate the quantities μ_ε, μ and ν by $\widehat{\mu}_\varepsilon$, $\widehat{\mu}$ and $\widehat{\nu}$, respectively, and the theorem result follows. □

5.3.4 Semi-Parametric Estimation of Quintile Share Ratio Measure of Inequality

Recall from Section 2.3.5 that the quintile share ratio (QSR) is defined by

$$\eta = \frac{\int_{Q_F(0.8)}^{\infty} x dF(x)}{\int_0^{Q_F(0.2)} x dF(x)} = \frac{EXI\{X > Q_F(0.8)\}}{EXI\{X \leq Q_F(0.2)\}}, \tag{5.124}$$

where $I\{.\}$ is an indicator function. In this section we derive semi-parametric estimators for η.

Estimation of η when F_θ is Derived from a Generalized Pareto Distribution

The following theorem gives a semi-parametric estimator for η in the case where the parametric distribution is derived from a GPD.

Theorem 5.10. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the GPD is a reasonable approximation to the distribution of the exceedances $X_{n-k+1,n} - X_{n-k,n}, X_{n-k+2,n} - X_{n-k,n}, \ldots, X_{n,n} - X_{n-k,n}$ for a given k. A semi-parametric estimator for η is given by

$$\widehat{\eta}_{SP} = 0.2n \left[X_{n-k,n} + \frac{\widehat{\sigma}}{\widehat{\gamma}} \left(\frac{1}{1-\widehat{\gamma}} \left(\frac{0.2n}{k} \right)^{-\widehat{\gamma}} - 1 \right) \right] \Big/ \left[\sum_{i=1}^{[0.2n]} X_{i,n} \right], \; n-k \leq [0.8n], \tag{5.125}$$

and

$$\widehat{\eta}_{SP} = \left[\sum_{i=[0.8n]}^{n-k} X_{i,n} + k \left(X_{n-k,n} + \frac{\widehat{\sigma}}{1-\widehat{\gamma}} \right) \right] \Big/ \left[\sum_{i=1}^{[0.2n]} X_{i,n} \right], \; n-k > [0.8n], \tag{5.126}$$

where $\widehat{\sigma}$ and $\widehat{\gamma}$ are estimators for the unknown scale and shape parameters σ and γ of the GPD using the exceedances.

Proof. Estimating F by $\tilde{F}_n$ given in Equation (5.23), we can easily estimate the quantile function by

$$Q_{\tilde{F}_n}(\alpha) = \begin{cases} X_{[\alpha n],n} & \alpha \leq \frac{n-k}{n} \\ X_{n-k,n} + \frac{\widehat{\sigma}}{\widehat{\gamma}} \left[\left(\frac{n(1-\alpha)}{k} \right)^{-\widehat{\gamma}} - 1 \right], & \alpha > \frac{n-k}{n}, \end{cases} \tag{5.127}$$

where $[x]$ is the largest integer smaller than or equal to x, leading to

$$Q_{\tilde{F}_n}(0.2) = X_{[0.2n],n} \tag{5.128}$$

and

$$Q_{\tilde{F}_n}(0.8) = X_{n-k,n} + \frac{\widehat{\sigma}}{\widehat{\gamma}}\left[\left(\frac{0.2n}{k}\right)^{-\widehat{\gamma}} - 1\right] \text{ for } n-k \leq [0.8n]. \tag{5.129}$$

It follows that for $n-k \leq [0.8n]$,

$$\begin{aligned}
\int_{Q_{\tilde{F}_n}(0.8)}^{\infty} x d\tilde{F}_n(x) &= \frac{k}{n\widehat{\sigma}} \int_{Q_{\tilde{\widehat{F}}}(0.8)}^{\infty} x \left[1 + \frac{\widehat{\gamma}(x - X_{n-k,n})}{\widehat{\sigma}}\right]^{-\frac{1}{\widehat{\gamma}}-1} dx \\
&= \frac{k}{n}\left[Q_{\tilde{F}_n}(0.8) + \frac{\widehat{\sigma}}{1-\widehat{\gamma}} + \frac{\widehat{\gamma}}{1-\widehat{\gamma}}\left(Q_{\tilde{F}_n}(0.8) - X_{n-k,n}\right)\right] \\
&\quad \times \left[1 + \frac{\widehat{\gamma}}{\widehat{\sigma}}\left(Q_{\tilde{F}_n}(0.8) - X_{n-k,n}\right)\right]^{-1/\widehat{\gamma}},
\end{aligned}$$

which, by substituting Equation (5.129) for $Q_{\tilde{F}_n}(0.8)$ and rearranging, becomes

$$\int_{Q_{\tilde{F}_n}(0.8)}^{\infty} x d\tilde{F}_n(x) = 0.2\left[X_{n-k,n} + \frac{\widehat{\sigma}}{\widehat{\gamma}}\left(\frac{1}{1-\widehat{\gamma}}\left(\frac{0.2n}{k}\right)^{-\widehat{\gamma}} - 1\right)\right]. \tag{5.130}$$

Similarly, for $n-k > [0.8n]$ we get

$$\int_{Q_{\tilde{F}_n}(0.8)}^{\infty} x d\tilde{F}_n(x) = \frac{1}{n}\sum_{i=[0.8n]}^{n-k} X_{i,n} + \frac{k}{n}\left[X_{n-k,n} + \frac{\widehat{\sigma}}{1-\widehat{\gamma}}\right]. \tag{5.131}$$

On the other hand,

$$\int_{0}^{Q_{\tilde{F}_n}(0.2)} x d\tilde{F}_n(x) = \frac{1}{n}\sum_{i=1}^{[0.2n]} X_{i,n}. \tag{5.132}$$

Thus

$$\widehat{\eta}_{SP} = 0.2n\left[X_{n-k,n} + \frac{\widehat{\sigma}}{\widehat{\gamma}}\left(\frac{1}{1-\widehat{\gamma}}\left(\frac{0.2n}{k}\right)^{-\widehat{\gamma}} - 1\right)\right] \Big/ \left[\sum_{i=1}^{[0.2n]} X_{i,n}\right],\ n-k \leq [0.8n], \tag{5.133}$$

and

$$\widehat{\eta}_{SP} = \left[\sum_{i=[0.8n]}^{n-k} X_{i,n} + k\left(X_{n-k,n} + \frac{\widehat{\sigma}}{1-\widehat{\gamma}}\right)\right] \Big/ \left[\sum_{i=1}^{[0.2n]} X_{i,n}\right],\ n-k > [0.8n]. \tag{5.134}$$

This completes the proof. □

Estimation of η when F_θ is Derived from a Strict Pareto Distribution

The following theorem gives a semi-parametric estimator for η in the case where the parametric distribution is derived from a strict Pareto distribution.

Theorem 5.11. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the strict Pareto distribution is a reasonable approximation to the distribution of the relative exceedances

$$\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$$

for a given k. A semi-parametric estimator for η is given by

$$\widehat{\eta}_{SP} = \left[\frac{k}{1-\widehat{\gamma}}\left(\frac{0.2n}{k}\right)^{1-\widehat{\gamma}} X_{n-k,n}\right] / \left[\sum_{i=1}^{[0.2n]} X_{i,n}\right], \; n-k \leq [0.8n], \tag{5.135}$$

and

$$\widehat{\eta}_{SP} = \left[\sum_{i=[0.8n]}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{1-\widehat{\gamma}}\right] / \left[\sum_{i=1}^{[0.2n]} X_{i,n}\right], \; n-k > [0.8n], \tag{5.136}$$

where $\widehat{\gamma}$ is an estimator for the unknown shape parameter γ of the strict Pareto distribution using the exceedances.

Proof. In the case of the strict Pareto distribution, we can also easily estimate Q_F by

$$Q_{\tilde{F}_n}(\alpha) = \begin{cases} X_{[\alpha n],n}, & \alpha \leq \frac{n-k}{n} \\ X_{n-k,n}\left(\frac{n(1-\alpha)}{k}\right)^{-\widehat{\gamma}}, & \alpha > \frac{n-k}{n}, \end{cases} \tag{5.137}$$

leading to

$$Q_{\tilde{F}_n}(0.2) = X_{[0.2n],n} \tag{5.138}$$

and

$$Q_{\tilde{F}_n}(0.8) = X_{n-k,n}\left(\frac{0.2n}{k}\right)^{-\widehat{\gamma}} \text{ for } n-k \leq [0.8n]. \tag{5.139}$$

It follows that

$$\int_0^{Q_{\tilde{F}_n}(0.2)} x d\tilde{F}_n(x) = \frac{1}{n}\sum_{i=1}^{[0.2n]} X_{i,n}, \tag{5.140}$$

$$\int_{Q_{\tilde{F}_n}(0.8)}^{\infty} x d\tilde{F}_n(x) = \frac{kX_{n-k,n}}{n(1-\widehat{\gamma})}\left(\frac{0.2n}{k}\right)^{1-\widehat{\gamma}}, \; n-k \leq [0.8n] \tag{5.141}$$

and

$$\int_{Q_{\tilde{F}_n}(0.8)}^{\infty} x d\tilde{F}_n(x) = \frac{1}{n}\sum_{i=[0.8n]}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{n(1-\widehat{\gamma})}, \; n-k > [0.8n]. \tag{5.142}$$

Thus

$$\widehat{\eta}_{SP} = \left[\frac{k}{1-\widehat{\gamma}}\left(\frac{0.2n}{k}\right)^{1-\widehat{\gamma}} X_{n-k,n}\right] / \left[\sum_{i=1}^{[0.2n]} X_{i,n}\right], \; n-k \leq [0.8n], \tag{5.143}$$

and

$$\widehat{\eta}_{SP} = \left[\sum_{i=[0.8n]}^{n-k} X_{i,n} + \frac{kX_{n-k,n}}{1-\widehat{\gamma}}\right] / \left[\sum_{i=1}^{[0.2n]} X_{i,n}\right], \; n-k > [0.8n]. \tag{5.144}$$

This completes the proof. □

Estimation of η when F_θ is Derived from a Perturbed Pareto Distribution

The following theorem gives a semi-parametric estimator for η in the case where the parametric distribution is derived from a PPD.

Theorem 5.12. Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution F, with associated order statistics $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$. Assume that for a large n, the perturbed Pareto distribution is a reasonable approximation to the distribution of the relative exceedances

$$\frac{X_{n-k+1,n}}{X_{n-k,n}}, \frac{X_{n-k+2,n}}{X_{n-k,n}}, \ldots, \frac{X_{n,n}}{X_{n-k,n}}$$

for a given k. A semi-parametric estimator for η is given by

$$\widehat{\eta}_{SP} = \left[\frac{k(1-\widehat{c})}{1-\widehat{\gamma}} X_{n-k,n}^{1/\widehat{\gamma}} \widehat{Q}(0.8)^{-1/\widehat{\gamma}+1} + \frac{k\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)} X_{n-k,n}^{1/\widehat{\gamma}+\widehat{\tau}} \widehat{Q}(0.8)^{-1/\widehat{\gamma}-\widehat{\tau}+1}\right] / \left[\sum_{i=1}^{[0.2n]} X_{i,n}\right] \tag{5.145}$$

for $n-k \leq [0.8n]$ and

$$\widehat{\eta}_{SP} = \left[\sum_{i=[0.8n]}^{n-k} X_{i,n} + kX_{n-k,n}\left(\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right)\right] / \left[\sum_{i=1}^{[0.2n]} X_{i,n}\right], \; n-k > [0.8n], \tag{5.146}$$

where $\widehat{\gamma}$, $\widehat{c}$ and $\widehat{\tau}$ are estimators for the unknown parameters γ, c and τ in the PPD using the relative exceedances.

Proof. Recall that in the case of the PPD, F is estimated by

$$\tilde{F}_n(x) = \begin{cases} F_n(x), & x \leq X_{n-k,n}, \\ 1 - \frac{k}{n} X_{n-k,n}^{1/\widehat{\gamma}} \left[(1-\widehat{c})x^{-1/\widehat{\gamma}} + \widehat{c} X_{n-k,n}^{\widehat{\tau}} x^{-1/\widehat{\gamma}-\widehat{\tau}}\right], & x > X_{n-k,n}. \end{cases} \tag{5.147}$$

For each $\alpha > \frac{n-k}{n}$, denote by $\widehat{Q}(\alpha)$ the solution to the equation

$$1 - \frac{k}{n} X_{n-k,n}^{1/\widehat{\gamma}} \left[(1-\widehat{c})x^{-1/\widehat{\gamma}} + \widehat{c} X_{n-k,n}^{\widehat{\tau}} x^{-1/\widehat{\gamma}-\widehat{\tau}}\right] = \alpha, \tag{5.148}$$

i.e. the solution to

$$(1-\widehat{c})x^{-1/\widehat{\gamma}}+\widehat{c}X_{n-k,n}^{\widehat{\tau}}x^{-1/\widehat{\gamma}-\widehat{\tau}}=\frac{n}{k}X_{n-k,n}^{-1/\widehat{\gamma}}(1-\alpha). \tag{5.149}$$

We can then estimate the quantile function Q_F by

$$Q_{\tilde{F}_n}(\alpha)=\begin{cases} X_{[\alpha n],n}, & \alpha\leq\frac{n-k}{n},\\ \widehat{Q}(\alpha), & \alpha>\frac{n-k}{n}.\end{cases} \tag{5.150}$$

It follows that for $n-k\leq[0.8n]$,

$$\int_0^{Q_{\tilde{F}_n}(0.2)} x d\tilde{F}_n(x)=\frac{1}{n}\sum_{i=1}^{[0.2n]}X_{i,n} \tag{5.151}$$

and

$$\int_{Q_{\tilde{F}_n}(0.8)}^{\infty} x d\tilde{F}_n(x)=\frac{k(1-\widehat{c})}{n\widehat{\gamma}}X_{n-k,n}{}^{1/\widehat{\gamma}}\int_{\widehat{Q}(0.8)}^{\infty}x^{-1/\widehat{\gamma}}dx \\ +\frac{k\widehat{c}}{n}\left(\frac{1}{\widehat{\gamma}}+\widehat{\tau}\right)X_{n-k,n}^{1/\widehat{\gamma}+\widehat{\tau}}\int_{\widehat{Q}(0.8)}^{\infty}x^{-1/\widehat{\gamma}-\widehat{\tau}}dx.$$

But

$$\int_{\widehat{Q}(0.8)}^{\infty}x^{-1/\widehat{\gamma}}dx=\frac{\widehat{\gamma}}{1-\widehat{\gamma}}\widehat{Q}(0.8)^{-1/\widehat{\gamma}+1}$$

and

$$\int_{\widehat{Q}(0.8)}^{\infty}x^{-1/\widehat{\gamma}-\widehat{\tau}}dx=\frac{\widehat{\gamma}}{1+\widehat{\gamma}(\widehat{\tau}-1)}\widehat{Q}(0.8)^{-1/\widehat{\gamma}-\widehat{\tau}+1},$$

leading to

$$\int_{Q_{\tilde{F}_n}(0.8)}^{\infty} x d\tilde{F}_n(x)=\frac{k(1-\widehat{c})}{n(1-\widehat{\gamma})}X_{n-k,n}{}^{1/\widehat{\gamma}}\widehat{Q}(0.8)^{-1/\widehat{\gamma}+1} \\ +\frac{k\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{n(1+\widehat{\gamma}(\widehat{\tau}-1))}X_{n-k,n}^{1/\widehat{\gamma}+\widehat{\tau}}\widehat{Q}(0.8)^{-1/\widehat{\gamma}-\widehat{\tau}+1}. \tag{5.152}$$

On the other hand, for $n-k>[0.8n]$ we get

$$\int_{Q_{\tilde{F}_n}(0.8)}^{\infty} x d\tilde{F}_n(x)=\frac{1}{n}\sum_{i=[0.8n]}^{n-k}X_{i,n}+\frac{kX_{n-k,n}}{n}\left[\frac{1-\widehat{c}}{1-\widehat{\gamma}}+\frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)}\right]. \tag{5.153}$$

$$\widehat{\eta}_{SP}=\left[\frac{k(1-\widehat{c})}{1-\widehat{\gamma}}X_{n-k,n}^{1/\widehat{\gamma}}\widehat{Q}(0.8)^{-1/\widehat{\gamma}+1}+\frac{k\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)}X_{n-k,n}^{1/\widehat{\gamma}+\widehat{\tau}}\widehat{Q}(0.8)^{-1/\widehat{\gamma}-\widehat{\tau}+1}\right]\Big/\left[\sum_{i=1}^{[0.2n]}X_{i,n}\right] \tag{5.154}$$

for $n-k \leq [0.8n]$ and

$$\widehat{\eta}_{SP} = \left[\sum_{i=[0.8n]}^{n-k} X_{i,n} + kX_{n-k,n} \left(\frac{1-\widehat{c}}{1-\widehat{\gamma}} + \frac{\widehat{c}(1+\widehat{\gamma}\widehat{\tau})}{1+\widehat{\gamma}(\widehat{\tau}-1)} \right) \right] \Big/ \left[\sum_{i=1}^{[0.2n]} X_{i,n} \right], \; n-k > [0.8n]. \tag{5.155}$$

This completes the proof. □

5.4 Sampling Distributions of the Semi-Parametric Estimators of Inequality Measures

In this section we study the sampling distributions of the semi-parametric estimators of inequality measures developed above using simulation. We use the same underlying distributions as in Chapter 3, and we give the results in terms of Q-Q plots. These are given in Figures 5.1 to 5.3 for the Gini coefficient, and in Appendix C for other measures.

The following observations are made from investigating the sampling distributions of the semi-parametric estimators.

1. Semi-parametric Gini: See Figures 5.1 to 5.3 below. The sampling distributions are approximately normal, whether we fit the GPD, the strict Pareto or the PPD in the tails.

2. Semi-parametric GE0: See Figures C.1 to C.3 in Appendix C. The sampling distributions are approximately normal in all situations as in the case of the Gini.

3. Semi-parametric GE1: See Figures C.4 to C.6 in Appendix C. The sampling distributions are approximately normal when fitting either the GPD or the PPD in the tails, but the strict Pareto gives poor approximations though it appeared to be better than the nonparametric fit.

4. Semi-parametric GE1.3: See Figures C.7 to C.9 in Appendix C. The normal approximation is poor in many cases, but they are better than the nonparametric fit.

5. Semi-parametric A1: See Figures C.10 to C.12 in Appendix C. The sampling distributions are approximately normal, whether we fit the GPD, the strict Pareto or the PPD in the tails.

6. Semi-parametric A1.5: See Figures C.13 to C.15 in Appendix C. The sampling distributions are approximately normal, except in a few situations, especially when fitting the GPD and PPD in the tails.

7. Semi-parametric A2: See Figures C.16 to C.18 in Appendix C. The sampling distributions are approximately normal when the underlying distributions are Pa, Bu3, Fr1 and Fr2, and whether we fit the GPD, the strict Pareto or the PPD in the tails. The normal approximations are poor in the cases of Bu1, Bu2 and T2 although they are better than the nonparametric approximations.

8. Semi-parametric QSR: See Figures C.19 to C.21 in Appendix C. The sampling distributions are approximately normal in a few situations (e.g. Fr1, Fr2 and T2 distributions). Though poor in a number cases, the normal approximations are much better than those for the nonparametric procedure.

Overall we can say that the sampling distributions of the semi-parametric estimators of the inequality measures are approximately normal in a large number of cases. In cases where the normal approximations are poor, they appear to be better than those of nonparametric estimators as given in Chapter 3. Confidence intervals constructed based on the semi-parametric estimators using normal approximations, should therefore yield satisfactory results.

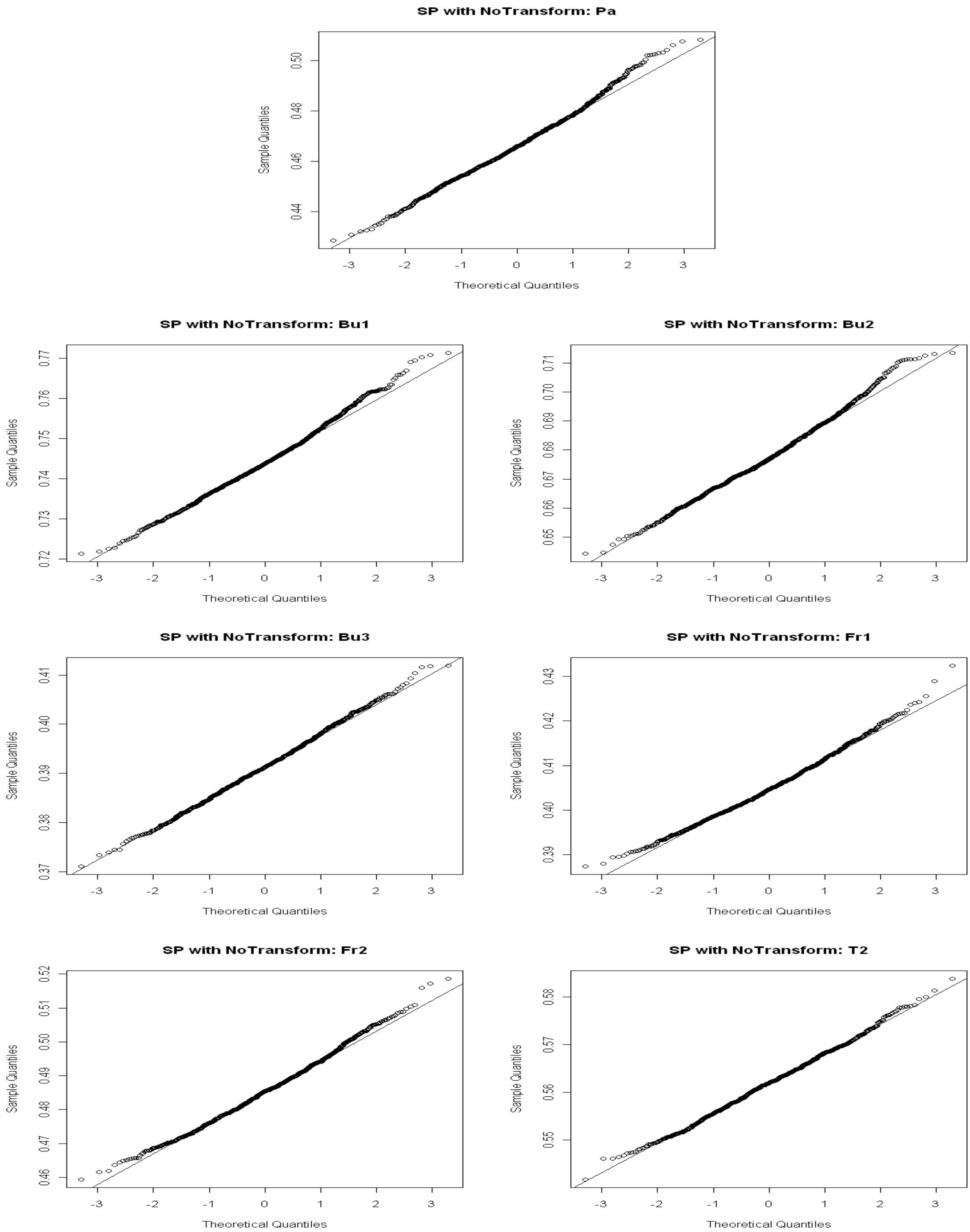

Figure 5.1: Sampling Distribution for SP Gini when Fitting the GPD to the Tails

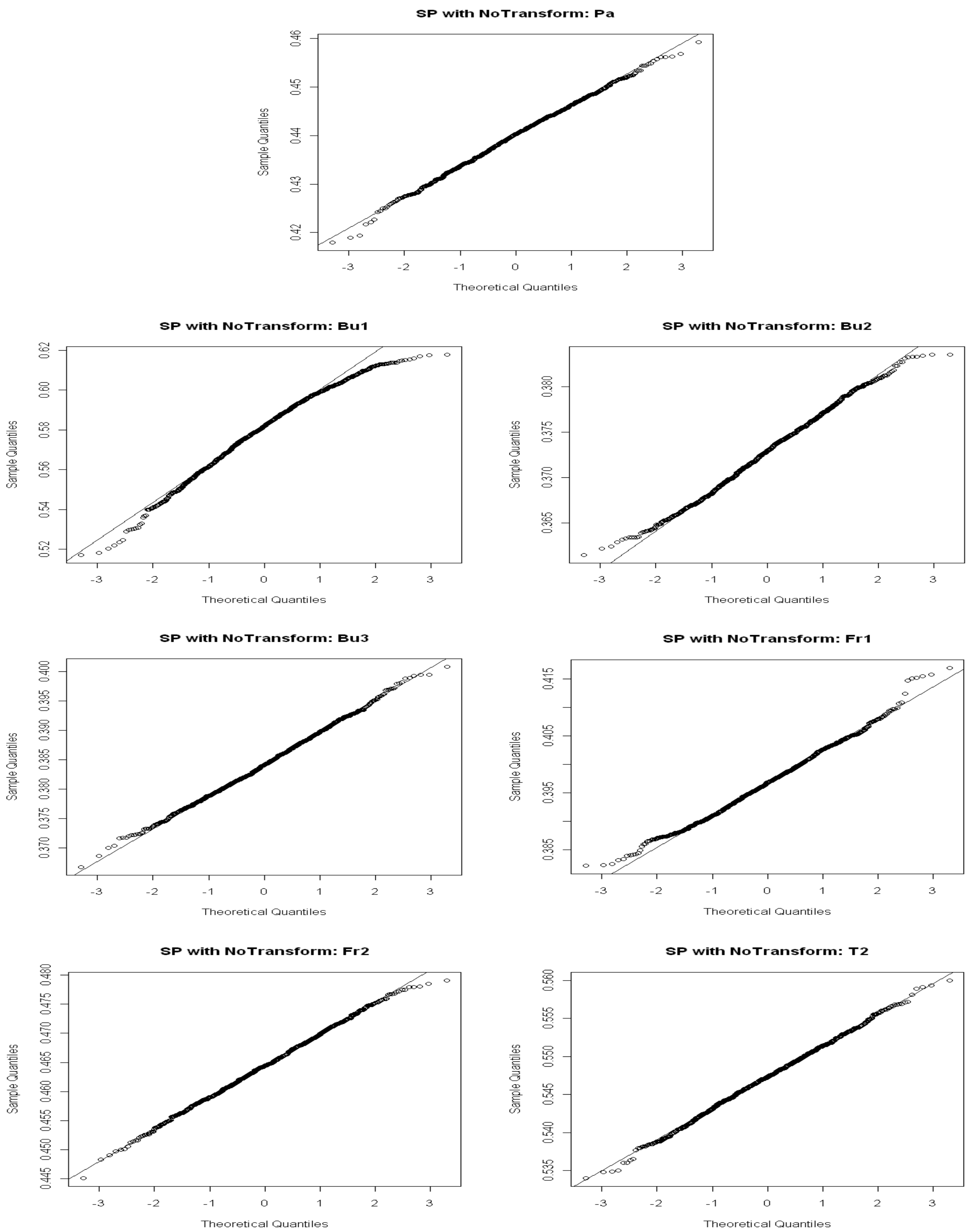

Figure 5.2: Sampling Distribution for SP Gini when Fitting the Strict Pareto to the Tails

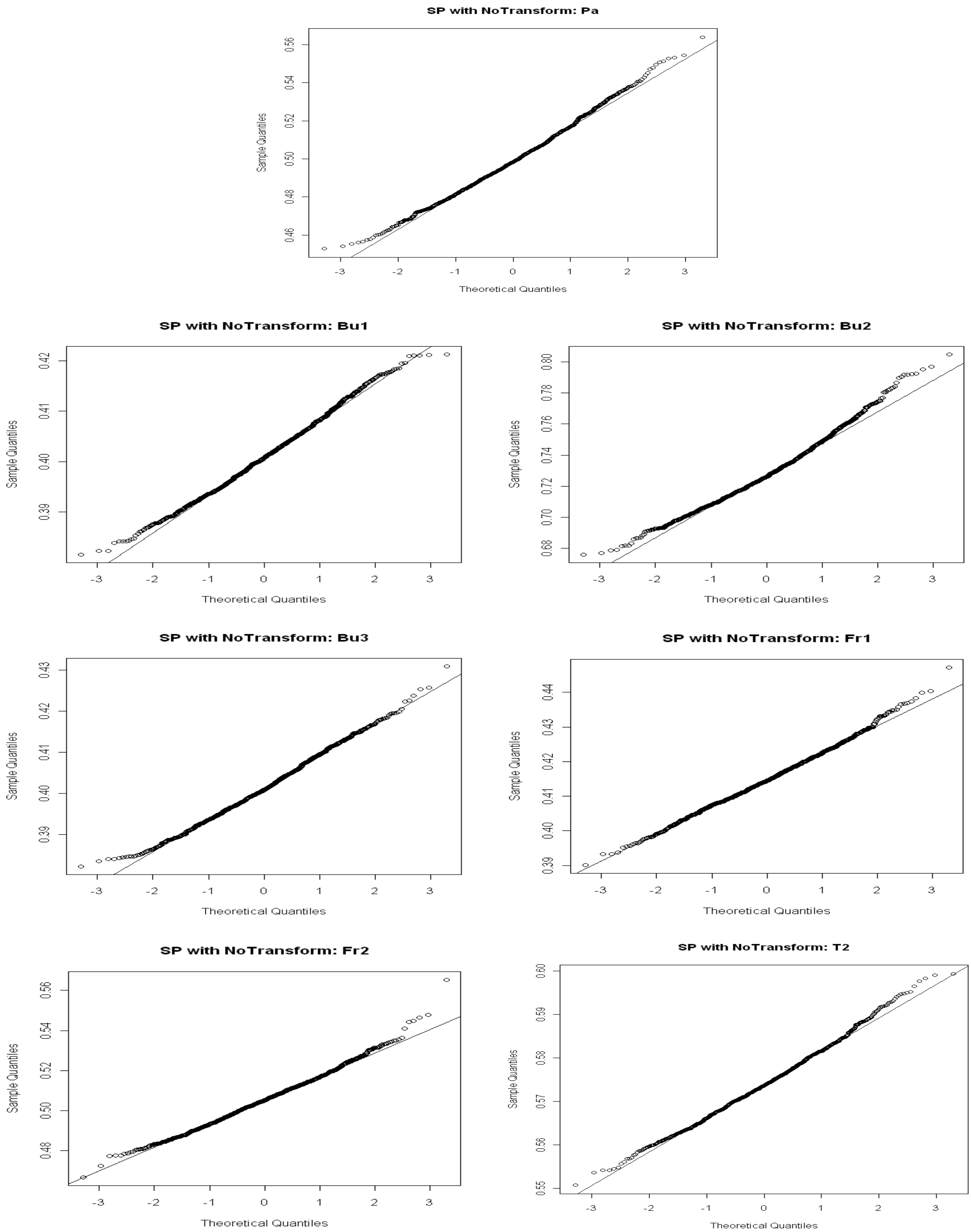

Figure 5.3: Sampling Distribution for SP Gini when Fitting the PPD to the Tails

Chapter 6

Confidence Intervals

In Chapter 5 point estimators for the measures of inequality were derived. A next step is to estimate the variance of such estimators and, furthermore, to obtain confidence intervals for the measures of inequality. A number of the best-known methods for constructing confidence intervals are discussed in general in this chapter. These methods are then applied to SP estimators in the second part of the chapter.

Definition 6.1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with distribution F and let $\theta = \theta(F)$ be the quantity of interest. A $100(1-\alpha)\%$ confidence interval for θ is the interval $[\widehat{\theta}_{low}, \widehat{\theta}_{up}]$ with

$$\widehat{\theta}_{up} = \widehat{\theta}_{up}(X_1, X_2, \ldots, X_n),$$

$$\widehat{\theta}_{low} = \widehat{\theta}_{low}(X_1, X_2, \ldots, X_n),$$

such that

$$P(\widehat{\theta}_{low} < \theta < \widehat{\theta}_{up}) = 1 - \alpha.$$

Note that the endpoints $\widehat{\theta}_{up}$ and $\widehat{\theta}_{low}$ are statistics (i.e. observable random variables) which are observed when the variables $X_1, X_2, \ldots, X_n$ are observed. In other words, $[\widehat{\theta}_{low}, \widehat{\theta}_{up}]$ is a random interval. The quantity $1-\alpha$ is called the confidence level.

A confidence interval is always qualified by a particular confidence level, often expressed as a percentage. The end points of the confidence interval are referred to as confidence limits, and the calculation of a confidence interval generally requires assumptions about the nature of the estimation process. Moreover, the wider a confidence interval, the less precise the estimator. Since precision often depends upon sample size, it follows that the larger the sample size of a study, the narrower the confidence interval and thus indicates a more reliable estimator.

6.1 Some Properties of Confidence Intervals

Given an observed sample from F, denote by $\widehat{\theta}_n$ a point estimator of θ and by $\left[\widehat{\theta}_{low}, \widehat{\theta}_{up}\right]$ the observed confidence interval for θ, where $\widehat{\theta}_{low}$ and $\widehat{\theta}_{up}$ are respectively the lower and the upper confidence limits. The confidence interval length is defined by

$$CI_l = \widehat{\theta}_{up} - \widehat{\theta}_{low} \tag{6.1}$$

and the shape of the interval by

$$CI_s = \frac{\widehat{\theta}_{up} - \widehat{\theta}_n}{\widehat{\theta}_n - \widehat{\theta}_{low}}. \tag{6.2}$$

The shape measures the asymmetry of the interval about the point estimator $\widehat{\theta}_n$. A shape greater than one indicates greater distance from $\widehat{\theta}_n$ to $\widehat{\theta}_{up}$ than from $\widehat{\theta}_{low}$ to $\widehat{\theta}_n$. Standard intervals (see Section 6.2) are symmetrical about $\widehat{\theta}_n$, having a shape of one by definition.

Now suppose we obtain by simulation N copies $[\widehat{\theta}^i_{low}, \widehat{\theta}^i_{up}]$, $i = 1, 2, \ldots, N$ of a confidence interval for a parameter θ. We define the following estimators:

1. Coverage probability: This is defined as $P(\widehat{\theta}_{low} < \theta < \widehat{\theta}_{up})$ and can be estimated by

$$CP = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}(\widehat{\theta}^i_{low} \leq \theta \leq \widehat{\theta}^i_{up}). \tag{6.3}$$

2. Lower non-coverage probability: This is defined as $P(\widehat{\theta}_{low} > \theta)$ and can be estimated by

$$LNCP = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}(\widehat{\theta}^i_{low} > \theta). \tag{6.4}$$

3. Upper non-coverage probability: This is defined as $P(\widehat{\theta}_{up} < \theta)$ and can be estimated by

$$UNCP = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}(\widehat{\theta}^i_{up} < \theta). \tag{6.5}$$

4. Average confidence interval length: This is defined as $E(\widehat{\theta}_{up}-\widehat{\theta}_{low})$ and can be estimated by

$$ACIL = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\theta}^{i}_{up}-\widehat{\theta}^{i}_{low}). \tag{6.6}$$

In the simulation study reported in the next chapter, these quantities will be calculated. Especially Equations (6.3) and (6.6) will be used to assess the performance of the confidence intervals.

There are several methods for constructing confidence intervals. A rule for constructing confidence intervals is typically related to a particular way of finding a point estimator of the quantity being considered. In the following sections we describe a number of different methods for constructing confidence intervals.

6.2 Confidence Intervals Based on Normal Approximation

Consider an i.i.d. sample $X_1, X_2, \ldots, X_n$ from a distribution F, and consider an estimator $\widehat{\theta}_n$ of some parameter θ based on $X_1, X_2, \ldots, X_n$. Assume that as the sample size n increases, the distribution of $\widehat{\theta}_n$ becomes normal, with approximate mean θ, and variance $(se)^2$ which can be estimated by $\widehat{se}^2$. Thus

$$\frac{\widehat{\theta}_n-\theta}{\widehat{se}} \mathrel{\dot\sim} N(0,1), \text{ for } n \text{ large.} \tag{6.7}$$

An approximate $100(1-\alpha)\%$ confidence interval for θ is then given by

$$\left[\widehat{\theta}_{low},\widehat{\theta}_{up}\right] = \left[\widehat{\theta}_n - z^{(1-\alpha/2)}\cdot\widehat{se}, \widehat{\theta}_n - z^{(\alpha/2)}\cdot\widehat{se}\right], \tag{6.8}$$

where $z^{(1-\alpha/2)}$ and $z^{(\alpha/2)}$ are respectively the $1-\alpha/2$ and the $\alpha/2$ standard normal quantiles. This interval is known as the standard interval. It is very useful and widely applicable because in many situations it turns out that for a large sample size, the distribution of the estimator can be approximated by the normal distribution.

6.3 Student t Confidence Intervals

Consider the same notation as in Section 6.2. For smaller n, a better approximation is often obtained by using the t distribution, viz.

$$\frac{\widehat{\theta}_n-\theta}{\widehat{se}} \mathrel{\dot\sim} t_{n-1}, \tag{6.9}$$

where t_{n-1} represents the Student t distribution with $n-1$ degrees of freedom.

Using this approximation, an approximate $100(1-\alpha)\%$ confidence interval for θ is given by

$$\left[\widehat{\theta}_{low}, \widehat{\theta}_{up}\right] = \left[\widehat{\theta}_n - t_{n-1}^{(1-\alpha/2)} \cdot \widehat{se}, \widehat{\theta}_n - t_{n-1}^{(\alpha/2)} \cdot \widehat{se}\right], \tag{6.10}$$

where $t_{n-1}^{(\alpha)}$ denotes the α^{th} percentile of the Student t distribution with $n-1$ degrees of freedom.

The use of the t distribution does not adjust the confidence interval to account for skewness in the underlying population or other errors that can result when $\widehat{\theta}_n$ is not the sample mean (see Efron and Tibshirani [26]). A solution to this problem is the bootstrap t method, a procedure which does adjust for these errors. This procedure will be discussed in Section 6.4.3.

Remark 6.1. Other approximations can be obtained using Edgeworth type expansions as extension of the normal (first order) approximation. We will not investigate these in the current study.

6.4 Bootstrap Confidence Intervals

Bootstrap methods are commonly used for constructing approximate confidence intervals. They are favored because they are free of model assumptions and they often yield reliable results for data with moderate sample sizes. In this section we describe several bootstrap confidence interval methods. Prior to that, we give some background ideas about the bootstrap. In what follows, we closely follow Efron and Tibshirani [26].

6.4.1 Basic Ideas of the Bootstrap

Consider a sample $X_1, X_2, \ldots, X_n$ from an unknown distribution F, and denote it by

$$\underline{X} = (X_1, X_2, \ldots, X_n)'. \tag{6.11}$$

In what follows, $\underline{X}$ will be considered as the original data set.

Bootstrap Sample and Bootstrap Replication

The bootstrap is a computer-based method for assigning measures of accuracy to statistical estimators (see [26]).

Definition 6.2. Bootstrap Sample.

A bootstrap sample is a random sample of size n drawn from $\underline{X}$ with replacement. We denote it by

$$\underline{X}^{\star} = (X_1^{\star}, X_2^{\star}, \ldots, X_n^{\star}). \tag{6.12}$$

Definition 6.3. Bootstrap Replication.

Suppose we wish to estimate a parameter of interest $\theta = t(F)$ on the basis of the sample $\underline{X}$, where t is the functional of interest. For this purpose, an estimator $\widehat{\theta}_n = s(\underline{X})$ is calculated from $\underline{X}$. A bootstrap replication (also called bootstrap replicate) of $\widehat{\theta}_n$ is the value of the statistic s evaluated for the bootstrap sample $\underline{X}^{\star}$. We denote it by $\widehat{\theta}^{\star} = s(\underline{X}^{\star})$, where $s(\underline{X}^{\star})$ is the result of applying the same function $s(.)$ to $\underline{X}^{\star}$ as was applied to $\underline{X}$.

To illustrate, if $s(\underline{X})$ is the sample mean $\overline{X}$, then $s(\underline{X}^{\star})$ is the mean of the bootstrap data set $\underline{X}^{\star}$, i.e.

$$\overline{X}^{\star} = \frac{1}{n}\sum_{i=1}^{n} X_i^{\star}. \tag{6.13}$$

Bootstrap Estimate of Standard Error

Denote by $se_F(\widehat{\theta}_n)$ the standard error of the estimator of θ.

Definition 6.4. The bootstrap estimate of $se_F(\widehat{\theta}_n)$ is a plug-in estimate that uses the empirical distribution function F_n in place of the unknown distribution F. Specifically, the bootstrap estimate of $se_F(\widehat{\theta}_n)$ is defined by

$$se_{F_n}(\widehat{\theta}^{\star}). \tag{6.14}$$

In other words, it is the standard error for $\widehat{\theta}_n$ using data sets of size n randomly selected from F_n.

The bootstrap procedure for estimating the standard error of $\widehat{\theta}_n = s(\underline{X})$ is given in the following algorithm (see [26]).

Algorithm 6.1. Bootstrap Algorithm for Estimating Standard Errors.

1. Select B bootstrap samples $\underline{X}^{\star 1}, \underline{X}^{\star 2}, \ldots, \underline{X}^{\star B}$, each consisting of n data values drawn with replacement from $\underline{X}$.

2. Calculate the bootstrap replication corresponding to each bootstrap sample, i.e.

$$\widehat{\theta}^{\star b} = s(\underline{X}^{\star b}),\ b = 1, 2, \ldots B. \tag{6.15}$$

3. Estimate the standard error $se_F(\widehat{\theta}_n)$ by the standard deviation of the B replications, i.e.

$$\widehat{se}_B = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}\left(\widehat{\theta}^{\star b} - \widehat{\theta}^{\star}(.)\right)^2}, \tag{6.16}$$

where

$$\widehat{\theta}^{\star}(.) = \frac{1}{B}\sum_{b=1}^{B}\widehat{\theta}^{\star b}. \tag{6.17}$$

Since $\widehat{se}_B$ is based on F_n (the nonparametric estimator of the population distribution function F), it is sometimes called a nonparametric bootstrap estimate of standard error. When the estimator of F is derived from a parametric model for the data, the corresponding estimate of the standard error is known as the parametric bootstrap estimate of the standard error, often defined by

$$se_{\widehat{F}_{par}}(\widehat{\theta}^{\star}), \tag{6.18}$$

where $\widehat{F}_{par}$ is the parametric estimator of F.

In the simulations we will use the previous algorithm quite often because of the complexity in finding formulas for the variance of some inequality measures. In fact, explicit formulas for standard error calculation exist for only very few statistics.

With this background, we next describe different methods for constructing bootstrap confidence intervals.

6.4.2 Bootstrap Percentile Intervals

Consider a situation where bootstrap samples $\underline{X}^{\star}$ are generated and bootstrap replications $\widehat{\theta}^{\star} = s(\underline{X}^{\star})$ are computed as described previously. Denote by $\widehat{G}$ the empirical distribution function of $\widehat{\theta}^{\star}$. The $1-\alpha$ percentile interval is defined by

$$\left[\widehat{\theta}_{low}, \widehat{\theta}_{up}\right] = \left[\widehat{G}^{-1}(\alpha/2), \widehat{G}^{-1}(1-\alpha/2)\right], \tag{6.19}$$

where $\widehat{G}^{-1}(\alpha)$ is the $100\alpha^{th}$ percentile of $\widehat{G}$, denoted by $\widehat{\theta}^{\star(\alpha)}$. This interval can also be written as

$$\left[\widehat{\theta}_{low}, \widehat{\theta}_{up}\right] = \left[\widehat{\theta}^{\star(\alpha/2)}, \widehat{\theta}^{\star(1-\alpha/2)}\right]. \tag{6.20}$$

In practice, the procedure for deriving percentile confidence intervals is described in the following algorithm.

Algorithm 6.2. Algorithm for Constructing Bootstrap Percentile Intervals.

1. Generate B bootstrap samples
$$\underline{X}^{\star 1}, \underline{X}^{\star 2}, \ldots, \underline{X}^{\star B}$$
and compute the corresponding bootstrap replications
$$\widehat{\theta}^{\star b} = s(\underline{X}^{\star b}),\ b = 1, 2, \ldots B.$$

2. Take $\widehat{\theta}_B^{\star(\alpha)}$ as the $B\alpha^{th}$ value in the ordered list of the B replications of $\widehat{\theta}^{\star}$. If $B\alpha$ is not an integer, the following procedure can be used. Let $m = [(B+1)\alpha]$, the largest integer less than or equal to $(B+1)\alpha$. Then define the empirical α and $1-\alpha$ quantiles by the m^{th} and the $(B+1-m)^{th}$ largest values of $\widehat{\theta}^{\star}$.

3. Construct the $1-\alpha$ bootstrap percentile confidence interval by
$$\left[\widehat{\theta}_{low}, \widehat{\theta}_{up}\right] = \left[\widehat{\theta}_B^{\star(\alpha/2)}, \widehat{\theta}_B^{\star(1-\alpha/2)}\right]. \tag{6.21}$$

The percentile intervals are preferred to the standard normal intervals, especially when the sample size is small. In fact, when the sample size is small, the normal approximation is typically not accurate.

6.4.3 Bootstrap t Confidence Intervals

Since the assumption of an approximate normal distribution is easily violated when the sample size is small, an alternative approach was to obtain the percentiles from the t distribution. Similarly, the bootstrap t method makes use of a t statistic defined by
$$Z = \frac{\widehat{\theta} - \theta}{\widehat{se}}, \tag{6.22}$$
and is described in the following algorithm.

Algorithm 6.3. Algorithm for Constructing Bootstrap t Confidence Intervals.

1. Generate B bootstrap samples
$$\underline{X}^{\star 1}, \underline{X}^{\star 2}, \ldots, \underline{X}^{\star B}$$

and compute the corresponding bootstrap replications

$$Z^{\star b} = \frac{\widehat{\theta}^{\star b} - \widehat{\theta}}{\widehat{se}^{\star b}},\ b = 1, 2, \ldots B,$$

where $\widehat{\theta}^{\star b}$ is the b^{th} bootstrap replicate for $\widehat{\theta}$, and $\widehat{se}^{\star b}$ is the estimated standard error of $\widehat{\theta}^{\star b}$ using the bootstrap sample $\underline{X}^{\star b}$. In order to obtain $\widehat{se}^{\star b}$, bootstrap samples are taken from $\underline{X}^{\star b}$ and not from the original sample $\underline{X}$. These are used to obtain second level bootstrap replications. Finally, the standard deviation of these bootstrap replications gives $\widehat{se}^{\star b}$.

2. Estimate the α^{th} percentile of $Z^{\star}$ by the value of $\widehat{t}^{(\alpha)}$ such that

$$\frac{\#\{Z^{\star b} \leq \widehat{t}^{(\alpha)}\}}{B} = \alpha. \tag{6.23}$$

If $B\alpha$ is not an integer, one can use the same procedure as in step 2 of Algorithm 6.2.

3. Construct the bootstrap t interval by

$$\left[\widehat{\theta}_{low}, \widehat{\theta}_{up}\right] = \left[\widehat{\theta} - \widehat{t}^{(1-\alpha/2)} \cdot \widehat{se}^{\star}, \widehat{\theta} - \widehat{t}^{(\alpha/2)} \cdot \widehat{se}^{\star}\right], \tag{6.24}$$

where $\widehat{se}^{\star}$ is the bootstrap estimate of the standard error of $\widehat{\theta}$.

It has been shown (see Efron and Tibshirani [26]) that the coverage of the bootstrap t interval tends to be closer to the desired level than that of the standard interval. Unfortunately the gain in accuracy goes hand in hand with a loss in generality, since the bootstrap t interval applies only to the given sample. Moreover, there is a major computational difficulty with the use of the bootstrap t intervals, due to the fact that one has to estimate the standard error for each bootstrap replication using a second level bootstrap. A faster way is to use the jackknife on second level to estimate the standard error $\widehat{se}^{\star b}$ from each first level bootstrap sample. The jackknife method for estimating the standard error is described in the next section.

6.4.4 Bias-Corrected and Accelerated (BCa) Confidence Intervals

When the estimator for the parameter of interest is biased, the bootstrap percentile method may not work well. A modified percentile method was therefore proposed by Efron [25], called the bias-corrected and accelerated bootstrap. This method often provides more accurate confidence intervals. It will be denoted by BCa CIs. Before describing the method, we first give some background on the jackknife.

Jackknife Sample and Jackknife Replication

Definition 6.5. Jackknife Sample.

Given a data set $\underline{X} = (X_1, X_2, \ldots, X_n)'$, the i^{th} jackknife sample $\underline{X}_{(i)}$ is defined to be $\underline{X}$ with the i^{th} data point removed, i.e.

$$\underline{X}_{(i)} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)'. \tag{6.25}$$

Definition 6.6. Jackknife Replication.

The i^{th} jackknife replication $\widehat{\theta}_{(i)}$ of the statistic $\widehat{\theta} = s(\underline{X})$ is $s(.)$ evaluated at $\underline{X}_{(i)}$, i.e.

$$\widehat{\theta}_{(i)} = s(\underline{X}_{(i)}). \tag{6.26}$$

Note that in this case $s(.)$ must also be definable on $n-1$ arguments. For the plug-in statistics $\widehat{\theta} = t(F_n)$, we have

$$\widehat{\theta}_{(i)} = t(\widehat{F}_{(i)}), \tag{6.27}$$

where $\widehat{F}_{(i)}$ is the empirical distribution of the $n-1$ points in $\underline{X}_{(i)}$.

Jackknife Estimate of Standard Error

The jackknife estimate of standard error is given by

$$\widehat{se}_{jack} = \sqrt{\frac{n-1}{n}\sum_{i=1}^{n}\left(\widehat{\theta}_{(i)} - \widehat{\theta}_{(.)}\right)^2}, \tag{6.28}$$

where

$$\widehat{\theta}_{(.)} = \frac{1}{n}\sum_{i=1}^{n}\widehat{\theta}_{(i)}. \tag{6.29}$$

Note that in the case of the bootstrap t intervals, the jackknife method can be used to estimate the standard error of $\widehat{\theta}^{\star b},\ b = 1, 2, \ldots, B$.

With this background, we now describe the BCa confidence intervals. Let

$$\alpha_1 = \Phi\left(\widehat{z}_0 + \frac{\widehat{z}_0 + z^{(\alpha/2)}}{1 - \widehat{a}(\widehat{z}_0 + z^{(\alpha/2)})}\right), \tag{6.30}$$

$$\alpha_2 = \Phi\left(\widehat{z}_0 + \frac{\widehat{z}_0 + z^{(\alpha/2)}}{1 - \widehat{a}(\widehat{z}_0 + z^{(1-\alpha/2)})}\right), \tag{6.31}$$

where Φ is the standard normal distribution function,

$$\widehat{z}_0 = \Phi^{-1}\left(\frac{\#\{\widehat{\theta}^{\star b} \leq \widehat{\theta}\}}{B}\right), \tag{6.32}$$

$$z^{(\alpha/2)} = \Phi^{-1}(\alpha/2), \tag{6.33}$$

$$z^{(1-\alpha/2)} = \Phi^{-1}(1-\alpha/2) \tag{6.34}$$

and

$$\widehat{a} = \left[\frac{1}{6}\sum_{i=1}^{n}\left(\widehat{\theta}_{(.)} - \widehat{\theta}_{(i)}\right)^3\right] \Big/ \left[\sum_{i=1}^{n}\left(\widehat{\theta}_{(.)} - \widehat{\theta}_{(i)}\right)^2\right]^{3/2}. \tag{6.35}$$

The $(1-\alpha)$ BCa confidence interval for θ is defined by

$$\left[\widehat{\theta}_{low}, \widehat{\theta}_{up}\right] = \left[\widehat{\theta}^{\star(\alpha_1)}, \widehat{\theta}^{\star(\alpha_2)}\right], \tag{6.36}$$

where $\widehat{\theta}^{\star(\alpha)}$ is the α^{th} percentile of the ordered values $\widehat{\theta}^{\star b}$.

As mentioned at the beginning of the section, the BCa method often provides a more accurate CI than the standard normal and the percentile methods, even when the estimation of the parameter of interest is biased (see Efron and Tibshirani [26]). However, the BCa method requires a large number of bootstrap replications.

Remark 6.2. Other proposals, e.g. the approximate bootstrap confidence (ABC) interval, are approximations to the BCa and the bootstrap t intervals. They are complex and do not add much information, and so they will not be considered in this study.

6.4.5 Semi-Parametric Bootstrap Confidence Intervals

The semi-parametric bootstrap method of constructing confidence intervals is described as follows (see Davidson and Flachaire [18]).

1. Construct n independent Bernoulli variables $U_i,\ i = 1, 2, \ldots, n$, each equal to 1 with a probability P_{tail} and 0 with probability $1 - P_{tail}$ (we assume here that we fit a semi-parametric distribution to the original data: the nonparametric distribution is used for the $n(1 - P_{tail})$ lower values and a parametric distribution is used in the upper tail). The value $X_i^{\star b}$ of the bootstrap sample is a

simulated value from the parametric distribution if $U_i = 1$ and from the empirical distribution of the $n(1-P_{tail})$ smallest order statistics if $U_i = 0$.

2. For the bootstrap sample $\underline{X}^{\star b}$, compute the value of the statistic $s(\underline{X}^{\star b})$ and denote the result by $\widehat{\theta}^{\star b}$.

3. Repeat Steps 1 and 2 B times to obtain $\widehat{\theta}^{\star 1}, \widehat{\theta}^{\star 2}, \ldots, \widehat{\theta}^{\star B}$, and estimate the standard error of $\widehat{\theta}^{\star}$ (denoted $\widehat{se}^{\star}$) by the standard deviation of $\widehat{\theta}^{\star 1}, \widehat{\theta}^{\star 2}, \ldots, \widehat{\theta}^{\star B}$.

4. Construct the semi-parametric bootstrap confidence interval by

$$\left[\widehat{\theta}_{low}, \widehat{\theta}_{up}\right] = \left[\widehat{\theta}_n - \widehat{se}^{\star}\Phi^{-1}\left(\frac{1-\alpha}{2}\right), \widehat{\theta}_n + \widehat{se}^{\star}\Phi^{-1}\left(\frac{1-\alpha}{2}\right)\right]. \tag{6.37}$$

Remark 6.3. The semi-parametric method can be used to construct bootstrap t intervals. The procedure is as in Algorithm 6.3, but now one should take the bootstrap samples as in Step 1 above.

In the case of BCa intervals, the standard bootstrap is used to find the quantity $\widehat{z}_0$ in Equation (6.32). This can also be done by taking the bootstrap samples as in Step 1 above.

Remark 6.4. In the next chapter, confidence intervals will be studied for the measures of inequality. Not many such studies have been carried out in the existing literature. For example, Qin et al. [45] mentions that confidence intervals for Gini have not been studied by previous authors, with the exception of Sandström et al. [49], where 95% normal approximation confidence intervals based on three variance estimators were briefly discussed.

In order to construct asymptotic confidence intervals, we now discuss the asymptotic normality for the traditional estimators of the measures of inequality.

6.5 Asymptotic Normality for the Gini, the GE and the Atkinson Measures

We have the following asymptotic normality for the Gini, the GE and the Atkinson measures.

1. **Gini** (see Davidson [17]):

$$n^{1/2}(\widehat{I}_G - I_G) \xrightarrow{D} N\left(0, \sigma_G^2\right), \tag{6.38}$$

where σ_G^2 is given by

$$\sigma_G^2 = \frac{1}{\mu^2} \operatorname{Var}\left(-(I_G+1)X + 2\left(XF(X) - \int_0^X x dF(x) \right) \right), \tag{6.39}$$

and can be estimated by

$$\widehat{\sigma}_G^2 = \frac{1}{n\widehat{\mu}^2} \sum_{i=1}^{n} (\widehat{z}_i - \bar{z})^2, \tag{6.40}$$

with

$$\widehat{z}_i = -(\widehat{I}_G + 1) X_{i,n} + \frac{2i-1}{n} X_{i,n} - \frac{2}{n} \sum_{j=1}^{i} X_{j,n},$$

and

$$\bar{z} = \frac{1}{n} \sum_{i=1}^{n} \widehat{z}_i.$$

2. **Generalized Entropy** (see Schluter and van Garderen [52] and Mills and Zandvakili [42]):

$$n^{1/2}(\widehat{I}_E^\alpha - I_E^\alpha) \xrightarrow{D} N\left(0, \sigma_E^2\right), \tag{6.41}$$

where σ_E^2 is given by

$$\sigma_E^2 = \frac{1}{(\alpha^2-\alpha)^2} \frac{1}{\mu_1^{2\alpha+2}} \left[\mu_1^2 \mu_{2\alpha} + \alpha^2 \mu_\alpha^2 \mu_2 - 2\alpha \mu_\alpha \mu_1 \mu_{\alpha+1} - (1-\alpha)^2 \mu_\alpha^2 \mu_1^2 \right], \tag{6.42}$$

and can be estimated by

$$\widehat{\sigma}_E^2 = \frac{1}{(\alpha^2-\alpha)^2} \frac{1}{m_1^{2\alpha+2}} \left[m_1^2 m_{2\alpha} + \alpha^2 m_\alpha^2 m_2 - 2\alpha m_\alpha m_1 m_{\alpha+1} - (1-\alpha)^2 m_\alpha^2 m_1^2 \right], \tag{6.43}$$

$\alpha \neq 0, 1$, with

$$m_\alpha = \frac{1}{n} \sum_{i=1}^{n} X_i^\alpha.$$

The asymptotic variances σ_0^2 and σ_1^2 for I_E^0 and I_E^1 can be estimated respectively by

$$\widehat{\sigma}_0^2 = \frac{t_{24} - t_{11}^2}{(n-1)m_{11}^2} + \theta^2 \left(\frac{m_{22}}{m_{11}^2} + 1 \right) - \frac{2\theta}{n-1} \left(\frac{t_{22}}{m_{11}^2} + 1 \right) \tag{6.44}$$

and

$$\widehat{\sigma}_1^2 = \frac{t_{23} - t_{10}^2}{n-1} + \left(\frac{m_{22}}{m_{11}^2} + 1 - \frac{2}{n-1} \left(\frac{t_{21}}{m_{11}} - \phi \right) \right), \tag{6.45}$$

where $m_{11} = m_1$, $m_{22} = m_2$,

$$t_{10} = \frac{1}{n}\sum_{i=1}^{n}\log X_i,\ t_{11} = \frac{1}{n}\sum_{i=1}^{n} X_i \log X_i,\ t_{21} = t_{11},\ t_{22} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \log X_i,$$

$$t_{23} = \frac{1}{n}\sum_{i=1}^{n}(\log X_i)^2,\ t_{24} = \frac{1}{n}\sum_{i=1}^{n} X_i^2(\log X_i)^2,\ \phi = t_{10} - 1 \text{ and } \theta = 1 + \frac{t_{11}}{m_{11}}.$$

3. **Atkinson** (see Chotikapanich and Creedy [8]):

$$n^{1/2}(\widehat{I_A^{\varepsilon}} - I_A^{\varepsilon}) \xrightarrow{D} N\left(0, \sigma_A^2\right), \tag{6.46}$$

where σ_G^2 is given by

$$\sigma_A^2 = \left(\frac{1 - I_A^{\varepsilon}}{\varepsilon \mu_{\varepsilon}}\right)^2 \left[(\mu_{2\varepsilon} - \mu_{\varepsilon}^2) - 2\left(\frac{\varepsilon \mu_{\varepsilon}}{\mu_1}\right)(\mu_{\varepsilon+1} - \mu_{\varepsilon}\mu_1) + \left(\frac{\varepsilon \mu_{\varepsilon}}{\mu_1}\right)^2 (\mu_2 - \mu_1^2)\right], \tag{6.47}$$

and can be estimated by

$$\widehat{\sigma}_A^2 = \left(\frac{1 - I_A^{\varepsilon}}{\varepsilon m_{\varepsilon}}\right)^2 \left[(m_{2\varepsilon} - m_{\varepsilon}^2 - 2\left(\frac{\varepsilon m_{\varepsilon}}{m_1}\right)(m_{\varepsilon+1} - m_{\varepsilon} m_1) + \left(\frac{\varepsilon m_{\varepsilon}}{m_1}\right)^2 (m_2 - m_1^2)\right]. \tag{6.48}$$

6.6 Asymptotic Normality for the Quintile Share Ratio

The limiting normal distribution of the QSR has not yet been obtained in the literature. We have derived this and although it is not central to our work, we do supply an outline of the proof in Appendix A. With $\eta \equiv \eta(F)$, the nonparametric estimator is given by $\widehat{\eta} \equiv \eta(F_n)$, with F_n the empirical distribution function.

Now, let $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ denote the order statistics, then

$$\widehat{\eta} = \left[\frac{1}{n}\sum_{i=[0.8n]+1}^{n} X_{i,n}\right] \Big/ \left[\frac{1}{n}\sum_{i=1}^{[0.2n]} X_{i,n}\right]. \tag{6.49}$$

The limiting distribution of $\widehat{\eta}$ is given in the following theorem.

Theorem 6.1. Assuming that the underlying distribution is of Pareto-type, i.e. it satisfies

$$1 - F(x) = x^{-1/\gamma} l_F(x),\ \gamma > 0,$$

we have for $\gamma < \frac{1}{2}$,

$$n^{1/2}(\widehat{\eta} - \eta) \xrightarrow{D} N\left(0, \sigma_\eta^2\right), \tag{6.50}$$

with

$$\sigma_\eta^2 = \frac{D_{0.2}^2 \sigma^2(0.8, 1) + N_{0.8}^2 \sigma^2(0, 0.2) - 2D_{0.2}N_{0.8}(0.2Q(0.2) - D_{0.2})(N_{0.8} - 0.2Q(0.8))}{D_{0.2}^4}, \tag{6.51}$$

where

$$N_\beta = \int_{Q(\beta)}^{\infty} x dF(x) = \int_\beta^1 Q(s) ds, \tag{6.52}$$

$$D_\alpha = \int_0^{Q(\alpha)} x dF(x) = \int_0^\alpha Q(s) ds, \tag{6.53}$$

and $\sigma^2(s,t)$ is defined as

$$\sigma^2(s,t) = \int_s^t \int_s^t (u \wedge v - uv) dQ(u) dQ(v). \tag{6.54}$$

Proof. See Appendix A. □

Remark 6.5. The limiting variance in Theorem 6.1 can be estimated by plugging in the obvious estimators in Equation (6.51).

Remark 6.6. In the case where $\gamma = \frac{1}{2}$, asymptotic normality is still obtained, although the proof needs a slight modification to hold in that case. For the case $\frac{1}{2} < \gamma < 1$, a limiting stable distribution is obtained due to the very heavy tails of F. Although the latter theorem can also be proved, a much more careful analysis is needed and not pursued in this research.

Remark 6.7. The asymptotic variances of the semi-parametric estimators can also be found in principle, but are much more difficult to obtain. This will be dealt with in future research.

Chapter 7

Simulation Study

In the previous chapter we developed computational formulas for estimating inequality measures in semi-parametric settings. The measures considered were the Gini coefficient, the generalized entropy, the Atkinson and the quintile share ratio. The use of a parametric distribution was involved in each of the methods and so it is important to decide which distribution to use in the tail. In this chapter we conduct a simulation study to test the performance of our estimators. We generate samples from different distributions (Pareto, Burr, Fréchet, Student t). For each sample, we choose the appropriate threshold, depending on whether we fit the Pareto (Pa), the Generalized Pareto (GPD) or the Perturbed Pareto (PPD). We also choose the threshold applying the method suggested by Cowell and Flachaire [10], which consists of fitting the strict Pareto to the data in the tail rather than the exceedances. In order to estimate the parameters, we use the maximum likelihood method as well as the robust estimation methods described in Sections 4.2.1, 4.2.2 and 4.2.3. Furthermore we study the properties of different confidence intervals described in Chapter 6.

The same underlying distributions as in Chapter 3 will be used:

1. The Pareto distribution with $x_0 = 0.1$ and $\beta = 1.5$ (we will refer to it as Pa);
2. The Burr distribution with $\alpha = 2$, $\tau = 0.83$ and $\lambda = 1$ (we will refer to it as Bu1);
3. The Burr distribution with $\alpha = 1$, $\tau = 1.4$ and $\lambda = 1$ (we will refer to it as Bu2);
4. The Burr distribution with $\alpha = 0.5$, $\tau = 4$ and $\lambda = 1$ (we will refer to it as Bu3);
5. The Fréchet distribution with $\alpha = 2$ (we will refer to it as Fr1);
6. The Fréchet distribution with $\alpha = 1.7$ (we will refer to it as Fr2);

7. The $|t_2|$ distribution (we will refer to it as T2).

Before carrying out the simulation, we first calculated the population values for the inequality measures in the case of the above distributions. The results are given in Table 7.1.

	Gini	GE0	GE1	GE1.3	A1	A1.5	A2	QSR
Pa (γ=0.67)	0.5000	0.4319	0.9014	2.0463	0.3508	0.4074	0.4444	8.1583
Bu1 (γ=0.96)	0.7621	1.4611	1.5789	2.6688	0.7680	0.9243	1.0000	157.4449
Bu2 (γ=0.71)	0.7143	1.0544	1.7309	5.9718	0.6516	0.7753	0.8786	52.4419
Bu3 (γ=0.25)	0.4009	0.2708	0.3826	0.5002	0.2372	0.3065	0.3633	6.7015
Fr1 (γ=0.50)	0.4142	0.2837	0.4093	0.5381	0.2471	0.3133	0.3634	6.8124
Fr2 (γ=0.59)	0.5034	0.4275	0.6906	1.0927	0.4515	0.5319	0.5865	15.3868
T2 (γ=0.50)	0.5708	0.6931	0.6931	0.8726	0.5000	0.7091	.	.

Table 7.1: Population Values for Measures of Inequality

Remark 7.1. The γ values of the respective distributions are included in the table to give an indication of the tail heaviness. Note especially the large values of QSR for the very heavy-tailed Bu1 and Bu2.

7.1 Mean Squared Errors and Sensitivity to Contaminations

We use simple random samples of size 500, 1000, 5000 and 10000 to estimate the mean squared errors (MSEs) for each measure of inequality. We denote the estimators by SPGPD (when fitting the GPD in the tail), SPPa (when fitting the strict Pareto), and SPPPD (when fitting the PPD). The nonparametric estimator is denoted by NP and the Cowell estimator by SPCo. We also studied the sensitivity of our estimators to outliers in terms of Relative Impact of Contamination (RIC). Note that in their semi-parametric procedure Cowell and Flachaire [10] fitted the strict Pareto to the data in the tail rather than the exceedances. They put emphasis on the GE measures in their semi-parametric approach. In this study, we apply their approach to the other measures that we consider, i.e. the Gini, Atkinson and quintile share ratio.

7.1.1 Mean Squared Errors for the Estimators

The results obtained in the case of the mean squared errors are given in Table 7.2 for the Gini coefficient. These results are discussed following Table 7.3. The MSE estimates for the other measures are given in Appendix D (Tables D.1 to D.3 for the GE measures, Tables D.4 to D.6 for the Atkinson measures and Table D.7 for the QSR measure). Note that the values in brackets are the standard errors of the MSEs. In each row of the tables, the value in bold is the minimum MSE.

Gini (n=500)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0049 (0.0003)	0.0048 (0.0004)	0.0052 (0.0007)	**0.0017** (0.0001)	0.0028 (0.0006)
Bu1	0.0019 (0.0001)	**0.0016** (0.0004)	0.0025 (0.0003)	0.0580 (0.0086)	**0.0016** (0.0003)
Bu2	0.0044 (0.0002)	**0.0043** (0.0004)	0.0081 (0.001)	0.0578 (0.0078)	0.0063 (0.0003)
Bu3	0.0016 (0.0001)	0.0015 (0.0001)	0.0010 (0.0009)	0.0014 (0.0001)	**0.0009** (0.0001)
Fr1	0.0015 (0.0009)	0.0013 (0.0001)	**0.0011** (0.0001)	0.0015 (0.0001)	**0.0011** (0.0001)
Fr2	0.0028 (0.0011)	0.0026 (0.0002)	**0.0023** (0.0003)	0.0054 (0.0001)	0.0032 (0.0002)
T2	0.0010 (0.0005)	0.0012 (0.0002)	**0.0009** (0.0001)	0.0025 (0.0001)	0.0017 (0.0001)
Average	0.0026 (0.0002)	**0.0025** (0.0001)	0.0030 (0.0002)	0.0183 (0.0017)	**0.0025** (0.0001)

Gini (n=1000)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0045 (0.0003)	0.0093 (0.0001)	0.0045 (0.0002)	0.0033 (0.0001)	**0.0011** (0.0001)
Bu1	**0.0017** (0.0001)	0.0217 (0.0002)	0.0019 (0.0002)	0.0423 (0.0025)	0.0041 (0.0003)
Bu2	**0.0033** (0.0002)	0.0512 (0.0003)	0.0058 (0.0003)	0.0393 (0.0011)	0.0045 (0.0000)
Bu3	0.0014 (0.0001)	0.0006 (0.0001)	0.0006 (0.0001)	0.0010 (0.0000)	**0.0005** (0.0000)
Fr1	0.0011 (0.0006)	0.0006 (0.0001)	0.0006 (0.0001)	0.0011 (0.0000)	**0.0004** (0.0001)
Fr2	0.0021 (0.0009)	0.0019 (0.0001)	0.0015 (0.0000)	0.0040 (0.0000)	**0.0014** (0.0002)
T2	0.0008 (0.0002)	0.0008 (0.0001)	**0.0005** (0.0000)	0.0017 (0.0000)	**0.0005** (0.0001)
Average	0.0021 (0.0002)	0.0123 (0.0001)	0.0022 (0.0001)	0.0132 (0.0004)	**0.0018** (0.0001)

Gini (n=5000)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0017 (0.0000)	0.0044 (0.0001)	0.0013 (0.0001)	0.0036 (0.0000)	**0.0002** (0.0000)
Bu1	**0.0004** (0.0000)	0.0085 (0.0001)	0.0005 (0.0001)	0.0206 (0.0003)	0.0019 (0.0001)
Bu2	**0.0011** (0.0001)	0.0109 (0.0001)	0.0020 (0.0002)	0.0225 (0.0003)	**0.0011** (0.0001)
Bu3	0.0002 (0.0000)	0.0002 (0.0000)	0.0002 (0.0000)	0.0004 (0.0000)	**0.0001** (0.0000)
Fr1	0.0002 (0.0005)	0.0002 (0.0000)	0.0006 (0.0000)	0.0005 (0.0000)	**0.0001** (0.0000)
Fr2	0.0006 (0.0006)	0.0008 (0.0000)	0.0006 (0.0000)	0.0021 (0.0000)	**0.0003** (0.0000)
T2	0.0006 (0.0002)	0.0003 (0.0000)	0.0002 (0.0000)	0.0008 (0.0000)	**0.0001** (0.0000)
Average	0.0007 (0.0001)	0.0036 (0.0000)	0.0008 (0.0000)	0.0072 (0.0000)	**0.0005** (0.0000)

Gini (n=10000)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0017 (0.0001)	0.0044 (0.0001)	0.0014 (0.0000)	0.0043 (0.0000)	**0.0003** (0.0000)
Bu1	**0.0002** (0.0000)	0.0085 (0.0001)	0.0004 (0.0000)	0.0161 (0.0001)	0.0017 (0.0001)
Bu2	0.0010 (0.0001)	0.0107 (0.0001)	0.0014 (0.0000)	0.0184 (0.0001)	**0.0007** (0.0001)
Bu3	**0.0001** (0.0000)	0.0002 (0.0000)	**0.0001** (0.0000)	0.0003 (0.0000)	**0.0001** (0.0000)
Fr1	0.0002 (0.0004)	0.0002 (0.0000)	**0.0001** (0.0000)	0.0003 (0.0000)	**0.0001** (0.0000)
Fr2	0.0005 (0.0003)	0.0008 (0.0000)	0.0004 (0.0000)	0.0015 (0.0000)	**0.0001** (0.0000)
T2	**0.0001** (0.0001)	0.0002 (0.0000)	**0.0001** (0.0000)	0.0006 (0.0000)	**0.0001** (0.0000)
Average	0.0005 (0.0001)	0.0036 (0.0000)	0.0006 (0.0000)	0.0059 (0.0000)	**0.0004** (0.0000)

Table 7.2: Mean Squared Errors for Gini Measure

Note that in Table 7.2, the MSEs are the same in the first two columns for $n = 500$. This is due to the fact that the Cowell method reduces to the nonparametric method for $n = 500$.

7.1.2 Sensitivity of Inequality Measures to Outliers

Let $I(F)$ be an inequality measure. In order to study the sensitivity of $I(F)$ to outliers, Cowell and Flachaire [10] made use of contamination of the original data, e.g. by multiplying the largest value of the data by 10.

Other ways of contaminating the data are by replacing a proportion (e.g. $1\%, 2\%, \ldots$) of the data by a fixed value, and instead of just the maximum, multiplying a given number of larger order statistics by a constant.

Consider a random sample $\underline{X} = (X_1, X_2, \ldots, X_n)$ from F and let $\underline{X}^\star$ be the contaminated sample. Denote by $\widehat{F}$ an estimate of F using $\underline{X}$, and by $\widehat{F}^\star$ the corresponding estimate using $\underline{X}^\star$. In order to check the effect of contamination, one has to compare the estimates $I(\widehat{F})$ and $I(\widehat{F}^\star)$.

A measure of sensitivity called relative impact of contamination is then defined as (see [10])

$$RIC(I) = \frac{I(\widehat{F}) - I(\widehat{F}^\star)}{I(\widehat{F})}, \tag{7.1}$$

with small values (near zero) indicating insensitivity to contamination.

We apply this approach to the measures considered in our study. We generate data sets from each of the distributions, contaminate each one of them in a similar way and calculate the relative impact of contamination for each inequality measure. One of the ways to contaminate the data mentioned above is to multiply a reasonable proportion p of them by a certain constant c. In our simulation we chose a random proportion $p = 1\%$ of the data and $c = 10$. We confine ourselves to giving the results for Gini in Table 7.3. The results for other measures are given in Appendix E (Tables E.1 to E.3 for the GE measures, Tables E.4 to E.6 for the Atkinson measures and Table E.7 for the QSR measure).

Gini (n=500)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0694	0.0694	0.0603	0.0606	**0.0601**
	(0.004)	(0.004)	(0.0156)	(0.0092)	(0.0024)
Bu1	0.0172	0.0172	**0.0150**	0.0156	0.0152
	(0.0016)	(0.0016)	(0.0106)	(0.0639)	(0.0009)
Bu2	0.0250	0.0250	**0.0133**	0.0227	0.0280
	(0.0022)	(0.0022)	(0.0099)	(0.0251)	(0.0012)
Bu3	0.1050	0.1050	0.0669	0.0677	**0.0644**
	(0.0038)	(0.0038)	(0.0094)	(0.0017)	(0.0013)
Fr1	0.0661	0.0661	0.0441	0.0363	**0.0312**
	(0.0036)	(0.0036)	(0.006)	(0.0022)	(0.0017)
Fr2	0.0702	0.0702	0.0386	**0.0374**	0.0386
	(0.0035)	(0.0035)	(0.0079)	(0.003)	(0.0016)
T2	0.0457	0.0457	0.0364	0.0381	**0.0353**
	(0.0024)	(0.0024)	(0.0075)	(0.0012)	(0.0008)
Average	0.0569	0.0569	0.0392	0.0398	**0.0390**
	(0.0012)	(0.0012)	(0.0038)	(0.0099)	(0.0006)

Gini (n=1000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0729	0.0462	0.0478	**0.0401**	0.0480
	(0.0038)	(0.0023)	(0.0056)	(0.002)	(0.0019)
Bu1	0.0868	0.0479	0.0494	0.0471	**0.0427**
	(0.0013)	(0.0074)	(0.0047)	(0.009)	(0.0008)
Bu2	0.0726	0.0404	0.0278	0.0291	**0.0259**
	(0.0017)	(0.0044)	(0.0051)	(0.0057)	(0.001)
Bu3	0.1064	0.0872	**0.0560**	0.0626	0.0610
	(0.0033)	(0.0016)	(0.0039)	(0.0012)	(0.0007)
Fr1	0.0676	0.0457	0.0481	**0.0325**	0.0349
	(0.003)	(0.0013)	(0.0119)	(0.001)	(0.001)
Fr2	0.0654	0.0541	0.0435	**0.0327**	0.0349
	(0.0029)	(0.0013)	(0.0065)	(0.0012)	(0.0011)
T2	0.0480	0.0397	0.0256	**0.0205**	0.0274
	(0.0019)	(0.0009)	(0.0027)	(0.0006)	(0.0006)
Average	0.0742	0.0516	0.0426	**0.0378**	0.0393
	(0.001)	(0.0013)	(0.0024)	(0.0016)	(0.0004)

Gini (n=5000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0706	0.0465	0.0400	**0.0364**	0.0392
	(0.0024)	(0.001)	(0.0034)	(0.0009)	(0.0006)
Bu1	0.0367	0.0203	0.0098	**0.0142**	0.0188
	(0.0009)	(0.002)	(0.0018)	(0.0022)	(0.0005)
Bu2	0.0252	0.0249	**0.0201**	0.0278	0.0212
	(0.0014)	(0.0018)	(0.0042)	(0.0019)	(0.0005)
Bu3	0.1063	0.0942	**0.0521**	0.0635	0.0594
	(0.0019)	(0.0009)	(0.0006)	(0.0006)	(0.0003)
Fr1	0.0655	0.0504	0.0377	0.0393	**0.0313**
	(0.0017)	(0.0007)	(0.0065)	(0.0005)	(0.0004)
Fr2	0.0674	0.0514	0.0347	0.0397	**0.0315**
	(0.0017)	(0.0007)	(0.0018)	(0.0006)	(0.0004)
T2	0.0489	0.0358	0.0228	**0.0211**	0.0235
	(0.0012)	(0.0005)	(0.0004)	(0.0004)	(0.0002)
Average	0.0601	0.0462	**0.0310**	0.0346	0.0321
	(0.0006)	(0.0005)	(0.0013)	(0.0005)	(0.0002)

Gini (n=10000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0704	0.0446	**0.0360**	0.0379	0.0382
	(0.002)	(0.0008)	(0.0005)	(0.0007)	(0.0004)
Bu1	0.0199	0.0187	**0.0183**	0.0192	0.0193
	(0.0008)	(0.0014)	(0.0003)	(0.0015)	(0.0003)
Bu2	0.0243	**0.0195**	0.0211	0.0204	0.0199
	(0.001)	(0.0011)	(0.0043)	(0.0012)	(0.0003)
Bu3	0.1059	0.0937	**0.0519**	0.0667	0.0589
	(0.0013)	(0.0006)	(0.0004)	(0.0005)	(0.0002)
Fr1	0.0668	0.0506	0.0429	0.0431	**0.0408**
	(0.0015)	(0.0005)	(0.0003)	(0.0004)	(0.0003)
Fr2	0.0661	0.0504	0.0438	0.0428	**0.0406**
	(0.0013)	(0.0005)	(0.0003)	(0.0004)	(0.0003)
T2	0.0482	0.0359	**0.0232**	0.0287	0.0235
	(0.0008)	(0.0004)	(0.0003)	(0.0003)	(0.0002)
Average	0.0574	0.0448	**0.0339**	0.0370	0.0345
	(0.0005)	(0.0003)	(0.0006)	(0.0003)	(0.0001)

Table 7.3: Relative Impact of Contamination on Gini Measure

Discussion

We see from Table 7.2 that the MSEs for Gini are smaller for most of the semi-parametric estimators developed in this study (SPGPD, SPPa and SPPPD). These methods outperform both the nonparametric method and the semi-parametric method by Cowell and Flachaire [10]. The fact that the standard errors are quite small shows that the estimation methods are quite accurate.

Table 7.3 shows that semi-parametric estimators of the inequality measures are less sensitive to data contamination compared to the nonparametric estimators. The results confirm the conclusions reached by Cowell and Flachaire [10] for the measures they considered in their study. A comparison of their estimation method to the methods developed in this research shows similarities in a number of cases, but a clear improvement over their methods is seen throughout the simulation study in that the smallest RICs are obtained for SPGPD, SPPa and SPPPD in most cases. Once again the standard errors are quite small, showing the accuracy of the estimation procedures.

From Tables 7.2 and 7.3 we conclude that the semi-parametric estimators developed in this research for the Gini measure of inequality are reliable, in that not only do they perform well in terms of the MSEs, but also that they are less sensitive to outliers than the nonparametric methods and the Cowell and Flachaire [10] method. This provides the users with more possibilities in assessing the inequality in income, even though there exists some complexity in the computation of SPGPD, SPPa and SPPPD compared to nonparametric methods.

The above observations apply also to the estimates for the generalized entropy (see Tables D.1 to D.3, Tables E.1 to E.3), the Atkinson (see Tables D.4 to D.6, Tables E.4 to E.6) and the QSR (see Table D.7 and Table E.7) measures of inequality.

7.2 Confidence Intervals

In the previous section we studied the mean squared errors of the estimators of inequality measures and considered their sensitivity to outliers. The next step is to construct confidence intervals for these measures. In this section we study the properties of different confidence intervals described in Chapter 6, via simulation. As before, we generate samples of size 500, 1000, 5000 and 10000 from the aforementioned distributions (Pareto, Burr, Fréchet, Student t) and apply the methods discussed in Chapter 6 to obtain approximate confidence intervals. In particular, we consider the coverage probabilities, the rate of missing left endpoint (lower non-coverage probability), the rate of missing right endpoint (upper non-coverage probability) and the average confidence interval length. We denote by I an inequality measure, by $\widehat{I}_{NP}$ its nonparametric estimator and by $\widehat{I}_{SP}$ its semi-parametric estimator. Below is the complete list of confidence intervals that are constructed for each measure of inequality.

1. Standard confidence interval (denoted by SNI):

$$[I_{low}, I_{up}] = \left[\widehat{I}_{NP} - z^{(1-\alpha/2)} \cdot \widehat{se}, \widehat{I}_{NP} - z^{(\alpha/2)} \cdot \widehat{se}\right].$$

2. Student t interval (denoted by STI): Instead of the normal percentiles $z^{(\alpha)}$ in the previous cases, we use the Student t percentiles $t_{n-1}^{(\alpha)}$.

3. Bootstrap percentile interval (denoted by BPI): It is constructed using Algorithm 6.2.

4. Power transformation based bootstrap percentile interval (denoted by PTBPI): Algorithm 6.2 is used to obtain a confidence interval for $T = I^{-c}$. Denoting the interval by $[T_{lo}, T_{up}]$, the corresponding confidence interval for I is given by

$$[I_{low}, I_{up}] = \left[T_{up}^{-1/c}, T_{lo}^{-1/c}\right].$$

5. Bootstrap t interval (denoted by BTI): It is constructed using Algorithm 6.3.

6. Bootstrap calibrated and accelerated intervals (denoted by BCAI): It is constructed as described in Section 6.4.4.

7. Power transformation based bootstrap calibrated and accelerated interval (denoted by PTBCAI): The method in Section 6.4.4 is used to construct $[T_{lo}, T_{up}]$ for $T = I^{-c}$. Applying a transformation leads to a confidence interval

$$[I_{low}, I_{up}] = \left[T_{up}^{-1/c}, T_{lo}^{-1/c}\right]$$

for I.

8. Bootstrap percentile SPGPD interval (denoted by BPGPDI): Algorithm 6.2 is applied to the semi-parametric estimator of the measure when fitting the GPD in the tails.

9. Bootstrap percentile SPPa interval (denoted by BPPI): Algorithm 6.2 is applied to the semi-parametric estimator of the measure when fitting the strict Pareto in the tails.

10. Bootstrap t SPGPD interval (denoted by BTGPDI): Algorithm 6.3 is applied to the semi-parametric estimator of the measure when fitting the GPD in the tails.

11. Bootstrap t SPPa interval (denoted by BTPI): Algorithm 6.3 is applied to the semi-parametric estimator of the measure when fitting the strict Pareto in the tails.

Remark 7.2.

1. The asymptotic variance of the estimator $\widehat{T}_{NP}$ for $T = I^{-c}$ is found by using the delta method as follows. Let $g(x) = x^{-c}$. We have $T = g(I)$ and $\widehat{T}_{NP} = g(\widehat{I}_{NP})$. Furthermore we have $E\widehat{I}_{NP} \approx I$. Denoting by σ_I^2 the asymptotic variance of $\widehat{I}_{NP}$, it follows that the asymptotic variance of $\widehat{T}_{NP}$ is given by

$$\sigma_T^2 \approx [g'(I)]^2 \sigma_I^2 = c^2 I^{-2c-2} \sigma_I^2, \tag{7.2}$$

which can be estimated by

$$\widehat{\sigma}_T^2 = c^2 \widehat{I}_{NP}^{-2c-2} \widehat{\sigma}_I^2. \tag{7.3}$$

2. In the methods given from 8 to 11 above, the bootstrap samples are taken as follows. Denote by $X_1, X_2, \ldots, X_n$ the original sample and by $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ the order statistics. Suppose we fit the nonparametric distribution to $n-k$ lower values and a parametric distribution to the remaining k data points. The bootstrap sample consists of taking $n-k$ observations from the sample $(X_{1,n}, X_{2,n}, \ldots, X_{n-k,n})$ with replacement and k observations from the appropriate parametric distribution.

3. The perturbed Pareto distribution involves three parameters that need to satisfy a certain number of conditions to make the estimation possible. This requires using only simulated samples that make possible such conditions. Handling this issue makes the simulation more complicated, especially since we make an extensive use of the bootstrap. Thus we did not include the intervals based on the PPD estimators in our simulation study.

Remark 7.3. Given N estimates of a measure T, the standard error is obtained as follows:

- Split the N values into r identical blocks of size $[N/r]$.
- Find the average estimate for each block.
- Calculate the standard error over the r blocks.

Throughout the simulation we use $r = 10$.

7.2.1 Simulation Results for Confidence Intervals

In this subsection we give in tabular form the confidence interval simulation results for the Gini coefficient. The Coverage Probability (CP in Equation (6.3)) and Average Confidence Interval Length (ACIL in Equation (6.6)) results are discussed in the next subsection. Although we do not discuss the lower and upper non-coverage probabilities (LNCP and UNCP), we include them to indicate to what extent the confidence intervals are skewed. We confine ourselves to giving the results for the Gini. Results for the other measures are given in Appendix F although they will also be discussed in the next subsection. Take note that throughout the simulation we use $B = 1000$ bootstrap samples wherever the bootstrap is used.

		Gini	(Pa, n=500)	
	ACIL	CP	LNCP	UNCP
SNI	0.1949	0.9156	0.0104	0.0740
STI	0.2043	0.9145	0.0179	0.0676
BPI	0.1942	0.9140	0.0200	0.0660
PTBPI	0.1954	0.9225	0.0302	0.0473
BTI	0.3058	0.9116	0.0319	0.0565
BCAI	0.1708	0.9043	0.0233	0.0724
PTBCAI	0.1667	0.9110	0.0283	0.0607
BPGPDI	0.2013	0.9266	0.0263	0.0471
BTGPDI	0.3665	0.9166	0.0431	0.0403
BPPI	0.1635	0.8373	0.0375	0.1252
BTPI	0.3065	0.8654	0.0024	0.1322

		Gini	(Pa, n=1000)	
	ACIL	CP	LNCP	UNCP
SNI	0.1674	0.9241	0.0101	0.0658
STI	0.1725	0.9291	0.0152	0.0557
BPI	0.1604	0.9306	0.0038	0.0656
PTBPI	0.1596	0.9372	0.0213	0.0415
BTI	0.2551	0.9271	0.0251	0.0478
BCAI	0.1458	0.9157	0.0248	0.0595
PTBCAI	0.1463	0.9015	0.0341	0.0644
BPGPDI	0.1661	0.9615	0.0121	0.0264
BTGPDI	0.2941	0.9227	0.0392	0.0381
BPPI	0.0987	0.9475	0.0030	0.0495
BTPI	0.0714	0.9321	0.0200	0.0479

		Gini	(Pa, n=5000)	
	ACIL	CP	LNCP	UNCP
SNI	0.1110	0.9370	0.0049	0.0581
STI	0.1101	0.9414	0.0037	0.0549
BPI	0.1022	0.9394	0.0012	0.0594
PTBPI	0.1078	0.9437	0.0012	0.0551
BTI	0.1786	0.9199	0.0311	0.0490
BCAI	0.0970	0.9113	0.0320	0.0567
PTBCAI	0.1001	0.9045	0.0314	0.0641
BPGPDI	0.0824	0.9520	0.0460	0.0020
BTGPDI	0.2261	0.9456	0.0024	0.0520
BPPI	0.0325	0.9351	0.0010	0.0639
BTPI	0.0270	0.9516	0.0030	0.0454

		Gini	(Pa, n=10000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0892	0.9487	0.0036	0.0477
STI	0.0927	0.9415	0.0037	0.0548
BPI	0.0867	0.9493	0.0203	0.0304
PTBPI	0.0849	0.9499	0.0024	0.0477
BTI	0.1377	0.9138	0.0371	0.0491
BCAI	0.0808	0.9093	0.0355	0.0552
PTBCAI	0.0803	0.9062	0.0402	0.0536
BPGPDI	0.0563	0.9560	0.0230	0.0210
BTGPDI	0.0690	0.9480	0.0100	0.0420
BPPI	0.0246	0.9557	0.0220	0.0223
BTPI	0.0211	0.9415	0.0027	0.0558

Table 7.4: Properties of Confidence Intervals for Gini when Samples come from Pa

		Gini	(Bu1, n=500)	
	ACIL	CP	LNCP	UNCP
SNI	0.1185	0.9009	0.0310	0.0681
STI	0.1225	0.9031	0.0320	0.0649
BPI	0.1210	0.9207	0.0112	0.0681
PTBPI	0.1210	0.9327	0.0300	0.0373
BTI	0.1606	0.9148	0.0402	0.0450
BCAI	0.1016	0.9097	0.0276	0.0627
PTBCAI	0.1013	0.9169	0.0235	0.0596
BPGPDI	0.1303	0.9498	0.0362	0.0140
BTGPDI	0.2131	0.9316	0.0441	0.0243
BPPI	0.1556	0.8748	0.0325	0.0927
BTPI	0.1292	0.9320	0.0300	0.0380

		Gini	(Bu1, n=1000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0996	0.9388	0.0120	0.0492
STI	0.1021	0.9287	0.0194	0.0519
BPI	0.0974	0.9415	0.0049	0.0536
PTBPI	0.0975	0.9479	0.0200	0.0321
BTI	0.1333	0.9248	0.0317	0.0435
BCAI	0.0854	0.9195	0.0312	0.0493
PTBCAI	0.0854	0.9195	0.0249	0.0556
BPGPDI	0.1084	0.9297	0.0683	0.0020
BTGPDI	0.1680	0.9321	0.0357	0.0322
BPPI	0.1516	0.9409	0.0200	0.0391
BTPI	0.1410	0.9585	0.0110	0.0305

		Gini	(Bu1, n=5000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0621	0.9436	0.0170	0.0394
STI	0.0620	0.9574	0.0046	0.0380
BPI	0.0587	0.9556	0.0023	0.0421
PTBPI	0.0616	0.9528	0.0024	0.0448
BTI	0.0828	0.9287	0.0345	0.0368
BCAI	0.0538	0.9125	0.0426	0.0449
PTBCAI	0.0557	0.9217	0.0306	0.0477
BPGPDI	0.0503	0.9580	0.0220	0.0200
BTGPDI	0.0616	0.9580	0.0220	0.0200
BPPI	0.0982	0.9504	0.0240	0.0256
BTPI	0.0990	0.9519	0.0200	0.0281

		Gini	(Bu1, n=10000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0491	0.9527	0.0145	0.0328
STI	0.0505	0.9574	0.0058	0.0368
BPI	0.0484	0.9607	0.0012	0.0381
PTBPI	0.0476	0.9622	0.0023	0.0355
BTI	0.0630	0.9289	0.0357	0.0354
BCAI	0.0436	0.9158	0.0421	0.0421
PTBCAI	0.0433	0.9241	0.0365	0.0394
BPGPDI	0.0338	0.9599	0.0303	0.0098
BTGPDI	0.0303	0.9590	0.0120	0.0290
BPPI	0.0691	0.9609	0.0081	0.0310
BTPI	0.0675	0.9517	0.0250	0.0233

Table 7.5: Properties of Confidence Intervals for Gini when Samples come from Bu1

	Gini	(Bu2, n=500)		
	ACIL	CP	LNCP	UNCP
SNI	0.1640	0.8806	0.0222	0.0972
STI	0.1684	0.8858	0.0296	0.0846
BPI	0.1668	0.9081	0.0400	0.0519
PTBPI	0.1666	0.8944	0.0500	0.0556
BTI	0.2596	0.8926	0.0349	0.0725
BCAI	0.1427	0.8829	0.0155	0.1016
PTBCAI	0.1391	0.8802	0.0223	0.0975
BPGPDI	0.1755	0.9263	0.0300	0.0437
BTGPDI	0.5369	0.9289	0.0194	0.0517
BPPI	0.1333	0.8902	0.0859	0.0239
BTPI	0.1081	0.9278	0.0212	0.0510

	Gini	(Bu2, n=1000)		
	ACIL	CP	LNCP	UNCP
SNI	0.1435	0.8971	0.0201	0.0828
STI	0.1474	0.8940	0.0215	0.0845
BPI	0.1398	0.9061	0.0041	0.0898
PTBPI	0.1401	0.9102	0.0400	0.0498
BTI	0.2262	0.9010	0.0317	0.0673
BCAI	0.1233	0.8907	0.0085	0.1008
PTBCAI	0.1229	0.8916	0.0276	0.0808
BPGPDI	0.1485	0.9472	0.0200	0.0328
BTGPDI	0.2467	0.9371	0.0211	0.0418
BPPI	0.1329	0.9643	0.0197	0.0160
BTPI	0.1107	0.9343	0.0185	0.0472

	Gini	(Bu2, n=5000)		
	ACIL	CP	LNCP	UNCP
SNI	0.1005	0.9123	0.0118	0.0759
STI	0.1002	0.9110	0.0118	0.0772
BPI	0.0936	0.9176	0.0400	0.0424
PTBPI	0.0973	0.9281	0.0013	0.0706
BTI	0.1631	0.8967	0.0344	0.0689
BCAI	0.0864	0.9079	0.0249	0.0672
PTBCAI	0.0891	0.9108	0.0229	0.0663
BPGPDI	0.0830	0.9550	0.0300	0.0150
BTGPDI	0.1622	0.9437	0.0512	0.0051
BPPI	0.0914	0.9682	0.0308	0.0010
BTPI	0.0812	0.9312	0.0520	0.0168

	Gini	(Bu2, n=10000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0838	0.9226	0.0114	0.0660
STI	0.0866	0.9162	0.0116	0.0722
BPI	0.0817	0.9258	0.0300	0.0442
PTBPI	0.0796	0.9343	0.0300	0.0357
BTI	0.1322	0.9004	0.0353	0.0643
BCAI	0.0747	0.9141	0.0301	0.0558
PTBCAI	0.0726	0.9152	0.0324	0.0524
BPGPDI	0.0587	0.9550	0.0420	0.0030
BTGPDI	0.0979	0.9538	0.0412	0.0050
BPPI	0.0647	0.9170	0.0430	0.0400
BTPI	0.0591	0.9558	0.0191	0.0251

Table 7.6: Properties of Confidence Intervals for Gini when Samples come from Bu2

	Gini	(Bu3, n=500)		
	ACIL	CP	LNCP	UNCP
SNI	0.1119	0.9351	0.0234	0.0415
STI	0.1183	0.9305	0.0279	0.0416
BPI	0.1131	0.9375	0.0222	0.0403
PTBPI	0.1141	0.9338	0.0345	0.0317
BTI	0.1414	0.9314	0.0382	0.0304
BCAI	0.1005	0.9243	0.0390	0.0367
PTBCAI	0.1021	0.9224	0.0510	0.0266
BPGPDI	0.1179	0.9360	0.0320	0.0320
BTGPDI	0.1443	0.9100	0.0480	0.0420
BPPI	0.0787	0.9193	0.0300	0.0507
BTPI	0.0696	0.9398	0.0241	0.0361

	Gini	(Bu3, n=1000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0877	0.9390	0.0256	0.0354
STI	0.0907	0.9393	0.0266	0.0341
BPI	0.0868	0.9433	0.0189	0.0378
PTBPI	0.0864	0.9525	0.0222	0.0253
BTI	0.1057	0.9452	0.0318	0.0230
BCAI	0.0796	0.9220	0.0502	0.0278
PTBCAI	0.0808	0.9289	0.0433	0.0278
BPGPDI	0.0901	0.9440	0.0260	0.0300
BTGPDI	0.1444	0.9478	0.0502	0.0020
BPPI	0.0574	0.9094	0.0403	0.0503
BTPI	0.0519	0.9442	0.0200	0.0358

	Gini	(Bu3, n=5000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0473	0.9474	0.0243	0.0283
STI	0.0477	0.9441	0.0265	0.0294
BPI	0.0459	0.9575	0.0219	0.0206
PTBPI	0.0480	0.9531	0.0186	0.0283
BTI	0.0584	0.9337	0.0457	0.0206
BCAI	0.0425	0.9219	0.0528	0.0253
PTBCAI	0.0431	0.9264	0.0541	0.0195
BPGPDI	0.0372	0.9510	0.0280	0.0210
BTGPDI	0.0413	0.9560	0.0340	0.0100
BPPI	0.0297	0.9567	0.0302	0.0131
BTPI	0.0263	0.9594	0.0300	0.0106

	Gini	(Bu3, n=10000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0355	0.9675	0.0143	0.0182
STI	0.0359	0.9676	0.0153	0.0171
BPI	0.0356	0.9697	0.0143	0.0160
PTBPI	0.0348	0.9667	0.0185	0.0148
BTI	0.0387	0.9273	0.0556	0.0171
BCAI	0.0314	0.9236	0.0370	0.0394
PTBCAI	0.0319	0.9337	0.0468	0.0195
BPGPDI	0.0263	0.9640	0.0160	0.0200
BTGPDI	0.0223	0.9778	0.0202	0.0020
BPPI	0.0221	0.9620	0.0140	0.0240
BTPI	0.0291	0.9582	0.0204	0.0214

Table 7.7: Properties of Confidence Intervals for Gini when Samples come from Bu3

		Gini	(Fr1, n=500)	
	ACIL	CP	LNCP	UNCP
SNI	0.1202	0.9464	0.0234	0.0302
STI	0.1184	0.9497	0.0200	0.0303
BPI	0.1155	0.9464	0.0220	0.0316
PTBPI	0.1120	0.9491	0.0205	0.0304
BTI	0.1484	0.9348	0.0348	0.0304
BCAI	0.1031	0.9113	0.0478	0.0409
PTBCAI	0.1031	0.9239	0.0420	0.0341
BPGPDI	0.1242	0.9537	0.0443	0.0020
BTGPDI	0.2244	0.9462	0.0497	0.0041
BPPI	0.0757	0.9246	0.0722	0.0032
BTPI	0.0649	0.9412	0.0543	0.0045

		Gini	(Fr1, n=1000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0926	0.9513	0.0233	0.0254
STI	0.0916	0.9589	0.0133	0.0278
BPI	0.0889	0.9663	0.0034	0.0303
PTBPI	0.0920	0.9640	0.0056	0.0304
BTI	0.1214	0.9421	0.0350	0.0229
BCAI	0.0791	0.9232	0.0413	0.0355
PTBCAI	0.0811	0.9238	0.0471	0.0291
BPGPDI	0.0925	0.9560	0.0420	0.0020
BTGPDI	0.1086	0.9759	0.0221	0.0020
BPPI	0.0553	0.9540	0.0250	0.0210
BTPI	0.0493	0.9660	0.0180	0.0160

		Gini	(Fr1, n=5000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0484	0.9557	0.0221	0.0222
STI	0.0493	0.9674	0.0143	0.0183
BPI	0.0467	0.9616	0.0155	0.0229
PTBPI	0.0480	0.9708	0.0132	0.0160
BTI	0.0560	0.9297	0.0461	0.0242
BCAI	0.0428	0.9309	0.0461	0.0230
PTBCAI	0.0432	0.9294	0.0499	0.0207
BPGPDI	0.0387	0.9500	0.0300	0.0200
BTGPDI	0.0366	0.9760	0.0210	0.0030
BPPI	0.0291	0.9560	0.0130	0.0310
BTPI	0.0253	0.9580	0.0230	0.0190

		Gini	(Fr1, n=10000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0373	0.9684	0.0122	0.0194
STI	0.0364	0.9640	0.0165	0.0195
BPI	0.0362	0.9675	0.0154	0.0171
PTBPI	0.0368	0.9723	0.0117	0.0160
BTI	0.0408	0.9338	0.0456	0.0206
BCAI	0.0320	0.9382	0.0412	0.0206
PTBCAI	0.0323	0.9317	0.0476	0.0207
BPGPDI	0.0271	0.9620	0.0360	0.0020
BTGPDI	0.0234	0.9720	0.0160	0.0120
BPPI	0.0221	0.9620	0.0130	0.0250
BTPI	0.0243	0.9680	0.0170	0.0150

Table 7.8: Properties of Confidence Intervals for Gini when Samples come from Fr1

		Gini	(Fr2, n=500)	
	ACIL	CP	LNCP	UNCP
SNI	0.1573	0.9491	0.0059	0.0450
STI	0.1533	0.9482	0.0082	0.0436
BPI	0.1485	0.9523	0.0200	0.0277
PTBPI	0.1436	0.9485	0.0024	0.0491
BTI	0.2185	0.9306	0.0312	0.0382
BCAI	0.1324	0.9065	0.0369	0.0566
PTBCAI	0.1298	0.9175	0.0347	0.0478
BPGPDI	0.1596	0.9256	0.0463	0.0281
BTGPDI	0.2077	0.9379	0.0314	0.0307
BPPI	0.0997	0.9494	0.0259	0.0247
BTPI	0.0930	0.9419	0.0247	0.0334

		Gini	(Fr2, n=1000)	
	ACIL	CP	LNCP	UNCP
SNI	0.1264	0.9585	0.0046	0.0369
STI	0.1246	0.9570	0.0035	0.0395
BPI	0.1195	0.9524	0.0012	0.0464
PTBPI	0.1233	0.9505	0.0059	0.0436
BTI	0.1871	0.9370	0.0326	0.0304
BCAI	0.1062	0.9263	0.0273	0.0464
PTBCAI	0.1074	0.9176	0.0389	0.0435
BPGPDI	0.1228	0.9576	0.0404	0.0020
BTGPDI	0.2253	0.9456	0.0263	0.0281
BPPI	0.0605	0.9576	0.0200	0.0224
BTPI	0.0464	0.9498	0.0245	0.0257

		Gini	(Fr2, n=5000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0736	0.9609	0.0100	0.0291
STI	0.0746	0.9613	0.0046	0.0341
BPI	0.0691	0.9596	0.0023	0.0381
PTBPI	0.0708	0.9675	0.0022	0.0303
BTI	0.1022	0.9292	0.0354	0.0354
BCAI	0.0647	0.9249	0.0410	0.0341
PTBCAI	0.0643	0.9235	0.0411	0.0354
BPGPDI	0.0563	0.9640	0.0260	0.0100
BTGPDI	0.0697	0.9764	0.0116	0.0120
BPPI	0.0279	0.9637	0.0253	0.0110
BTPI	0.0237	0.9667	0.0203	0.0130

		Gini	(Fr2, n=10000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0590	0.9648	0.0023	0.0329
STI	0.0579	0.9649	0.0023	0.0328
BPI	0.0564	0.9687	0.0022	0.0291
PTBPI	0.0572	0.9681	0.0066	0.0253
BTI	0.0797	0.9233	0.0451	0.0316
BCAI	0.0514	0.9355	0.0328	0.0317
PTBCAI	0.0513	0.9208	0.0464	0.0328
BPGPDI	0.0376	0.9760	0.0140	0.0100
BTGPDI	0.0324	0.9640	0.0160	0.0200
BPPI	0.0217	0.9680	0.0210	0.0110
BTPI	0.0188	0.9720	0.0110	0.0170

Table 7.9: Properties of Confidence Intervals for Gini when Samples come from Fr2

Gini (T2, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.1043	0.9431	0.0278	0.0291
STI	0.1072	0.9446	0.0289	0.0265
BPI	0.1025	0.9435	0.0223	0.0342
PTBPI	0.1089	0.9476	0.0233	0.0291
BTI	0.1288	0.9277	0.0445	0.0278
BCAI	0.0909	0.9309	0.0362	0.0329
PTBCAI	0.0936	0.9261	0.0398	0.0341
BPGPDI	0.1117	0.9496	0.0384	0.0120
BTGPDI	0.1699	0.9440	0.0240	0.0320
BPPI	0.0710	0.9499	0.0261	0.0240
BTPI	0.0529	0.9425	0.0294	0.0281

Gini (T2, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.0836	0.9530	0.0178	0.0292
STI	0.0841	0.9568	0.0166	0.0266
BPI	0.0802	0.9564	0.0145	0.0291
PTBPI	0.0808	0.9654	0.0055	0.0291
BTI	0.0957	0.9252	0.0458	0.0290
BCAI	0.0712	0.9155	0.0604	0.0241
PTBCAI	0.0739	0.9357	0.0340	0.0303
BPGPDI	0.0847	0.9640	0.0160	0.0200
BTGPDI	0.1093	0.9660	0.0320	0.0020
BPPI	0.0450	0.9619	0.0171	0.0210
BTPI	0.0360	0.9686	0.0193	0.0121

Gini (T2, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.0449	0.9667	0.0195	0.0138
STI	0.0438	0.9606	0.0211	0.0183
BPI	0.0422	0.9606	0.0176	0.0218
PTBPI	0.0434	0.9730	0.0076	0.0194
BTI	0.0468	0.9419	0.0398	0.0183
BCAI	0.0390	0.9318	0.0429	0.0253
PTBCAI	0.0397	0.9301	0.0470	0.0229
BPGPDI	0.0352	0.9680	0.0220	0.0100
BTGPDI	0.0311	0.9640	0.0160	0.0200
BPPI	0.0220	0.9680	0.0210	0.0110
BTPI	0.0286	0.9645	0.0151	0.0204

Gini (T2, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.0343	0.9685	0.0132	0.0183
STI	0.0343	0.9685	0.0132	0.0183
BPI	0.0322	0.9653	0.0175	0.0172
PTBPI	0.0326	0.9768	0.0095	0.0137
BTI	0.0347	0.9406	0.0400	0.0194
BCAI	0.0297	0.9581	0.0201	0.0218
PTBCAI	0.0299	0.9568	0.0214	0.0218
BPGPDI	0.0333	0.9700	0.0200	0.0100
BTGPDI	0.0380	0.9699	0.0150	0.0151
BPPI	0.0240	0.9610	0.0250	0.0140
BTPI	0.0236	0.9605	0.0171	0.0224

Table 7.10: Properties of Confidence Intervals for Gini when Samples come from T2

7.2.2 Discussion of Simulation Results for Confidence Intervals

In this subsection we discuss the confidence interval simulation results for the Gini, GE, Atkinson and QSR measures of inequality.

Gini Coefficient

1. Pareto distribution (Pa): See Table 7.4 above. The performance is unsatisfactory for samples of size 500 in terms of the average confidence interval lengths (ACIL) and coverage probabilities (CP). For samples of size 1000, the BPGPDI and BPPI are quite good in that the CPs are about the desired coverage of 95%. As the sample size increases (see e.g. $n = 5000$ and $n = 10000$), the intervals based on semi-parametric estimators (BPGPDI, BTGPDI, BPPI and BTPI) outperform the usual intervals in that they have reasonable ACILs and the CPs are more or less around the desired coverage of 95%.

2. Samples from the Burr distributions (Bu1, Bu2 and Bu3): See Tables 7.5 to 7.7 above. The performance is unsatisfactory for small samples, but as the sample size increases, the intervals based on semi-parametric estimators perform quite well. The bootstrap t and the BCa intervals perform poorly compared to other intervals.

3. Fréchet distributions (Fr1 and Fr2): See Tables 7.8 and 7.9 above. Once again the intervals based on semi-parametric estimators perform better than other intervals in terms of ACILs as well as CPs in most cases. Apart from BTI, BCAI and PTBCAI, all the intervals seem to do well for samples less than 1000 in terms of CPs, but with larger ACILs.

4. Student t distribution (T2): See Table 7.10 above. The results are similar to those obtained in the case of Fr1 and Fr2.

Generalized Entropy with Parameter 0 (GE0)

We now consider Tables F.1 to F.7 in Appendix F.

1. Pareto distribution (Pa): See Table F.1. We can see that the ACILs are larger than those for Gini. As the sample size increases the intervals based on semi-parametric estimators (BPGPDI, BTGPDI, BPPI and BTPI) outperform the usual intervals in that they have reasonable ACILs and the CPs are more or less around the desired coverage of 95%. We also see that the bootstrap t interval (BTI) performs quite well.

2. Burr distributions (Bu1, Bu2 and Bu3): See Tables F.2 to F.4. The performance is unsatisfactory for small samples, but as the sample size increases, the intervals based on semi-parametric estimators perform quite well in terms of ACILs and CPs (see e.g. $n = 10000$). The bootstrap t intervals perform well even for small samples. Also, notice that the power transformation gives improvement over the nonparametric method in the case of the BCa intervals (compare BCAI and PTBCAI in the tables).

3. Fréchet distributions (Fr1 and Fr2): See Tables F.5 and F.6. The intervals based on semi-parametric estimators perform better than other intervals in terms of ACILs as well as CPs in most cases for large samples, with BTGPDI, BPPI and BTPI being the overall best ones. We also see that some of the usual methods (e.g. SNI, STI, BTI) have good coverage probabilities but their ACILs are higher than the intervals based on semi-parametric estimators.

4. Student t distribution (T2): See Table F.7. Once again the intervals based on semi-parametric estimators outperform other intervals in that the CPs are more or less around the desired coverage of 95%. Moreover, the power transformation gives an improvement of the BCa and the bootstrap percentile intervals (compare BPI and PTBPI, BCAI and PTBCAI) for most of the sample sizes.

Generalized Entropy with Parameter 1 (GE1)

We now consider Tables F.8 to F.14 in Appendix F.

1. Pareto distribution (Pa): See Table F.8. The confidence intervals based on semi-parametric estimators (BPGPDI, BTGPDI BPPI and BTPI) outperform the intervals based on the usual methods for large samples. See e.g. $n = 5000$ an $n = 10000$.

2. Burr distributions (Bu1, Bu2 and Bu3): See Tables F.9 to F.11. The results are similar to those of GE0 and thus show that the intervals based on semi-parametric estimators outperform other intervals, especially for large samples.

3. Fréchet distributions (Fr1 and Fr2): See Tables F.12 and F.13. The intervals based on semi-parametric estimators perform better than other intervals in terms of ACILs as well as CPs in most cases for large samples. We also see that although some of the usual methods yield acceptable coverage probabilities, their ACILs are considerably larger than the intervals based on semi-parametric estimators.

4. Student t distribution (T2): See Table F.14. The intervals based on semi-parametric estimators outperform the other intervals.

Generalized Entropy with Parameter 1.3 (GE1.3)

We now consider Tables F.15 to F.21 in Appendix F.

1. Pareto distribution (Pa): See Table F.15. The performance is unsatisfactory for samples of size 500 in terms of the average confidence interval lengths (ACIL) and coverage probabilities (CP) for all the intervals. For samples of size 1000, there is an improvement of the SNI, BTI, BPPI and BTPI although the ACILs are higher in the cases of SNI and BTI. As the sample size increases the intervals based on semi-parametric estimators (BPGPDI, BTGPDI, BPPI and BTPI) outperform the usual intervals in that they have reasonable ACILs and the CPs are more or less around the desired coverage of 95%.

2. Burr distributions (Bu1, Bu2 and Bu3): See Tables F.16 to F.18. The performance is unsatisfactory for small samples, but as the sample size increases, the intervals based on semi-parametric estimators perform quite well in terms of ACILs and CPs.

3. Fréchet distributions (Fr1 and Fr2): See Tables F.19 and F.20. The intervals based on semi-parametric estimators perform better than other intervals in terms of ACILs as well as CPs in most cases for large samples. Note that even in the cases where the CPs are not satisfactory, the ACILs are smaller than the ACILs for the standard methods.

4. Student t distribution (T2): See Table F.21. Once again the intervals based on semi-parametric estimators outperform other intervals in that the CPs are more or less around the desired coverage of 95% for large samples. See e.g. $n = 5000$ and $n = 10000$.

Atkinson Coefficient with Parameter 1 (A1)

We now consider Tables F.22 to F.28 in Appendix F.

1. Pareto distribution (Pa): Table F.22. Most of the intervals perform well for large samples, but the performance is unsatisfactory for samples smaller than 5000. However BPPI and BTPI are quite good for $n = 1000$ and $n = 5000$.

2. Burr distributions (Bu1, Bu2 and Bu3): See Tables F.23 to F.25. The results are similar to the Pareto case. The performance is better for large samples, with the intervals based on semi-parametric estimators being among the best ones. In fact they have reasonably small ACILs in most cases.

3. Fréchet distributions (Fr1 and Fr2): See Tables F.26 and F.27. Again the intervals based on semi-parametric estimators perform quite well for large samples in terms of ACILs as well as CPs in most cases for large samples.

4. Student t distribution (T2): See Table F.28. In this case we see that the intervals based on semi-parametric estimators perform quite well for $n = 500$ and $n = 1000$. For $n = 5000$, the performance is poorer than other intervals such us SNI, BPI and PTBPI. For $n = 10000$, the confidence intervals based on semi-parametric estimators (BPGPDI, BTGPDI, BPPI and BTPI) outperform the intervals based on the usual methods.

Atkinson Coefficient with Parameter 1.5 (A1.5)

We now consider Tables F.29 to F.35 in Appendix F.

1. Pareto distribution (Pa): See Table F.29. All the methods perform unsatisfactorily for small samples. For large samples the confidence intervals based on semi-parametric estimators outperform the intervals based on the usual methods.

2. Burr distributions (Bu1, Bu2 and Bu3): See Tables F.30 to F.32. The performance is unsatisfactory for small samples in all cases, but as the sample size increases, the intervals based on semi-parametric estimators outperform other intervals.

3. Fréchet distributions (Fr1 and Fr2): See Tables F.33 and F.34. The intervals based on semi-parametric estimators perform better than other intervals in terms of ACILs as well as CPs in most cases for large samples, with good results for $n = 10000$.

4. Student t distribution (T2): See Table F.35. The intervals based on semi-parametric estimators perform quite well for $n = 10000$. Also BPI, PTBPI and PTBCAI are quite good for large samples. It is interesting to see that in cases where the semi-parametric methods are not satisfactory, the coverage probabilities are not too far from the desired coverage of 95%.

Atkinson Coefficient with Parameter 2 (A2)

We now consider Tables F.36 to F.41 in Appendix F.

1. Pareto distribution (Pa): Table F.36 shows that the performance is improved as the sample size increases in most cases. Though the CPs seem a bit higher in some cases for the confidence intervals based on semi-parametric estimators (BPGPDI, BTGPDI, BPPI and BTPI), we can see that their performance is acceptable overall.

2. Burr distributions (Bu1, Bu2 and Bu3): See Tables F.37 to F.39. The intervals based on semi-parametric estimators outperform other intervals in terms of ACILs as well as CPs for large samples. The performance is unsatisfactory for small samples across the methods.

3. Fréchet distributions (Fr1 and Fr2): See Tables F.40 and F.41. All the intervals do quite well for $n = 5000$ and $n = 10000$, but the performance is unsatisfactory for $n = 500$. Once again the intervals based on semi-parametric estimators are quite satisfactory in that they have reasonably small ACILs with CPs around the desired coverage of 95%.

Quintile Share Ratio (QSR)

We now consider Tables F.42 to F.47 in Appendix F.

1. Pareto distribution (Pa): See Table F.42. We can see that in all cases, the ACILs are quite large due to the fact that the variance for the estimator of QSR can be quite large. However, the lengths are somewhat shorter in the cases that the CIs are based on semi-parametric estimators. The BPPI and the BTPI particularly do well in terms of coverage probabilities, especially for large samples.

2. Burr distributions (Bu1, Bu2 and Bu3): See Tables F.43 to F.45. Once again the performance is reasonable for large samples in the cases of intervals based on semi-parametric estimators, with BPPI and BTPI having coverage probabilities around the desired coverage of 95%. Although the performance is unsatisfactory for small samples in most cases, the ACILs appear to be smaller for the semi-parametric methods. The ACILs are very large for Bu1 and Bu2. They are much smaller for Bu3. Remember that the EVI for Bu3 is much smaller than that of Bu1 and Bu2.

3. Fréchet distributions (Fr1 and Fr2): See Tables F.46 and F.47. The same as before is noticed, with the semi-parametric methods doing reasonably well in the case of Fr1 for large samples (see e.g. $n = 5000$ and $n = 10000$). Note that the performance of Fr1 is better than that of Fr2. This is partly due to the fact that the EVI of Fr1 is smaller than that of Fr2.

Remark 7.4. The Atkinson measure with parameter 2 and the QSR do not exist for the $T2$ distribution and so they have not been considered for the confidence intervals when simulating from that distribution.

In order to give an idea of the variability of the estimation procedures, we also estimated the standard errors for the coverage probabilities and the average confidence interval lengths for all the measures of inequality, using the method described in Remark 7.3. Once again we confine ourselves to giving the results for the Gini (see Table 7.11 and Table 7.12). Results for other measures are given in Appendix G (Tables G.1 to G.6 for the GE measures, Tables G.7 to G.12 for the Atkinson measures and Tables G.13 and G.14 for the QSR measure).

Gini				Pa Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0201	0.0207	0.0224	0.0224	0.0130	0.0150	0.0197	0.0107	0.0178	0.0489	0.0450	0.0232
1000	0.0210	0.0216	0.0224	0.0218	0.0124	0.0178	0.0150	0.0109	0.0155	0.0204	0.0245	0.0185
5000	0.0164	0.0149	0.0154	0.0151	0.0123	0.0132	0.0131	0.0261	0.0129	0.0146	0.0115	0.0150
10000	0.0155	0.0171	0.0164	0.0154	0.0121	0.0126	0.0111	0.0126	0.0173	0.0125	0.0111	0.0140

Gini				Bu1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0128	0.0136	0.0223	0.0227	0.0125	0.0162	0.0194	0.0221	0.0296	0.0738	0.0797	0.0295
1000	0.0134	0.0139	0.0224	0.0208	0.0119	0.0126	0.0112	0.0254	0.0230	0.0693	0.0819	0.0278
5000	0.0159	0.0149	0.0161	0.0161	0.0120	0.0122	0.0123	0.0179	0.0140	0.0221	0.0231	0.0161
10000	0.0162	0.0153	0.0158	0.0162	0.0119	0.0129	0.0130	0.0120	0.0111	0.0198	0.0149	0.0145

Gini				Bu2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0280	0.0390	0.0304	0.0321	0.0395	0.0353	0.0282	0.0207	0.0201	0.0418	0.0496	0.0332
1000	0.0294	0.0384	0.0281	0.0309	0.0166	0.0203	0.0219	0.0109	0.0235	0.0376	0.0366	0.0267
5000	0.0211	0.0213	0.0237	0.0224	0.0186	0.0202	0.0187	0.0105	0.0191	0.0149	0.0147	0.0187
10000	0.0199	0.0211	0.0127	0.0127	0.0182	0.0198	0.0187	0.0109	0.0172	0.0176	0.0117	0.0164

Gini				Bu3 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0276	0.0274	0.0272	0.0275	0.0185	0.0219	0.0228	0.0173	0.0127	0.0245	0.0258	0.0230
1000	0.0279	0.0279	0.0219	0.0221	0.0191	0.0188	0.0213	0.0192	0.0188	0.0317	0.0320	0.0237
5000	0.0223	0.0215	0.0219	0.0214	0.0178	0.0181	0.0165	0.0131	0.0158	0.0219	0.0184	0.0190
10000	0.0224	0.0220	0.0220	0.0212	0.0188	0.0168	0.0159	0.0202	0.0168	0.0225	0.0175	0.0196

Gini				Fr1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0274	0.0271	0.0278	0.0274	0.0247	0.0237	0.0237	0.0089	0.0219	0.0224	0.0152	0.0227
1000	0.0217	0.0221	0.0274	0.0218	0.0188	0.0246	0.0236	0.0057	0.0182	0.0151	0.0223	0.0201
5000	0.0221	0.0217	0.0215	0.0216	0.0186	0.0161	0.0160	0.0100	0.0037	0.0157	0.0157	0.0166
10000	0.0222	0.0218	0.0222	0.0218	0.0178	0.0185	0.0165	0.0029	0.0058	0.0086	0.0068	0.0150

Gini				Fr2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0269	0.0268	0.0273	0.0272	0.0238	0.0251	0.0220	0.0205	0.0103	0.0273	0.0281	0.0241
1000	0.0268	0.0275	0.0277	0.0279	0.0250	0.0246	0.0245	0.0166	0.0204	0.0280	0.0259	0.0250
5000	0.0273	0.0276	0.0280	0.0274	0.0244	0.0233	0.0229	0.0184	0.0217	0.0141	0.0009	0.0215
10000	0.0276	0.0273	0.0278	0.0272	0.0240	0.0244	0.0230	0.0168	0.0091	0.0144	0.0020	0.0203

Gini				T2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0272	0.0222	0.0274	0.0275	0.0248	0.0229	0.0226	0.0029	0.0138	0.0212	0.0324	0.0223
1000	0.0216	0.0216	0.0269	0.0216	0.0194	0.0230	0.0239	0.0139	0.0152	0.0154	0.0174	0.0200
5000	0.0215	0.0222	0.0218	0.0224	0.0168	0.0178	0.0164	0.0127	0.0147	0.0220	0.0191	0.0189
10000	0.0219	0.0222	0.0219	0.0219	0.0177	0.0222	0.0183	0.0128	0.0068	0.0190	0.0152	0.0182

Table 7.11: Standard Errors for the CPs for Gini Index

Gini				Pa Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0230	0.0231	0.0225	0.0230	0.0377	0.0201	0.0195	0.0212	0.0906	0.0243	0.0668	0.0338
1000	0.0184	0.0177	0.0191	0.0182	0.0345	0.0176	0.0171	0.0172	0.1211	0.0126	0.0094	0.0275
5000	0.0122	0.0119	0.0112	0.0113	0.0257	0.0106	0.0107	0.0090	0.0464	0.0039	0.0033	0.0142
10000	0.0105	0.0105	0.0098	0.0100	0.0193	0.0086	0.0090	0.0076	0.0150	0.0033	0.0027	0.0097
Gini				Bu1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0127	0.0129	0.0139	0.0141	0.0191	0.0119	0.0109	0.0135	0.3341	0.0282	0.0249	0.0451
1000	0.0108	0.0106	0.0113	0.0103	0.0156	0.0094	0.0101	0.0112	0.0258	0.0287	0.0230	0.0152
5000	0.0068	0.0066	0.0064	0.0065	0.0095	0.0059	0.0059	0.0053	0.0125	0.0127	0.0114	0.0081
10000	0.0058	0.0058	0.0057	0.0057	0.0085	0.0047	0.0049	0.0045	0.0099	0.0112	0.0107	0.0070
Gini				Bu2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0188	0.0193	0.0209	0.0198	0.0336	0.0166	0.0160	0.0183	0.0614	0.0194	0.0142	0.0235
1000	0.0167	0.0163	0.0167	0.0161	0.0285	0.0147	0.0148	0.0155	0.2981	0.0173	0.0144	0.0426
5000	0.0118	0.0117	0.0110	0.0111	0.0205	0.0102	0.0102	0.0088	0.0573	0.0088	0.0074	0.0153
10000	0.0100	0.0102	0.0098	0.0097	0.0171	0.0085	0.0085	0.0072	0.0365	0.0076	0.0069	0.0120
Gini				Bu3 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0124	0.0126	0.0122	0.0124	0.0168	0.0110	0.0110	0.0720	0.0679	0.0090	0.0080	0.0223
1000	0.0092	0.0091	0.0094	0.0092	0.0142	0.0087	0.0089	0.0466	0.0704	0.0065	0.0064	0.0181
5000	0.0049	0.0049	0.0048	0.0048	0.0073	0.0043	0.0044	0.0038	0.0043	0.0039	0.0035	0.0046
10000	0.0043	0.0042	0.0041	0.0041	0.0051	0.0035	0.0036	0.0034	0.0040	0.0033	0.0029	0.0039
Gini				Fr1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0128	0.0129	0.0123	0.0123	0.0197	0.0115	0.0117	0.0126	0.0686	0.0074	0.0065	0.0171
1000	0.0097	0.0095	0.0089	0.0088	0.0106	0.0087	0.0087	0.0094	0.0163	0.0056	0.0049	0.0092
5000	0.0047	0.0052	0.0047	0.0047	0.0056	0.0044	0.0044	0.0038	0.0133	0.0029	0.0025	0.0051
10000	0.0043	0.0044	0.0041	0.0041	0.0051	0.0038	0.0039	0.0033	0.0035	0.0025	0.0021	0.0037
Gini				Fr2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0173	0.0170	0.0160	0.0160	0.0295	0.0147	0.0148	0.0162	0.2648	0.0111	0.0081	0.0387
1000	0.0141	0.0136	0.0128	0.0128	0.0181	0.0117	0.0117	0.0129	0.0289	0.0063	0.0052	0.0135
5000	0.0076	0.0083	0.0075	0.0073	0.0111	0.0069	0.0070	0.0056	0.0093	0.0028	0.0024	0.0069
10000	0.0069	0.0070	0.0065	0.0063	0.0101	0.0061	0.0063	0.0049	0.0054	0.0024	0.0021	0.0058
Gini				T2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0113	0.0108	0.0111	0.0112	0.0137	0.0097	0.0100	0.0120	0.0164	0.0072	0.0055	0.0108
1000	0.0095	0.0083	0.0084	0.0083	0.0100	0.0078	0.0075	0.0086	0.0184	0.0046	0.0036	0.0086
5000	0.0044	0.0046	0.0043	0.0044	0.0109	0.0040	0.0039	0.0035	0.0031	0.0022	0.0019	0.0043
10000	0.0039	0.0039	0.0039	0.0039	0.0051	0.0035	0.0033	0.0030	0.0026	0.0019	0.0016	0.0033

Table 7.12: Standard Errors for the ACILs for Gini Index

We can see from these tables that the standard errors are reasonably small, showing that there is not much variation in the estimation of the CP and ACIL for Gini, regardless of the underlying distribution. This shows that the conclusions reached by comparing all the different CI methods are reliable in that the estimation is reasonably accurate. The same applies to all other measures (see Tables G.1 to G.14).

Overall Conclusions

Throughout the simulation results we see that the confidence intervals based on semi-parametric estimators perform very well compared to the intervals based on the usual methods. The results show that the semi-parametric methods yield reasonable confidence interval lengths with quite satisfactory coverage probabilities (more or less around the desired coverage of 95%). In cases where the coverage probability is not very satisfactory, we can see that it is still closer to the desired coverage for the SP methods than for the nonparametric methods for most of the measures, especially for large sample sizes. Over all the measures and all the underlying distributions, the performance is not very satisfactory for small samples. The bootstrap percentile intervals perform quite well in many cases in terms of the coverage probability although the confidence interval lengths are large in some cases. A further improvement is often obtained by applying a power transformation (compare e.g. BPI and PTBPI, BCAI and PTBCAI).

Given the fact that confidence intervals are obtained for complex measures, the proposed procedures do remarkably well. The performance of the bootstrap is also remarkably good given the complexity of the statistics underlying the bootstrap procedures. Furthermore, the fact that the standard errors are reasonably small shows that the estimation procedures are quite accurate and the conclusions drawn are reliable.

Chapter 8

Applications

In the previous chapter we studied the properties of the methods developed for statistical inference for the Gini, the GE, the Atkinson and the quintile share ratio measures of inequality. Such a study was conducted by simulation, generating the data from a number of well known heavy-tailed distributions (Pareto, Burr, Fréchet and Student t). In practice, however, one has to deal with real data sets. In this chapter we illustrate how the techniques proposed can be used in practice. This is done by considering claims data from a South African short term insurer, a Norwegian fire insurance data set, and a sample from the 2005 South African income and expenditure survey data. In the analyses, for completeness, all three the parametric distributions are applied, i.e. GPD, strict Pareto and PPD. Since it is not expected that one parametric distribution will be "best" overall, a method is needed for choosing a distribution best suited for a given data set. This is done in the final section where a measure of representativeness of a sample to a particular distribution is discussed, in order to determine which of the three distributions (GPD, strict Pareto or PPD) gives a better approximation of the tail.

8.1 Description of the Data

1. **Claims data**: The data consist of a portfolio of claims from a South African short term insurer, from 1 July 2004 to 21 July 2006. The dates used, were the dates the claims occurred, and not the dates the claims were registered. The claim amounts were the total claim amounts and ignored any excesses paid by the client. The claim amounts were adjusted for inflation to July 2006 as base month. Finally, any negative or zero claim amounts were deleted from the data sets. Negative amounts occur if, for instance, the value of the items salvaged from the wreck exceeds the claim amount in value. The final sample size is 16104. We will refer to the data set

as Portfolio. This data set is analyzed in Berning [5].

2. **Norwegian fire insurance data**: The data consist of 375 claims paid by Storbränder, a Norwegian fire insurance company, during 1980 (see Berning [5]). We will refer to these data as Nordata.

3. **2005 South African income and expenditure survey data**: The data consist of a sample of size 5000 from the 2005 South African income and expenditure survey data. Note that although the original survey data came from a complex sampling procedure, we took a simple random sample from it. The data set we work with will therefore be considered as a simple random sample from the survey data considered as the population. We will refer to it as IESdata.

The histograms and the boxplots for the data are shown in Figure 8.1, giving an idea of the tails of the distributions.

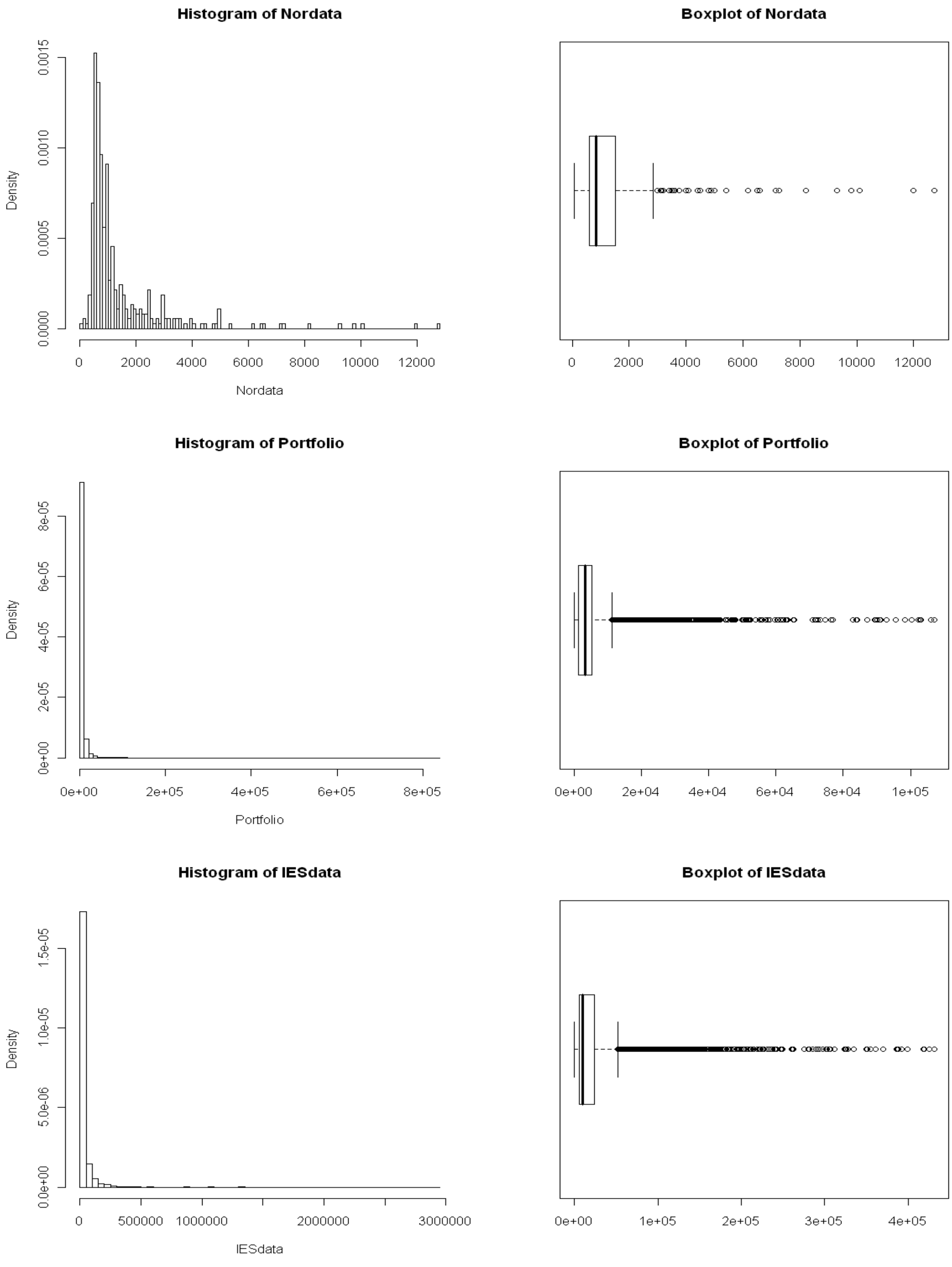

Figure 8.1: Histograms and Boxplots for Nordata, Portfolio and IESdata

We see from Figure 8.1 that all three data sets have heavy tails and so we can use them to illustrate our methods. Table 8.1 gives some descriptive statistics (Median, Median Absolute Deviation (MAD) and maximum value) for each data set.

	Sample Size n	Median	MAD	Maximum
Nordata	374	850.0000	433.6605	12725.0000
Portfolio	16104	3268.5050	2951.3525	835567.7000
IESdata	5000	9823.6330	8763.3317	2910826.1000

Table 8.1: Descriptive Statistics for the Data Sets

These three data sets cover a spectrum of sizes, small (Nordata), medium (IESdata) and large (Portfolio).

8.2 Estimation of Inequality Measures

In this section we estimate the measures of inequality using the data sets described above. For each data set, we estimate the extreme value index (using the Hill estimator with 10% of the data) and we find the nonparametric estimates of the inequality measures under consideration. The results are given in Table 8.2 (we denote by $\widehat{\gamma}$ the estimate of the EVI).

Data	$\hat{\gamma}$	Gini	GE0	GE1	GE1.3	A1	A1.5	A2	QSR
Nordata	0.4858	0.4487	0.3297	0.3930	0.4398	0.2809	0.3619	0.4247	7.8764
Portfolio	0.5895	0.5767	0.7070	0.8174	1.1214	0.5069	0.6884	0.8929	34.8292
IESdata	0.7339	0.6990	0.9661	1.1531	1.5007	0.6194	0.7373	0.8376	40.6089

Table 8.2: EVI and Nonparametric Estimates of Inequality Measures

The estimates of the extreme value index are all less than 1, justifying the use of the data sets for illustration. Recall that the semi-parametric methods were developed under the restriction that the extreme value index is less than 1.

The semi-parametric estimates for the inequality measures using the methods described in Chapter 5 (fitting the GPD, the strict Pareto and the PPD in the tails) are now calculated. We use 10% of the data (upper order statistics) in each case both to estimate the parameters in the tail distribution and to fit that distribution. For comparison purposes we put together both the nonparametric and semi-parametric estimates. The results are given in Table 8.3.

Gini	NP	SPGPD	SPPa	SPPPD
Nordata	0.4487	0.4203	0.4339	0.4671
Portfolio	0.5767	0.5923	0.5145	0.5850
IESdata	0.6990	0.7132	0.6838	0.6860

A1	NP	SPGPD	SPPa	SPPPD
Nordata	0.2809	0.2510	0.3039	0.2969
Portfolio	0.5069	0.5276	0.5034	0.5072
IESdata	0.6194	0.6366	0.6137	0.6177

GE0	NP	SPGPD	SPPa	SPPPD
Nordata	0.3297	0.2890	0.3622	0.2018
Portfolio	0.7070	0.7499	0.6999	0.7453
IESdata	0.9661	1.0122	0.9621	0.9636

A1.5	NP	SPGPD	SPPa	SPPPD
Nordata	0.3619	0.3296	0.3827	0.3759
Portfolio	0.6884	0.6992	0.6862	0.6888
IESdata	0.7373	0.7463	0.7318	0.7395

GE1	NP	SPGPD	SPPa	SPPPD
Nordata	0.3930	2.5062	0.3810	0.3930
Portfolio	0.8174	1.1905	0.7951	0.7234
IESdata	1.1531	1.5312	1.1294	1.1279

A2	NP	SPGPD	SPPa	SPPPD
Nordata	0.4247	0.3931	0.4432	0.4369
Portfolio	0.8929	0.8963	0.8922	0.8931
IESdata	0.8376	0.8429	0.8339	0.8344

GE1.3	NP	SPGPD	SPPa	SPPPD
Nordata	0.4398	0.2259	0.6357	0.5931
Portfolio	1.1214	0.5109	1.2282	1.2940
IESdata	1.5007	1.6626	1.4349	1.4410

QSR	NP	SPGPD	SPPa	SPPPD
Nordata	7.8764	7.0874	8.3567	8.1857
Portfolio	34.8292	36.6990	34.4200	34.9122
IESdata	40.6089	42.4122	41.8292	41.9375

Table 8.3: Nonparametric and Semi-Parametric Estimates of Inequality Measures

We see from Table 8.3 that in some cases an estimate seems to drift away from the others (see e.g. SPGDP for Nordata and Portfolio in the case of GE1.3). This indicates that the corresponding tail distribution is not appropriate for that particular case, emphasizing the necessity to choose the tail distribution which best fits the data. This will be done in the final section using a measure of representativeness of a sample to a particular distribution. Also note that the high values of the estimates are due to the heaviness of the tails.

8.3 Confidence Intervals

In this section we estimate confidence intervals for all the measures using the different data sets. We are particularly interested in comparing the estimated average confidence interval lengths (ACILs) and the estimated coverage probabilities (CPs) in the sense of the simulation carried out in Chapter 7. The confidence intervals considered in the simulation are also considered here.

In order to estimate the CPs and the ACILs, we use the data set as our (bootstrap) population. For this "population" each of the inequality measures is calculated for each method. These are then taken as (bootstrap) population values. For this "population" and these "population values", a simulation study is carried out analogous to that of Chapter 7. We take a (bootstrap) sample from this population and use it to obtain a confidence interval for each (confidence interval) method considered. This process is repeated B times, in order to find B confidence intervals for each method. Based on these B intervals we obtain an estimated average confidence interval length and coverage probability. For the latter, the (bootstrap) population values are used as "true" values. The procedure followed for each method is summarized in Algorithm 8.1 below.

Algorithm 8.1. Denote the data by $\underline{X} = (X_1, X_2, \ldots, X_n)$ (bootstrap population) and let I be a measure of inequality. Estimate I by $\widehat{I}$ and then perform the following steps.

1. Take a bootstrap sample $\underline{X}^\star$ and use a confidence interval method to obtain an interval $(L^\star, U^\star)$.
2. Repeat 1 B times to obtain B confidence intervals $(L^{\star(b)}, U^{\star(b)}), b = 1, 2, \ldots, B$.
3. Calculate $\widehat{ACIL}^\star$ as
$$\widehat{ACIL}^\star = \frac{1}{B} \sum_{b=1}^{B} \left(U^{\star(b)} - L^{\star(b)} \right). \tag{8.1}$$
4. Calculate $\widehat{CP}^\star$ as
$$\widehat{CP}^\star = \frac{1}{B} \sum_{b=1}^{B} \mathbf{1} \left(L^{\star(b)} \leq \widehat{I} \leq U^{\star(b)} \right). \tag{8.2}$$

Remark 8.1. Note that in step 1 in the algorithm, when the confidence interval method is based on the bootstrap, B bootstrap samples are taken from $\underline{X}^\star$ (thus a second level bootstrap). See Chapter 6 for the different confidence interval methods. Recall that in this thesis we use $B = 1000$ in each level of bootstrap.

8.3.1 Results for Confidence Intervals and Discussion

The results are given in Tables 8.4 to 8.11. The values in brackets are the standard errors of the estimators, obtained in the same way as described in Remark 7.3. CP(NP) denotes the CP when using the nonparametric estimate as population value. Similarly, CP(GPD), CP(Pa) and CP(PPD) denote the CP when using the SP estimate with GPD, Pa and PPD in the tail respectively. In the discussion of all the results, we compare the CPs with respect to the estimates used as population values. For example, if the SP PPD estimate is used as population value then we consider the CPs in the last column of each table. We also mention how the performances compare with respect to the population values.

Gini Coefficient

Nordata					
Gini	**ACIL**	**CP (NP)**	**CP(GPD)**	**CP(Pa)**	**CP(PPD)**
SNI	0.0793 (0.0071)	0.9392 (0.0063)	0.9345 (0.0057)	0.9417 (0.0056)	0.9330 (0.0059)
STI	0.0841 (0.0064)	0.9315 (0.0072)	0.8984 (0.0069)	0.9053 (0.0064)	0.8969 (0.0073)
BPI	0.0721 (0.0069)	0.9355 (0.0067)	0.9205 (0.0071)	0.9276 (0.0059)	0.9191 (0.0068)
PTBPI	0.0798 (0.0075)	0.9311 (0.0066)	0.9178 (0.0065)	0.9249 (0.0059)	0.9164 (0.0067)
BTI	0.0709 (0.0073)	0.9227 (0.0064)	0.9088 (0.0065)	0.9158 (0.0057)	0.9074 (0.0065)
BCAI	0.0596 (0.0066)	0.9087 (0.0065)	0.8956 (0.0066)	0.9024 (0.0058)	0.8941 (0.0064)
PTBCAI	0.0712 (0.0072)	0.9283 (0.0062)	0.9238 (0.0067)	0.9309 (0.0055)	0.9224 (0.0063)
BPGPDI	0.0588 (0.0043)	0.9397 (0.0063)	0.9284 (0.0063)	0.9355 (0.0056)	0.9269 (0.0068)
BTGPDI	0.0547 (0.0039)	0.9353 (0.0068)	0.9158 (0.0073)	0.9329 (0.0068)	0.9344 (0.0071)
BPPI	0.0449 (0.0032)	0.9471 (0.0066)	0.9042 (0.0068)	0.9461 (0.0059)	0.9457 (0.0067)
BTPI	0.0507 (0.0033)	0.9457 (0.0067)	0.9283 (0.0071)	0.9354 (0.0062)	0.9368 (0.0076)

Portfolio					
Gini	**ACIL**	**CP (NP)**	**CP(GPD)**	**CP(Pa)**	**CP(PPD)**
SNI	0.0379 (0.0038)	0.9319 (0.0043)	0.9142 (0.0044)	0.9212 (0.0039)	0.9128 (0.0044)
STI	0.0407 (0.0038)	0.8988 (0.0043)	0.9191 (0.0045)	0.9262 (0.0038)	0.9177 (0.0045)
BPI	0.0376 (0.0042)	0.9321 (0.0054)	0.9423 (0.0056)	0.9495 (0.0048)	0.9408 (0.0055)
PTBPI	0.0382 (0.0037)	0.9458 (0.006)	0.9441 (0.0062)	0.9514 (0.0053)	0.9476 (0.0063)
BTI	0.0345 (0.0038)	0.8850 (0.0056)	0.8879 (0.0058)	0.8948 (0.005)	0.8865 (0.0062)
BCAI	0.0319 (0.0034)	0.8847 (0.0039)	0.9179 (0.0041)	0.9250 (0.0035)	0.9165 (0.004)
PTBCAI	0.0382 (0.0032)	0.9418 (0.006)	0.9449 (0.0063)	0.9522 (0.0053)	0.9434 (0.0061)
BPGPDI	0.0173 (0.0038)	0.9442 (0.0045)	0.9519 (0.0046)	0.9593 (0.004)	0.9504 (0.0047)
BTGPDI	0.0292 (0.0073)	0.9377 (0.0045)	0.9470 (0.0046)	0.9543 (0.004)	0.9455 (0.0046)
BPPI	0.0282 (0.0008)	0.9456 (0.0039)	0.9381 (0.0043)	0.9453 (0.0034)	0.9466 (0.004)
BTPI	0.0224 (0.0042)	0.9447 (0.0038)	0.9413 (0.004)	0.9485 (0.0034)	0.9458 (0.0039)

IESdata					
Gini	**ACIL**	**CP (NP)**	**CP(GPD)**	**CP(Pa)**	**CP(PPD)**
SNI	0.0370 (0.0041)	0.9249 (0.0034)	0.9199 (0.0039)	0.9271 (0.0035)	0.9284 (0.004)
STI	0.0305 (0.0043)	0.9318 (0.0032)	0.9358 (0.0033)	0.9332 (0.0034)	0.9343 (0.0033)
BPI	0.0368 (0.0041)	0.9211 (0.0036)	0.9191 (0.0041)	0.9266 (0.0036)	0.9276 (0.004)
PTBPI	0.0375 (0.0042)	0.9328 (0.0032)	0.9476 (0.0041)	0.9352 (0.0035)	0.9361 (0.004)
BTI	0.0325 (0.0036)	0.9097 (0.0038)	0.8884 (0.0044)	0.8951 (0.0034)	0.8917 (0.0029)
BCAI	0.0312 (0.0035)	0.9176 (0.0041)	0.8907 (0.0038)	0.9176 (0.0034)	0.9193 (0.0039)
PTBCAI	0.0375 (0.0042)	0.9305 (0.004)	0.9476 (0.0041)	0.9349 (0.0036)	0.9361 (0.0041)
BPGPDI	0.0230 (0.0026)	0.9496 (0.0033)	0.9357 (0.0027)	0.9453 (0.0023)	0.9441 (0.0026)
BTGPDI	0.0419 (0.0047)	0.9413 (0.0037)	0.9336 (0.0034)	0.9431 (0.0032)	0.9412 (0.0036)
BPPI	0.0323 (0.0041)	0.9496 (0.0031)	0.9536 (0.0038)	0.9509 (0.0032)	0.9521 (0.0037)
BTPI	0.0371 (0.0036)	0.9451 (0.0029)	0.9429 (0.0025)	0.9497 (0.0023)	0.9414 (0.0026)

Table 8.4: Application ACILs and CPs for Gini

Consider Table 8.4 above. We see that the coverage is unsatisfactory in many cases for Nordata. However, the CPs for the confidence intervals based on semi-parametric estimators are close to the desired coverage of 95% (see e.g. BPPI and BTPI in each case). Keep in mind that Nordata is a (unrealistically) small data set for this type of analysis and we would not expect good results. With a power transformation, the bootstrap percentile method PTBPI does well in the case of Portfolio. The ACILs are also reasonably short over the methods, especially those based on semi-parametric estimators. The CPs are very satisfactory for the confidence intervals based on SP methods when using SP estimates as population values (see e.g. BPGPD, BTGPD, BPPI and BTPI in the last three columns for Portfolio and IESdata).

Generalized Entropy with Parameter 0 (GE0)

Nordata					
GE0	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0659	0.9461	0.9439	0.9412	0.9424
	(0.0079)	(0.0033)	(0.0023)	(0.0023)	(0.0024)
STI	0.0666	0.9029	0.9074	0.9144	0.9060
	(0.0086)	(0.0034)	(0.0028)	(0.0026)	(0.003)
BPI	0.0681	0.9280	0.9298	0.9471	0.9284
	(0.0086)	(0.0039)	(0.0029)	(0.0024)	(0.0028)
PTBPI	0.0520	0.9252	0.9271	0.9342	0.9456
	(0.0088)	(0.0029)	(0.0026)	(0.0024)	(0.0027)
BTI	0.0608	0.9150	0.9183	0.9350	0.9265
	(0.008)	(0.0037)	(0.0026)	(0.0023)	(0.0026)
BCAI	0.0648	0.8992	0.9046	0.9316	0.9032
	(0.0076)	(0.004)	(0.0027)	(0.0024)	(0.0026)
PTBCAI	0.0564	0.9317	0.9332	0.9403	0.9517
	(0.0089)	(0.003)	(0.0027)	(0.0022)	(0.0026)
BPGPDI	0.0661	0.9392	0.9477	0.9449	0.9362
	(0.0078)	(0.0023)	(0.0026)	(0.0023)	(0.0028)
BTGPDI	0.0791	0.9219	0.9351	0.9322	0.9236
	(0.0067)	(0.0028)	(0.003)	(0.0028)	(0.0029)
BPPI	0.0717	0.9468	0.9133	0.9403	0.9119
	(0.0041)	(0.0036)	(0.0028)	(0.0024)	(0.0027)
BTPI	0.0627	0.9380	0.9376	0.9478	0.9461
	(0.0087)	(0.0037)	(0.0029)	(0.0025)	(0.0031)

Portfolio					
GE0	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0786	0.9359	0.9235	0.9305	0.9223
	(0.0079)	(0.0039)	(0.0044)	(0.0039)	(0.0045)
STI	0.0787	0.8999	0.9284	0.9356	0.9269
	(0.0081)	(0.0039)	(0.0045)	(0.0039)	(0.0045)
BPI	0.0869	0.9479	0.9518	0.9591	0.9503
	(0.0086)	(0.0047)	(0.0056)	(0.0048)	(0.0055)
PTBPI	0.0879	0.9393	0.9436	0.9510	0.9521
	(0.0088)	(0.0048)	(0.0063)	(0.0054)	(0.0063)
BTI	0.0800	0.8761	0.8969	0.9138	0.8955
	(0.008)	(0.0044)	(0.0058)	(0.005)	(0.0063)
BCAI	0.0744	0.9241	0.9272	0.9343	0.9257
	(0.0076)	(0.0045)	(0.0042)	(0.0035)	(0.0041)
PTBCAI	0.0889	0.9557	0.9445	0.9518	0.9459
	(0.0089)	(0.0048)	(0.0063)	(0.0054)	(0.0062)
BPGPDI	0.0375	0.9466	0.9416	0.9489	0.9478
	(0.0038)	(0.0006)	(0.0047)	(0.004)	(0.0047)
BTGPDI	0.0456	0.9379	0.9366	0.9489	0.9511
	(0.0053)	(0.0041)	(0.0046)	(0.004)	(0.0046)
BPPI	0.0556	0.9466	0.9476	0.9548	0.9461
	(0.0035)	(0.0045)	(0.0043)	(0.0035)	(0.004)
BTPI	0.0447	0.9329	0.9418	0.9581	0.9493
	(0.0042)	(0.0043)	(0.004)	(0.0034)	(0.004)

IESdata					
GE0	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0409	0.9304	0.9307	0.9386	0.9388
	(0.0051)	(0.0023)	(0.0023)	(0.0021)	(0.0024)
STI	0.0464	0.9287	0.9349	0.9321	0.9334
	(0.0058)	(0.0028)	(0.0024)	(0.0021)	(0.0024)
BPI	0.0435	0.9424	0.9385	0.9458	0.9417
	(0.0042)	(0.0026)	(0.0031)	(0.0026)	(0.0023)
PTBPI	0.0482	0.9448	0.9502	0.9477	0.9488
	(0.0057)	(0.0025)	(0.0034)	(0.0029)	(0.0034)
BTI	0.0442	0.9147	0.9032	0.9202	0.9218
	(0.0054)	(0.0022)	(0.0031)	(0.0027)	(0.0034)
BCAI	0.0516	0.9331	0.9237	0.9308	0.9322
	(0.0049)	(0.0023)	(0.0022)	(0.0019)	(0.0022)
PTBCAI	0.0503	0.9267	0.9211	0.9385	0.9389
	(0.0047)	(0.0027)	(0.0034)	(0.0029)	(0.0033)
BPGPDI	0.0459	0.9570	0.9383	0.9457	0.9514
	(0.0042)	(0.0026)	(0.0025)	(0.0022)	(0.0025)
BTGPDI	0.0506	0.9462	0.9432	0.9455	0.9478
	(0.0051)	(0.0022)	(0.0025)	(0.0021)	(0.0024)
BPPI	0.0519	0.9435	0.9412	0.9515	0.9527
	(0.0049)	(0.0021)	(0.0023)	(0.0019)	(0.0021)
BTPI	0.0437	0.9463	0.9404	0.9448	0.9459
	(0.0041)	(0.0023)	(0.0022)	(0.0018)	(0.0021)

Table 8.5: Application ACILs and CPs for GE0

Consider Table 8.5 above. BPPI performs quite well for Nordata when using NP estimate as population value. Interestingly, BPI and PTBCAI perform satisfactorily for Portfolio. The intervals based on SP methods have reasonably short ACILs with CPs not too far from the desired coverage of 95%. This shows that they perform better than the usual methods in most cases, especially when using SP estimates as population values (see e.g. the last three columns of each table).

Generalized Entropy with Parameter 1 (GE1)

Nordata					
GE1	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0429	0.9159	0.9346	0.9227	0.9239
	(0.0086)	(0.005)	(0.0036)	(0.0035)	(0.0036)
STI	0.0409	0.9059	0.8984	0.9165	0.9282
	(0.0094)	(0.0051)	(0.0044)	(0.0039)	(0.0045)
BPI	0.0521	0.9171	0.9106	0.9285	0.9102
	(0.0093)	(0.0057)	(0.0045)	(0.0036)	(0.0042)
PTBPI	0.0563	0.9252	0.9179	0.9359	0.9271
	(0.0096)	(0.0044)	(0.0041)	(0.0036)	(0.0042)
BTI	0.0488	0.9282	0.9092	0.9267	0.9183
	(0.0087)	(0.0054)	(0.0041)	(0.0036)	(0.0041)
BCAI	0.0309	0.9431	0.8956	0.9133	0.8955
	(0.0082)	(0.0059)	(0.0041)	(0.0036)	(0.0039)
PTBCAI	0.0575	0.9260	0.9240	0.9219	0.9330
	(0.0097)	(0.0045)	(0.0042)	(0.0033)	(0.0039)
BPGPDI	0.0527	0.9219	0.9083	0.9264	0.9378
	(0.0085)	(0.0033)	(0.0039)	(0.0035)	(0.0042)
BTGPDI	0.0434	0.9319	0.9258	0.9339	0.9355
	(0.0073)	(0.0042)	(0.0045)	(0.0042)	(0.0045)
BPPI	0.0636	0.9338	0.9043	0.9419	0.9434
	(0.0045)	(0.0054)	(0.0042)	(0.0036)	(0.0042)
BTPI	0.0335	0.9417	0.9383	0.9492	0.9375
	(0.0095)	(0.0054)	(0.0045)	(0.0039)	(0.0048)

Portfolio					
GE1	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0. 0423	0.9359	0.9235	0.9305	0.9223
	(0.0086)	(0.0039)	(0.0044)	(0.0039)	(0.0045)
STI	0. 0430	0.8999	0.9284	0.9356	0.9269
	(0.0088)	(0.0039)	(0.0045)	(0.0039)	(0.0045)
BPI	0.0473	0.9479	0.9518	0.9591	0.9503
	(0.0093)	(0.0047)	(0.0056)	(0.0048)	(0.0055)
PTBPI	0.0504	0.9393	0.9436	0.9510	0.9521
	(0.0096)	(0.0048)	(0.0063)	(0.0054)	(0.0063)
BTI	0.0460	0.8761	0.8969	0.9138	0.8955
	(0.0087)	(0.0044)	(0.0058)	(0.005)	(0.0063)
BCAI	0.0473	0.9241	0.9272	0.9343	0.9257
	(0.0082)	(0.0045)	(0.0042)	(0.0035)	(0.0041)
PTBCAI	0.0456	0.9557	0.9445	0.9518	0.9459
	(0.0097)	(0.0048)	(0.0063)	(0.0054)	(0.0062)
BPGPDI	0.0391	0.9466	0.9416	0.9489	0.9478
	(0.0042)	(0.0006)	(0.0047)	(0.004)	(0.0047)
BTGPDI	0.0469	0.9379	0.9366	0.9489	0.9511
	(0.0058)	(0.0041)	(0.0046)	(0.004)	(0.0046)
BPPI	0.0438	0.9466	0.9476	0.9548	0.9461
	(0.0038)	(0.0045)	(0.0043)	(0.0035)	(0.004)
BTPI	0.0452	0.9329	0.9418	0.9581	0.9493
	(0.0046)	(0.0043)	(0.004)	(0.0034)	(0.004)

IESdata					
GE1	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0471	0.9017	0.9077	0.9357	0.9359
	(0.0035)	(0.0027)	(0.0024)	(0.0022)	(0.0025)
STI	0.0458	0.9272	0.9109	0.9393	0.9306
	(0.0034)	(0.0024)	(0.0025)	(0.0022)	(0.0025)
BPI	0.0491	0.9343	0.9256	0.9429	0.9388
	(0.0036)	(0.0021)	(0.0034)	(0.0028)	(0.0024)
PTBPI	0.0431	0.9359	0.9272	0.9448	0.9459
	(0.0036)	(0.0029)	(0.0037)	(0.0032)	(0.0037)
BTI	0.0414	0.8991	0.8805	0.9174	0.9190
	(0.0032)	(0.0037)	(0.0034)	(0.0029)	(0.0037)
BCAI	0.0461	0.9246	0.9108	0.9280	0.9294
	(0.0033)	(0.0024)	(0.0023)	(0.002)	(0.0023)
PTBCAI	0.0484	0.9359	0.9183	0.9457	0.9461
	(0.0031)	(0.0023)	(0.0037)	(0.0032)	(0.0036)
BPGPDI	0.0414	0.9318	0.9354	0.9528	0.9485
	(0.0029)	(0.0023)	(0.0026)	(0.0023)	(0.0026)
BTGPDI	0.0390	0.9389	0.9277	0.9426	0.9449
	(0.0035)	(0.0028)	(0.0026)	(0.0022)	(0.0025)
BPPI	0.0402	0.9478	0.9483	0.9528	0.9515
	(0.0032)	(0.0021)	(0.0024)	(0.002)	(0.0022)
BTPI	0.0424	0.9419	0.9375	0.9452	0.9489
	(0.0028)	(0.0024)	(0.0023)	(0.0019)	(0.0022)

Table 8.6: Application ACILs and CPs for GE1

Consider Table 8.6 above. The coverage probabilities are unsatisfactory in some of the cases, especially for the Nordata in the NP situations. There is some improvement for the Nordata when using the SP methods. Furthermore, it seems in general that the NP method improves when using SP estimates as population values. The CPs for the SP methods are the closest to the desired coverage when using SP estimates as population values, especially when using Pa and PPD for the latter. The performance is improved for Portfolio and IESdata when using SP estimates as population values. Also note the good performance of BPI and PTBPI in these cases.

Generalized Entropy with Parameter 1.3 (GE1.3)

Nordata					
GE1.3	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0510 (0.0095)	0.9367 (0.0018)	0.9386 (0.0035)	0.9259 (0.0035)	0.9371 (0.0036)
STI	0.0516 (0.0103)	0.8941 (0.0019)	0.9120 (0.0043)	0.9190 (0.0039)	0.9106 (0.0045)
BPI	0.0528 (0.0103)	0.9190 (0.0021)	0.9345 (0.0044)	0.9519 (0.0036)	0.9331 (0.0042)
PTBPI	0.0542 (0.0105)	0.9160 (0.0016)	0.9318 (0.004)	0.9489 (0.0036)	0.9504 (0.0041)
BTI	0.0562 (0.0095)	0.9049 (0.002)	0.9229 (0.004)	0.9397 (0.0035)	0.9312 (0.004)
BCAI	0.0603 (0.0091)	0.8902 (0.0022)	0.9091 (0.0041)	0.9363 (0.0036)	0.9077 (0.0039)
PTBCAI	0.0647 (0.0106)	0.9227 (0.0016)	0.9379 (0.0041)	0.9450 (0.0033)	0.9565 (0.0039)
BPGPDI	0.0424 (0.0094)	0.9294 (0.0012)	0.9325 (0.0039)	0.9496 (0.0035)	0.9409 (0.0042)
BTGPDI	0.0573 (0.008)	0.9137 (0.0016)	0.9398 (0.0045)	0.9369 (0.0042)	0.9282 (0.0044)
BPPI	0.0507 (0.0149)	0.9073 (0.002)	0.9179 (0.0042)	0.9450 (0.0036)	0.9265 (0.0041)
BTPI	0.0609 (0.0104)	0.9297 (0.002)	0.9323 (0.0044)	0.9326 (0.0038)	0.9409 (0.0047)

Portfolio					
GE1.3	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0545 (0.0095)	0.9162 (0.0042)	0.9088 (0.0044)	0.9259 (0.0039)	0.9173 (0.0045)
STI	0.0576 (0.0097)	0.9081 (0.0042)	0.9138 (0.0046)	0.9308 (0.0039)	0.9223 (0.0045)
BPI	0.0544 (0.0103)	0.9081 (0.0052)	0.9417 (0.0057)	0.9543 (0.0049)	0.9455 (0.0056)
PTBPI	0.0421 (0.0105)	0.9139 (0.0052)	0.9389 (0.0063)	0.9461 (0.0054)	0.9473 (0.0064)
BTI	0.0402 (0.0095)	0.9204 (0.0049)	0.9024 (0.0059)	0.9093 (0.0051)	0.9131 (0.0063)
BCAI	0.0601 (0.0091)	0.9026 (0.0049)	0.9025 (0.0042)	0.9296 (0.0036)	0.9211 (0.0041)
PTBCAI	0.0522 (0.0106)	0.9267 (0.0052)	0.9097 (0.0064)	0.9457 (0.0054)	0.9481 (0.0062)
BPGPDI	0.0470 (0.0046)	0.9443 (0.0057)	0.9367 (0.0047)	0.9441 (0.0041)	0.9452 (0.0048)
BTGPDI	0.0439 (0.0064)	0.9387 (0.0045)	0.9318 (0.0047)	0.9391 (0.0041)	0.9383 (0.0047)
BPPI	0.0473 (0.0042)	0.9042 (0.0049)	0.9428 (0.0044)	0.9501 (0.0035)	0.9413 (0.004)
BTPI	0.0521 (0.005)	0.9202 (0.0047)	0.9246 (0.0041)	0.9353 (0.0035)	0.9345 (0.004)

IESdata					
GE1.3	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0405 (0.0051)	0.8916 (0.0023)	0.9014 (0.0024)	0.9292 (0.0022)	0.9294 (0.0025)
STI	0.0460 (0.0057)	0.9196 (0.0028)	0.9046 (0.0025)	0.9228 (0.0022)	0.9241 (0.0025)
BPI	0.0431 (0.0042)	0.9332 (0.0026)	0.9291 (0.0033)	0.9363 (0.0027)	0.9323 (0.0024)
PTBPI	0.0478 (0.0056)	0.9355 (0.0025)	0.9307 (0.0036)	0.9382 (0.0031)	0.9393 (0.0036)
BTI	0.0438 (0.0054)	0.9057 (0.0022)	0.8942 (0.0033)	0.9110 (0.0028)	0.9126 (0.0036)
BCAI	0.0511 (0.0049)	0.9141 (0.0023)	0.9045 (0.0023)	0.9215 (0.002)	0.9229 (0.0023)
PTBCAI	0.0499 (0.0047)	0.9276 (0.0027)	0.9119 (0.0036)	0.9391 (0.0031)	0.9395 (0.0035)
BPGPDI	0.0455 (0.0042)	0.9476 (0.0026)	0.9289 (0.0026)	0.9462 (0.0023)	0.9419 (0.0026)
BTGPDI	0.0501 (0.0051)	0.9369 (0.0022)	0.9312 (0.0026)	0.9360 (0.0022)	0.9383 (0.0025)
BPPI	0.0514 (0.0049)	0.9441 (0.0021)	0.9318 (0.0024)	0.9462 (0.002)	0.9479 (0.0022)
BTPI	0.0433 (0.0041)	0.9469 (0.0023)	0.9310 (0.0023)	0.9454 (0.0019)	0.9464 (0.0022)

Table 8.7: Application ACILs and CPs for GE1.3

Consider Table 8.7 above. The performance of all the methods is quite unsatisfactory for Nordata when considering the NP and SP GPD estimates as population values. Recall that Nordata is a small data set and we would not expect good results. However, we have a great improvement when using the SP Pa and PPD estimates as population values. For Portfolio and IESdata, BPGPDI and BPPI perform quite well, especially with SP estimates as population values. For Portfolio, BPI also gives satisfactory CPs with SP estimates as population values. In the latter case, PTBPI is also quite good with NP, SP Pa and SP PPD estimates as population values. We also see in some cases that although the SP methods do not give satisfactory results, their CPs are still closer to the desired coverage of 95% than the nonparametric methods.

Atkinson Coefficient with Parameter 1 (A1)

Nordata					
A1	**ACIL**	**CP (NP)**	**CP(GPD)**	**CP(Pa)**	**CP(PPD)**
SNI	0.0920 (0.0072)	0.9323 (0.0062)	0.9158 (0.0058)	0.9361 (0.0063)	0.9358 (0.006)
STI	0.1041 (0.0065)	0.8963 (0.0071)	0.8804 (0.007)	0.8999 (0.0072)	0.8996 (0.0074)
BPI	0.0906 (0.007)	0.9384 (0.0066)	0.9221 (0.0072)	0.9321 (0.0067)	0.9318 (0.0069)
PTBPI	0.0883 (0.0076)	0.9357 (0.0065)	0.9395 (0.0066)	0.9394 (0.0066)	0.9421 (0.0068)
BTI	0.0954 (0.0074)	0.9067 (0.0063)	0.8906 (0.0066)	0.9234 (0.0064)	0.9192 (0.0066)
BCAI	0.0755 (0.0067)	0.8935 (0.0064)	0.8776 (0.0067)	0.9071 (0.0065)	0.8968 (0.0065)
PTBCAI	0.0741 (0.0073)	0.9217 (0.0061)	0.9054 (0.0068)	0.9354 (0.0062)	0.9351 (0.0064)
BPGPDI	0.0961 (0.0043)	0.9262 (0.0062)	0.9398 (0.0064)	0.9343 (0.0063)	0.9327 (0.0069)
BTGPDI	0.0712 (0.0039)	0.9137 (0.0067)	0.8975 (0.0074)	0.9344 (0.0068)	0.9371 (0.0072)
BPPI	0.0836 (0.0032)	0.9321 (0.0065)	0.9361 (0.0069)	0.9457 (0.0066)	0.9451 (0.0068)
BTPI	0.0826 (0.0033)	0.9261 (0.0066)	0.9097 (0.0072)	0.9299 (0.0067)	0.9234 (0.0077)

Portfolio					
A1	**ACIL**	**CP (NP)**	**CP(GPD)**	**CP(Pa)**	**CP(PPD)**
SNI	0.0506 (0.0038)	0.9121 (0.0043)	0.8959 (0.0044)	0.9249 (0.0039)	0.9155 (0.0045)
STI	0.0542 (0.0038)	0.9170 (0.0043)	0.9008 (0.0045)	0.9199 (0.0039)	0.9104 (0.0045)
BPI	0.0427 (0.0042)	0.9451 (0.0053)	0.9235 (0.0056)	0.9434 (0.0048)	0.9436 (0.0055)
PTBPI	0.0434 (0.0037)	0.9489 (0.0059)	0.9252 (0.0063)	0.9452 (0.0054)	0.9454 (0.0063)
BTI	0.0391 (0.0038)	0.8859 (0.0055)	0.8702 (0.0058)	0.8983 (0.005)	0.8892 (0.0063)
BCAI	0.0363 (0.0034)	0.9158 (0.0039)	0.8996 (0.0042)	0.9287 (0.0035)	0.9192 (0.0041)
PTBCAI	0.0435 (0.0032)	0.9427 (0.0059)	0.9263 (0.0063)	0.9516 (0.0054)	0.9462 (0.0062)
BPGPDI	0.0199 (0.0038)	0.9498 (0.0044)	0.9429 (0.0047)	0.9431 (0.004)	0.9513 (0.0047)
BTGPDI	0.0324 (0.0074)	0.9448 (0.0044)	0.9281 (0.0046)	0.9581 (0.004)	0.9484 (0.0046)
BPPI	0.0276 (0.0008)	0.9459 (0.0038)	0.9393 (0.0043)	0.9497 (0.0035)	0.9494 (0.004)
BTPI	0.0241 (0.0043)	0.9391 (0.0038)	0.9465 (0.004)	0.9323 (0.0034)	0.9326 (0.004)

IESdata					
A1	**ACIL**	**CP (NP)**	**CP(GPD)**	**CP(Pa)**	**CP(PPD)**
SNI	0.0374 (0.004)	0.9308 (0.0034)	0.9263 (0.0042)	0.9336 (0.0038)	0.9349 (0.0044)
STI	0.0308 (0.0042)	0.9378 (0.0032)	0.9424 (0.0036)	0.9397 (0.0037)	0.9408 (0.0036)
BPI	0.0372 (0.004)	0.9375 (0.0036)	0.9255 (0.0045)	0.9431 (0.0039)	0.9461 (0.0044)
PTBPI	0.0379 (0.0041)	0.9493 (0.0032)	0.9442 (0.0045)	0.9497 (0.0038)	0.9485 (0.0044)
BTI	0.0328 (0.0035)	0.9061 (0.0038)	0.8946 (0.0048)	0.9014 (0.0037)	0.9079 (0.0044)
BCAI	0.0315 (0.0034)	0.9240 (0.0041)	0.8969 (0.0041)	0.9240 (0.0037)	0.9257 (0.0042)
PTBCAI	0.0379 (0.0041)	0.9457 (0.004)	0.9342 (0.0045)	0.9484 (0.0039)	0.9469 (0.0045)
BPGPDI	0.0232 (0.0025)	0.9361 (0.0033)	0.9422 (0.0029)	0.9319 (0.0025)	0.9327 (0.0028)
BTGPDI	0.0423 (0.0046)	0.9418 (0.0037)	0.9401 (0.0037)	0.9497 (0.0035)	0.9478 (0.0039)
BPPI	0.0326 (0.004)	0.9542 (0.0031)	0.9443 (0.0041)	0.9516 (0.0035)	0.9508 (0.004)
BTPI	0.0375 (0.0035)	0.9417 (0.0029)	0.9495 (0.0027)	0.9463 (0.0025)	0.9480 (0.0028)

Table 8.8: Application ACILs and CPs for A1

Consider Table 8.8 above. For Nordata, the performance of most of the methods is unsatisfactory, with BPPI having satisfactory CPs when using SP Pa and PPD estimates as population values. This could again be due to the small sample size. The CIs based on SP estimators perform satisfactorily for Portfolio and IESdata, especially when using SP estimates as population values. Also the CPs are quite satisfactory in the case of BPI and PTBPI for both the data sets. PTBCAI also does well for IESdata. The ACILs are reasonably short in most cases.

Atkinson Coefficient with Parameter 1.5 (A1.5)

Nordata

A1.5	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0520 (0.0073)	0.9524 (0.0056)	0.9240 (0.0054)	0.9445 (0.0059)	0.9442 (0.0056)
STI	0.0522 (0.0066)	0.9160 (0.0064)	0.8883 (0.0055)	0.9108 (0.0056)	0.9077 (0.0058)
BPI	0.0503 (0.0071)	0.9380 (0.0059)	0.9102 (0.0056)	0.9304 (0.0052)	0.9301 (0.0054)
PTBPI	0.0634 (0.0077)	0.9357 (0.0059)	0.9076 (0.0051)	0.9377 (0.0051)	0.9374 (0.0053)
BTI	0.0654 (0.0075)	0.9260 (0.0057)	0.8986 (0.0051)	0.9317 (0.005)	0.9275 (0.0051)
BCAI	0.0574 (0.0068)	0.9129 (0.0058)	0.8855 (0.0052)	0.9153 (0.0051)	0.9149 (0.0051)
PTBCAI	0.0517 (0.0074)	0.9412 (0.0055)	0.9135 (0.0053)	0.9438 (0.0048)	0.9334 (0.005)
BPGPDI	0.0540 (0.0044)	0.9464 (0.0056)	0.9180 (0.005)	0.9467 (0.0049)	0.9451 (0.0054)
BTGPDI	0.0518 (0.004)	0.9335 (0.006)	0.9056 (0.0058)	0.9428 (0.0053)	0.9455 (0.0056)
BPPI	0.0682 (0.0033)	0.9227 (0.0059)	0.8941 (0.0054)	0.9441 (0.0051)	0.9439 (0.0053)
BTPI	0.0738 (0.0034)	0.9467 (0.0059)	0.9179 (0.0056)	0.9383 (0.0052)	0.9417 (0.006)

Portfolio

A1.5	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0629 (0.0039)	0.9413 (0.0038)	0.9304 (0.004)	0.9432 (0.0035)	0.9437 (0.0041)
STI	0.0409 (0.0039)	0.9261 (0.0038)	0.9089 (0.0041)	0.9282 (0.0035)	0.9186 (0.0041)
BPI	0.0370 (0.0043)	0.9461 (0.0048)	0.9318 (0.0051)	0.9519 (0.0044)	0.9521 (0.005)
PTBPI	0.0379 (0.0038)	0.9562 (0.0053)	0.9335 (0.0057)	0.9537 (0.0049)	0.9539 (0.0057)
BTI	0.0333 (0.0039)	0.9087 (0.005)	0.8780 (0.0053)	0.9064 (0.0045)	0.8972 (0.0057)
BCAI	0.0312 (0.0035)	0.9284 (0.0035)	0.9197 (0.0038)	0.9371 (0.0032)	0.9275 (0.0037)
PTBCAI	0.0382 (0.0033)	0.9524 (0.0053)	0.9346 (0.0057)	0.9562 (0.0049)	0.9547 (0.0056)
BPGPDI	0.0191 (0.0039)	0.9462 (0.004)	0.9413 (0.0042)	0.9516 (0.0037)	0.9499 (0.0043)
BTGPDI	0.0236 (0.0075)	0.9350 (0.004)	0.9365 (0.0042)	0.9367 (0.0036)	0.9369 (0.0042)
BPPI	0.0235 (0.0008)	0.9459 (0.0034)	0.9374 (0.0039)	0.9476 (0.0031)	0.9479 (0.0036)
BTPI	0.0250 (0.0044)	0.9387 (0.0034)	0.9108 (0.0037)	0.9398 (0.0031)	0.9331 (0.0036)

IESdata

A1.5	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0381 (0.0039)	0.9402 (0.0039)	0.9328 (0.0041)	0.9401 (0.0037)	0.9414 (0.0038)
STI	0.0314 (0.0041)	0.9374 (0.0031)	0.9249 (0.0035)	0.9363 (0.0036)	0.9374 (0.0032)
BPI	0.0379 (0.0039)	0.9378 (0.0035)	0.9320 (0.0033)	0.9396 (0.0038)	0.9366 (0.0043)
PTBPI	0.0386 (0.004)	0.9497 (0.0031)	0.9408 (0.0035)	0.9483 (0.0037)	0.9493 (0.0043)
BTI	0.0335 (0.0034)	0.9261 (0.0037)	0.9009 (0.0043)	0.9277 (0.0036)	0.9243 (0.0043)
BCAI	0.0321 (0.0033)	0.9342 (0.0040)	0.9032 (0.0041)	0.9305 (0.0036)	0.9322 (0.0041)
PTBCAI	0.0386 (0.004)	0.9453 (0.0039)	0.9407 (0.0043)	0.9480 (0.0038)	0.9493 (0.0039)
BPGPDI	0.0237 (0.0025)	0.9434 (0.0032)	0.9418 (0.0028)	0.9484 (0.0024)	0.9492 (0.0027)
BTGPDI	0.0432 (0.0045)	0.9420 (0.0036)	0.9367 (0.0036)	0.9463 (0.0034)	0.9454 (0.0038)
BPPI	0.0333 (0.0039)	0.9568 (0.0039)	0.9439 (0.0041)	0.9583 (0.0034)	0.9541 (0.0039)
BTPI	0.0382 (0.0034)	0.9422 (0.0028)	0.9361 (0.0026)	0.9429 (0.0024)	0.9446 (0.0027)

Table 8.9: Application ACILs and CPs for A1.5

Consider Table 8.9 above. Once again we see that the CIs based on SP estimators outperform the usual methods in terms of CPs and ACILs for most cases, especially when using SP estimates as population values.

Atkinson Coefficient with Parameter 2 (A2)

Nordata					
A2	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0859 (0.0068)	0.8439 (0.0062)	0.8405 (0.0059)	0.8411 (0.0064)	0.8438 (0.0061)
STI	0.0851 (0.0061)	0.9167 (0.0071)	0.9045 (0.006)	0.9172 (0.0061)	0.9141 (0.0063)
BPI	0.0746 (0.0066)	0.9060 (0.0066)	0.9366 (0.0061)	0.9069 (0.0057)	0.9066 (0.0059)
PTBPI	0.0676 (0.0072)	0.9483 (0.0065)	0.9314 (0.0056)	0.9453 (0.0056)	0.9491 (0.0058)
BTI	0.0640 (0.007)	0.9083 (0.0063)	0.9449 (0.0056)	0.9182 (0.0054)	0.9124 (0.0056)
BCAI	0.0976 (0.0063)	0.8977 (0.0064)	0.8917 (0.0057)	0.8917 (0.0056)	0.9013 (0.0056)
PTBCAI	0.0769 (0.0069)	0.9485 (0.0061)	0.9199 (0.0058)	0.9434 (0.0052)	0.9459 (0.0054)
BPGPDI	0.0574 (0.0041)	0.9264 (0.0062)	0.9244 (0.0054)	0.9453 (0.0053)	0.9417 (0.0059)
BTGPDI	0.0800 (0.0037)	0.9087 (0.0067)	0.9119 (0.0063)	0.9394 (0.0058)	0.9321 (0.0061)
BPPI	0.0809 (0.0031)	0.9436 (0.0065)	0.9204 (0.0059)	0.9517 (0.0056)	0.9495 (0.0058)
BTPI	0.0862 (0.0032)	0.9377 (0.0066)	0.9453 (0.0061)	0.9349 (0.0057)	0.9383 (0.0065)

Portfolio					
A2	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0461 (0.0036)	0.8825 (0.0043)	0.9369 (0.0043)	0.8998 (0.0038)	0.9303 (0.0044)
STI	0.0509 (0.0036)	0.9065 (0.0043)	0.9153 (0.0045)	0.9247 (0.0038)	0.9251 (0.0044)
BPI	0.0587 (0.004)	0.9443 (0.0053)	0.9383 (0.0056)	0.9586 (0.0047)	0.9588 (0.0055)
PTBPI	0.0584 (0.0036)	0.9457 (0.0059)	0.9356 (0.0062)	0.9454 (0.0053)	0.9416 (0.0062)
BTI	0.0518 (0.0036)	0.9179 (0.0055)	0.9141 (0.0057)	0.9127 (0.0049)	0.9135 (0.0062)
BCAI	0.0469 (0.0033)	0.9279 (0.0039)	0.9261 (0.0041)	0.9437 (0.0035)	0.9342 (0.004)
PTBCAI	0.0590 (0.0031)	0.9478 (0.0059)	0.9411 (0.0062)	0.9499 (0.0053)	0.9494 (0.0061)
BPGPDI	0.0435 (0.0036)	0.9474 (0.0044)	0.9479 (0.0046)	0.9583 (0.004)	0.9565 (0.0047)
BTGPDI	0.0436 (0.007)	0.9436 (0.0044)	0.9431 (0.0046)	0.9493 (0.004)	0.9435 (0.0046)
BPPI	0.0568 (0.0008)	0.9511 (0.0038)	0.9441 (0.0043)	0.9542 (0.0034)	0.9445 (0.0039)
BTPI	0.0291 (0.0041)	0.9374 (0.0038)	0.9372 (0.004)	0.9363 (0.0034)	0.9396 (0.0039)

IESdata					
A2	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	0.0384 (0.0038)	0.9388 (0.0036)	0.9319 (0.004)	0.9392 (0.0036)	0.9405 (0.0037)
STI	0.0316 (0.004)	0.9362 (0.0034)	0.9240 (0.0034)	0.9354 (0.0035)	0.9365 (0.0033)
BPI	0.0382 (0.0038)	0.9471 (0.0038)	0.9311 (0.0032)	0.9487 (0.0037)	0.9459 (0.0038)
PTBPI	0.0389 (0.0039)	0.9452 (0.0034)	0.9399 (0.0034)	0.9474 (0.0036)	0.9484 (0.0042)
BTI	0.0337 (0.0034)	0.9354 (0.004)	0.9245 (0.0041)	0.9368 (0.0037)	0.9334 (0.0042)
BCAI	0.0324 (0.0033)	0.9435 (0.0043)	0.9323 (0.004)	0.9466 (0.0039)	0.9463 (0.004)
PTBCAI	0.0389 (0.0039)	0.9468 (0.0042)	0.9498 (0.0038)	0.9471 (0.0037)	0.9484 (0.0038)
BPGPDI	0.0239 (0.0024)	0.9459 (0.0035)	0.9409 (0.0027)	0.9475 (0.0027)	0.9483 (0.0026)
BTGPDI	0.0435 (0.0044)	0.9414 (0.0039)	0.9358 (0.0035)	0.9454 (0.0036)	0.9445 (0.0037)
BPPI	0.0335 (0.0038)	0.9464 (0.0033)	0.9403 (0.0039)	0.9513 (0.0041)	0.9527 (0.0038)
BTPI	0.0385 (0.0034)	0.9418 (0.0031)	0.9352 (0.0025)	0.9429 (0.0023)	0.9437 (0.0026)

Table 8.10: Application ACILs and CPs for A2

Consider Table 8.10 above. For all the data sets, PTBPI and PTBCAI perform quite satisfactorily. The CIs based on SP estimators perform well in many cases. Note that even in the cases where the performance is unsatisfactory, the SP methods lead to CPs not too far from the desired coverage. Once again, the performance is quite good when using SP estimates as population values.

Quintile Share Ratio (QSR)

Nordata					
QSR	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	2.5146	0.9306	0.9414	0.9287	0.9399
	(0.0091)	(0.0076)	(0.0063)	(0.0062)	(0.0065)
STI	2.5033	0.9214	0.9147	0.9218	0.9133
	(0.0082)	(0.0086)	(0.0076)	(0.0071)	(0.008)
BPI	2.6760	0.9412	0.9373	0.9448	0.9359
	(0.0089)	(0.008)	(0.0078)	(0.0065)	(0.0075)
PTBPI	2.7237	0.9454	0.9346	0.9318	0.9373
	(0.0096)	(0.0079)	(0.0072)	(0.0065)	(0.0074)
BTI	2.4042	0.9523	0.9257	0.9325	0.9334
	(0.0094)	(0.0077)	(0.0072)	(0.0063)	(0.0072)
BCAI	2.2873	0.9040	0.9118	0.9391	0.9104
	(0.0085)	(0.0078)	(0.0073)	(0.0064)	(0.0071)
PTBCAI	2.6854	0.9546	0.9427	0.9478	0.9494
	(0.0093)	(0.0074)	(0.0074)	(0.006)	(0.0069)
BPGPDI	2.2479	0.9433	0.9353	0.9525	0.9437
	(0.0055)	(0.0076)	(0.0069)	(0.0062)	(0.0075)
BTGPDI	1.9607	0.9495	0.9426	0.9497	0.9431
	(0.005)	(0.0082)	(0.008)	(0.0075)	(0.0078)
BPPI	2.3130	0.9506	0.9387	0.9478	0.9493
	(0.0041)	(0.0079)	(0.0075)	(0.0065)	(0.0074)
BTPI	2.1277	0.9497	0.9351	0.9454	0.9437
	(0.0042)	(0.008)	(0.0078)	(0.0068)	(0.0084)

Portfolio					
QSR	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	5.5216	0.9214	0.9115	0.9287	0.9201
	(0.0118)	(0.0046)	(0.0048)	(0.0042)	(0.0049)
STI	5.6461	0.9269	0.9165	0.9336	0.9251
	(0.0121)	(0.0046)	(0.005)	(0.0042)	(0.0049)
BPI	5.7214	0.9462	0.9475	0.9572	0.9483
	(0.0128)	(0.0056)	(0.0062)	(0.0053)	(0.006)
PTBPI	5.7540	0.9520	0.9497	0.9489	0.9512
	(0.0131)	(0.0057)	(0.0068)	(0.0059)	(0.0069)
BTI	5.0738	0.8702	0.9051	0.8912	0.9058
	(0.0119)	(0.0053)	(0.0063)	(0.0055)	(0.0068)
BCAI	4.7864	0.8986	0.9052	0.9324	0.9239
	(0.0113)	(0.0054)	(0.0046)	(0.0039)	(0.0045)
PTBCAI	5.7453	0.9592	0.9124	0.9485	0.9517
	(0.0133)	(0.0057)	(0.0069)	(0.0059)	(0.0068)
BPGPDI	2.4322	0.9556	0.9395	0.9569	0.9535
	(0.0157)	(0.0062)	(0.0051)	(0.0044)	(0.0051)
BTGPDI	3.1466	0.9432	0.9346	0.9419	0.9411
	(0.018)	(0.0049)	(0.0051)	(0.0044)	(0.0051)
BPPI	3.4145	0.9513	0.9456	0.9531	0.9491
	(0.0152)	(0.0054)	(0.0047)	(0.0037)	(0.0044)
BTPI	2.1695	0.9485	0.9274	0.9481	0.9473
	(0.0163)	(0.0051)	(0.0044)	(0.0037)	(0.0043)

IESdata					
QSR	ACIL	CP (NP)	CP(GPD)	CP(Pa)	CP(PPD)
SNI	8.0842	0.8921	0.9148	0.9351	0.9348
	(0.0113)	(0.0039)	(0.0037)	(0.0041)	(0.0038)
STI	8.3657	0.8973	0.8794	0.9017	0.9086
	(0.0125)	(0.004)	(0.0038)	(0.0038)	(0.004)
BPI	7.2500	0.9459	0.9311	0.9471	0.9488
	(0.0176)	(0.0042)	(0.0038)	(0.0036)	(0.0037)
PTBPI	6.9348	0.9439	0.9485	0.9449	0.9458
	(0.0114)	(0.0037)	(0.0035)	(0.0035)	(0.0036)
BTI	8.3294	0.9266	0.9396	0.9224	0.9182
	(0.0163)	(0.0039)	(0.0035)	(0.0034)	(0.0035)
BCAI	7.9821	0.9302	0.9166	0.9361	0.9358
	(0.0106)	(0.0044)	(0.0036)	(0.0035)	(0.0035)
PTBCAI	7.1492	0.9388	0.9454	0.9394	0.9411
	(0.0166)	(0.0037)	(0.0036)	(0.0033)	(0.0034)
BPGPDI	7.1935	0.9399	0.9188	0.9372	0.9356
	(0.0134)	(0.0041)	(0.0034)	(0.0034)	(0.0037)
BTGPDI	8.0124	0.9367	0.9465	0.9434	0.9456
	(0.0163)	(0.0038)	(0.004)	(0.0036)	(0.0038)
BPPI	7.9313	0.9472	0.9252	0.9447	0.9495
	(0.0106)	(0.0044)	(0.0037)	(0.0035)	(0.0036)
BTPI	7.2095	0.9448	0.9387	0.9469	0.9473
	(0.0166)	(0.0035)	(0.0038)	(0.0036)	(0.0041)

Table 8.11: Application ACILs and CPs for QSR

Consider Table 8.11 above. We have satisfactory results for most of SP methods in terms of CPs and ACILs. It is interesting to note that BPI, PTBPI and PTBCAI perform quite well in terms of CPs. The performance of the CIs based on SP methods is improved when using SP estimates as population values.

Overall Conclusions

The results from the real life data sets confirm those obtained in simulation. Although in some cases the performance is not totally satisfactory, the intervals based on SP estimators proved to outperform the CIs based on the usual methods in terms of CPs and ACILs. Even when they are not entirely satisfactory, the SP methods lead to CPs that tend to be not too far away from the desired coverage of 95%. A better performance is obtained when using SP estimates as population values rather than the NP values. Also, the usual bootstrap methods, especially the bootstrap percentile, the bias corrected and accelerated, and their power transformations perform quite well in a number of cases.

Overall we see that the procedures proposed in this study do remarkably well and can be used in practice to reach satisfactory conclusions. Also, given the complexity of the statistics underlying the bootstrap procedures, we can say that such procedures do quite well.

Remark 8.2. The illustration given here is to show how the methods developed in this thesis can be applied to practical data sets. Given a data set, if one determines which of the three distributions (GPD, strict Pareto or PPD) gives a better approximation of the tail, then the corresponding semi-parametric method should be the preferred method to use. Recall that semi-parametric estimators not only perform well in terms of MSE, but are also less sensitive to outliers. Therefore the conclusions based on such estimators are expected to be more reliable than those based on nonparametric methods. However, an important issue is the choice of the tail distribution, choosing between the three candidate parametric forms. A possible way of doing this is to use a measure of representativeness proposed by Bertino [6]. This is discussed in the following subsection.

Measure of Representativeness

As mentioned above, it is clear from this study that the proposed semi-parametric methods are more reliable than the traditional ones. Not only do they perform well in terms of MSEs and CIs, but they are also less sensitive to outliers. In order to decide which of the three parametric distributions (GPD, strict Pareto or PPD) to use in a specific application, one can measure the representativeness of the sample to each of the distributions.

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a simple random sample of size n from a distribution F and denote by $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ its associated order statistics. A representativeness index is given by (see Bertino [6])

$$R(\underline{X}, F) = 1 - \frac{12n}{4n^2 - 1} \sum_{i=1}^{n} \left(F(X_{i.n}) - \frac{2i-1}{2n} \right)^2 . \tag{8.3}$$

This index is used to measure how well the sample $\underline{X}$ represents its parent distribution. Therefore, it can be used to select the model one should use when having to choose between a number of different models.

In our case, we use the measure for a particular data set to decide between the three parametric choices, GPD, strict Pareto and PPD. To do this we calculate $R(\underline{X}, GPD)$, $R(\underline{X}, Pa)$ and $R(\underline{X}, PPD)$, and the largest value determines the preferred distribution to use. Applying the measure of representativeness to 10% of the upper values of the data sets considered, leads to the results in Table 8.12 below.

X	**R(X,GPD)**	**R(X,Pa)**	**R(X,PPD)**
Nordata	0.94656	0.99457	0.99309
Portfolio	0.97362	0.99987	0.99988
IESdata	0.96404	0.99657	0.99485

Table 8.12: Representativeness Index for the GPD, the strict Pareto and the PPD

These results show that we would prefer the strict Pareto distribution for Nordata and IESdata, but that the PPD is the preferred choice in the case of Portfolio. For all three data sets, the strict Pareto and the PPD are well represented by the data, with GPD lagging behind to a slight extent. The corresponding estimators are the most reliable, and should be considered for assessing the inequality in a given situation.

Remark 8.3. Given a practical data set, in order to use a semi-parametric method, a two-step approach would be to first determine which of the three parametric distributions to use in the tail estimation. The corresponding semi-parametric procedure can then be used as a preferred choice to estimate the desired measures.

Chapter 9

Conclusions and Further Work

Measures of inequality play an extremely important role in many areas of science and its applications. Over time a large number of such measures have been proposed and studied intensively. The oldest and most well-known one is the Gini index, already proposed in 1912. Other popular measures are those in the class of generalized entropy measures and Atkinson measures. A more recent measure is the quintile share ratio. This measure has become popular in the European Union since the European Council decided in 2001 that income inequality in member states should be described using two indices: the Gini Index and the quintile share ratio. Measures of inequality are by their very nature dependent on the tails of the underlying distribution, in our case the right hand tail, and thus also sensitive to the tail length, as can be seen from their unbounded influence functions. Traditional estimators for these measures are obtained as plug-in estimators, i.e. by plugging-in the empirical distribution function in the functional defining the measure. Such estimators are clearly of a nonparametric nature. However, they are typically also sensitive to outlying values and thus there is a need for more robust estimators of these measures. Two main approaches exist for robustifying the estimators, either by trimming the data or by modifying the estimator of the underlying distribution. There is a strong need to improve on current robust procedures, especially regarding the second approach, being the more sensible approach to follow.

The first major aim of this thesis was therefore to propose robust estimators based on modifying the estimator of the underlying distribution. This was done in a semi-parametric fashion, using the empirical distribution function where the bulk of the data resides and a parametric component in the tail of the distribution. The parametric component was obtained using limiting results from extreme value theory. This led to three possible parametric distributions, viz. the generalized Pareto, the strict Pareto and the perturbed Pareto. The parameters of these distributions are obtained by fitting the distributions to the exceedances of a threshold, i.e. using a number of upper order statistics in maximum likelihood estimation. The proposals were applied to the measures of inequality mentioned above and the necessary expressions were derived under the three distribution types. For the quintile share ratio the influence function and asymptotic normality were also derived, since these are not available yet in the literature.

The second aim of the thesis was to study the performance of the proposed semi-parametric estimators and to compare them to their nonparametric counterparts. This was done through simulation studies under a range of different heavy-tailed underlying distributions and a number of different sample sizes. Two major simulation studies were carried out. In the first one, the performance of the estimators was studied through their mean squared errors and their sensitivity to outliers. In the second one they were used as basis to construct confidence intervals for the corresponding measures and these confidence intervals were compared based on their coverage probabilities and average confidence interval lengths. A number of different asymptotic and bootstrap methods were used to construct these intervals. Second level bootstraps were also used in order to obtain standard errors for these estimators. In both these simulation studies, the proposed estimators showed a great improvement over the traditional nonparametric ones, mostly over all three choices of parametric distributions. For the latter, no clear overall best choice of parametric distribution was obtained. In all cases small standard errors were obtained, indicating a high degree of reliability of the estimators and procedures. The quintile share ratio is perhaps a bit of an outlier. Its value is clearly influenced by the typically large value of the numerator, generally leading to large values of the measure and its estimates. Its behavior also seems to be less stable than that of the other measures.

A final aim of the thesis was to illustrate the use of the proposed methods in practice. The methods were applied to three sets of data, viz. claims data from a short term South African insurer, Norwegian fire insurance data and 2005 South African income and expenditure survey data. In order to choose between the three parametric distributions, it was proposed that a measure of sample representativeness be used. This was applied to the three data sets to choose the appropriate parametric distribution to use in the tail estimation.

This thesis proposed and studied a number of semi-parametric estimators and found them to perform well. However, there are a number of issues that could and should be investigated in further research. Some of the main ones are the following.

1. Research on the asymptotic behavior of semi-parametric estimators of measures of inequality, e.g. their asymptotic distributions and in particular, their asymptotic variances. Since the es-

timators depend in quite a complex way on the sample size, this will be a rather difficult task. However, the results could be very valuable in improving some of the inference procedures.

2. As an extension of the ideas in 1, the saddle point method could be investigated as a means of improving the approximate distribution of the estimators. The saddle point is known to give accurate approximations in many instances and could lead to improvements in the present situation.

3. In the application of the methods used in this research a choice of threshold was based on a restricted simulation study. A further in-depth study of the "best" choice of this threshold could be useful. This is a broad topic to be considered and justifies a separate study by itself.

4. The three parametric distributions used were the Pareto, the generalized Pareto and the perturbed Pareto. These were choices made on the basis of results from extreme value theory and generally gave extremely good results. For a particular data set, a proposal was made to use a measure of sample representativeness in order to choose between the three parametric distributions. Further investigation is needed to see if this method can be improved upon, using other measures of model choice, including results based on goodness-of-fit tests.

5. In this thesis only data based on independent, identically distributed samples were considered. Many surveys, however, are based on complex sampling with weighting. An important extension will be to adapt the current results so as to apply also to complex samples. This will considerably extend the applicability of the current results.

6. In the current work, estimation of unknown parameters was based on maximum likelihood. Some alternatives were considered, but found not to give worthwhile improvement over maximum likelihood. However, there are a number of other candidate estimators that could be considered. Examples include probability weighted moments, minimum distance estimation, Bayes estimation, and some of these may very well give better results. Future research will compare different estimation procedures.

7. In many cases some prior information may be available that could be incorporated into the analysis. This could especially relate to the tail of the underlying distribution. Such information could then be applied in a Bayesian approach and could lead to enhanced inference.

8. It was previously mentioned that there are two main approaches to robustify estimators of the measures of inequality. In this thesis we used one approach, which to our mind is the most sensible to follow. Future research should, however, also consider the other approach and compare its performance to that of the current one.

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Appendix A

Outline of Proof of Theorem 6.1

Define, for $0 < \alpha < \frac{1}{2} < \beta < 1$,

$$N_\beta = \int_{Q(\beta)}^{\infty} x dF(x) = \int_{\beta}^{1} Q(s) ds \tag{A.1}$$

and

$$D_\alpha = \int_{0}^{Q(\alpha)} x dF(x) = \int_{0}^{\alpha} Q(s) ds. \tag{A.2}$$

The QSR is then defined as

$$\eta = \frac{N_{0.8}}{D_{0.2}}. \tag{A.3}$$

Also define the corresponding nonparametric estimators as

$$N_{n,\beta} = \frac{1}{n} \sum_{i=[n\beta]+1}^{n} X_{i,n} \tag{A.4}$$

and

$$D_{n,\alpha} = \frac{1}{n} \sum_{i=1}^{[n\alpha]} X_{i,n}, \tag{A.5}$$

with $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ the order statistics of the sample. Then

$$\hat{\eta} = \frac{N_{n,0.8}}{D_{n,0.2}}, \tag{A.6}$$

and we consider, more generally, the asymptotic behavior of

$$\frac{N_{n,\beta}}{D_{n,\alpha}} - \frac{N_\beta}{D_\alpha} \equiv T_n(\alpha, \beta) - T(\alpha, \beta). \tag{A.7}$$

Note that for the lower sum we always have

$$D_{n,\alpha} = \frac{1}{n}\sum_{i=1}^{[n\alpha]} X_{i,n} \xrightarrow{P} D_\alpha, \text{ as } n \to \infty. \tag{A.8}$$

Using Slutsky, we can therefore write for large n

$$\begin{aligned} T_n(\alpha,\beta) - T(\alpha,\beta) &= \frac{N_{n,\beta}D_\alpha - D_{n,\alpha}N_\beta}{D_{n,\alpha}D_\alpha} \\ &= \frac{(N_{n,\beta}-N_\beta)D_\alpha - (D_{n,\alpha}-D_\alpha)N_\beta}{D_{n,\alpha}D_\alpha} \\ &\approx \frac{(N_{n,\beta}-N_\beta)D_\alpha - (D_{n,\alpha}-D_\alpha)N_\beta}{D_\alpha^2}. \end{aligned} \tag{A.9}$$

We consider the two terms in the numerator separately.

Let $U_1, U_2, \ldots, U_n$ be a sample from a $U(0,1)$ distribution and let $U_{1,n} < U_{2,n} < \ldots < U_{n,n}$ be the corresponding order statistics. Also define $U_{0,n} = 0$ and $U_{n+1,n} = 1$. Let $G_n(t)$ be the uniform empirical distribution function, i.e.

$$G_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(U_i \le t). \tag{A.10}$$

Using the probability integral transform, we write

$$D(X_{1,n}, X_{2,n}, \ldots, X_{n,n}) = D(Q(U_{1,n}), Q(U_{2,n}), \ldots, Q(U_{n,n})).$$

The idea of the proof is to apply the probability integral transform, decompose the relevant sums of order statistics into sums of Lebesgue-Stieltjes integrals involving $G_n(t)$, integrating by parts to obtain integrals of the uniform empirical process and then applying the well-known Brownian bridge approximation to the latter. See e.g. Csörgő, Haeusler and Mason [14] and Mason [41] for details of this type of decomposition.

For integers $0 \le m < n-k \le n$

$$\begin{aligned} \frac{1}{n}\sum_{i=m+1}^{n-k} X_{i,n} - \int_{m/n}^{1-k/n} Q(s)ds &= \int_{m/n}^{1-k/n} (s - G_n(s))dQ(s) + \int_{m/n}^{U_{m,n}} \left(G_n(s) - \frac{m}{n}\right) dQ(s) \\ &\quad + \int_{U_{n-k,n}}^{1-k/n} \left(G_n(s) - \frac{n-k}{n}\right) dQ(s). \end{aligned}$$

For the lower sum, this gives

$$D_{n,\alpha} - \int_0^{[n\alpha]/n} Q(s)ds = \int_0^{[n\alpha]/n} (s - G_n(s))dQ(s) + \int_{U_{[n\alpha],n}}^{[n\alpha]/n} \left(G_n(s) - \frac{[n\alpha]}{n}\right) dQ(s). \tag{A.11}$$

Clearly $\frac{[n\alpha]}{n} \to \alpha$ and $U_{[n\alpha]} \xrightarrow{P} \alpha$, at a rate of at least $n^{-1/2}$, so it follows that

$$D_{n,\alpha} - D_\alpha = \int_0^\alpha (s - G_n(s))dQ(s) + o_P(n^{-1/2}).$$

It is well-known that the process $\sqrt{n}(u - G_n(u))$ can be approximated on a special probability space by a Brownian bridge process $B_n(u)$. See e.g. Csörgő, Csörgő, Horváth and Mason [13] in this respect. Applying this to the above, gives

$$\sqrt{n}(D_{n,\alpha} - D_\alpha) = \int_0^\alpha B_n(s)dQ(s) + o_P(1). \tag{A.12}$$

Clearly the first term on the right hand side has a $N\left(0, \sigma^2(0,\alpha)\right)$ distribution, with

$$\sigma^2(s,t) = \operatorname{Var}\left(\int_s^t B_n(u)dQ(u)\right) = \int_s^t \int_s^t (u \wedge v - uv)dQ(u)dQ(v). \tag{A.13}$$

For the upper sum $N_{n,\beta}$ one has to be more careful in order to avoid terms diverging to infinity. Remember that we work under the assumption that $1 - F(x) = x^{-1/\gamma} l_F(x)$, with $\gamma < 1$.

Now, for $\gamma > \frac{1}{2}$, this upper sum converges to a stable distribution of index $1/\gamma$. See e.g. Csörgő, Horváth and Mason [15] for results applicable to this case. Our interest is to derive results for the limiting normal case, so we will not consider this case.

The case $\gamma = \frac{1}{2}$ does give a limiting normal distribution, but may have $\sigma^2\left(\frac{1}{2},1\right) = \infty$. In such cases we need to work with $\sigma^2\left(\frac{1}{2}, 1 - \frac{1}{n}\right)$ and the results still go through in a slightly modified form. However, for ease of exposition we will assume that $\sigma^2\left(\frac{1}{2},1\right) < \infty$. Under that assumption it then follows, in a fashion similar to the lower sum, that

$$\sqrt{n}(N_{n,\beta} - N_\beta) = \int_\beta^1 B_n(s)dQ(s) + o_P(1). \tag{A.14}$$

Clearly the first term on the right hand side has a $N\left(0, \sigma^2(\beta,1)\right)$ distribution.

Using (A.12) and (A.14) in (A.9), gives

$$\sqrt{n}(T_n(\alpha,\beta) - T(\alpha,\beta)) = \frac{D_\alpha Z_1 - N_\beta Z_2}{D_\alpha^2} + o_P(1), \tag{A.15}$$

with

$$Z_1 = \int_\beta^1 B_n(s)dQ(s) \sim N(0, \sigma^2(\beta,1))$$

and

$$Z_2 = \int_0^{\alpha} B_n(s)dQ(s) \sim N\left(0, \sigma^2(0,\alpha)\right).$$

We still need to find the covariance between Z_1 and Z_2. For this, use the properties of the Brownian bridge

$$\begin{aligned}
EZ_1Z_2 &= E\int_{\beta}^{1} B_n(s)dQ(s)\int_0^{\alpha} B_n(s)dQ(s) \\
&= \int_{\beta}^{1}\int_0^{\alpha} EB_n(s)B_n(t)dQ(s)dQ(t) \\
&= \int_{\beta}^{1}\int_0^{\alpha} s(1-t)dQ(s)dQ(t) \\
&= \int_0^{\alpha} sdQ(s)\int_{\beta}^{1}(1-t)dQ(t).
\end{aligned}$$

Using partial integration, it follows that

$$\int_0^{\alpha} sdQ(s) = \alpha Q(\alpha) - D_{\alpha}$$

and

$$\int_{\beta}^{1}(1-t)dQ(t) = N_{\beta} - (1-\beta)Q(\beta).$$

It follows that

$$\begin{aligned}
\operatorname{Var}\left(\frac{D_{\alpha}Z_1 - N_{\beta}Z_2}{D_{\alpha}^2}\right) &= D_{\alpha}^{-4}(D_{\alpha}^2\operatorname{Var}(Z_1) + N_{\beta}^2\operatorname{Var}(Z_2) - 2D_{\alpha}N_{\beta}EZ_1Z_2) \\
&= \frac{D_{\alpha}^2\sigma^2(\beta,1) + N_{\beta}^2\sigma^2(0,\alpha) - 2D_{\alpha}N_{\beta}(\alpha Q(\alpha) - D_{\alpha})(N_{\beta} - (1-\beta)Q(\beta))}{D_{\alpha}^4} \\
&= \sigma_{\eta}^2(\alpha,\beta).
\end{aligned}$$

We conclude that as $n \to \infty$

$$\sqrt{n}(T_n(\alpha,\beta) - T(\alpha,\beta)) \xrightarrow{D} N\left(0, \sigma_{\eta}^2(\alpha,\beta)\right). \tag{A.16}$$

Taking $\alpha = 0.2$ and $\beta = 0.8$ completes the proof.

Appendix B

Sampling Distributions for Nonparametric Estimators of Inequality Measures

The following graphs display the normal Q-Q plots of the sampling distributions of the nonparametric estimators of the generalized entropy, the Atkinson and the QSR measures of inequality.

B.1 Generalized Entropy with Parameter 0 (GE0)

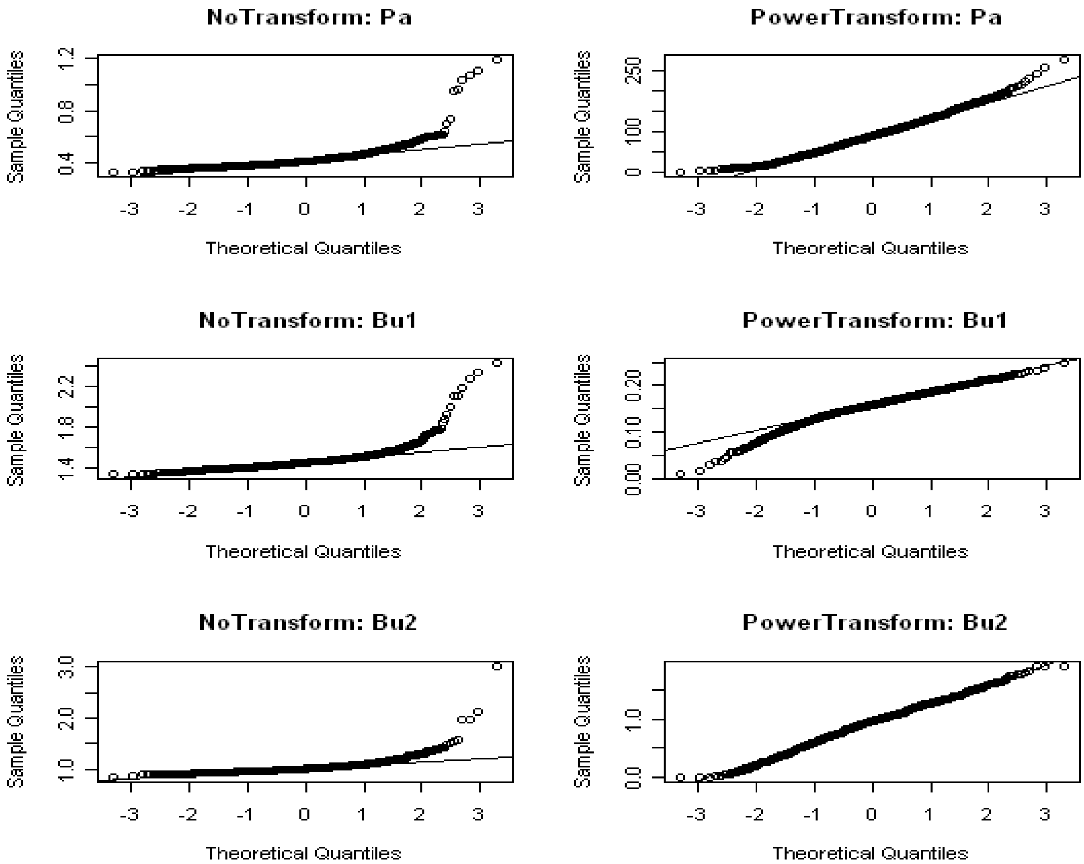

Figure B.1: Sampling Distribution for GE0 (Samples from Pa, Bu1, Bu2 Distributions)

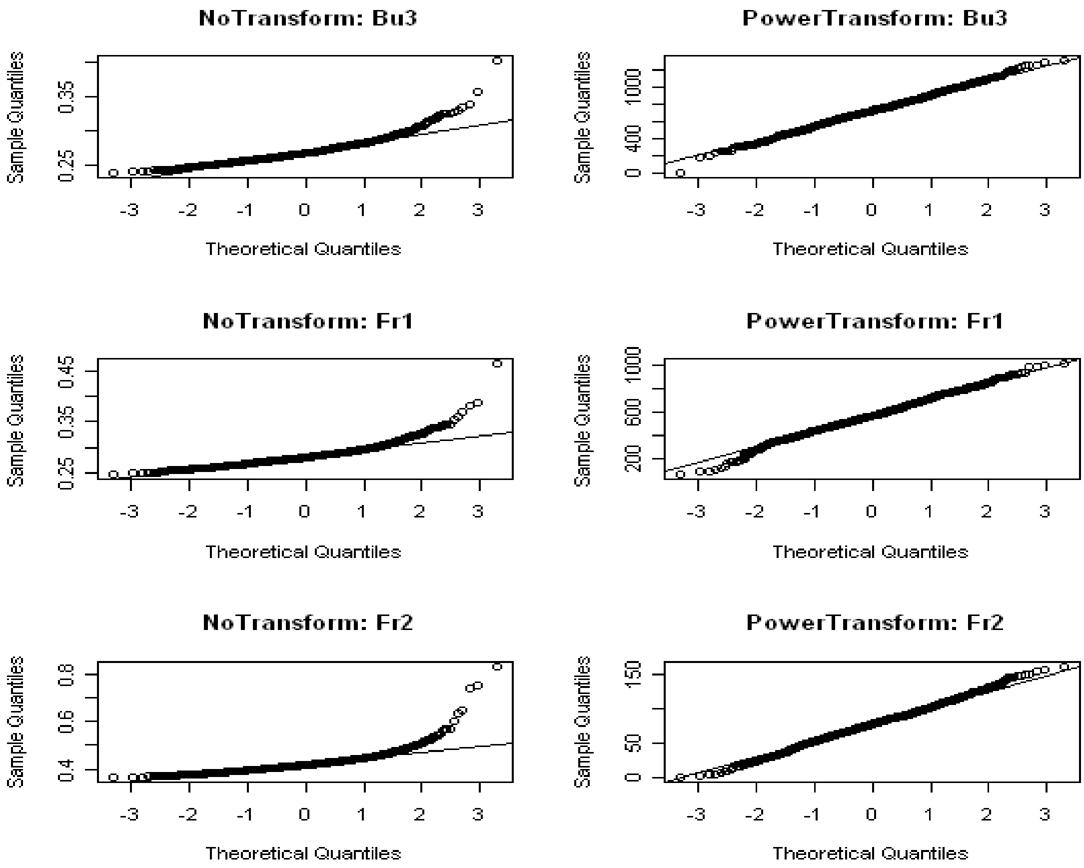

Figure B.2: Sampling Distribution for GE0 (Samples from Bu3, Fr1, Fr2 Distributions)

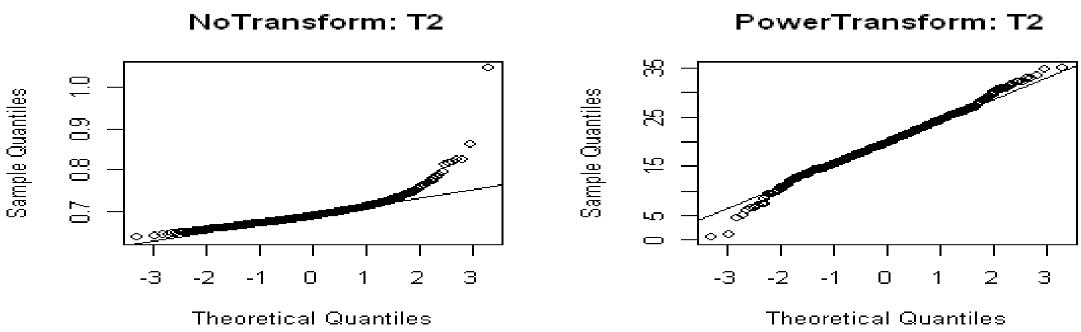

Figure B.3: Sampling Distribution for GE0 (Samples from T2 Distribution)

B.2 Generalized Entropy with Parameter 1 (GE1)

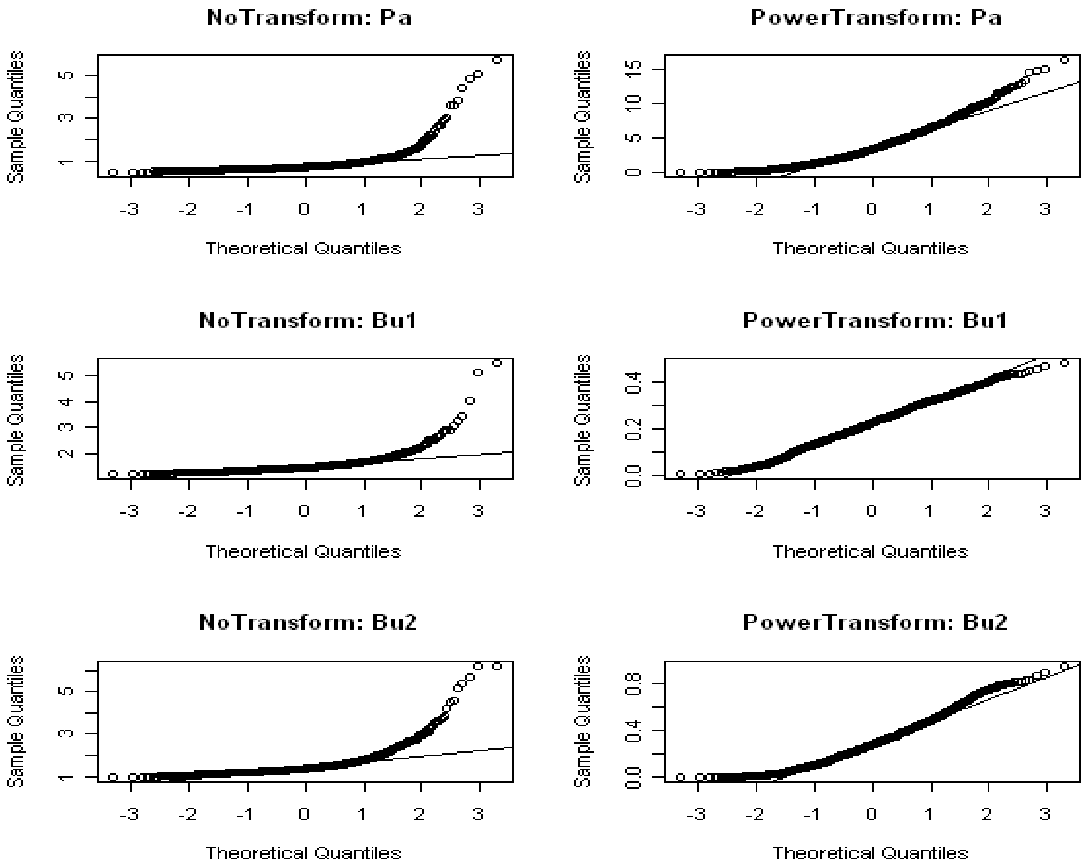

Figure B.4: Sampling Distribution for GE1 (Samples from Pa, Bu1, Bu2 Distributions)

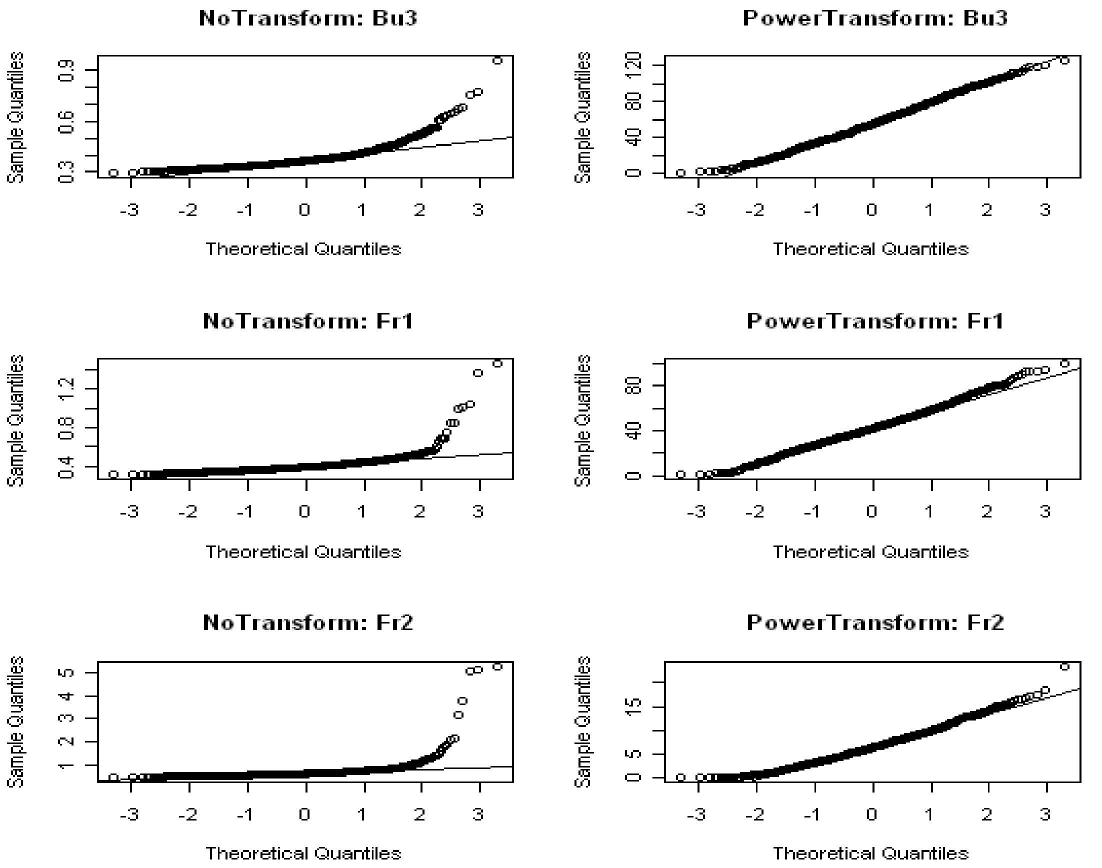

Figure B.5: Sampling Distribution for GE1 (Samples from Bu3, Fr1, Fr2 Distributions)

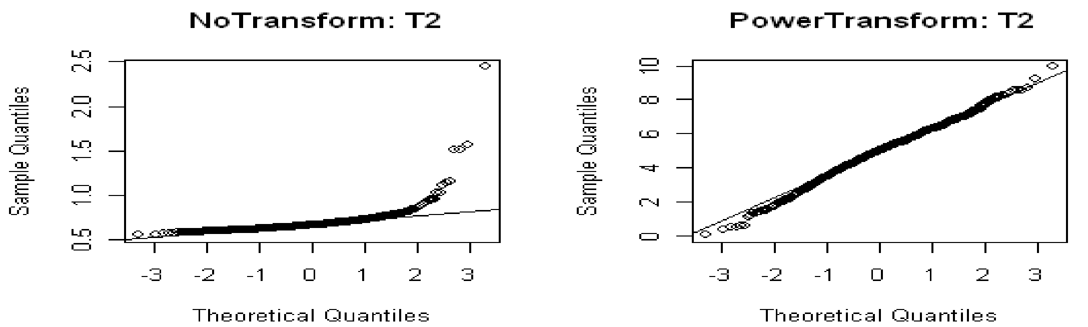

Figure B.6: Sampling Distribution for GE1 (Samples from T2 Distribution)

B.3 Generalized Entropy with Parameter 1.3 (GE1.3)

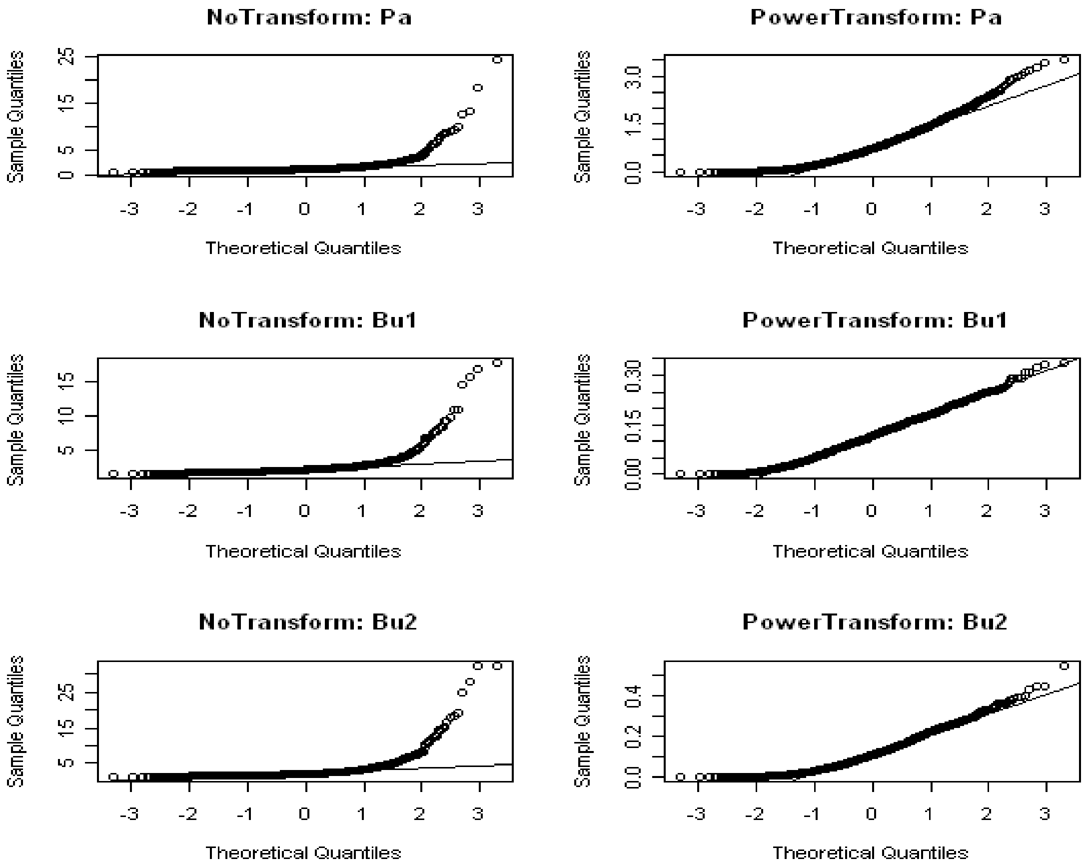

Figure B.7: Sampling Distribution for GE1.3 (Samples from Pa, Bu1, Bu2 Distributions)

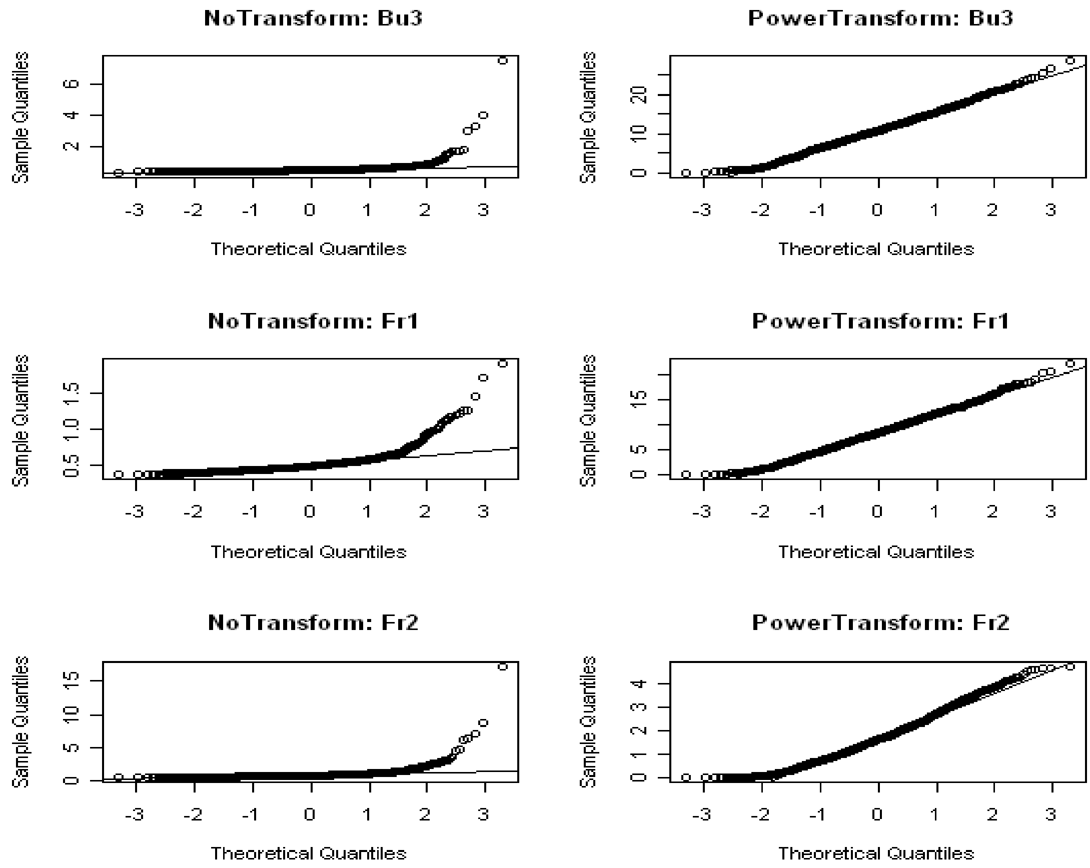

Figure B.8: Sampling Distribution for GE1.3 (Samples from Bu3, Fr1, Fr2 Distributions)

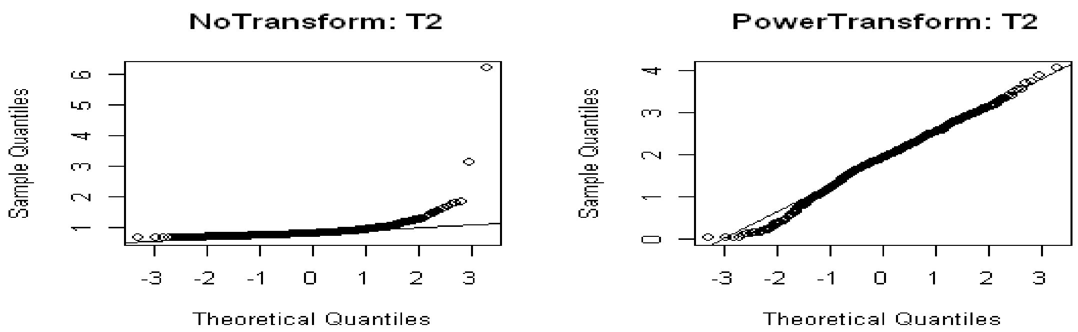

Figure B.9: Sampling Distribution for GE1.3 (Samples from T2 Distribution)

B.4 Atkinson Coefficient with Parameter 1 (A1)

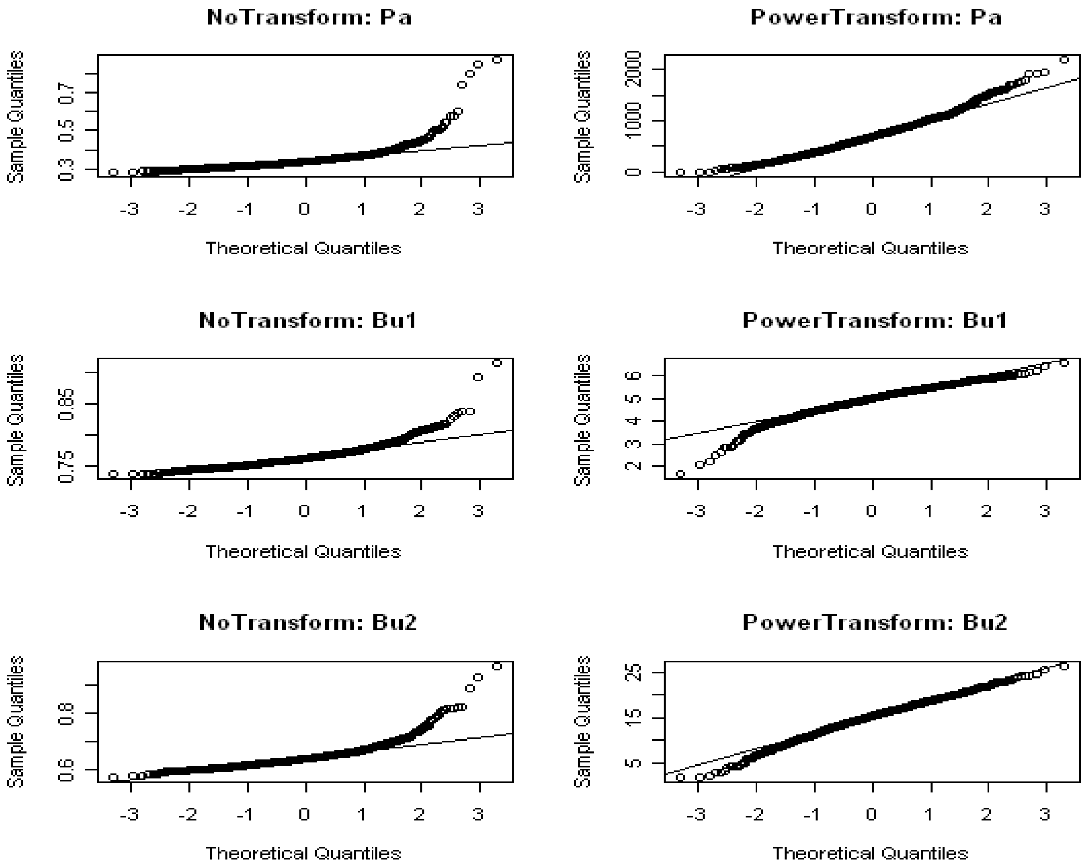

Figure B.10: Sampling Distribution for A1 (Samples from Pa, Bu1, Bu2 Distributions)

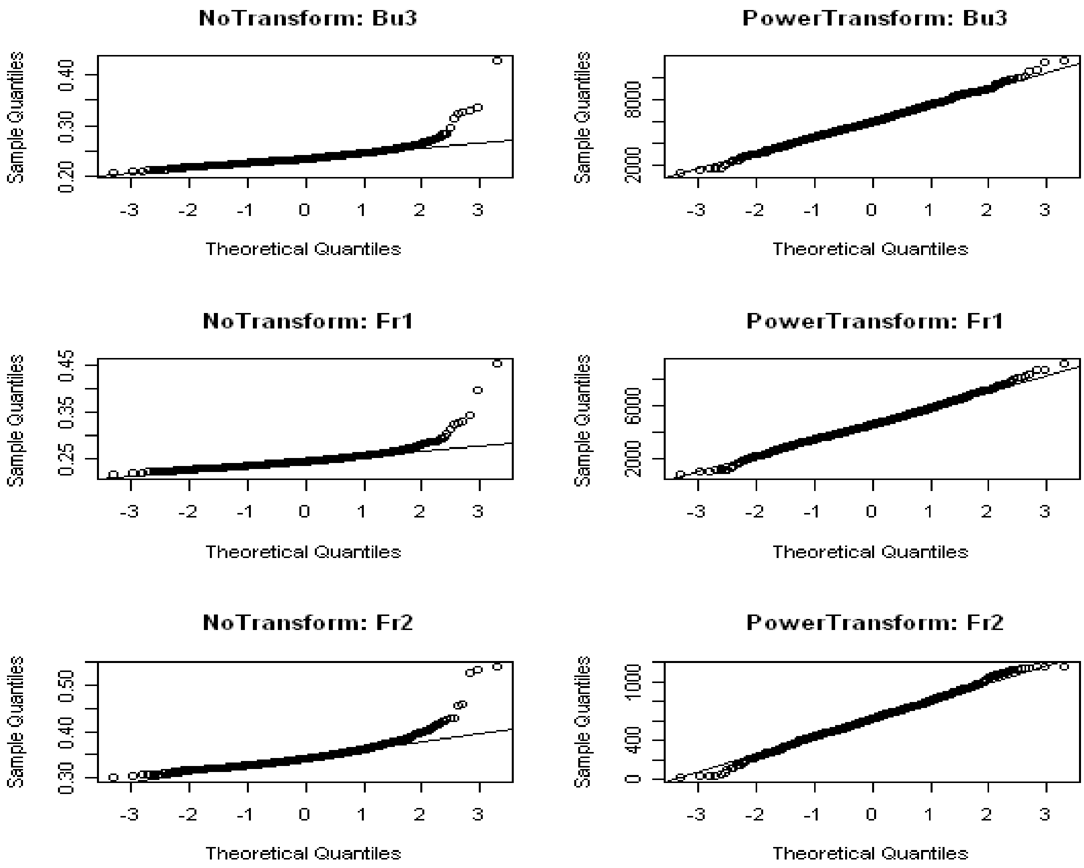

Figure B.11: Sampling Distribution for A1 (Samples from Bu3, Fr1, Fr2 Distributions)

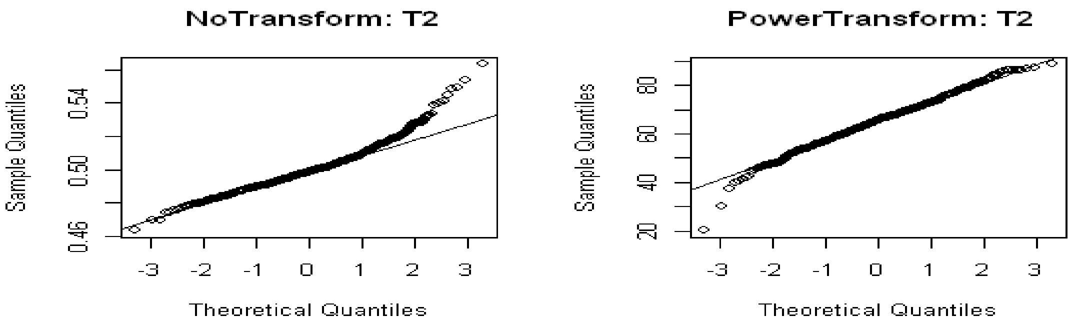

Figure B.12: Sampling Distribution for A1 (Samples from T2 Distribution)

B.5 Atkinson Coefficient with Parameter 1.5 (A1.5)

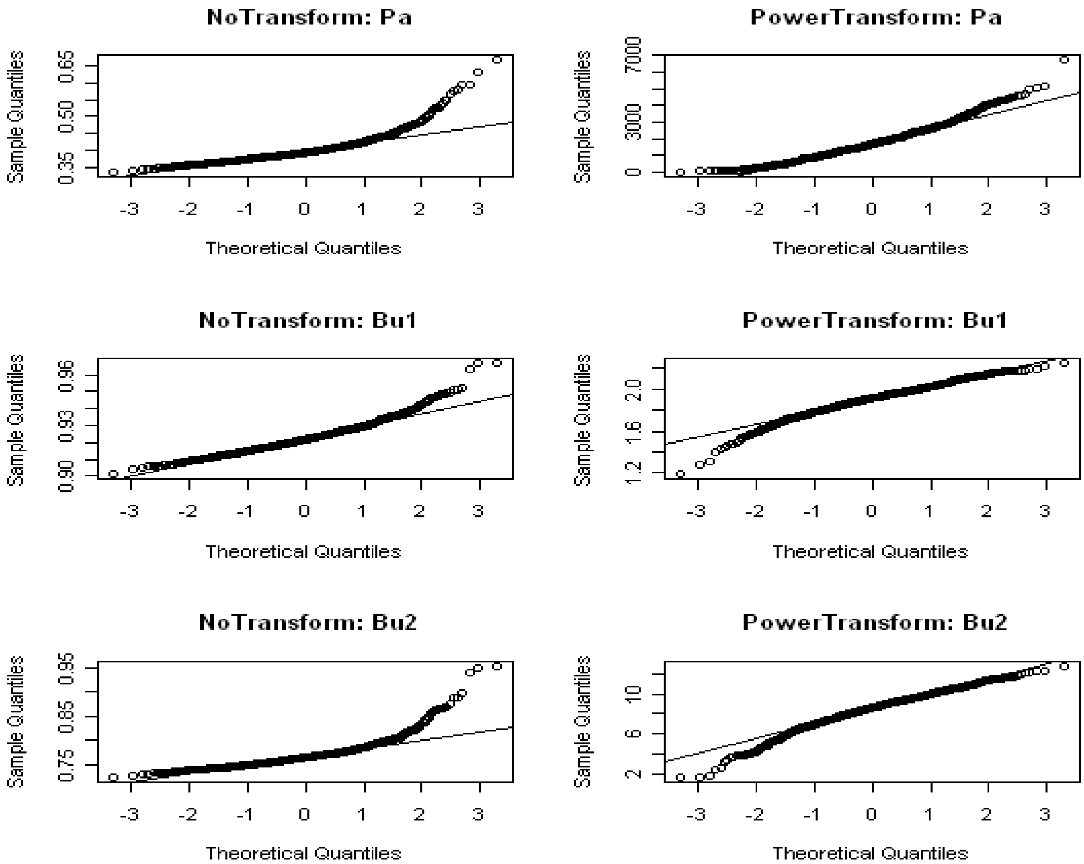

Figure B.13: Sampling Distribution for A1.5 (Samples from Pa, Bu1, Bu2 Distributions)

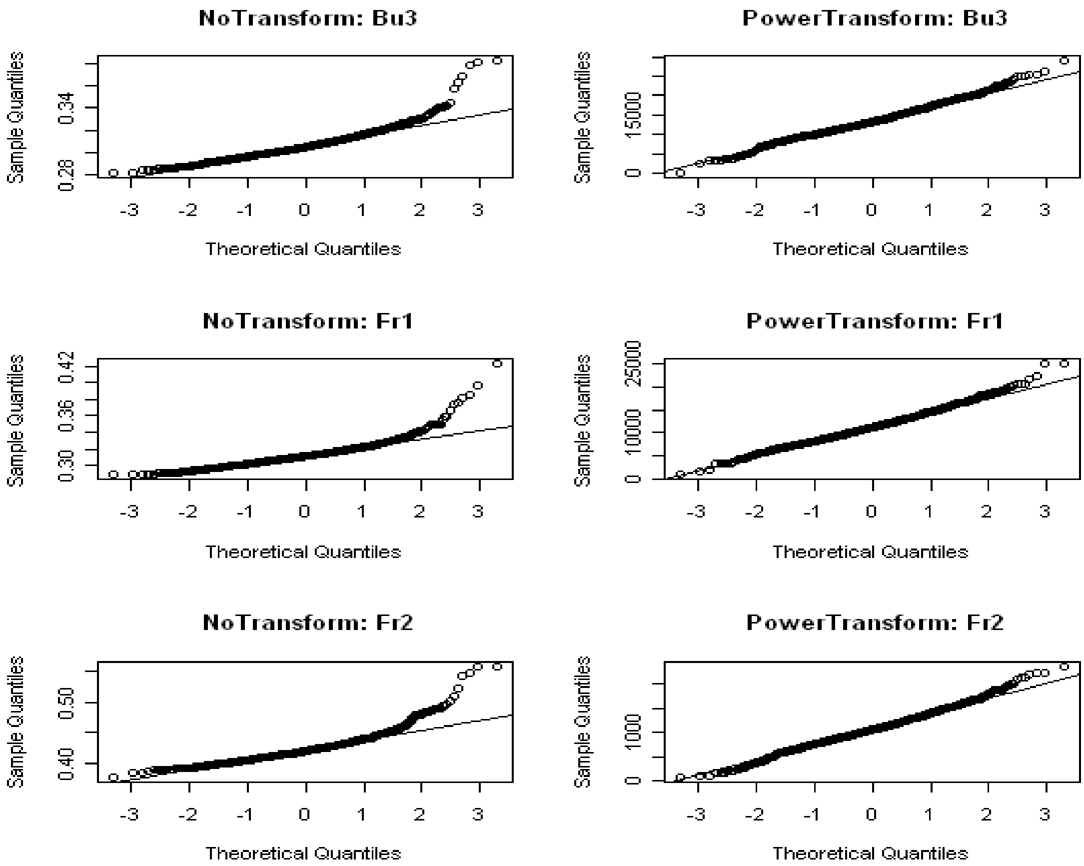

Figure B.14: Sampling Distribution for A1.5 (Samples from Bu3, Fr1, Fr2 Distributions)

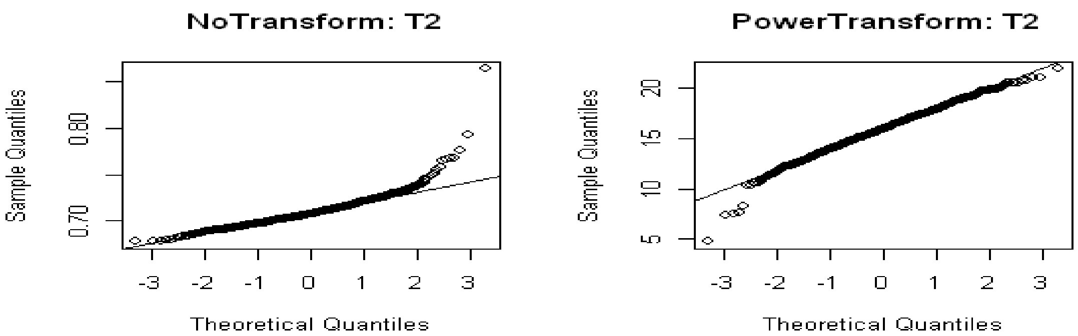

Figure B.15: Sampling Distribution for A1.5 (Samples from T2 Distribution)

B.6 Atkinson Coefficient with Parameter 2 (A2)

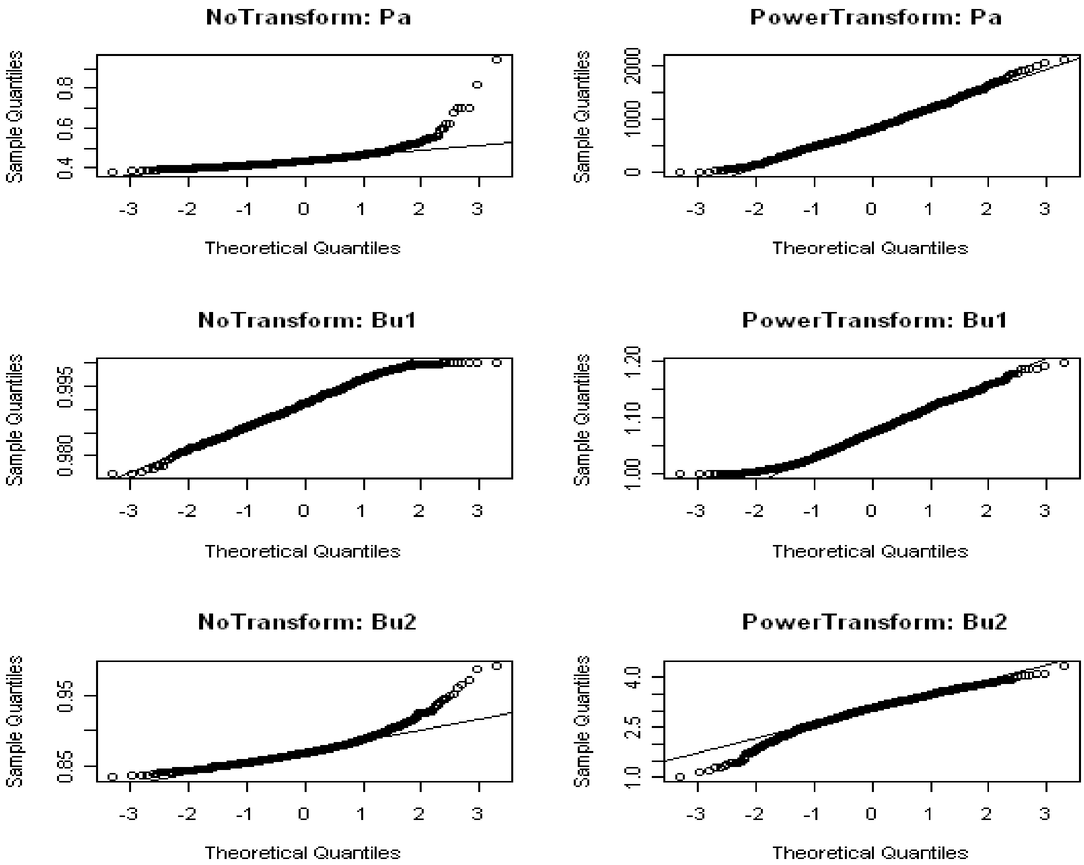

Figure B.16: Sampling Distribution for A2 (Samples from Pa, Bu1, Bu2 Distributions)

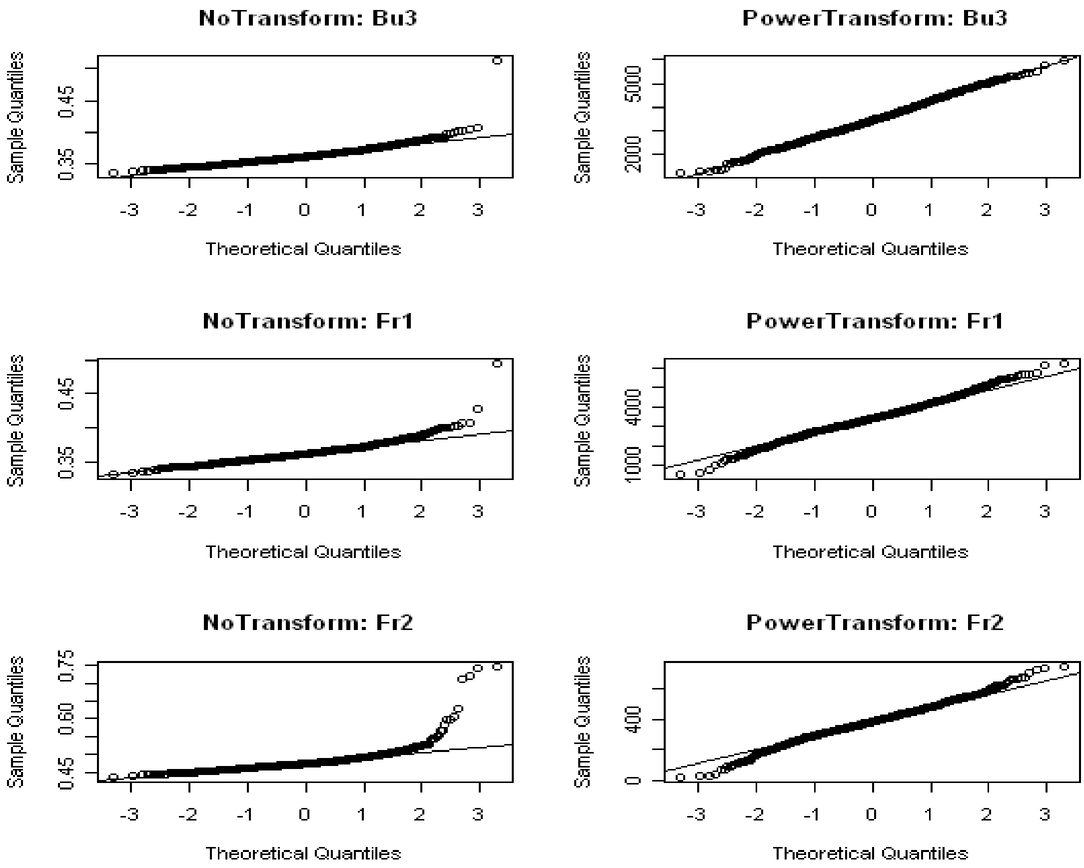

Figure B.17: Sampling Distribution for A2 (Samples from Bu3, Fr1, Fr2 Distributions)

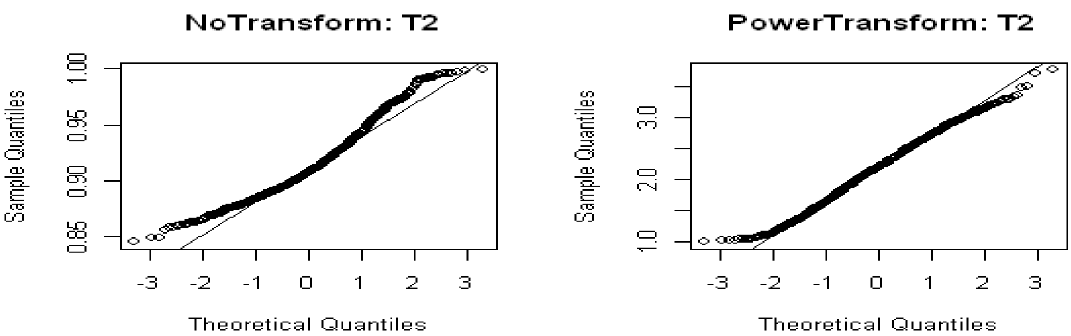

Figure B.18: Sampling Distribution for A2 (Samples from T2 Distribution)

Appendix C

Sampling Distributions for Semi-Parametric Estimators of Inequality Measures

The following graphs display the normal Q-Q plots of the sampling distributions of the semi-parametric estimators of the generalized entropy, the Atkinson and the QSR measures of inequality.

C.1 Generalized Entropy with Parameter 0 (GE0)

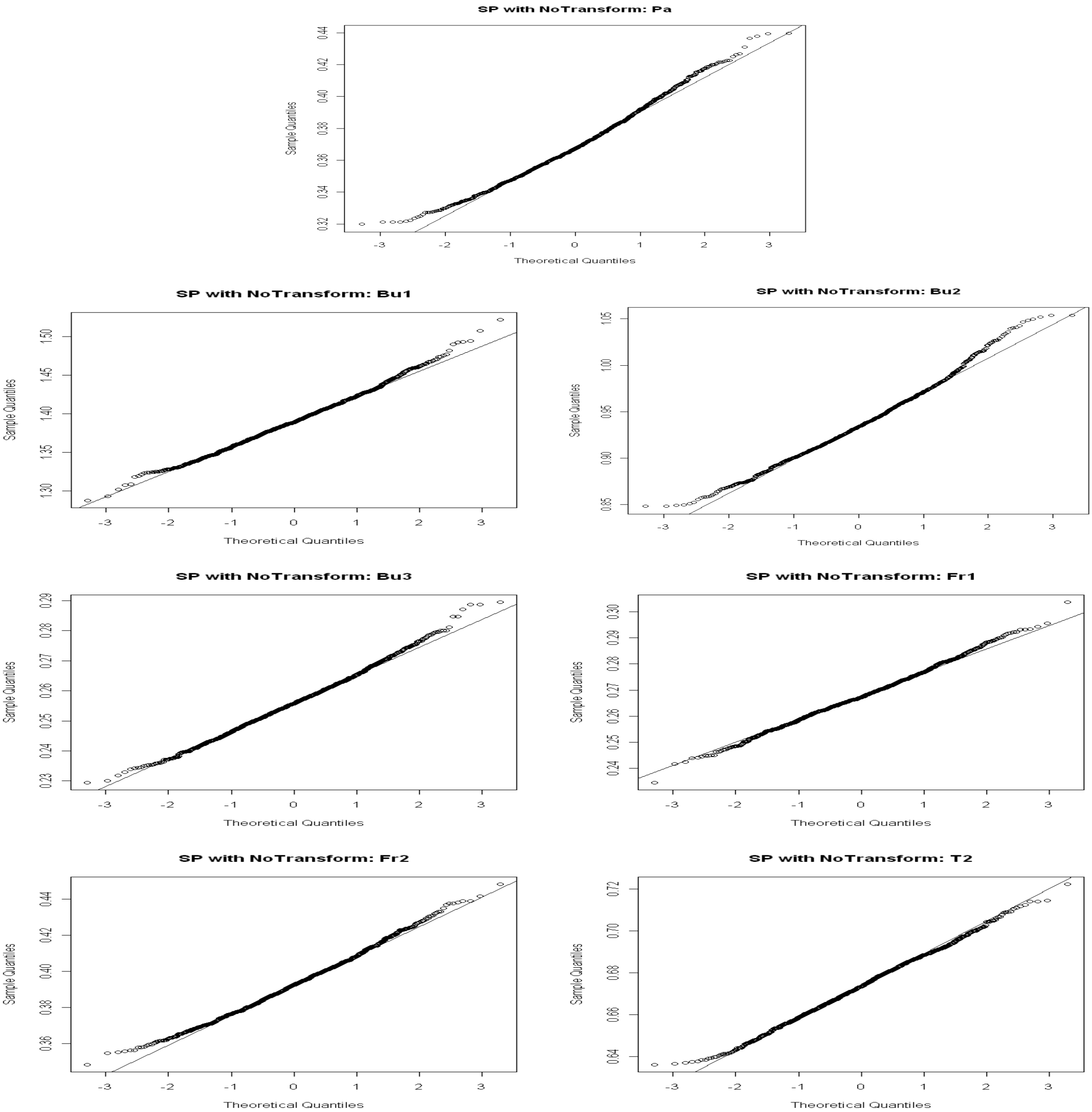

Figure C.1: Sampling Distribution for SP GE0 when Fitting the GPD to the Tails

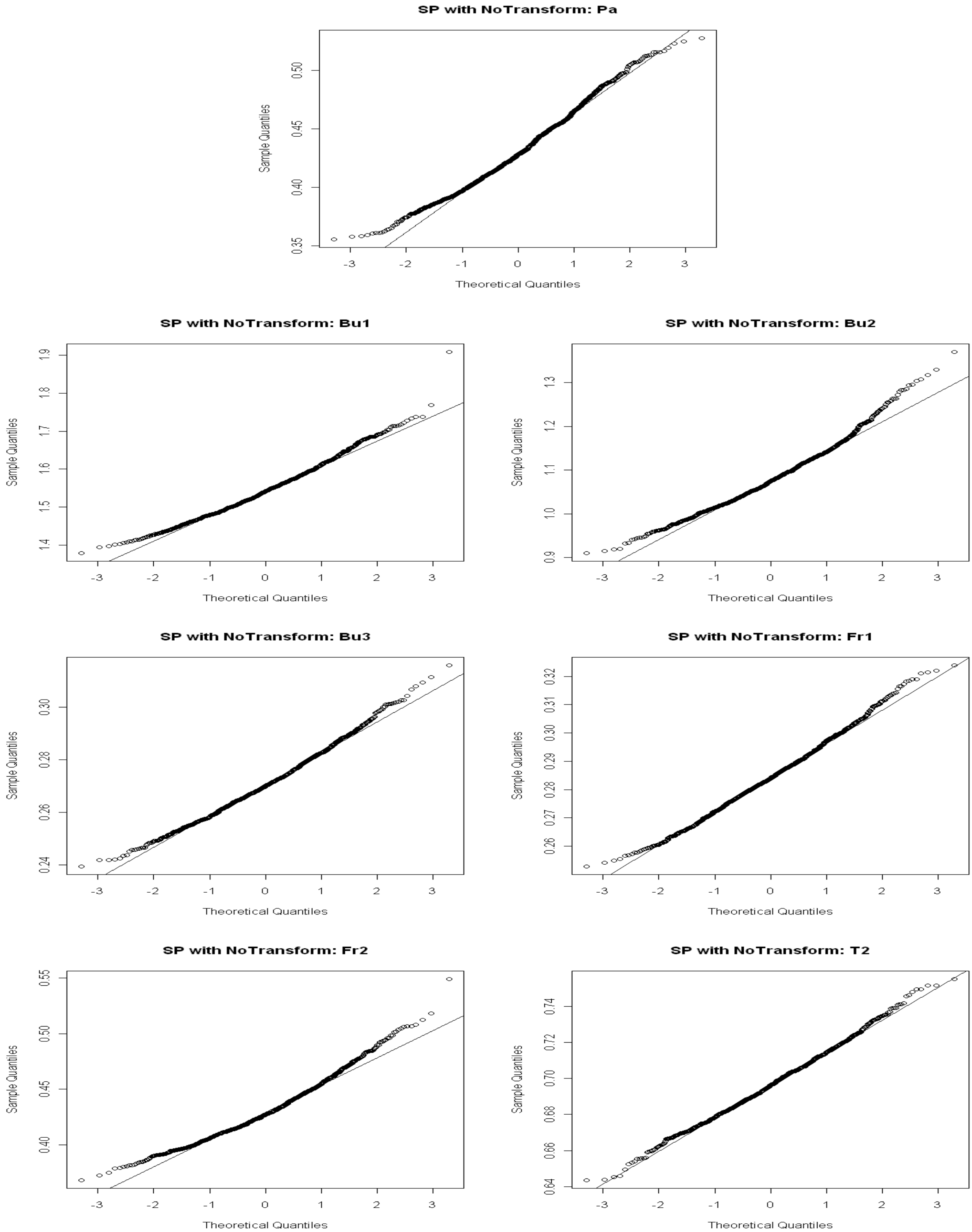

Figure C.2: Sampling Distribution for SP GE0 when Fitting the Strict Pareto to the Tails

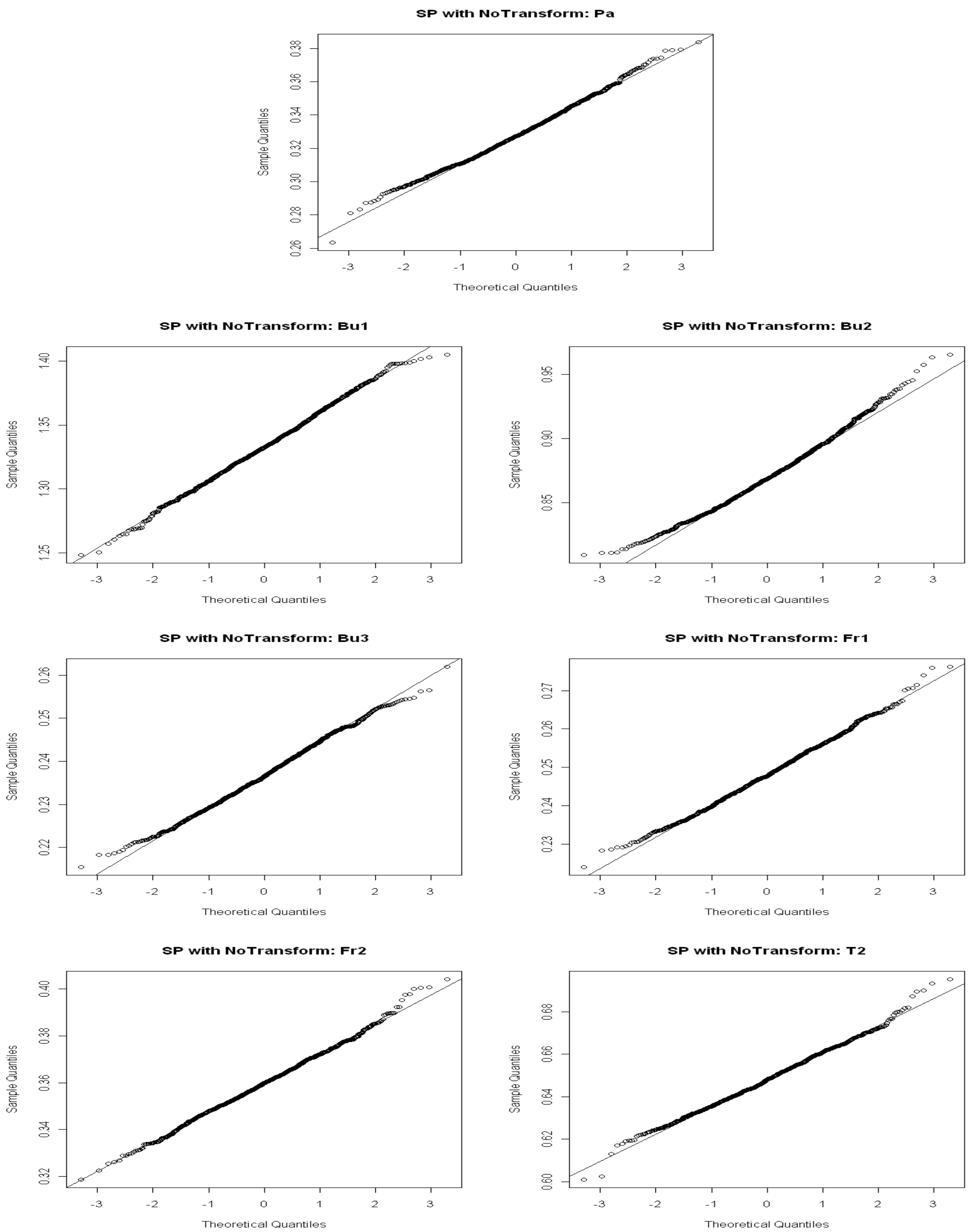

Figure C.3: Sampling Distribution for SP GE0 when Fitting the PPD to the Tails

C.2 Generalized Entropy with Parameter 1 (GE1)

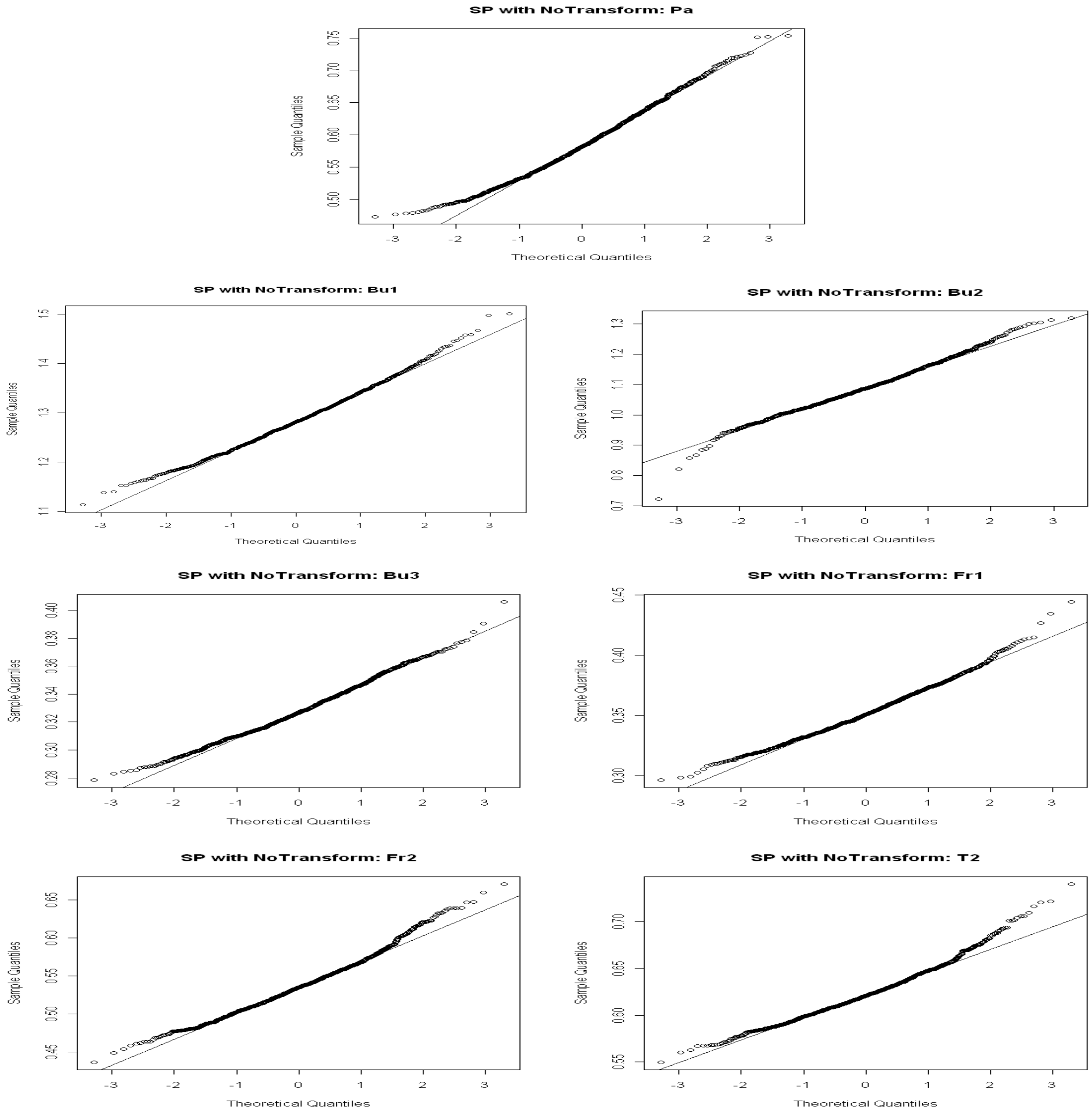

Figure C.4: Sampling Distribution for SP GE1 when Fitting the GPD to the Tails

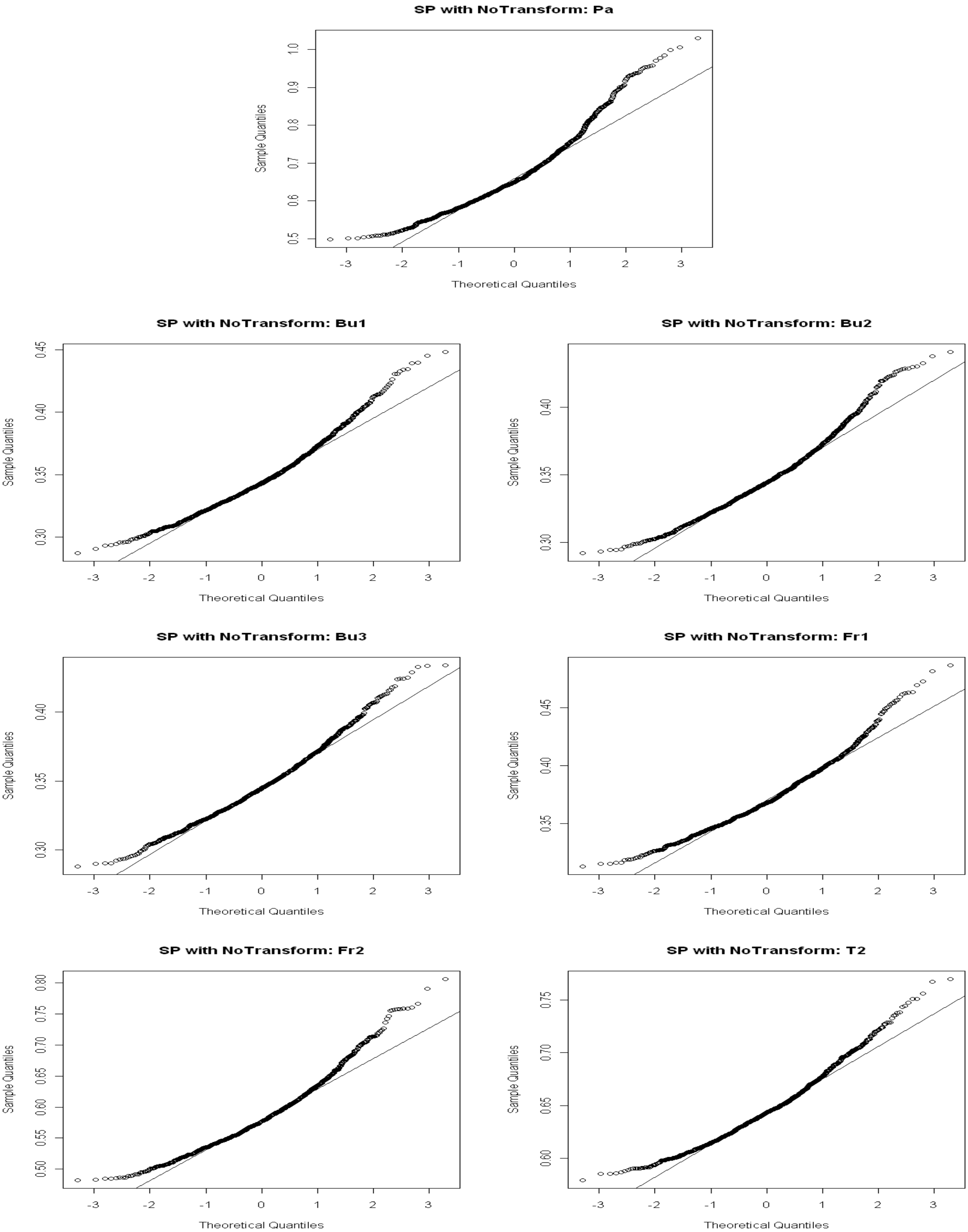

Figure C.5: Sampling Distribution for SP GE1 when Fitting the Strict Pareto to the Tails

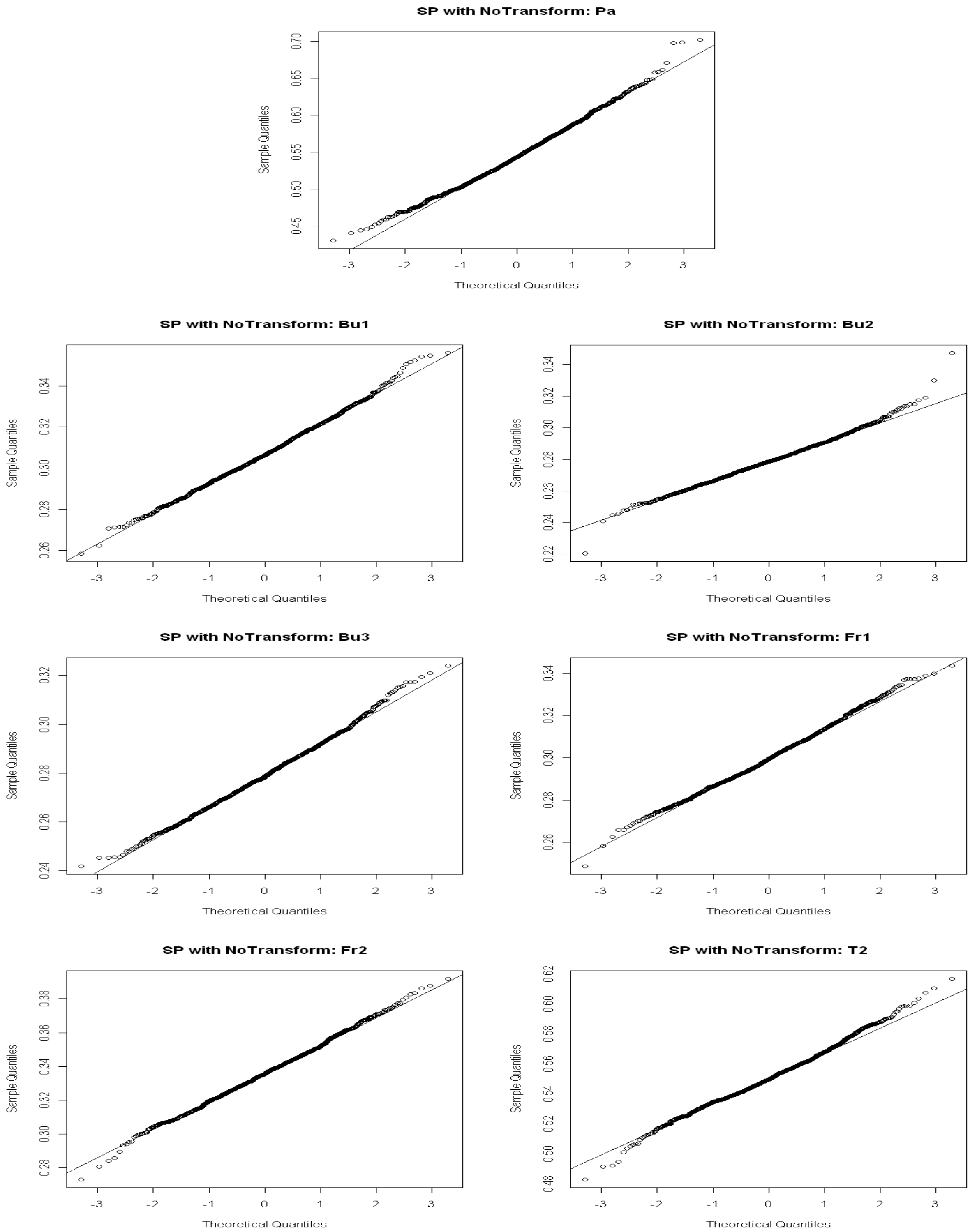

Figure C.6: Sampling Distribution for SP GE1 when Fitting the PPD to the Tails

C.3 Generalized Entropy with Parameter 1.3 (GE1.3)

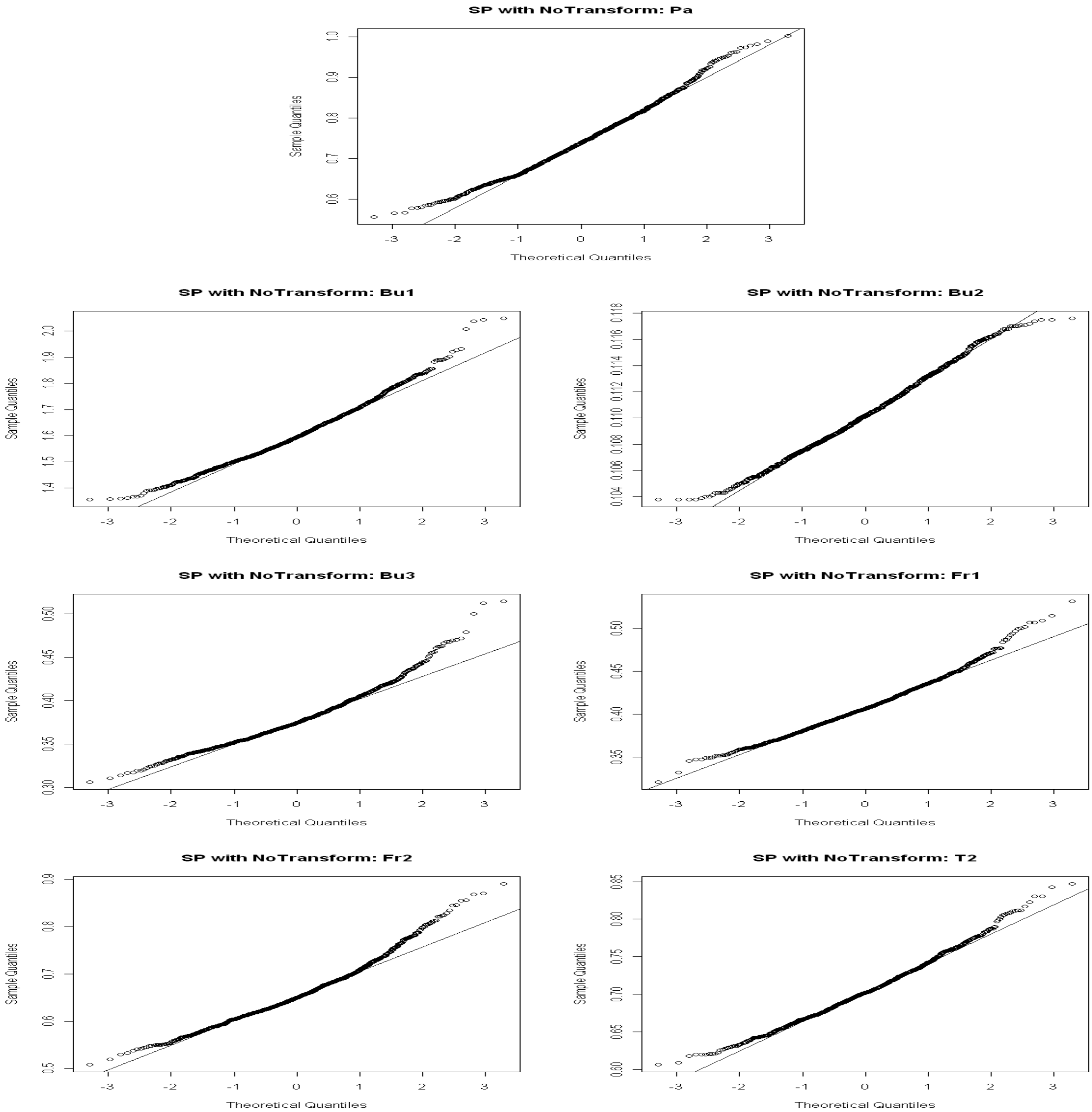

Figure C.7: Sampling Distribution for SP GE1.3 when Fitting the GPD to the Tails

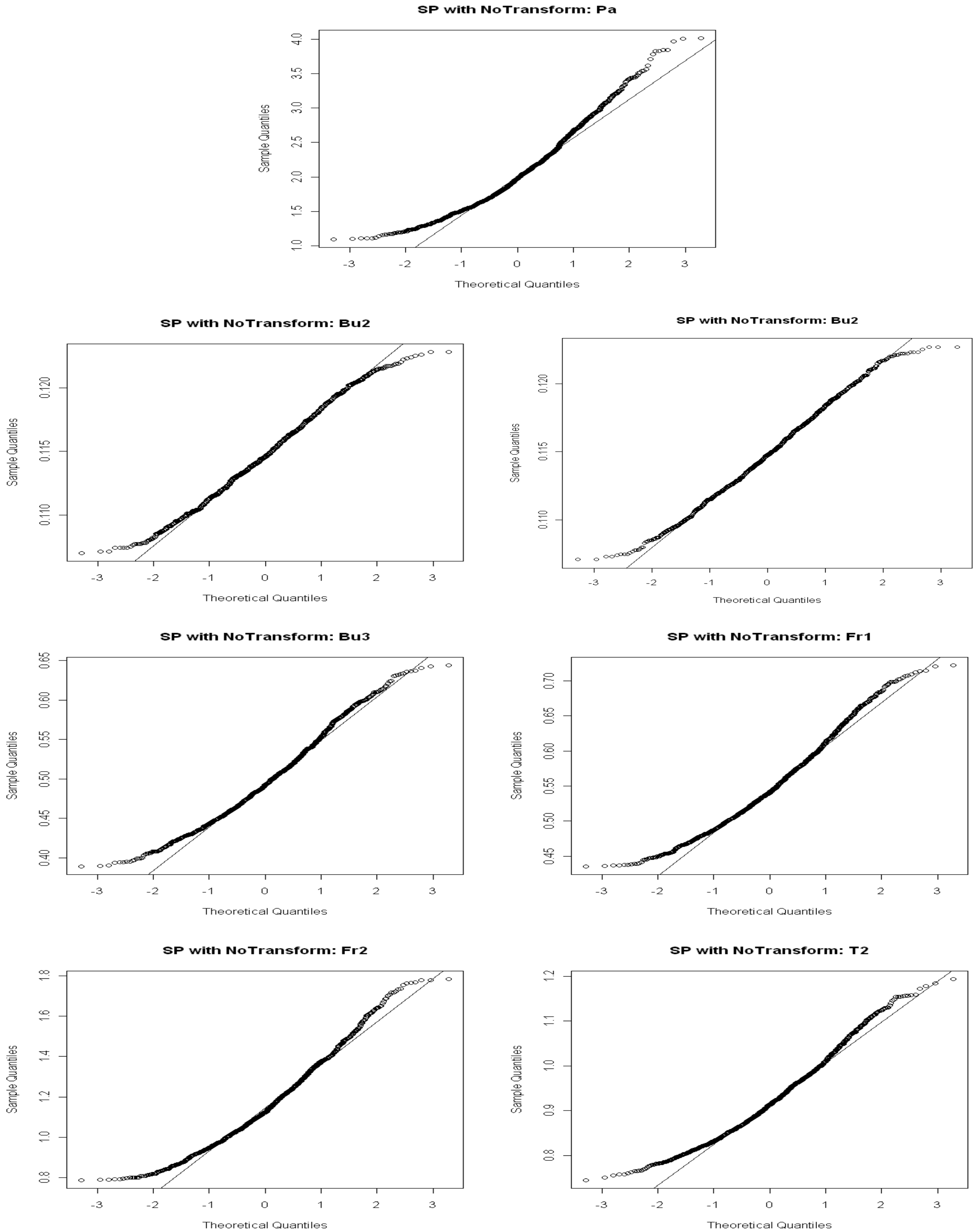

Figure C.8: Sampling Distribution for SP GE1.3 when Fitting the Strict Pareto to the Tails

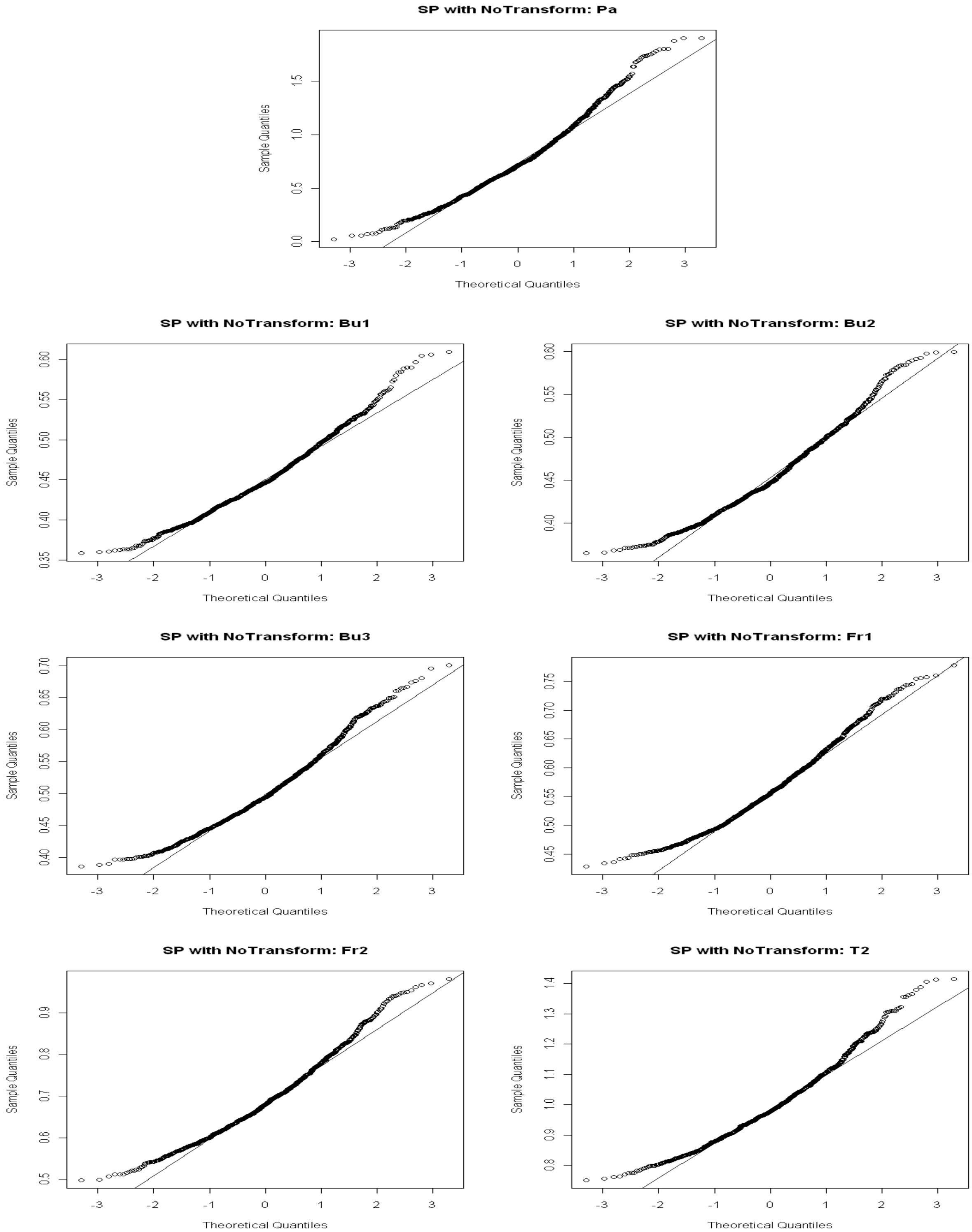

Figure C.9: Sampling Distribution for SP GE1.3 when Fitting the PPD to the Tails

C.4 Atkinson Coefficient with Parameter 1 (A1)

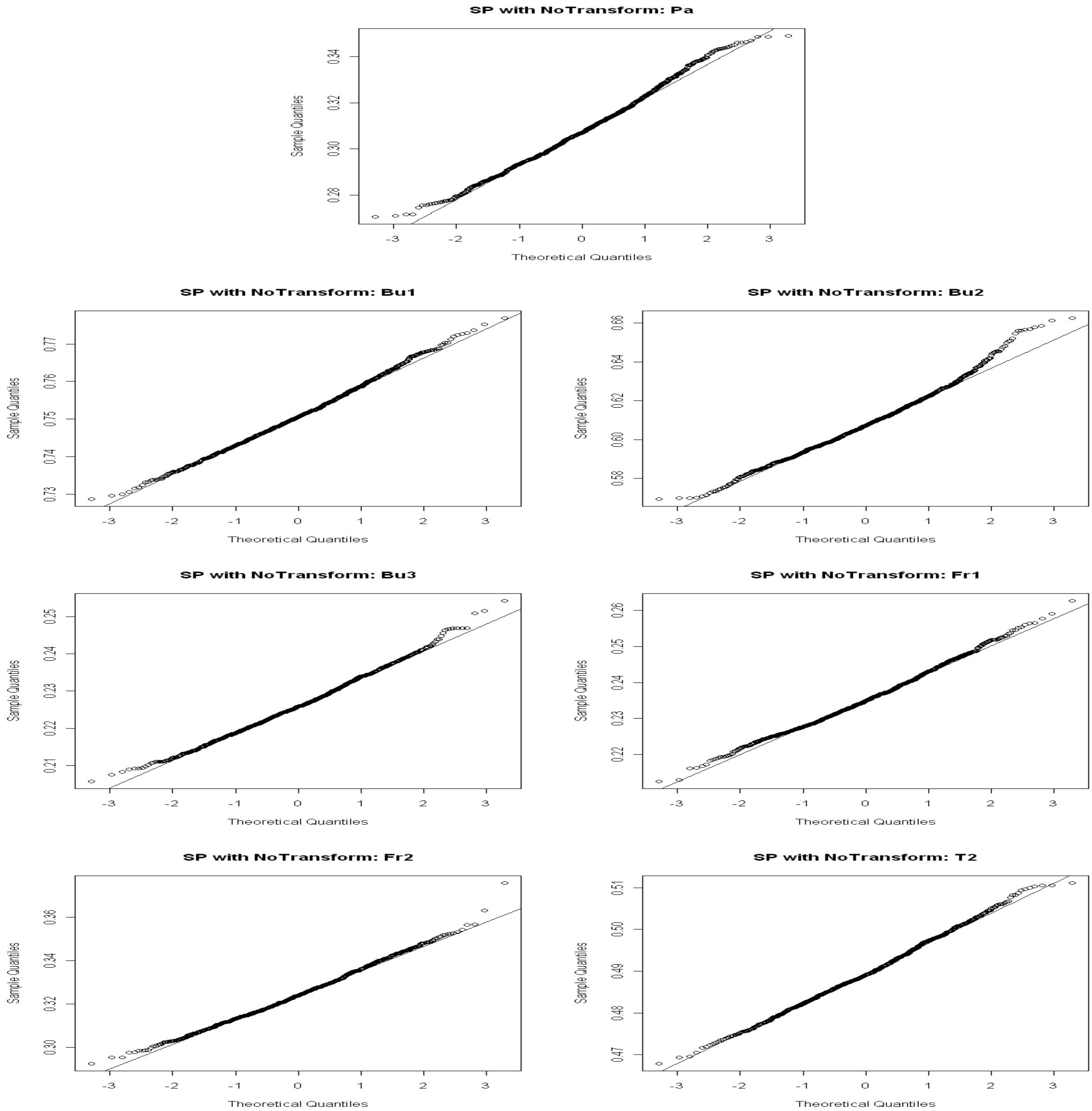

Figure C.10: Sampling Distribution for SP A1 when Fitting the GPD to the Tails

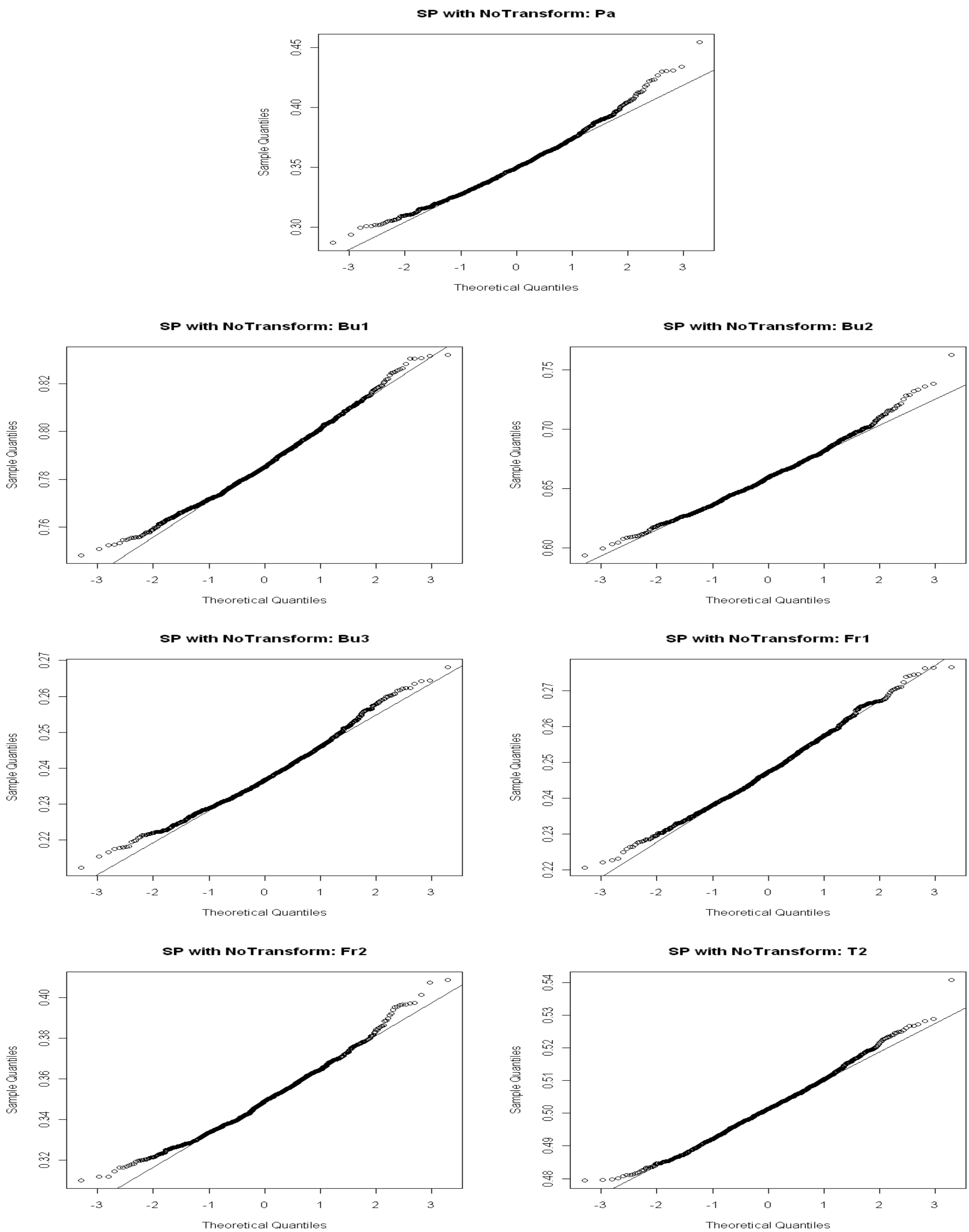

Figure C.11: Sampling Distribution for SP A1 when Fitting the Strict Pareto to the Tails

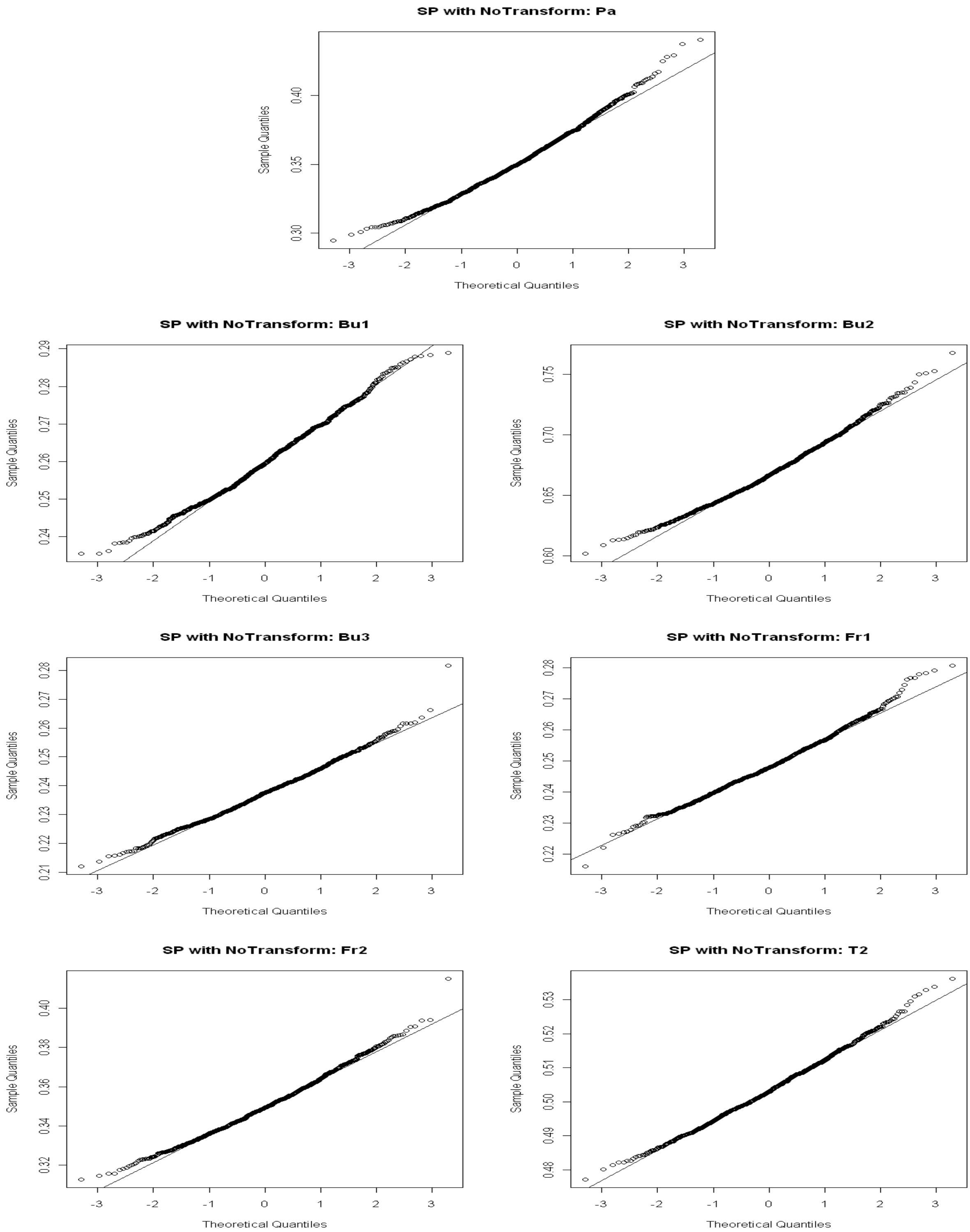

Figure C.12: Sampling Distribution for SP A1 when Fitting the PPD to the Tails

C.5 Atkinson Coefficient with Parameter 1.5 (A1.5)

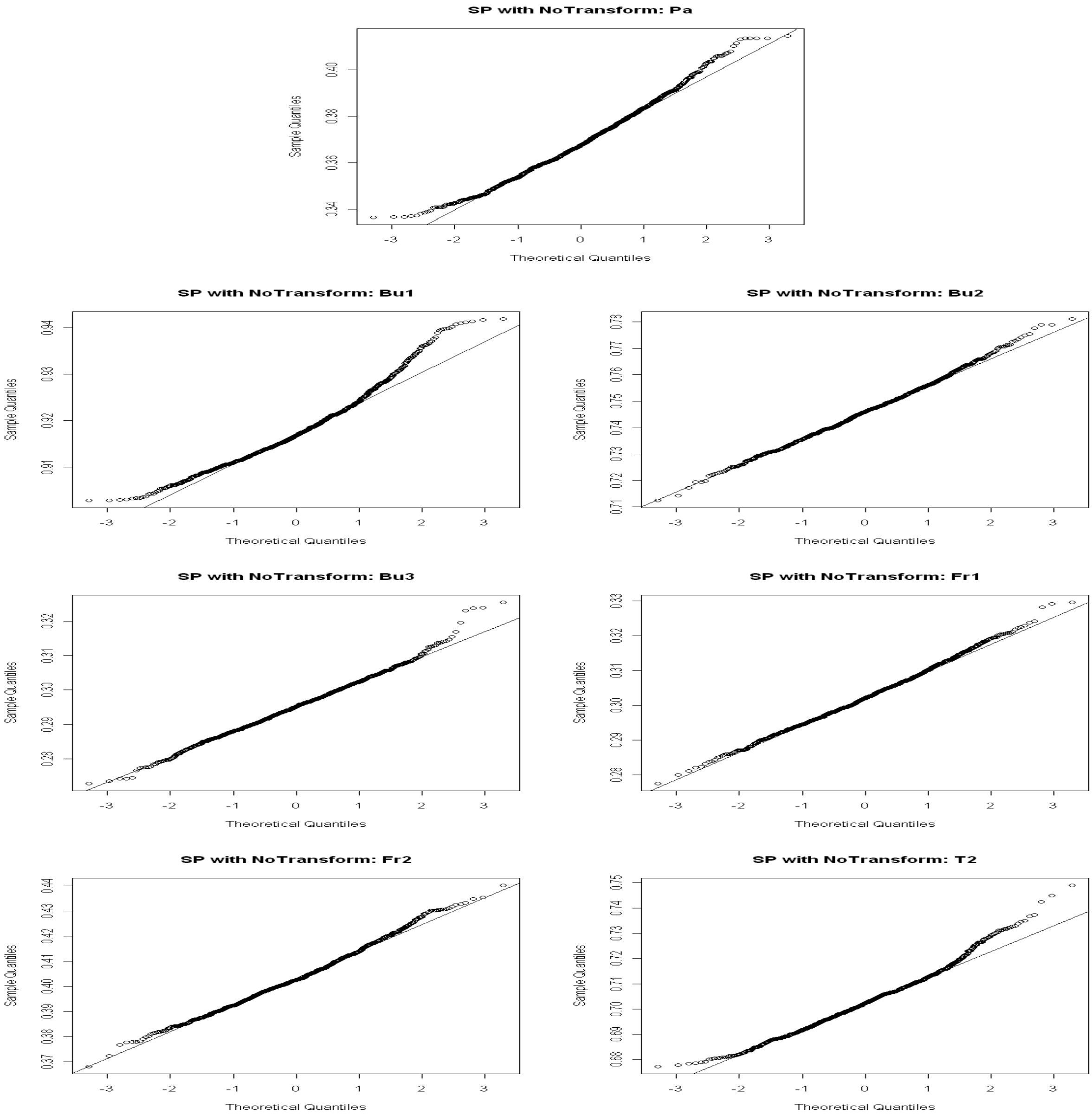

Figure C.13: Sampling Distribution for SP A1.5 when Fitting the GPD to the Tails

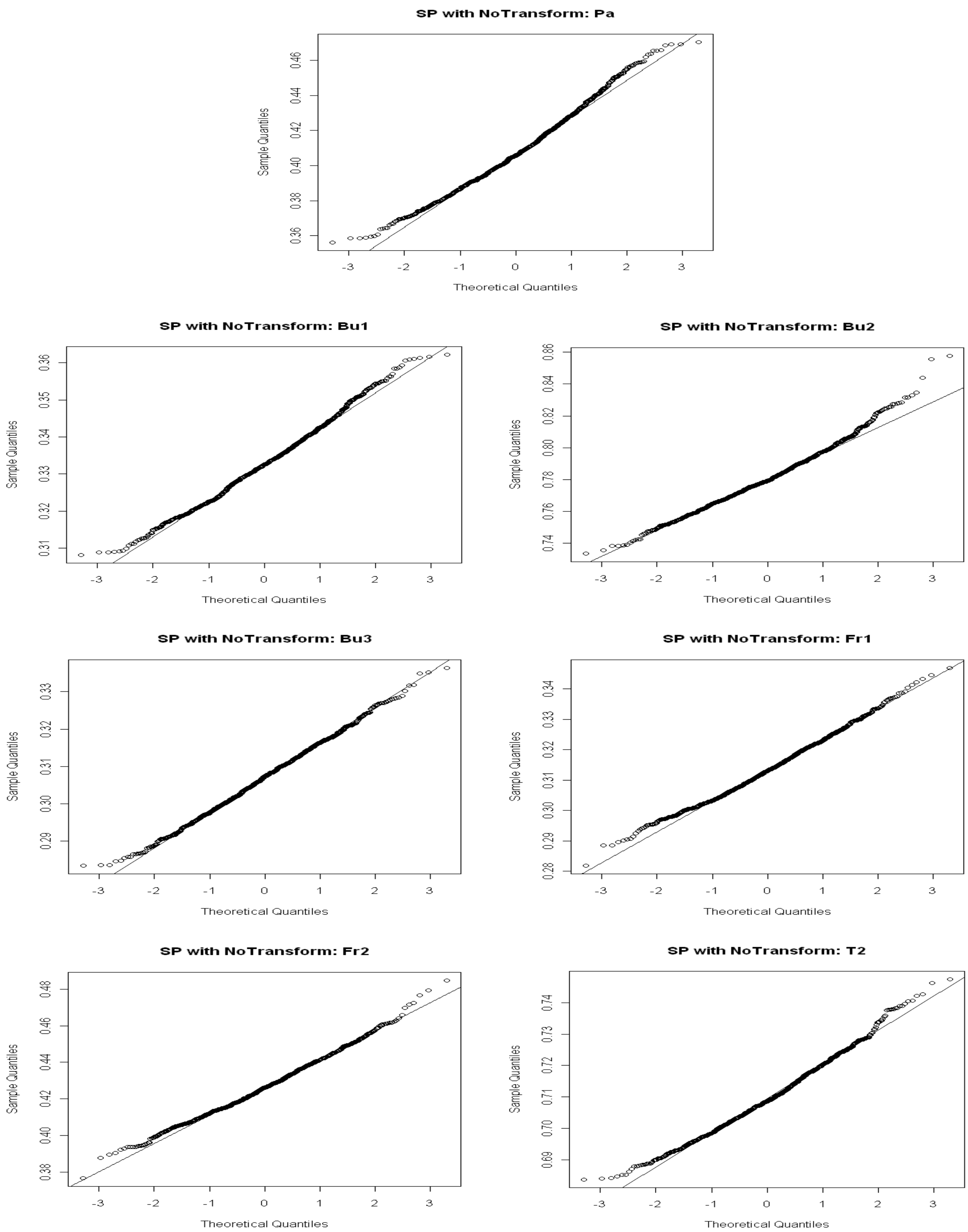

Figure C.14: Sampling Distribution for SP A1.5 when Fitting the Strict Pareto to the Tails

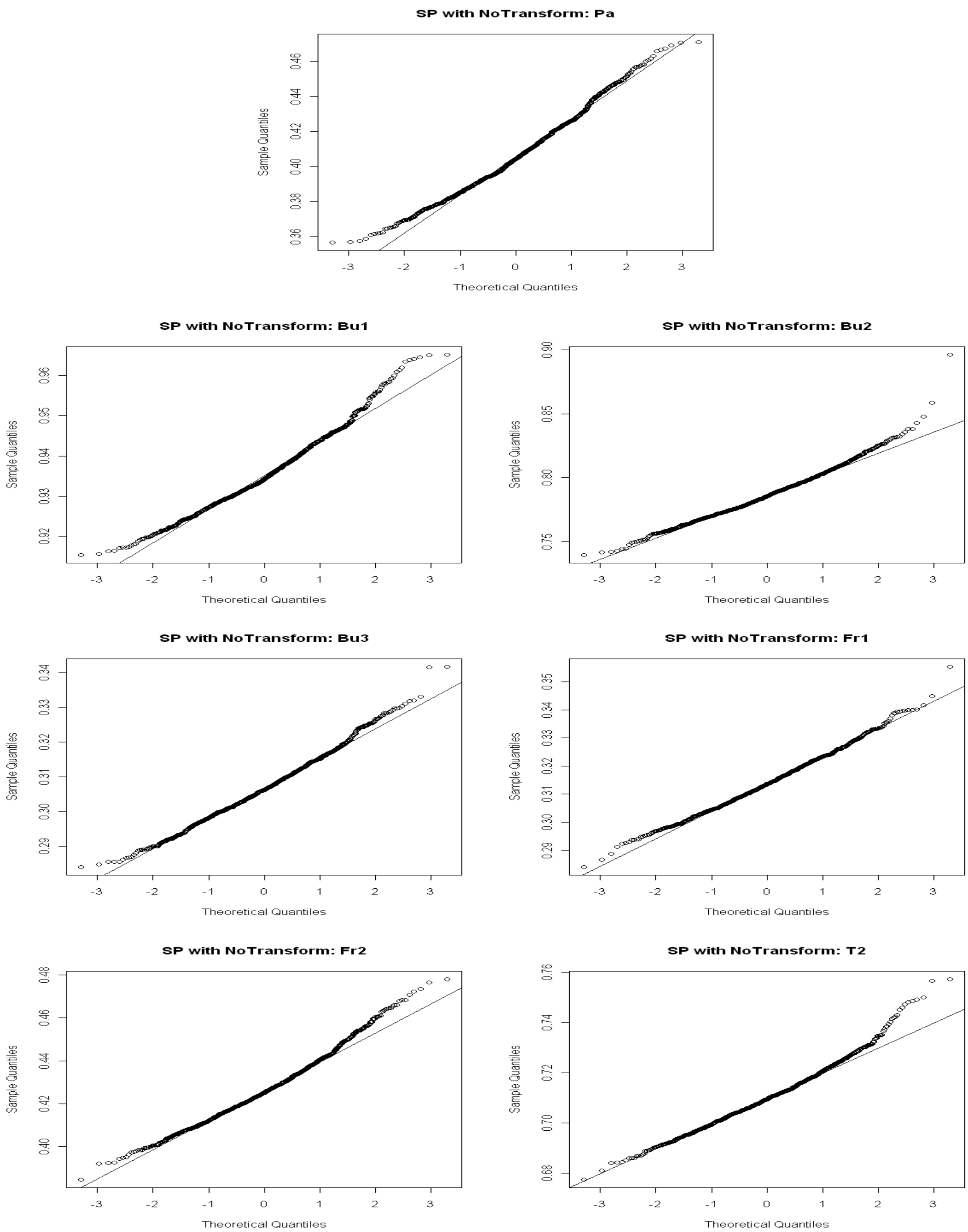

Figure C.15: Sampling Distribution for SP A1.5 when Fitting the PPD to the Tails

C.6 Atkinson Coefficient with Parameter 2 (A2)

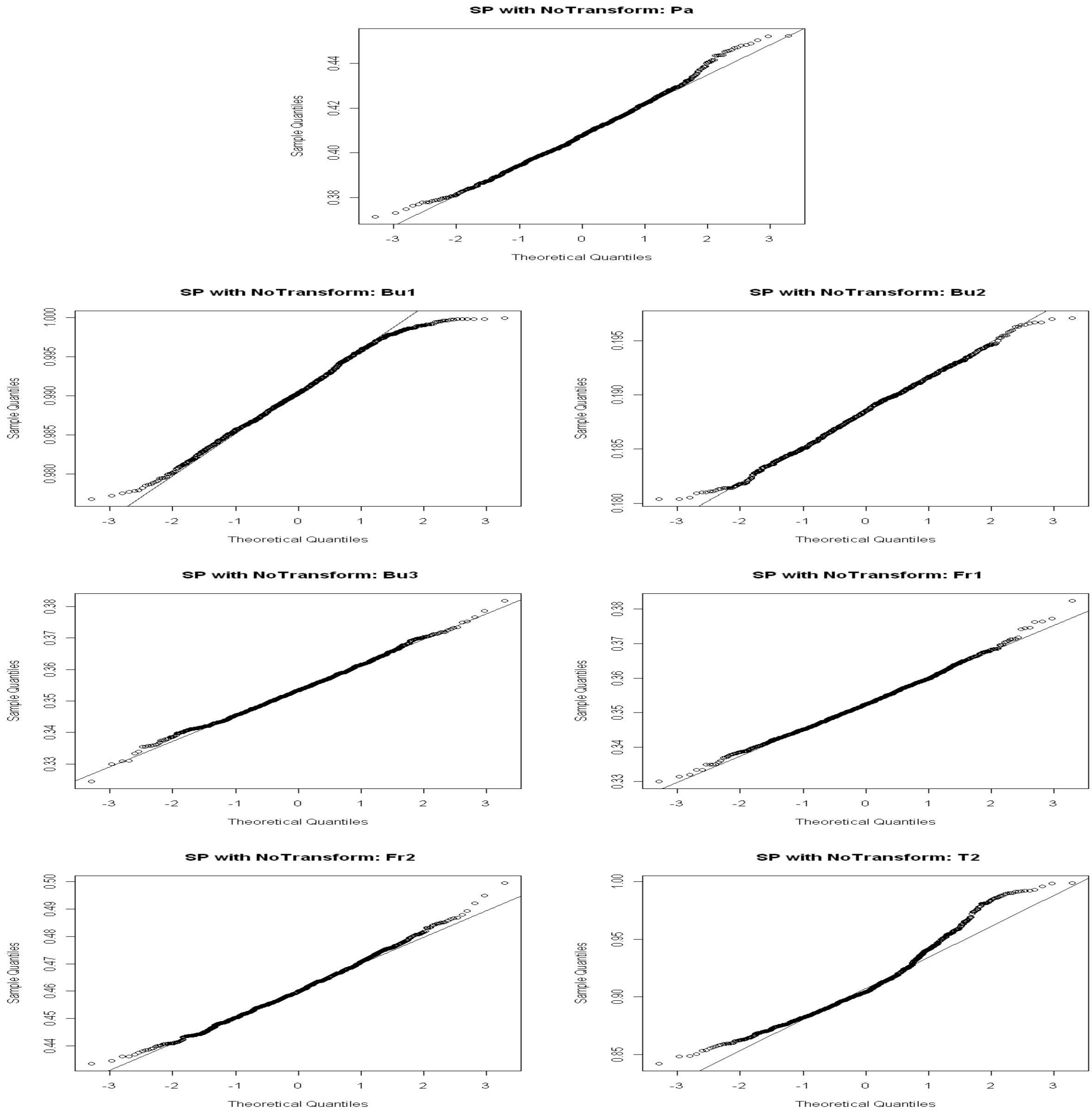

Figure C.16: Sampling Distribution for SP A2 when Fitting the GPD to the Tails

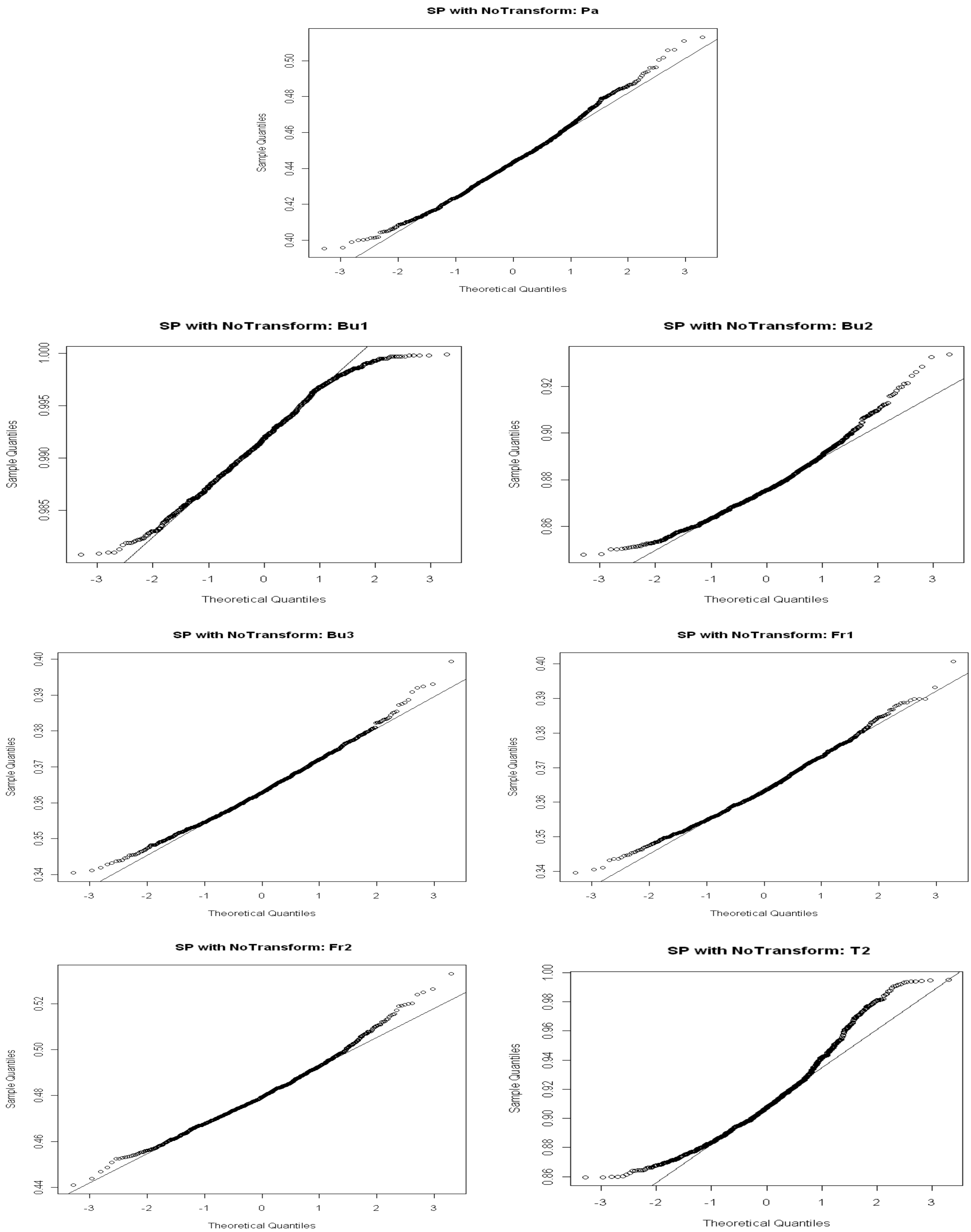

Figure C.17: Sampling Distribution for SP A2 when Fitting the Strict Pareto to the Tails

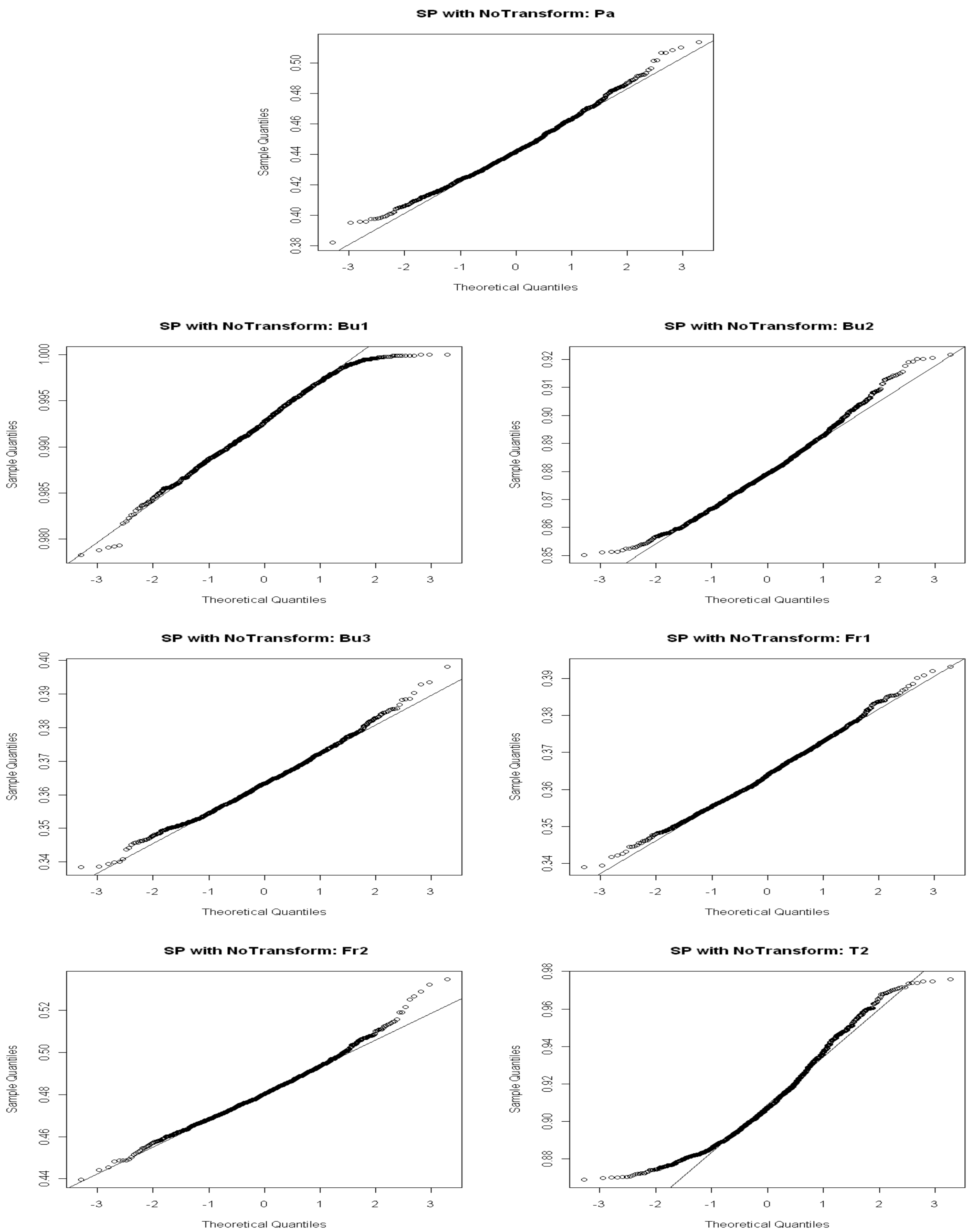

Figure C.18: Sampling Distribution for SP A2 when Fitting the PPD to the Tails

C.7 Quintile Share Ratio (QSR)

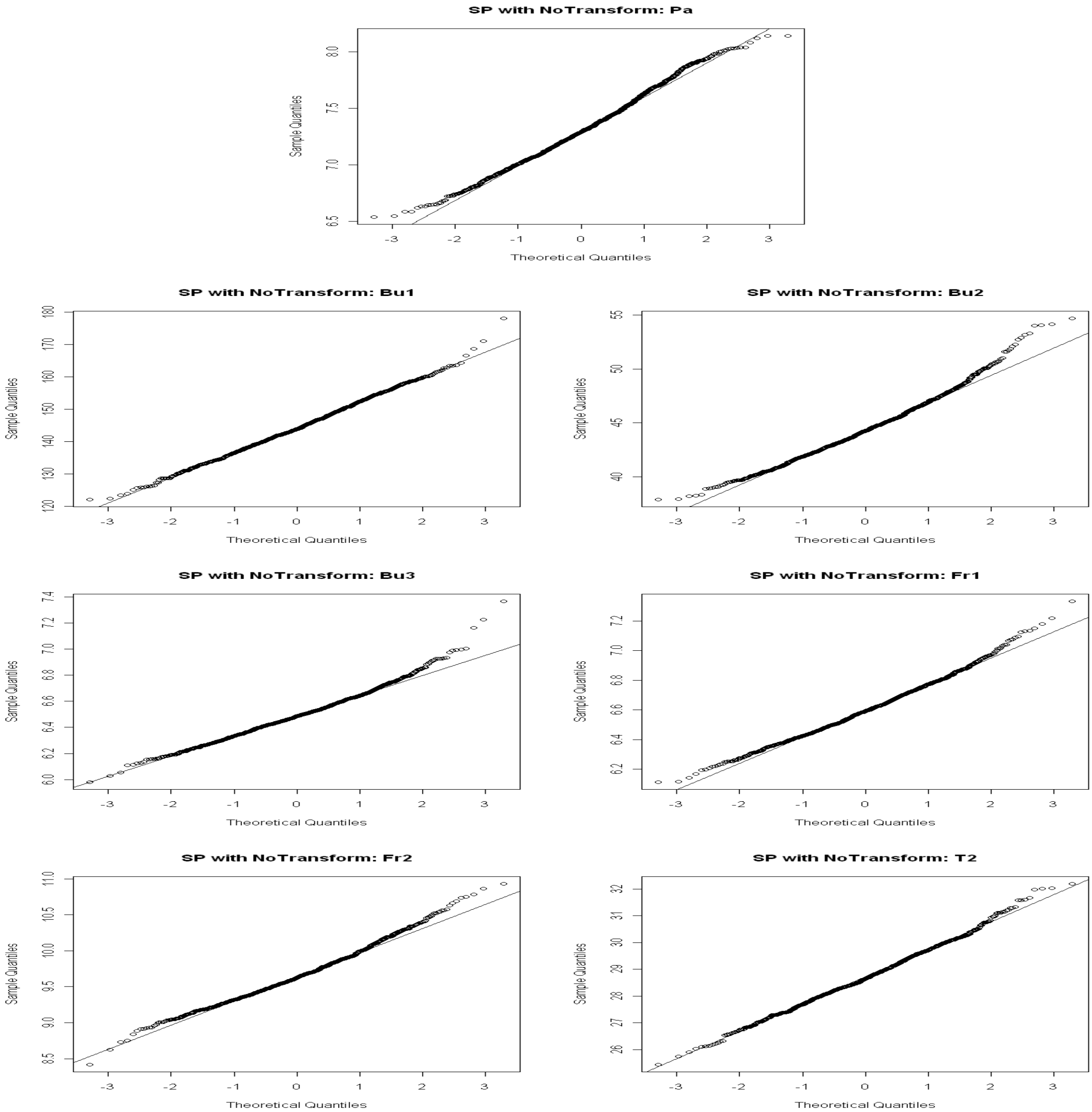

Figure C.19: Sampling Distribution for SP QSR when Fitting the GPD to the Tails

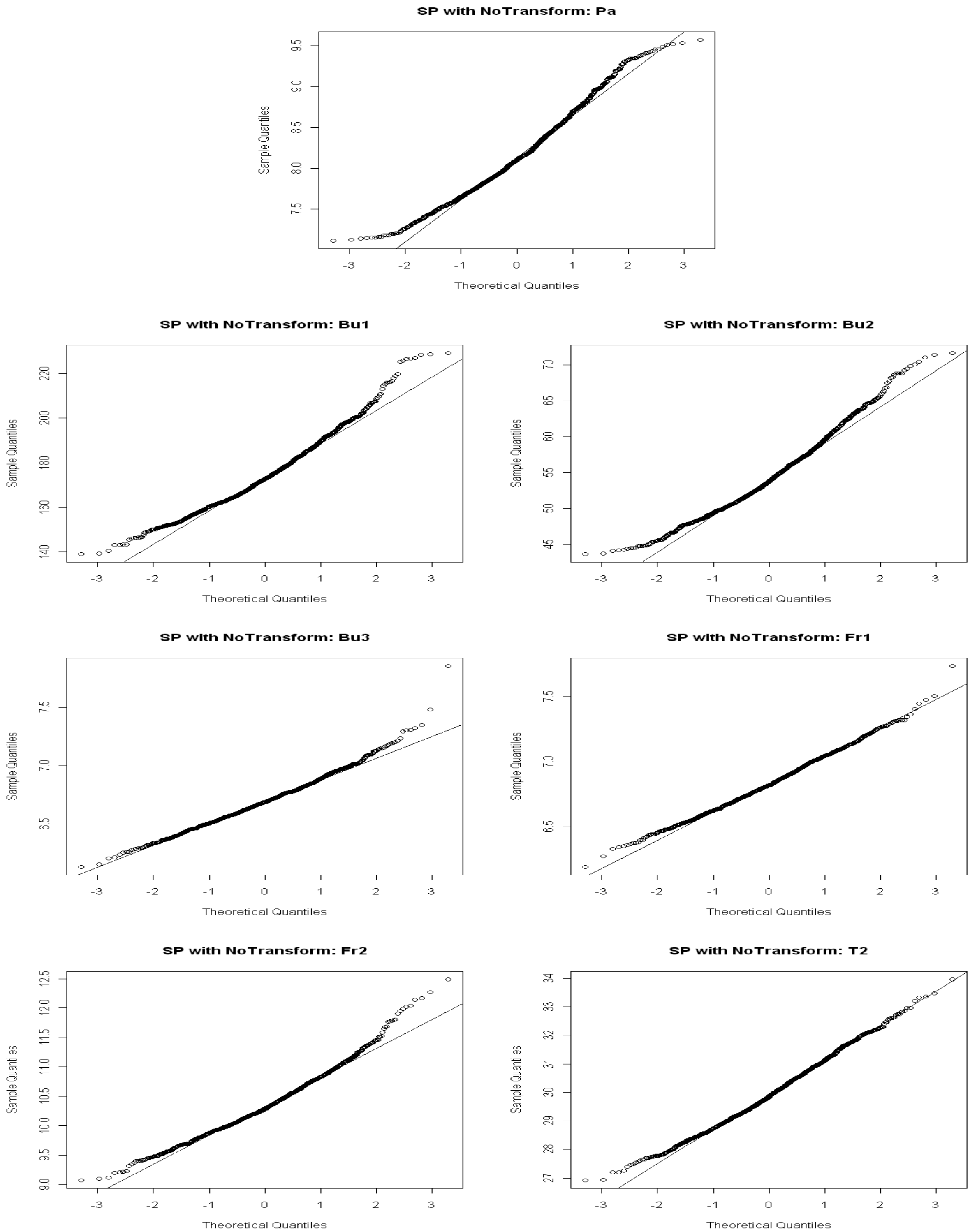

Figure C.20: Sampling Distribution for SP QSR when Fitting the Pa to the Tails

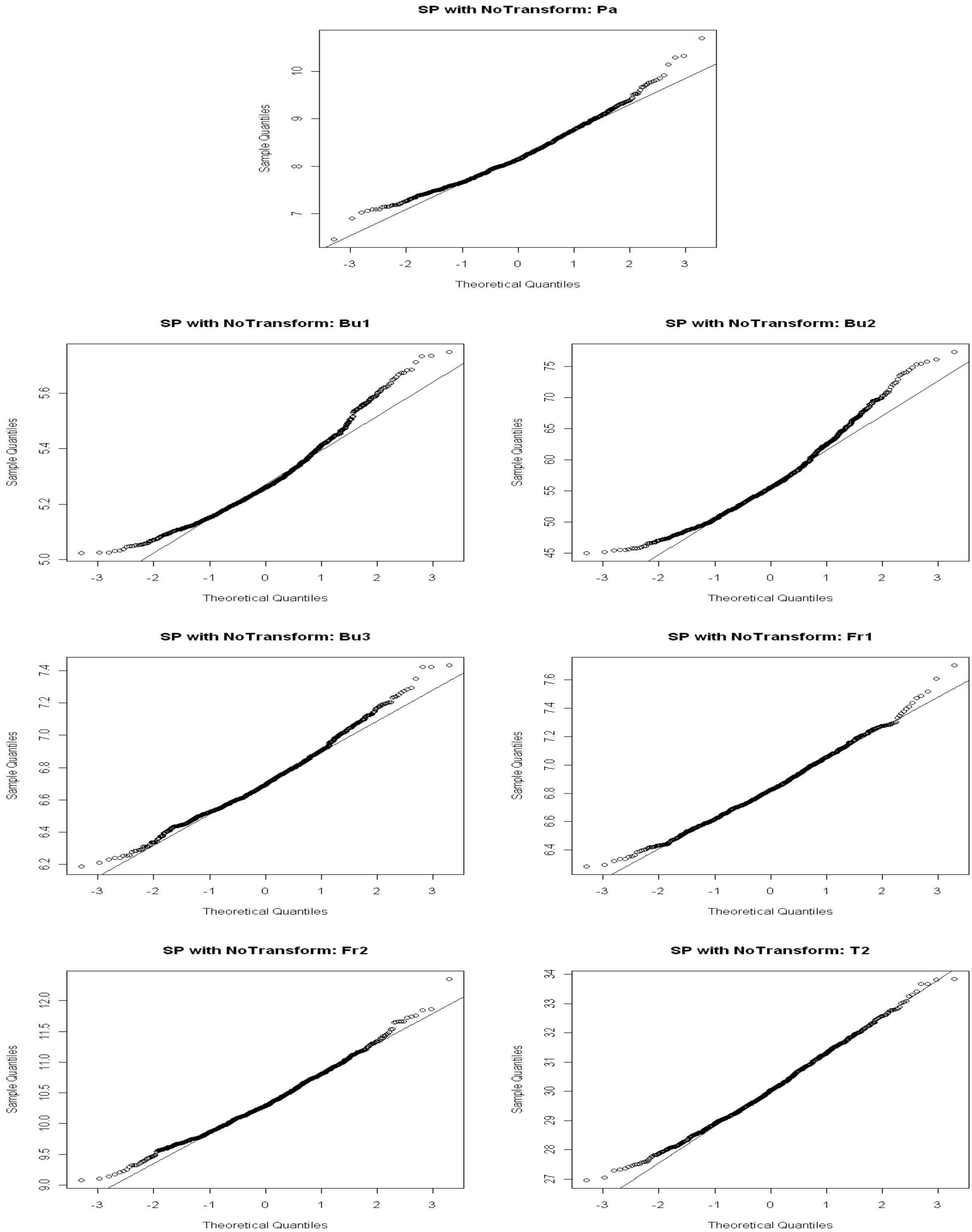

Figure C.21: Sampling Distribution for SP QSR when Fitting the PPD to the Tails

Appendix D

Mean Squared Errors for Estimators of Inequality Measures

The following tables display the MSEs for the estimators of the generalized entropy, the Atkinson and the QSR measures of inequality.

D.1 Generalized Entropy with Parameter 0 (GE0)

GE0 (n=500)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.0425	0.0424	0.0232	**0.0096**	0.0294
	(0.0025)	(0.0069)	(0.0044)	(0.0005)	(0.0005)
Bu1	**0.0384**	0.0391	0.0396	0.2053	0.1013
	(0.0086)	(0.007)	(0.0035)	(0.0091)	(0.0095)
Bu2	0.0754	0.0759	**0.0513**	0.1392	0.0853
	(0.0109)	(0.0088)	(0.0245)	(0.0057)	(0.0004)
Bu3	0.0029	0.0031	0.0036	**0.0020**	0.0052
	(0.0004)	(0.0004)	(0.0008)	(0.0003)	(0.0001)
Fr1	0.0041	0.0042	0.0029	**0.0025**	0.0056
	(0.0026)	(0.0014)	(0.0008)	(0.0002)	(0.0001)
Fr2	0.0129	0.0122	0.0096	**0.0078**	0.0155
	(0.0016)	(0.0071)	(0.0014)	(0.001)	(0.0003)
T2	0.0067	0.0066	**0.0053**	0.0062	0.0102
	(0.0047)	(0.0031)	(0.0025)	(0.003)	(0.0009)
Average	0.0261	0.0262	**0.0194**	0.0532	0.0361
	(0.0022)	(0.0022)	(0.0036)	(0.0016)	(0.0014)

GE0 (n=1000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.0167	0.0129	0.0154	**0.0050**	0.0234
	(0.0022)	(0.0014)	(0.0011)	(0.0004)	(0.0004)
Bu1	0.0345	0.0835	**0.0187**	0.0956	0.0451
	(0.0061)	(0.006)	(0.0011)	(0.0017)	(0.0064)
Bu2	**0.0373**	0.0719	0.0460	0.0617	0.0690
	(0.0061)	(0.0023)	(0.0179)	(0.001)	(0.0001)
Bu3	0.0024	0.0016	0.0019	**0.0010**	0.0037
	(0.0003)	(0.0001)	(0.0003)	(0.0001)	(0.0001)
Fr1	0.0024	0.0017	0.0022	**0.0013**	0.0041
	(0.0016)	(0.0005)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0107	0.0058	0.0057	**0.0040**	0.0120
	(0.0009)	(0.0004)	(0.0002)	(0.0003)	(0.0002)
T2	0.0050	0.0035	**0.0029**	0.0031	0.0071
	(0.0019)	(0.002)	(0.0018)	(0.0012)	(0.0008)
Average	0.0156	0.0258	**0.0133**	0.0245	0.0235
	(0.0013)	(0.0010)	(0.0026)	(0.0003)	(0.0009)

GE0 (n=5000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.0112	0.0041	0.0070	**0.0012**	0.0137
	(0.0008)	(0.0003)	(0.0002)	(0.0000)	(0.0001)
Bu1	**0.0089**	0.0123	0.0123	0.0179	0.0232
	(0.0018)	(0.0007)	(0.0006)	(0.0008)	(0.0007)
Bu2	0.0371	0.0155	0.0260	**0.0119**	0.0414
	(0.0053)	(0.001)	(0.0011)	(0.0004)	(0.0001)
Bu3	0.0005	0.0003	0.0004	**0.0002**	0.0017
	(0.0001)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Fr1	0.0006	0.0004	0.0005	**0.0003**	0.0019
	(0.0009)	(0.0002)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0031	0.0015	0.0020	**0.0009**	0.0060
	(0.0002)	(0.0001)	(0.0001)	(0.0001)	(0.0000)
T2	0.0011	0.0008	**0.0005**	0.0007	0.0031
	(0.0012)	(0.0013)	(0.0008)	(0.0008)	(0.0006)
Average	0.0089	0.0050	0.0070	**0.0047**	0.0130
	(0.0008)	(0.0003)	(0.0002)	(0.0002)	(0.0001)

GE0 (n=10000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.0084	0.0027	0.0042	**0.0007**	0.0109
	(0.0008)	(0.0001)	(0.0001)	(0.0000)	(0.0001)
Bu1	**0.0085**	0.0088	**0.0085**	0.0105	0.0171
	(0.0005)	(0.0002)	(0.0002)	(0.0003)	(0.0003)
Bu2	0.0253	0.0064	0.0159	**0.0042**	0.0339
	(0.0032)	(0.0002)	(0.0005)	(0.0002)	(0.0001)
Bu3	0.0004	0.0002	0.0003	**0.0001**	0.0012
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Fr1	0.0002	0.0002	0.0003	**0.0001**	0.0013
	(0.0008)	(0.0002)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0014	0.0006	0.0015	**0.0005**	0.0046
	(0.0001)	(0.0001)	(0.0000)	(0.0000)	(0.0000)
T2	0.0010	0.0004	0.0006	**0.0003**	0.0022
	(0.0004)	(0.0007)	(0.0005)	(0.0005)	(0.0004)
Average	0.0065	0.0028	0.0045	**0.0023**	0.0102
	(0.0005)	(0.0001)	(0.0001)	(0.0001)	(0.0001)

Table D.1: Mean Squared Errors for GE0 Measure

D.2 Generalized Entropy with Parameter 1 (GE1)

GE1 (n=500)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.3225	0.2834	**0.1849**	0.2014	0.1916
	(0.0526)	(0.0436)	(0.0098)	(0.0050)	(0.0360)
Bu1	0.2483	0.2396	0.2303	0.2303	**0.2193**
	(0.0378)	(0.0143)	(0.0721)	(0.0097)	(0.0103)
Bu2	0.0052	**0.0048**	0.0081	0.0578	0.0063
	(0.0555)	(0.0278)	(0.0156)	(0.0089)	(0.0185)
Bu3	0.0253	0.0266	**0.0125**	0.0127	0.0195
	(0.0062)	(0.0069)	(0.0008)	(0.0003)	(0.0008)
Fr1	0.0228	0.0197	**0.0125**	0.0126	0.0301
	(0.0034)	(0.0086)	(0.0009)	(0.0003)	(0.0030)
Fr2	0.1040	0.1102	**0.0638**	0.0666	0.0945
	(0.0095)	(0.0191)	(0.0071)	(0.0014)	(0.0086)
T2	0.0437	0.0512	**0.0206**	0.0226	0.0376
	(0.0117)	(0.0115)	(0.0021)	(0.0022)	(0.0058)
Average	0.1103	0.1051	**0.0761**	0.0863	0.0856
	(0.0124)	(0.0084)	(0.0107)	(0.0020)	(0.0062)

GE1 (n=1000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.2966	0.2570	0.1643	**0.1620**	0.1844
	(0.0512)	(0.0195)	(0.0053)	(0.0025)	(0.0028)
Bu1	**0.1573**	0.9193	0.1939	0.1939	0.2174
	(0.0212)	(0.0122)	(0.045)	(0.002)	(0.0093)
Bu2	**0.0045**	0.0512	0.0064	0.0393	0.0053
	(0.0451)	(0.0497)	(0.0076)	(0.0052)	(0.0163)
Bu3	0.0161	0.0129	**0.0081**	0.0084	0.0108
	(0.0032)	(0.0007)	(0.0002)	(0.0002)	(0.0002)
Fr1	0.0169	0.0149	**0.0078**	0.0087	0.0212
	(0.0028)	(0.0027)	(0.0006)	(0.0002)	(0.0003)
Fr2	0.0825	0.0717	0.0492	**0.0479**	0.0884
	(0.0083)	(0.0167)	(0.0017)	(0.0008)	(0.0071)
T2	0.0190	0.0233	**0.0147**	0.0151	0.0291
	(0.0097)	(0.0071)	(0.0018)	(0.0012)	(0.0007)
Average	0.0847	0.1929	**0.0635**	0.0679	0.0795
	(0.0104)	(0.0083)	(0.0066)	(0.0009)	(0.0029)

GE1 (n=5000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.2070	**0.0634**	0.1168	0.0840	0.1248
	(0.0375)	(0.0051)	(0.0013)	(0.0025)	(0.0019)
Bu1	0.1454	0.4395	0.4395	0.1154	**0.1098**
	(0.0125)	(0.012)	(0.0016)	(0.0015)	(0.0064)
Bu2	0.0040	0.0109	0.0020	0.0225	**0.0011**
	(0.0332)	(0.0183)	(0.0051)	(0.002)	(0.0104)
Bu3	0.0099	**0.0028**	0.0040	0.0032	0.0104
	(0.0020)	(0.0002)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.0069	0.0042	0.0036	**0.0030**	0.0102
	(0.0020)	(0.0004)	(0.0001)	(0.0001)	(0.0002)
Fr2	0.0712	0.0219	0.0398	**0.0216**	0.0709
	(0.0068)	(0.0019)	(0.0005)	(0.0005)	(0.0007)
T2	0.0104	**0.0053**	0.0076	0.0054	0.0198
	(0.0065)	(0.0048)	(0.0005)	(0.001)	(0.0006)
Average	0.0650	0.0783	0.0876	**0.0364**	0.0496
	(0.0075)	(0.0033)	(0.0008)	(0.0005)	(0.0018)

GE1 (n=10000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1084	**0.0334**	**0.0334**	0.0625	0.1241
	(0.0367)	(0.0015)	(0.0009)	(0.0017)	(0.0006)
Bu1	0.0575	0.2743	0.2745	0.0904	**0.0201**
	(0.0115)	(0.0117)	(0.001)	(0.0011)	(0.0013)
Bu2	0.0040	0.0107	0.0015	0.0166	**0.0007**
	(0.0201)	(0.0105)	(0.0025)	(0.0019)	(0.0092)
Bu3	0.0051	**0.0013**	0.0032	0.0020	0.0104
	(0.0016)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.0058	**0.0018**	0.0028	0.0180	0.0102
	(0.0011)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0536	**0.0089**	0.0229	0.0147	0.0701
	(0.0023)	(0.0008)	(0.0001)	(0.0001)	(0.0003)
T2	0.0055	**0.0031**	0.0054	0.0036	0.0198
	(0.0016)	(0.0014)	(0.0004)	(0.001)	(0.0002)
Average	0.0343	0.0476	0.0491	**0.0297**	0.0365
	(0.0062)	(0.0023)	(0.0004)	(0.0004)	(0.0013)

Table D.2: Mean Squared Errors for GE1 Measure

D.3 Generalized Entropy with Parameter 1.3 (GE1.3)

GE1.3 (n=500)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1769	0.1706	**0.1703**	0.1745	0.1741
	(0.0913)	(0.0291)	(0.0299)	(0.0203)	(0.0116)
Bu1	0.0585	0.0475	0.0524	0.0344	**0.0279**
	(0.0339)	(0.0091)	(0.0088)	(0.0093)	(0.0097)
Bu2	0.4386	0.4295	**0.2005**	0.2454	0.2731
	(0.0815)	(0.0821)	(0.0917)	(0.0813)	(0.0991)
Bu3	0.1022	0.0890	0.0417	0.0405	**0.0157**
	(0.0191)	(0.0005)	(0.0006)	(0.0004)	(0.0004)
Fr1	0.1318	0.1075	0.0442	0.0874	**0.0367**
	(0.0055)	(0.0003)	(0.0004)	(0.0003)	(0.0006)
Fr2	0.0388	0.0379	0.0357	0.0196	**0.0174**
	(0.0816)	(0.0020)	(0.0019)	(0.0022)	(0.0033)
T2	0.1844	0.1422	**0.0793**	1.1149	3.3160
	(0.1082)	(0.0008)	(0.001)	(0.0011)	(0.0016)
Average	0.1616	0.1463	**0.0892**	0.2452	0.5516
	(0.0267)	(0.0125)	(0.0138)	(0.0120)	(0.0143)

GE1.3 (n=1000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1735	0.1844	**0.0890**	0.1638	0.1712
	(0.0850)	(0.0232)	(0.0213)	(0.0121)	(0.0110)
Bu1	0.0536	0.0392	0.0307	0.0318	**0.0272**
	(0.0311)	(0.0087)	(0.0073)	(0.0089)	(0.0091)
Bu2	0.4366	0.3860	**0.2217**	0.2418	0.2700
	(0.0670)	(0.0606)	(0.0881)	(0.0718)	(0.0917)
Bu3	0.0724	0.0423	0.0313	**0.0151**	0.0318
	(0.0093)	(0.0004)	(0.0005)	(0.0003)	(0.0003)
Fr1	0.1069	0.0992	0.0349	**0.0340**	0.1101
	(0.0047)	(0.0002)	(0.0004)	(0.0002)	(0.0003)
Fr2	0.0384	0.0325	0.0350	0.0131	**0.0065**
	(0.0755)	(0.0010)	(0.0012)	(0.0016)	(0.0017)
T2	0.1428	0.1615	**0.0595**	0.1632	2.7732
	(0.0937)	(0.0006)	(0.0008)	(0.0009)	(0.0010)
Average	0.1463	0.1350	**0.0717**	0.0947	0.4843
	(0.0236)	(0.0094)	(0.0130)	(0.0105)	(0.0133)

GE1.3 (n=5000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1566	0.1780	**0.0890**	0.1363	0.1499
	(0.0811)	(0.0081)	(0.0089)	(0.0069)	(0.0080)
Bu1	0.0531	0.0078	0.0278	**0.0099**	0.0265
	(0.0258)	(0.0071)	(0.0069)	(0.0067)	(0.0080)
Bu2	0.3955	0.3285	**0.0516**	0.2409	0.1985
	(0.0577)	(0.0463)	(0.0479)	(0.0527)	(0.0623)
Bu3	0.0230	0.0118	0.0188	**0.0040**	0.0047
	(0.006)	(0.0003)	(0.0003)	(0.0002)	(0.0002)
Fr1	0.0982	0.0131	0.0209	**0.0067**	0.0095
	(0.0031)	(0.0002)	(0.0003)	(0.0002)	(0.0003)
Fr2	0.0369	0.0260	0.0317	0.0016	**0.0010**
	(0.0599)	(0.0006)	(0.0007)	(0.0009)	(0.0010)
T2	0.1296	0.0358	0.0374	**0.0227**	0.0517
	(0.0883)	(0.0004)	(0.0003)	(0.0002)	(0.0002)
Average	0.1276	0.0859	**0.0396**	0.0603	0.0631
	(0.0212)	(0.0068)	(0.0070)	(0.0077)	(0.0090)

GE1.3 (n=10000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1346	0.1517	**0.0563**	0.1146	0.1171
	(0.0792)	(0.0065)	(0.0071)	(0.0047)	(0.0073)
Bu1	0.0523	**0.0066**	0.0243	0.0091	0.0265
	(0.0109)	(0.0063)	(0.0053)	(0.0049)	(0.0057)
Bu2	0.3856	0.3155	**0.0516**	0.2399	0.1984
	(0.0502)	(0.0411)	(0.0428)	(0.0433)	(0.0515)
Bu3	0.0152	0.0054	0.0152	**0.0019**	0.0023
	(0.0022)	(0.0002)	(0.0002)	(0.0001)	(0.0002)
Fr1	0.0256	0.0066	0.0169	**0.0035**	0.0044
	(0.0012)	(0.0001)	(0.0002)	(0.0002)	(0.0001)
Fr2	0.0362	0.0224	0.0317	**0.0007**	0.0009
	(0.0563)	(0.0005)	(0.0005)	(0.0006)	(0.0008)
T2	0.0481	0.0165	0.0311	**0.0133**	0.0268
	(0.0791)	(0.0002)	(0.0002)	(0.0001)	(0.0001)
Average	0.0997	0.0750	**0.0324**	0.0547	0.0538
	(0.0193)	(0.006)	(0.0062)	(0.0063)	(0.0075)

Table D.3: Mean Squared Errors for GE1.3 Measure

D.4 Atkinson Coefficient with Parameter 1 (A1)

A1 (n=500)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0081	0.0076	0.0105	**0.0037**	0.0040
	(0.0006)	(0.0007)	(0.016)	(0.0002)	(0.0002)
Bu1	**0.0018**	**0.0018**	0.0205	0.0065	0.0062
	(0.0002)	(0.0001)	(0.0181)	(0.0004)	(0.0003)
Bu2	0.0067	**0.0060**	0.0244	0.0093	0.0107
	(0.0004)	(0.0007)	(0.0013)	(0.0005)	(0.001)
Bu3	0.0022	0.0020	0.0013	**0.0009**	0.0011
	(0.0005)	(0.0004)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.0019	0.0022	0.0014	**0.0012**	0.0015
	(0.0011)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0179	0.0174	0.0245	0.0075	**0.0069**
	(0.002)	(0.0004)	(0.0009)	(0.0002)	(0.0004)
T2	0.0014	0.0015	**0.0013**	0.0015	0.0023
	(0.0003)	(0.0002)	(0.0001)	(0.0001)	(0.0003)
Average	0.0057	0.0055	0.0120	**0.0044**	0.0047
	(0.0004)	(0.0002)	(0.0035)	(0.0001)	(0.0002)

A1 (n=1000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0058	0.0051	0.0059	0.0021	**0.0019**
	(0.0003)	(0.0003)	(0.0003)	(0.0001)	(0.0001)
Bu1	**0.0011**	0.0029	0.0014	0.0026	0.0060
	(0.0001)	(0.0001)	(0.0010)	(0.0002)	(0.0002)
Bu2	**0.0042**	0.0056	0.0140	0.0049	0.0080
	(0.0003)	(0.0003)	(0.0010)	(0.0004)	(0.0004)
Bu3	0.0011	0.0008	0.0008	**0.0005**	**0.0005**
	(0.0003)	(0.0002)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.0009	0.0011	0.0009	**0.0007**	**0.0007**
	(0.0010)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0147	0.0135	0.0209	0.0073	**0.0040**
	(0.0018)	(0.0003)	(0.0002)	(0.0002)	(0.0002)
T2	0.0009	0.0008	**0.0007**	0.0008	0.0008
	(0.0002)	(0.0001)	(0.0000)	(0.0000)	(0.0001)
Average	0.0041	0.0043	0.0064	**0.0027**	0.0031
	(0.0003)	(0.0001)	(0.0002)	(0.0001)	(0.0001)

A1 (n=5000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0031	0.0015	0.0028	0.0005	**0.0004**
	(0.0003)	(0.0001)	(0.0001)	(0.0000)	(0.0001)
Bu1	**0.0004**	0.0006	0.0013	0.0008	0.0027
	(0.0000)	(0.0001)	(0.0008)	(0.0001)	(0.0001)
Bu2	0.0022	0.0014	0.0028	**0.0010**	0.0018
	(0.0002)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Bu3	0.0003	0.0002	0.0003	**0.0001**	**0.0001**
	(0.0001)	(0.0001)	(0.0000)	(0.0000)	(0.0001)
Fr1	0.0004	0.0002	0.0003	**0.0002**	**0.0002**
	(0.0005)	(0.0000)	(0.0001)	(0.0000)	(0.0000)
Fr2	0.0122	0.0113	0.0175	0.0070	**0.0037**
	(0.0009)	(0.0002)	(0.0002)	(0.0001)	(0.0000)
T2	**0.0002**	**0.0002**	**0.0002**	**0.0002**	**0.0002**
	(0.0002)	(0.0001)	(0.0000)	(0.0000)	(0.0000)
Average	0.0027	0.0022	0.0036	0.0014	**0.0013**
	(0.0002)	(0.0000)	(0.0001)	(0.0000)	(0.0000)

A1 (n=10000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0020	0.0007	0.0022	0.0003	**0.0002**
	(0.0002)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Bu1	**0.0003**	0.0004	**0.0003**	0.0005	0.0016
	(0.0000)	(0.0001)	(0.0000)	(0.0000)	(0.0001)
Bu2	0.0015	0.0007	0.0023	**0.0005**	0.0011
	(0.0001)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Bu3	**0.0001**	**0.0001**	**0.0001**	**0.0001**	**0.0001**
	(0.0001)	(0.0001)	(0.0000)	(0.0000)	(0.0000)
Fr1	0.0002	0.0002	0.0002	**0.0001**	**0.0001**
	(0.0005)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Fr2	0.0120	0.0112	0.0166	0.0070	**0.0030**
	(0.0008)	(0.0001)	(0.0001)	(0.0001)	(0.0000)
T2	**0.0001**	**0.0001**	**0.0001**	**0.0001**	**0.0001**
	(0.0001)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Average	0.0023	0.0019	0.0031	0.0012	**0.0009**
	(0.0001)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Table D.4: Mean Squared Errors for A1 Measure

D.5 Atkinson Coefficient with Parameter 1.5 (A1.5)

A1.5 (n=500)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.0070	0.0074	0.0085	**0.0036**	0.0040
	(0.0005)	(0.0004)	(0.001)	(0.0002)	(0.0003)
Bu1	**0.0007**	**0.0007**	0.0008	0.0008	0.0008
	(0.0003)	(0.0001)	(0.0003)	(0.0001)	(0.0001)
Bu2	**0.0029**	0.0031	0.0054	0.0041	0.0047
	(0.0003)	(0.0003)	(0.0007)	(0.0008)	(0.0003)
Bu3	0.0020	0.0018	0.0015	**0.0011**	0.0012
	(0.0003)	(0.0003)	(0.0003)	(0.0002)	(0.0001)
Fr1	0.0018	0.0017	0.0014	**0.0013**	0.0016
	(0.0012)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0178	0.0181	0.0233	0.0118	**0.0064**
	(0.0010)	(0.0003)	(0.0012)	(0.0004)	(0.0004)
T2	**0.0018**	0.0019	0.0022	**0.0018**	0.0021
	(0.0009)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
Average	0.0049	0.0050	0.0062	0.0035	**0.0030**
	(0.0003)	(0.0001)	(0.0003)	(0.0001)	(0.0001)

A1.5 (n=1000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.0057	0.0045	0.0058	**0.0017**	0.0019
	(0.0004)	(0.0003)	(0.0003)	(0.0001)	(0.0001)
Bu1	**0.0004**	0.0005	0.0005	0.0004	0.0007
	(0.0003)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Bu2	0.0024	0.0024	0.0034	**0.0022**	0.0030
	(0.0002)	(0.0001)	(0.0007)	(0.0002)	(0.0002)
Bu3	0.0012	0.0008	0.0008	**0.0006**	**0.0006**
	(0.0002)	(0.0002)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.0014	0.0009	**0.0005**	0.0007	0.0008
	(0.0009)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0155	0.0132	0.0207	0.0105	**0.0044**
	(0.0009)	(0.0002)	(0.0003)	(0.0003)	(0.0001)
T2	0.0011	0.0012	**0.0007**	0.0011	0.0010
	(0.0008)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Average	0.0040	0.0034	0.0046	0.0025	**0.0018**
	(0.0002)	(0.0001)	(0.0001)	(0.0001)	(0.0000)

A1.5 (n=5000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.0023	0.0012	0.0025	0.0005	**0.0004**
	(0.0004)	(0.0000)	(0.0001)	(0.0000)	(0.0000)
Bu1	**0.0001**	0.0001	0.0001	0.0001	0.0003
	(0.0001)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Bu2	0.0009	0.0006	0.0013	**0.0004**	0.0007
	(0.0002)	(0.0000)	(0.0004)	(0.0001)	(0.0001)
Bu3	0.0003	0.0002	0.0003	**0.0001**	**0.0001**
	(0.0001)	(0.0001)	(0.0000)	(0.0001)	(0.0001)
Fr1	0.0003	**0.0002**	0.0003	**0.0002**	**0.0002**
	(0.0009)	(0.0000)	(0.0001)	(0.0000)	(0.0000)
Fr2	0.0129	0.0119	0.0179	0.0100	**0.0041**
	(0.0008)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
T2	**0.0003**	**0.0003**	**0.0003**	**0.0003**	**0.0003**
	(0.0004)	(0.0000)	(0.0001)	(0.0001)	(0.0001)
Average	0.0024	0.0021	0.0032	0.0017	**0.0009**
	(0.0002)	(0.0000)	(0.0001)	(0.0000)	(0.0000)

A1.5 (n=10000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.0015	0.0006	0.0018	**0.0002**	**0.0002**
	(0.0003)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Bu1	**0.0001**	**0.0001**	**0.0001**	**0.0001**	0.0002
	(0.0001)	(0.0001)	(0.0000)	(0.0000)	(0.0000)
Bu2	0.0007	0.0003	0.0010	**0.0001**	0.0004
	(0.0001)	(0.0000)	(0.0001)	(0.0000)	(0.0000)
Bu3	0.0003	0.0001	0.0002	**0.0001**	**0.0001**
	(0.0001)	(0.0001)	(0.0000)	(0.0001)	(0.0001)
Fr1	0.0003	**0.0001**	0.0002	**0.0001**	**0.0001**
	(0.0005)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Fr2	0.0123	0.0116	0.0169	0.0100	**0.0041**
	(0.0005)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
T2	0.0002	**0.0001**	0.0002	0.0002	**0.0001**
	(0.0003)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Average	0.0023	0.0021	0.0031	0.0018	**0.0008**
	(0.0001)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Table D.5: Mean Squared Errors for A1.5 Measure

D.6 Atkinson Coefficient with Parameter 2 (A2)

A2 (n=500)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0072	0.0068	0.0098	0.0050	**0.0035**
	(0.0004)	(0.0008)	(0.0128)	(0.0003)	(0.0003)
Bu1	0.0006	0.0006	0.0007	**0.0002**	**0.0002**
	(0.0002)	(0.0001)	(0.0001)	(0.0003)	(0.0001)
Bu2	0.0023	0.0025	0.0033	**0.0020**	0.0021
	(0.0003)	(0.0005)	(0.0001)	(0.0001)	(0.0002)
Bu3	0.0017	0.0017	0.0013	**0.0011**	0.0012
	(0.0001)	(0.0001)	(0.0005)	(0.0001)	(0.0001)
Fr1	0.0019	0.0020	0.0014	**0.0013**	0.0016
	(0.0014)	(0.0006)	(0.0003)	(0.0001)	(0.0001)
Fr2	0.0169	0.0165	0.0217	0.0087	**0.0080**
	(0.0009)	(0.0003)	(0.001)	(0.0002)	(0.0003)
Average	0.0051	0.0050	0.0064	0.0031	**0.0028**
	(0.0003)	(0.0002)	(0.0018)	(0.0001)	(0.0001)

A2 (n=1000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0046	0.0051	0.0066	0.0027	**0.0018**
	(0.0004)	(0.0004)	(0.0038)	(0.0001)	(0.0001)
Bu1	0.0004	0.0003	0.0003	**0.0001**	**0.0001**
	(0.0002)	(0.0001)	(0.0000)	(0.0001)	(0.0000)
Bu2	0.0015	0.0013	0.0023	**0.0011**	0.0014
	(0.0003)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Bu3	0.0009	0.0010	0.0008	**0.0005**	0.0006
	(0.0001)	(0.0001)	(0.0001)	(0.0000)	(0.0001)
Fr1	0.0009	0.0011	0.0008	**0.0006**	0.0008
	(0.0009)	(0.0001)	(0.0002)	(0.0000)	(0.0001)
Fr2	0.0147	0.0134	0.0197	0.0084	**0.0048**
	(0.0005)	(0.0002)	(0.0009)	(0.0002)	(0.0003)
Average	0.0038	0.0037	0.0051	0.0022	**0.0016**
	(0.0002)	(0.0001)	(0.0006)	(0.0000)	(0.0001)

A2 (n=5000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0019	0.0018	0.0022	0.0007	**0.0004**
	(0.0002)	(0.0002)	(0.0002)	(0.0001)	(0.0000)
Bu1	0.0001	0.0001	0.0001	**0.0000**	**0.0000**
	(0.0001)	(0.0001)	(0.0000)	(0.0001)	(0.0000)
Bu2	0.0006	0.0005	0.0009	**0.0004**	**0.0004**
	(0.0003)	(0.0001)	(0.0000)	(0.0000)	(0.0001)
Bu3	0.0003	0.0003	0.0002	**0.0001**	**0.0001**
	(0.0000)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Fr1	0.0003	0.0003	0.0003	**0.0001**	0.0002
	(0.0005)	(0.0001)	(0.0001)	(0.0000)	(0.0001)
Fr2	0.0127	0.0118	0.0168	0.0080	**0.0045**
	(0.0003)	(0.0002)	(0.0003)	(0.0001)	(0.0001)
Average	0.0027	0.0025	0.0034	0.0016	**0.0009**
	(0.0001)	(0.0000)	(0.0001)	(0.0000)	(0.0000)

A2 (n=10000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0016	0.0006	0.0016	**0.0002**	**0.0002**
	(0.0002)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Bu1	0.0001	0.0001	0.0001	**0.0000**	**0.0000**
	(0.0001)	(0.0001)	(0.0000)	(0.0000)	(0.0000)
Bu2	0.0004	0.0003	0.0006	**0.0002**	0.0003
	(0.0002)	(0.0001)	(0.0000)	(0.0000)	(0.0000)
Bu3	**0.0001**	**0.0001**	**0.0001**	**0.0001**	**0.0001**
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Fr1	0.0002	**0.0001**	0.0002	**0.0001**	**0.0001**
	(0.0005)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Fr2	0.0121	0.0114	0.0161	0.0080	**0.0045**
	(0.0003)	(0.0001)	(0.0001)	(0.0000)	(0.0001)
Average	0.0024	0.0021	0.0031	0.0014	**0.0009**
	(0.0001)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Table D.6: Mean Squared Errors for A2 Measure

D.7 Quintile Share Ratio (QSR)

QSR (n=500)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.4517	0.4482	0.4529	**0.3362**	0.3404
	(0.0233)	(0.0152)	(0.0118)	(0.012)	(0.0114)
Bu1	0.5162	0.5416	**0.3311**	0.3371	0.6318
	(0.0371)	(0.034)	(0.0323)	(0.0343)	(0.0423)
Bu2	0.4660	0.4939	**0.3567**	0.4109	0.6912
	(0.0311)	(0.025)	(0.0315)	(0.0313)	(0.0259)
Bu3	0.4211	0.3823	0.5031	0.2628	**0.2027**
	(0.0186)	(0.0174)	(0.0181)	(0.0125)	(0.0164)
Fr1	0.4362	0.4370	0.5394	0.2422	**0.1998**
	(0.012)	(0.0111)	(0.0181)	(0.0125)	(0.0204)
Fr2	0.8141	0.8299	0.7152	**0.5361**	0.7957
	(0.0133)	(0.0121)	(0.0115)	(0.0133)	(0.0132)
Average	0.5176	0.5222	0.4831	**0.3542**	0.4769
	(0.0085)	(0.0073)	(0.0078)	(0.0075)	(0.0084)

QSR (n=1000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.4038	0.4155	0.5198	0.3242	**0.3203**
	(0.0191)	(0.0095)	(0.0101)	(0.0098)	(0.0070)
Bu1	0.5691	0.5572	**0.3313**	0.3360	0.5779
	(0.0350)	(0.0337)	(0.0312)	(0.0341)	(0.0417)
Bu2	0.4850	0.6185	**0.2120**	0.2225	0.5911
	(0.0298)	(0.0219)	(0.0305)	(0.0311)	(0.0241)
Bu3	0.2306	0.2034	0.3206	0.1479	**0.1094**
	(0.0167)	(0.0158)	(0.0137)	(0.0123)	(0.0137)
Fr1	0.2699	0.2383	0.3502	0.1388	**0.0895**
	(0.0107)	(0.0088)	(0.0129)	(0.0080)	(0.0132)
Fr2	0.7377	0.9366	0.6818	**0.5157**	0.7828
	(0.0115)	(0.0117)	(0.0111)	(0.0130)	(0.0127)
Average	0.4494	0.4949	0.4026	**0.2809**	0.4118
	(0.0078)	(0.0067)	(0.0071)	(0.0073)	(0.0077)

QSR (n=5000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.3698	0.2788	0.5040	0.2105	**0.1945**
	(0.0173)	(0.0055)	(0.0098)	(0.0054)	(0.0056)
Bu1	0.5582	0.5226	**0.3084**	0.3258	0.5338
	(0.0311)	(0.0321)	(0.0295)	(0.0208)	(0.0339)
Bu2	0.4736	0.5484	0.1586	**0.1318**	0.5901
	(0.0247)	(0.0207)	(0.0291)	(0.0297)	(0.0216)
Bu3	0.0594	0.0515	0.1042	0.0342	**0.0289**
	(0.0113)	(0.0037)	(0.0031)	(0.0030)	(0.0053)
Fr1	0.0604	0.0554	0.1176	0.0253	**0.0167**
	(0.0087)	(0.0051)	(0.0041)	(0.0067)	(0.0070)
Fr2	0.7055	0.7018	0.5451	**0.5072**	0.7201
	(0.0089)	(0.0098)	(0.0092)	(0.0110)	(0.0099)
Average	0.3712	0.3598	0.2897	**0.2058**	0.3474
	(0.0066)	(0.0058)	(0.0063)	(0.0056)	(0.0061)

QSR (n=10000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.2816	0.2210	0.5040	0.1409	**0.1238**
	(0.0129)	(0.0039)	(0.0056)	(0.0034)	(0.0037)
Bu1	0.4454	0.5132	0.3082	0.3258	**0.2113**
	(0.0297)	(0.0299)	(0.0217)	(0.0206)	(0.0295)
Bu2	0.4736	0.5484	0.1586	**0.1318**	0.4785
	(0.0201)	(0.0181)	(0.0208)	(0.0215)	(0.0203)
Bu3	0.0348	0.0322	0.0736	0.0227	**0.0222**
	(0.0099)	(0.002)	(0.0021)	(0.0021)	(0.0022)
Fr1	0.0330	0.0275	0.0728	0.0140	**0.0133**
	(0.0038)	(0.0021)	(0.0022)	(0.0025)	(0.0041)
Fr2	0.6981	0.7018	0.5069	**0.5016**	0.7126
	(0.0053)	(0.0057)	(0.0056)	(0.0091)	(0.0073)
Average	0.3278	0.3407	0.2707	**0.1895**	0.2603
	(0.0057)	(0.0051)	(0.0045)	(0.0045)	(0.0053)

Table D.7: Mean Squared Errors for QSR Measure

Appendix E

Relative Impact of Contamination for Estimators of Inequality Measures

The following tables display the RICs for the estimators of the generalized entropy, the Atkinson and the QSR measures of inequality.

E.1 Generalized Entropy with Parameter 0 (GE0)

GE0 (n=500)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1613	0.1613	0.0563	0.0717	**0.0538**
	(0.0018)	(0.0018)	(0.0066)	(0.0007)	(0.0031)
Bu1	0.0372	0.0372	**0.0125**	0.0179	0.0169
	(0.0006)	(0.0006)	(0.0019)	(0.0004)	(0.0003)
Bu2	0.0582	0.0582	0.0342	0.0272	**0.0165**
	(0.001)	(0.001)	(0.0012)	(0.0005)	(0.0013)
Bu3	0.2368	0.2368	0.0918	0.0776	**0.0715**
	(0.0016)	(0.0016)	(0.0047)	(0.0004)	(0.0002)
Fr1	0.2260	0.2260	0.1108	0.0745	**0.0694**
	(0.0017)	(0.0017)	(0.0017)	(0.0005)	(0.0002)
Fr2	0.1490	0.1490	0.0677	**0.0547**	0.0555
	(0.0013)	(0.0013)	(0.0069)	(0.0005)	(0.0002)
T2	0.0884	0.0884	0.0372	0.0275	**0.0258**
	(0.0008)	(0.0008)	(0.0009)	(0.0002)	(0.0001)
Average	0.1367	0.1367	0.0586	0.0502	**0.0442**
	(0.0005)	(0.0005)	(0.0016)	(0.0002)	(0.0005)

GE0 (n=1000)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1591	0.1183	0.0478	0.0635	**0.0435**
	(0.0015)	(0.0016)	(0.0064)	(0.0004)	(0.0001)
Bu1	0.0810	0.0666	0.0209	0.0180	**0.0076**
	(0.0006)	(0.001)	(0.0005)	(0.0002)	(0.0001)
Bu2	0.1593	0.0897	0.0328	0.0256	**0.0197**
	(0.0019)	(0.001)	(0.0028)	(0.0003)	(0.0015)
Bu3	0.2392	0.1758	0.1234	0.0758	**0.0532**
	(0.0013)	(0.0014)	(0.0021)	(0.0003)	(0.0003)
Fr1	0.2803	0.2528	0.1143	0.0692	**0.0499**
	(0.0011)	(0.0012)	(0.0014)	(0.0003)	(0.0001)
Fr2	0.1468	0.1227	0.0619	0.0490	**0.0445**
	(0.0013)	(0.0012)	(0.0027)	(0.0003)	(0.0001)
T2	0.1184	0.1037	0.0401	0.0270	**0.0262**
	(0.0006)	(0.0007)	(0.0007)	(0.0002)	(0.0001)
Average	0.1691	0.1328	0.0630	0.0469	**0.0349**
	(0.0005)	(0.0005)	(0.0011)	(0.0001)	(0.0002)

GE0 (n=5000)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1528	0.1050	0.0674	0.0553	**0.0506**
	(0.0011)	(0.0008)	(0.0021)	(0.0002)	(0.0001)
Bu1	0.0440	0.0210	0.0193	**0.0165**	0.0184
	(0.0004)	(0.0005)	(0.0002)	(0.0001)	(0.0000)
Bu2	0.0605	0.0514	0.0282	**0.0231**	0.0241
	(0.0006)	(0.0006)	(0.001)	(0.0001)	(0.0000)
Bu3	0.2468	0.2021	0.0833	0.0680	**0.0641**
	(0.0007)	(0.0006)	(0.0002)	(0.0001)	(0.0001)
Fr1	0.2007	0.0993	0.0982	0.0657	**0.0616**
	(0.0007)	(0.0006)	(0.0002)	(0.0001)	(0.0001)
Fr2	0.1659	0.1130	0.0732	0.0471	**0.0457**
	(0.0006)	(0.0007)	(0.0002)	(0.0001)	(0.0002)
T2	0.0895	0.0786	0.0415	**0.0262**	0.0283
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0000)
Average	0.1372	0.0958	0.0587	0.0431	**0.0418**
	(0.0003)	(0.0002)	(0.0003)	(0.0001)	(0.0000)

GE0 (n=10000)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1458	0.0947	0.0778	0.0533	**0.0498**
	(0.0007)	(0.0007)	(0.0004)	(0.0001)	(0.0001)
Bu1	0.0423	0.0278	0.0189	**0.0153**	0.0184
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0000)
Bu2	0.0800	0.0438	0.0308	**0.0224**	0.0239
	(0.0006)	(0.0005)	(0.0004)	(0.0001)	(0.0000)
Bu3	0.2320	0.1736	0.1131	0.0665	**0.0647**
	(0.0007)	(0.0004)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.2392	0.2104	0.1075	0.0635	**0.0620**
	(0.0006)	(0.0005)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.1471	0.0724	0.0725	**0.0452**	0.0456
	(0.0006)	(0.0005)	(0.0001)	(0.0001)	(0.0000)
T2	0.0898	0.0991	0.0422	**0.0258**	0.0289
	(0.0003)	(0.0002)	(0.0001)	(0.0001)	(0.0000)
Average	0.1394	0.1031	0.0661	**0.0417**	0.0419
	(0.0002)	(0.0002)	(0.0001)	(0.0000)	(0.0000)

Table E.1: Relative Impact of Contamination on GE0 Measure

E.2 Generalized Entropy with Parameter 1 (GE1)

GE1 (n=500)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1797	0.1797	**0.0555**	0.0828	0.0599
	(0.003)	(0.003)	(0.0025)	(0.0033)	(0.001)
Bu1	0.0701	0.0701	0.0790	0.0749	**0.0234**
	(0.0017)	(0.0017)	(0.0041)	(0.0013)	(0.0006)
Bu2	0.0810	0.0810	0.0434	0.0935	**0.0270**
	(0.0021)	(0.0021)	(0.0005)	(0.0072)	(0.0007)
Bu3	0.3698	0.3698	0.1037	**0.1031**	0.1233
	(0.0045)	(0.0045)	(0.0028)	(0.0006)	(0.0015)
Fr1	0.3290	0.3290	0.1415	**0.0937**	0.1097
	(0.0035)	(0.0035)	(0.0073)	(0.0005)	(0.0012)
Fr2	0.1976	0.1976	0.1411	**0.0652**	0.0658
	(0.0028)	(0.0028)	(0.0096)	(0.0008)	(0.0009)
T2	0.1962	0.1962	0.0801	**0.0451**	0.0654
	(0.0028)	(0.0028)	(0.0003)	(0.0004)	(0.0009)
Average	0.2033	0.2033	**0.0568**	0.0798	0.0678
	(0.0012)	(0.0012)	(0.0019)	(0.0012)	(0.0004)

GE1 (n=1000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1840	0.1400	0.0692	**0.0605**	0.0656
	(0.0032)	(0.0015)	(0.0037)	(0.0005)	(0.0063)
Bu1	0.0753	0.1493	**0.0310**	0.3302	0.3600
	(0.0015)	(0.0002)	(0.0014)	(0.0067)	(0.005)
Bu2	0.0882	0.1038	**0.0256**	0.0282	0.0322
	(0.0021)	(0.0018)	(0.0028)	(0.0005)	(0.0088)
Bu3	0.3575	0.1575	**0.1041**	0.1050	0.1715
	(0.0036)	(0.0011)	(0.0022)	(0.0005)	(0.0007)
Fr1	0.3284	0.1479	0.0975	0.0971	**0.0829**
	(0.0032)	(0.0011)	(0.0012)	(0.0005)	(0.0005)
Fr2	0.1943	0.1080	0.1004	**0.0603**	0.0643
	(0.0025)	(0.0012)	(0.0025)	(0.0004)	(0.0078)
T2	0.1716	0.0806	0.2327	0.0482	**0.0429**
	(0.002)	(0.0007)	(0.0003)	(0.0003)	(0.0005)
Average	0.1999	0.1267	**0.0944**	0.1042	0.1170
	(0.001)	(0.0004)	(0.0009)	(0.001)	(0.002)

GE1 (n=5000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1586	0.1080	0.0552	0.0528	**0.0358**
	(0.0021)	(0.0008)	(0.0015)	(0.0003)	(0.0002)
Bu1	0.0820	0.0618	0.0291	**0.0221**	0.5276
	(0.0013)	(0.0006)	(0.0033)	(0.0002)	(0.002)
Bu2	0.0835	0.0641	0.0539	**0.0240**	0.0617
	(0.0019)	(0.0007)	(0.0056)	(0.0002)	(0.005)
Bu3	0.3244	0.1381	0.1062	0.1064	**0.0897**
	(0.0017)	(0.0005)	(0.0002)	(0.0003)	(0.0002)
Fr1	0.3202	0.1313	0.0989	0.0993	**0.0803**
	(0.0022)	(0.0006)	(0.0003)	(0.0003)	(0.0002)
Fr2	0.1882	0.0947	0.0610	0.0611	**0.0323**
	(0.0018)	(0.0006)	(0.0002)	(0.0003)	(0.0003)
T2	0.1832	0.0765	0.0516	0.0548	**0.0371**
	(0.0014)	(0.0004)	(0.0002)	(0.0002)	(0.0001)
Average	0.1914	0.0964	0.0651	0.0601	**0.0449**
	(0.0007)	(0.0002)	(0.001)	(0.0001)	(0.0008)

GE1 (n=10000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1513	0.1128	0.0538	0.0533	**0.0360**
	(0.0016)	(0.0006)	(0.0002)	(0.0003)	(0.0002)
Bu1	0.0817	0.0632	**0.0199**	0.0234	0.0687
	(0.0012)	(0.0005)	(0.0001)	(0.0002)	(0.0032)
Bu2	0.0701	0.0651	0.0316	**0.0245**	0.0516
	(0.0012)	(0.0005)	(0.003)	(0.0002)	(0.0025)
Bu3	0.3317	0.1400	0.1075	0.1088	**0.0897**
	(0.0014)	(0.0004)	(0.0002)	(0.0003)	(0.0002)
Fr1	0.3150	0.1337	0.0996	0.1016	**0.0789**
	(0.0015)	(0.0004)	(0.0002)	(0.0002)	(0.0002)
Fr2	0.1937	0.0926	0.0613	0.0625	**0.0328**
	(0.0019)	(0.0004)	(0.0002)	(0.0003)	(0.0002)
T2	0.1753	0.0745	0.0530	0.0550	**0.0370**
	(0.0011)	(0.0003)	(0.0001)	(0.0002)	(0.0001)
Average	0.1884	0.0974	0.0609	0.0613	**0.0220**
	(0.0005)	(0.0002)	(0.0004)	(0.0001)	(0.0006)

Table E.2: Relative Impact of Contamination on GE1 Measure

E.3 Generalized Entropy with Parameter 1.3 (GE1.3)

GE1.3 (n=500)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.2250	0.2250	**0.0511**	0.2240	0.2549
	(0.0054)	(0.0054)	(0.003)	(0.0048)	(0.0057)
Bu1	0.1033	0.1033	0.0271	**0.0049**	0.1204
	(0.0028)	(0.0028)	(0.0025)	(0.0076)	(0.0027)
Bu2	0.1032	0.1032	**0.0217**	0.0324	0.0322
	(0.0032)	(0.0032)	(0.0008)	(0.0048)	(0.0074)
Bu3	0.4558	0.4558	**0.0913**	0.3026	0.3750
	(0.0065)	(0.0065)	(0.0018)	(0.0003)	(0.0081)
Fr1	0.4191	0.4191	0.0852	**0.0385**	0.7466
	(0.006)	(0.006)	(0.0077)	(0.0034)	(0.0078)
Fr2	0.2666	0.2666	**0.0518**	0.0692	0.3591
	(0.0055)	(0.0055)	(0.0026)	(0.0032)	(0.008)
T2	0.2542	0.2542	0.1438	0.1981	**0.1071**
	(0.0044)	(0.0044)	(0.0014)	(0.0088)	(0.0032)
Average	0.2610	0.2315	**0.0674**	0.1242	0.2122
	(0.0019)	(0.0019)	(0.0013)	(0.002)	(0.0025)

GE1.3 (n=1000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.3897	0.3362	**0.0717**	0.0941	0.3775
	(0.0041)	(0.0021)	(0.0073)	(0.0054)	(0.005)
Bu1	0.1132	0.0229	0.0161	**0.0132**	0.0833
	(0.0033)	(0.0036)	(0.0031)	(0.004)	(0.0043)
Bu2	0.2942	0.2176	0.0349	0.0884	**0.0312**
	(0.003)	(0.0059)	(0.0085)	(0.0051)	(0.0055)
Bu3	0.4416	0.2057	0.1801	0.2521	**0.1761**
	(0.0052)	(0.0022)	(0.0017)	(0.002)	(0.0091)
Fr1	0.3923	0.1857	0.1819	**0.1167**	0.1665
	(0.0048)	(0.0071)	(0.0017)	(0.0015)	(0.005)
Fr2	0.3504	0.3545	0.1696	**0.1606**	0.3137
	(0.0047)	(0.0011)	(0.0018)	(0.006)	(0.0021)
T2	0.2452	**0.1309**	0.1467	0.1519	0.1543
	(0.0041)	(0.0021)	(0.0013)	(0.0027)	(0.0079)
Average	0.3181	0.1455	0.1144	**0.1000**	0.1772
	(0.0016)	(0.0015)	(0.0017)	(0.0016)	(0.0023)

GE1.3 (n=5000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1981	0.0556	**0.0318**	0.0767	0.5480
	(0.0042)	(0.0071)	(0.0065)	(0.0069)	(0.01)
Bu1	0.3050	0.2449	**0.0358**	0.2254	0.2551
	(0.0024)	(0.0085)	(0.0072)	(0.0077)	(0.0051)
Bu2	0.0995	0.0164	0.0193	**0.0113**	0.1357
	(0.0026)	(0.0022)	(0.0053)	(0.0023)	(0.0079)
Bu3	0.4009	0.1675	**0.0922**	0.1987	0.1695
	(0.0038)	(0.0009)	(0.0003)	(0.0007)	(0.0007)
Fr1	0.3717	0.1602	**0.0854**	0.1894	0.1800
	(0.0032)	(0.0009)	(0.0002)	(0.0007)	(0.0009)
Fr2	0.2330	0.1781	**0.0531**	0.2130	0.1977
	(0.0039)	(0.0024)	(0.0004)	(0.0023)	(0.0088)
T2	0.2605	0.1071	**0.0480**	0.1209	0.1210
	(0.003)	(0.0007)	(0.0003)	(0.0008)	(0.0008)
Average	0.2670	0.0629	**0.0467**	0.0803	0.1567
	(0.0013)	(0.0017)	(0.0016)	(0.0016)	(0.0023)

GE1.3 (n=10000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1732	0.0969	**0.0550**	0.0613	0.0657
	(0.0034)	(0.0014)	(0.0002)	(0.001)	(0.0013)
Bu1	0.2312	0.1403	**0.0194**	0.2378	0.0538
	(0.0034)	(0.0038)	(0.0001)	(0.0056)	(0.0066)
Bu2	0.0966	**0.0256**	0.0445	0.0309	0.1164
	(0.0025)	(0.0042)	(0.007)	(0.0003)	(0.0064)
Bu3	0.4090	0.1644	**0.0941**	0.1870	0.1566
	(0.0033)	(0.0006)	(0.0002)	(0.0006)	(0.0005)
Fr1	0.3888	0.1577	**0.0862**	0.1793	0.1668
	(0.0037)	(0.0006)	(0.0002)	(0.0005)	(0.0006)
Fr2	0.2398	0.1707	**0.0534**	0.1918	0.3137
	(0.003)	(0.001)	(0.0002)	(0.0009)	(0.0093)
T2	0.2533	0.1046	**0.0497**	0.1114	0.1066
	(0.0022)	(0.0005)	(0.0001)	(0.0004)	(0.0004)
Average	0.2560	0.0755	**0.0575**	0.0748	0.0726
	(0.0012)	(0.0009)	(0.001)	(0.0008)	(0.0019)

Table E.3: Relative Impact of Contamination on GE1.3 Measure

E.4 Atkinson Coefficient with Parameter 1 (A1)

A1 (n=500)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1249	0.1249	0.0763	**0.0541**	0.0660
	(0.0012)	(0.0012)	(0.0001)	(0.0004)	(0.0008)
Bu1	0.0180	0.0180	0.0131	**0.0034**	0.0058
	(0.0003)	(0.0003)	(0.0007)	(0.001)	(0.0002)
Bu2	0.0335	0.0335	**0.0090**	0.0253	0.0127
	(0.0005)	(0.0005)	(0.0009)	(0.0004)	(0.0003)
Bu3	0.1959	0.1959	0.1321	**0.0646**	0.0647
	(0.0012)	(0.0012)	(0.0009)	(0.0003)	(0.0004)
Fr1	0.1918	0.1918	0.0874	**0.0609**	0.0623
	(0.0012)	(0.0012)	(0.0008)	(0.0003)	(0.0005)
Fr2	0.1181	0.1181	0.0537	**0.0403**	0.0447
	(0.0011)	(0.0011)	(0.0011)	(0.0003)	(0.0005)
T2	0.0602	0.0602	0.0290	**0.0185**	0.0195
	(0.0005)	(0.0005)	(0.0003)	(0.0001)	(0.0002)
Average	0.1061	0.1061	0.0572	**0.0382**	0.0394
	(0.0003)	(0.0003)	(0.0003)	(0.0002)	(0.0002)

A1 (n=1000)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1197	0.0823	0.0562	**0.0474**	0.0636
	(0.001)	(0.0006)	(0.0006)	(0.0003)	(0.0006)
Bu1	0.0175	0.0138	0.0083	0.0071	**0.0049**
	(0.0002)	(0.0002)	(0.0003)	(0.0001)	(0.0002)
Bu2	0.0328	0.0235	**0.0124**	0.0127	0.0143
	(0.0004)	(0.0003)	(0.0009)	(0.0001)	(0.0003)
Bu3	0.1963	0.1132	0.0914	**0.0516**	0.0590
	(0.001)	(0.0005)	(0.0003)	(0.0002)	(0.0002)
Fr1	0.1852	0.1071	0.1074	**0.0566**	0.0583
	(0.001)	(0.0005)	(0.001)	(0.0002)	(0.0003)
Fr2	0.1131	0.0702	0.0743	**0.0379**	0.0428
	(0.0008)	(0.0005)	(0.0004)	(0.0002)	(0.0003)
T2	0.0582	0.0345	0.0181	0.0179	**0.0176**
	(0.0004)	(0.0002)	(0.0099)	(0.0001)	(0.0001)
Average	0.1033	0.0635	0.0526	**0.0330**	0.0372
	(0.0003)	(0.0002)	(0.0014)	(0.0001)	(0.0001)

A1 (n=5000)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1174	0.0761	0.0635	**0.0423**	0.0518
	(0.0006)	(0.0004)	(0.0004)	(0.0001)	(0.0002)
Bu1	0.0181	0.0114	0.0100	**0.0063**	0.0078
	(0.0001)	(0.0001)	(0.0004)	(0.0000)	(0.0001)
Bu2	0.0330	0.0220	0.0175	**0.0123**	0.0144
	(0.0002)	(0.0001)	(0.0008)	(0.0001)	(0.0001)
Bu3	0.1993	0.1075	0.0979	0.0579	**0.0555**
	(0.0006)	(0.0002)	(0.0002)	(0.0001)	(0.0001)
Fr1	0.1824	0.1004	0.0914	0.0545	**0.0531**
	(0.0005)	(0.0003)	(0.0002)	(0.0001)	(0.0001)
Fr2	0.1144	0.0664	0.0575	**0.0357**	0.0371
	(0.0005)	(0.0002)	(0.0002)	(0.0001)	(0.0001)
T2	0.0602	0.0333	0.0284	0.0174	**0.0170**
	(0.0002)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Average	0.1035	0.0596	0.0523	**0.0324**	0.0338
	(0.0002)	(0.0001)	(0.0001)	(0.0000)	(0.0000)

A1 (n=10000)					
	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1167	0.0744	0.0618	**0.0403**	0.0473
	(0.0007)	(0.0002)	(0.0003)	(0.0001)	(0.0001)
Bu1	0.0181	0.0117	0.0086	**0.0062**	0.0078
	(0.0001)	(0.0001)	(0.0000)	(0.0000)	(0.0001)
Bu2	0.0330	0.0219	0.0171	**0.0116**	0.0131
	(0.0002)	(0.0001)	(0.0002)	(0.0000)	(0.0001)
Bu3	0.1947	0.1071	0.0971	0.0562	**0.0542**
	(0.0004)	(0.0002)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.1869	0.1024	0.0924	0.0539	**0.0526**
	(0.0004)	(0.0002)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.1145	0.0654	0.0573	**0.0347**	0.0359
	(0.0004)	(0.0002)	(0.0001)	(0.0001)	(0.0001)
T2	0.0601	0.0330	0.0285	0.0170	**0.0166**
	(0.0001)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Average	0.1034	0.0594	0.0518	**0.0314**	0.0325
	(0.0001)	(0.0001)	(0.0001)	(0.0000)	(0.0000)

Table E.4: Relative Impact of Contamination on A1 Measure

E.5 Atkinson Coefficient with Parameter 1.5 (A1.5)

A1.5 (n=500)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.1125	0.1125	0.0641	**0.0430**	0.0513
	(0.001)	(0.001)	(0.0021)	(0.0003)	(0.0006)
Bu1	0.0053	0.0053	0.0021	0.0027	**0.0015**
	(0.0001)	(0.0001)	(0.0005)	(0.0002)	(0.0001)
Bu2	0.0262	0.0262	0.0190	**0.0094**	0.0101
	(0.0003)	(0.0003)	(0.001)	(0.0001)	(0.0002)
Bu3	0.1625	0.1625	0.0998	0.0502	**0.0494**
	(0.001)	(0.001)	(0.007)	(0.0002)	(0.0003)
Fr1	0.1542	0.1542	0.0779	**0.0486**	0.0497
	(0.0009)	(0.0009)	(0.001)	(0.0002)	(0.0004)
Fr2	0.0968	0.0968	0.0842	**0.0328**	0.0382
	(0.0007)	(0.0007)	(0.0071)	(0.0002)	(0.0004)
T2	0.0293	0.0293	**0.0011**	0.0090	0.0095
	(0.0002)	(0.0002)	(0.0035)	(0.0001)	(0.0001)
Average	0.0838	0.0838	0.0497	**0.0279**	0.0300
	(0.0003)	(0.0003)	(0.0016)	(0.0001)	(0.0001)

A1.5 (n=1000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.1046	0.0399	**0.0314**	0.0427	0.0528
	(0.0008)	(0.0009)	(0.0023)	(0.0002)	(0.0005)
Bu1	0.0096	0.0068	0.0028	0.0019	**0.0013**
	(0.0001)	(0.0003)	(0.0001)	(0.0000)	(0.0001)
Bu2	0.0476	0.0351	0.0138	**0.0098**	0.0107
	(0.0003)	(0.0003)	(0.0014)	(0.0001)	(0.0001)
Bu3	0.1573	0.1771	0.0742	0.0482	**0.0468**
	(0.0007)	(0.0008)	(0.0012)	(0.0002)	(0.0002)
Fr1	0.1535	0.1744	**0.0093**	0.0462	0.0466
	(0.0007)	(0.0008)	(0.0006)	(0.0002)	(0.0003)
Fr2	0.0974	0.0349	0.0369	**0.0315**	0.0350
	(0.0006)	(0.0007)	(0.0034)	(0.0002)	(0.0003)
T2	0.0388	0.0322	0.0128	0.0084	**0.0083**
	(0.0002)	(0.0002)	(0.0004)	(0.0000)	(0.0001)
Average	0.0870	0.0715	**0.0232**	0.0270	0.0288
	(0.0002)	(0.0002)	(0.0006)	(0.0001)	(0.0001)

A1.5 (n=5000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.1340	0.1286	0.0555	**0.0360**	0.0429
	(0.0005)	(0.0005)	(0.0005)	(0.0001)	(0.0001)
Bu1	0.0155	0.0072	0.0026	**0.0019**	0.0023
	(0.0000)	(0.0000)	(0.0001)	(0.0000)	(0.0000)
Bu2	0.0471	0.0328	0.0105	**0.0090**	0.0097
	(0.0002)	(0.0002)	(0.0007)	(0.0000)	(0.0000)
Bu3	0.1567	0.0714	0.0773	0.0451	**0.0435**
	(0.0004)	(0.0004)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.1529	0.0757	0.0753	0.0442	**0.0432**
	(0.0004)	(0.0004)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0954	0.0869	0.0445	**0.0289**	0.0301
	(0.0004)	(0.0004)	(0.0009)	(0.0001)	(0.0001)
T2	0.0379	0.0300	0.0134	0.0080	**0.0078**
	(0.0001)	(0.0001)	(0.0000)	(0.0000)	(0.0000)
Average	0.0914	0.0618	0.0399	**0.0247**	0.0256
	(0.0001)	(0.0001)	(0.0002)	(0.0000)	(0.0000)

A1.5 (n=10000)					
	NP	**SPCo**	**SPGPD**	**SPPa**	**SPPPD**
Pa	0.1038	0.0494	0.0540	**0.0347**	0.0402
	(0.0005)	(0.0004)	(0.0001)	(0.0001)	(0.0001)
Bu1	0.0096	0.0071	0.0027	**0.0019**	0.0024
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Bu2	0.0471	0.0331	0.0128	**0.0088**	0.0093
	(0.0002)	(0.0001)	(0.0004)	(0.0000)	(0.0000)
Bu3	0.1573	0.0714	0.0777	0.0446	**0.0430**
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.1510	0.0847	0.0748	0.0430	**0.0422**
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0941	0.0576	0.0477	**0.0284**	0.0293
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
T2	0.0381	0.0305	0.0136	0.0080	**0.0078**
	(0.0001)	(0.0001)	(0.0000)	(0.0000)	(0.0000)
Average	0.0859	0.0477	0.0405	**0.0242**	0.0249
	(0.0001)	(0.0001)	(0.0001)	(0.0000)	(0.0000)

Table E.5: Relative Impact of Contamination on A1.5 Measure

E.6 Atkinson Coefficient with Parameter 2 (A2)

A2 (n=500)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0954	0.0954	**0.0189**	0.0397	0.0433
	(0.0008)	(0.0008)	(0.0081)	(0.0004)	(0.0005)
Bu1	0.0013	0.0013	0.0005	0.0004	**0.0003**
	(0.0000)	(0.0000)	(0.0001)	(0.0001)	(0.0000)
Bu2	0.0108	0.0108	0.0073	0.0084	**0.0045**
	(0.0001)	(0.0001)	(0.0017)	(0.0012)	(0.0001)
Bu3	0.1256	0.1256	0.0706	**0.0401**	0.0410
	(0.0007)	(0.0007)	(0.0022)	(0.0002)	(0.0002)
Fr1	0.1322	0.1322	0.0936	**0.0407**	0.0425
	(0.0007)	(0.0007)	(0.0051)	(0.0002)	(0.0003)
Fr2	0.0833	0.0833	0.0405	**0.0273**	0.0300
	(0.0006)	(0.0006)	(0.0021)	(0.0002)	(0.0003)
Average	0.0748	0.0748	0.0386	**0.0261**	0.0269
	(0.0002)	(0.0002)	(0.0015)	(0.0002)	(0.0001)

A2 (n=1000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0988	0.0566	0.0517	**0.0357**	0.0448
	(0.0007)	(0.0008)	(0.0007)	(0.0002)	(0.0004)
Bu1	0.0032	0.0011	0.0005	0.0004	**0.0003**
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Bu2	0.0111	0.0051	0.0090	0.0045	**0.0032**
	(0.0001)	(0.0003)	(0.0008)	(0.0000)	(0.0001)
Bu3	0.1298	0.0869	0.0642	0.0393	**0.0384**
	(0.0005)	(0.0006)	(0.0005)	(0.0001)	(0.0002)
Fr1	0.1316	0.0773	0.0786	0.0394	**0.0393**
	(0.0006)	(0.0006)	(0.0007)	(0.0001)	(0.0002)
Fr2	0.0807	0.0365	0.0344	0.0260	**0.0258**
	(0.0005)	(0.0006)	(0.0016)	(0.0001)	(0.0002)
Average	0.0759	0.0439	0.0397	**0.0242**	0.0253
	(0.0002)	(0.0002)	(0.0003)	(0.0000)	(0.0001)

A2 (n=5000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0922	0.0463	0.0467	**0.0324**	0.0378
	(0.0004)	(0.0004)	(0.0005)	(0.0001)	(0.0001)
Bu1	0.0017	0.0005	0.0004	**0.0002**	0.0003
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Bu2	0.0203	0.0133	0.0108	0.0038	**0.0034**
	(0.0001)	(0.0001)	(0.0016)	(0.0000)	(0.0000)
Bu3	0.1288	0.0403	0.0635	0.0371	**0.0358**
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.1298	0.0398	0.0635	0.0370	**0.0303**
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.0799	0.0402	0.0392	**0.0239**	0.0248
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
Average	0.0754	0.0301	0.0373	0.0224	**0.0221**
	(0.0001)	(0.0001)	(0.0002)	(0.0000)	(0.0000)

A2 (n=10000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.0919	0.0354	0.0478	**0.0308**	0.0316
	(0.0004)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
Bu1	0.0006	0.0003	0.0003	**0.0002**	0.0003
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Bu2	0.0103	0.0034	0.0051	0.0036	**0.0033**
	(0.0001)	(0.0001)	(0.0001)	(0.0000)	(0.0000)
Bu3	0.1284	0.0391	0.0635	0.0361	**0.0350**
	(0.0002)	(0.0002)	(0.0001)	(0.0000)	(0.0000)
Fr1	0.1288	0.0401	0.0635	0.0362	**0.0355**
	(0.0002)	(0.0002)	(0.0001)	(0.0000)	(0.0000)
Fr2	0.0790	0.0300	0.0400	**0.0237**	0.0244
	(0.0002)	(0.0002)	(0.0001)	(0.0000)	(0.0000)
Average	0.0732	0.0247	0.0367	0.0218	**0.0217**
	(0.0001)	(0.0001)	(0.0000)	(0.0000)	(0.0000)

Table E.6: Relative Impact of Contamination on A2 Measure

E.7 Quintile Share Ratio (QSR)

QSR (n=500)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1513	0.1513	**0.0302**	0.0843	0.1633
	(0.0022)	(0.0022)	(0.0001)	(0.0034)	(0.0052)
Bu1	0.0942	0.0942	**0.0051**	0.1209	0.1203
	(0.0021)	(0.0021)	(0.006)	(0.0073)	(0.0086)
Bu2	0.1112	0.1112	**0.0370**	0.0644	0.0451
	(0.002)	(0.002)	(0.0015)	(0.005)	(0.0056)
Bu3	0.1743	0.1743	0.2494	**0.0544**	0.0550
	(0.0011)	(0.0011)	(0.0082)	(0.0003)	(0.0005)
Fr1	0.1743	0.1743	0.0835	**0.0552**	0.0598
	(0.0015)	(0.0015)	(0.0018)	(0.0003)	(0.001)
Fr2	0.1430	0.1430	0.0729	**0.0524**	0.0993
	(0.0014)	(0.0014)	(0.003)	(0.0005)	(0.0091)
Average	0.1414	0.1414	0.0797	**0.0719**	0.0905
	(0.0006)	(0.0006)	(0.0016)	(0.0014)	(0.0021)

QSR (n=1000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1450	0.1120	**0.0456**	0.0609	0.1451
	(0.0013)	(0.002)	(0.0055)	(0.0005)	(0.0038)
Bu1	0.0991	0.1060	0.0635	**0.0622**	0.1724
	(0.0015)	(0.0046)	(0.0062)	(0.003)	(0.0055)
Bu2	0.1111	0.1008	0.0755	**0.0417**	0.1092
	(0.0029)	(0.0022)	(0.0061)	(0.0075)	(0.0066)
Bu3	0.1738	0.0939	0.1022	0.0538	**0.0523**
	(0.0012)	(0.0008)	(0.0066)	(0.0002)	(0.0002)
Fr1	0.1681	0.1001	0.0796	0.0517	**0.0514**
	(0.0009)	(0.0009)	(0.0006)	(0.0002)	(0.0002)
Fr2	0.1466	0.0799	0.0625	**0.0484**	0.0715
	(0.0011)	(0.0012)	(0.0028)	(0.0003)	(0.0026)
Average	0.1406	0.0988	0.0715	**0.0531**	0.1003
	(0.0006)	(0.0008)	(0.0018)	(0.0012)	(0.0014)

QSR (n=5000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1466	0.0845	0.0708	**0.0526**	0.0650
	(0.0009)	(0.0007)	(0.0006)	(0.0002)	(0.0003)
Bu1	0.0913	**0.0308**	0.0501	0.0359	0.0810
	(0.0007)	(0.0011)	(0.003)	(0.0002)	(0.0036)
Bu2	0.1018	0.0459	0.0526	**0.0411**	0.0587
	(0.0009)	(0.0011)	(0.004)	(0.0002)	(0.0009)
Bu3	0.1725	0.0884	0.0827	0.0501	**0.0482**
	(0.0004)	(0.0004)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.1694	0.0833	0.0799	0.0488	**0.0478**
	(0.0005)	(0.0004)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.1425	0.0658	0.0678	**0.0445**	0.0466
	(0.0005)	(0.0005)	(0.0002)	(0.0001)	(0.0001)
Average	0.1373	0.0665	0.0673	**0.0455**	0.0579
	(0.0002)	(0.0003)	(0.0007)	(0.0001)	(0.0005)

QSR (n=10000)	NP	SPCo	SPGPD	SPPa	SPPPD
Pa	0.1516	0.0864	0.0616	**0.0504**	0.0598
	(0.0013)	(0.0006)	(0.0023)	(0.0001)	(0.0002)
Bu1	0.0939	0.0389	0.0419	**0.0354**	0.0533
	(0.0007)	(0.0007)	(0.0001)	(0.0002)	(0.0006)
Bu2	0.1021	0.0508	0.0427	**0.0388**	0.0456
	(0.0006)	(0.0008)	(0.0018)	(0.0002)	(0.0002)
Bu3	0.1718	0.0876	0.0827	0.0487	**0.0471**
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
Fr1	0.1678	0.0839	0.0805	0.0479	**0.0470**
	(0.0003)	(0.0003)	(0.0001)	(0.0001)	(0.0001)
Fr2	0.1483	0.0669	0.0688	**0.0437**	0.0453
	(0.0008)	(0.0004)	(0.0001)	(0.0001)	(0.0001)
Average	0.1392	0.0691	0.0630	**0.0442**	0.0497
	(0.0003)	(0.0002)	(0.0004)	(0.0000)	(0.0001)

Table E.7: Relative Impact of Contamination on QSR Measure

Appendix F

Properties of Confidence Intervals for Inequality Measures

The following tables display the properties of confidence intervals for the estimators of the generalized entropy and the Atkinson measures of inequality.

F.1 Generalized Entropy with Parameter 0 (GE0)

	GE0 (Pa, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.8029	0.9091	0.0360	0.0549
STI	0.7378	0.9081	0.0500	0.0419
BPI	0.4162	0.9555	0.0014	0.0431
PTBPI	0.4090	0.9570	0.0200	0.0230
BTI	0.9230	0.9533	0.0177	0.0290
BCAI	0.3883	0.9338	0.0248	0.0414
PTBCAI	0.3836	0.9347	0.0222	0.0431
BPGPDI	0.3702	0.9533	0.0002	0.0465
BTGPDI	0.6971	0.9399	0.0061	0.0540
BPPI	0.7051	0.9540	0.0071	0.0389
BTPI	0.5454	0.9415	0.0025	0.0560

	GE0 (Pa, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.5785	0.9057	0.0400	0.0543
STI	0.5485	0.9177	0.0400	0.0423
BPI	0.3354	0.9614	0.0027	0.0359
PTBPI	0.3366	0.9610	0.0013	0.0377
BTI	0.7773	0.9480	0.0228	0.0292
BCAI	0.3236	0.9332	0.0294	0.0374
PTBCAI	0.3364	0.9319	0.0286	0.0395
BPGPDI	0.2959	0.9483	0.0400	0.0117
BTGPDI	0.5307	0.9395	0.0303	0.0302
BPPI	0.5828	0.9652	0.0052	0.0296
BTPI	0.3988	0.9589	0.0051	0.0360

	GE0 (Pa, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.4154	0.9372	0.0300	0.0328
STI	0.3303	0.9374	0.0200	0.0426
BPI	0.2124	0.9607	0.0100	0.0293
PTBPI	0.2284	0.9614	0.0125	0.0261
BTI	0.5689	0.9488	0.0237	0.0275
BCAI	0.2169	0.9429	0.0310	0.0261
PTBCAI	0.2145	0.9288	0.0389	0.0323
BPGPDI	0.1524	0.9600	0.0130	0.0270
BTGPDI	0.2459	0.9540	0.0300	0.0160
BPPI	0.3172	0.9641	0.0152	0.0207
BTPI	0.2025	0.9594	0.0110	0.0296

	GE0 (Pa, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.3461	0.9310	0.0060	0.0630
STI	0.3185	0.9307	0.0061	0.0632
BPI	0.2123	0.9690	0.0103	0.0207
PTBPI	0.2224	0.9678	0.0013	0.0309
BTI	0.7851	0.9492	0.0248	0.0260
BCAI	0.2260	0.9393	0.0317	0.0290
PTBCAI	0.2151	0.9381	0.0277	0.0342
BPGPDI	0.1526	0.9550	0.0131	0.0319
BTGPDI	0.2921	0.9672	0.0083	0.0245
BPPI	0.2101	0.9561	0.0285	0.0154
BTPI	0.1941	0.9507	0.0251	0.0242

Table F.1: Properties of Confidence Intervals for GE0 when Samples come from Pa

	GE0 (Bu1, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.8327	0.9406	0.0198	0.0396
STI	0.8302	0.9606	0.0196	0.0198
BPI	0.5858	0.9643	0.0112	0.0245
PTBPI	0.5795	0.9657	0.0112	0.0231
BTI	0.8150	0.9522	0.0303	0.0175
BCAI	0.5220	0.9362	0.0363	0.0275
PTBCAI	0.5315	0.9421	0.0362	0.0217
BPGPDI	0.6013	0.9200	0.0500	0.0300
BTGPDI	0.6878	0.9375	0.0312	0.0313
BPPI	0.9360	0.9592	0.0055	0.0353
BTPI	0.5108	0.9667	0.0000	0.0333

	GE0 (Bu1, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.6953	0.9671	0.0132	0.0197
STI	0.7094	0.9783	0.0145	0.0072
BPI	0.4614	0.9737	0.0048	0.0215
PTBPI	0.4597	0.9761	0.0024	0.0215
BTI	0.6894	0.9495	0.0288	0.0217
BCAI	0.4161	0.9370	0.0400	0.0230
PTBCAI	0.4276	0.9334	0.0436	0.0230
BPGPDI	0.4723	0.9433	0.0300	0.0267
BTGPDI	0.5303	0.9514	0.0207	0.0279
BPPI	0.9832	0.9573	0.0225	0.0202
BTPI	0.4822	0.9448	0.0301	0.0251

	GE0 (Bu1, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.4917	0.9516	0.0256	0.0228
STI	0.4844	0.9518	0.0255	0.0227
BPI	0.2661	0.9661	0.0124	0.0215
PTBPI	0.2818	0.9665	0.0147	0.0188
BTI	0.4324	0.9552	0.0287	0.0161
BCAI	0.2564	0.9440	0.0385	0.0175
PTBCAI	0.2617	0.9395	0.0375	0.0230
BPGPDI	0.2121	0.9667	0.0131	0.0202
BTGPDI	0.2487	0.9538	0.0200	0.0262
BPPI	0.3965	0.9500	0.0230	0.0270
BTPI	0.3505	0.9596	0.0387	0.0017

	GE0 (Bu1, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.4840	0.9613	0.0258	0.0129
STI	0.4831	0.9503	0.0268	0.0229
BPI	0.2617	0.9680	0.0158	0.0162
PTBPI	0.2732	0.9626	0.0200	0.0174
BTI	0.3356	0.9476	0.0321	0.0203
BCAI	0.2517	0.9325	0.0474	0.0201
PTBCAI	0.2677	0.9478	0.0234	0.0288
BPGPDI	0.2073	0.9567	0.0223	0.0210
BTGPDI	0.2826	0.9500	0.0200	0.0300
BPPI	0.3942	0.9547	0.0436	0.0017
BTPI	0.3495	0.9583	0.0401	0.0016

Table F.2: Properties of Confidence Intervals for GE0 when Samples come from Bu1

		GE0	(Bu2, n=500)	
	ACIL	CP	LNCP	UNCP
SNI	0.2531	0.9418	0.0324	0.0258
STI	0.2410	0.9463	0.0209	0.0328
BPI	0.5354	0.9540	0.0030	0.0430
PTBPI	0.5281	0.9500	0.0200	0.0300
BTI	0.9379	0.9310	0.0160	0.0530
BCAI	0.4811	0.9380	0.0150	0.0470
PTBCAI	0.4857	0.9540	0.0160	0.0300
BPGPDI	0.5821	0.9017	0.0000	0.0983
BTGPDI	0.6659	0.9717	0.0030	0.0253
BPPI	0.5618	0.9323	0.0062	0.0615
BTPI	0.6190	0.9333	0.0300	0.0367

		GE0	(Bu2, n=1000)	
	ACIL	CP	LNCP	UNCP
SNI	0.3027	0.9368	0.0368	0.0264
STI	0.2910	0.9153	0.0715	0.0132
BPI	0.4521	0.9620	0.0200	0.0180
PTBPI	0.4483	0.9640	0.0100	0.0260
BTI	0.9375	0.9480	0.0180	0.0340
BCAI	0.4142	0.9500	0.0170	0.0330
PTBCAI	0.4137	0.9460	0.0150	0.0390
BPGPDI	0.4673	0.9467	0.0033	0.0500
BTGPDI	0.5303	0.9417	0.0080	0.0503
BPPI	0.4463	0.9507	0.0292	0.0201
BTPI	0.3087	0.9557	0.0230	0.0213

		GE0	(Bu2, n=5000)	
	ACIL	CP	LNCP	UNCP
SNI	0.4878	0.9411	0.0389	0.0200
STI	0.4730	0.9666	0.0334	0.0000
BPI	0.3040	0.9020	0.0400	0.0580
PTBPI	0.3270	0.9170	0.0010	0.0820
BTI	0.7772	0.9310	0.0160	0.0530
BCAI	0.3053	0.9100	0.0160	0.0740
PTBCAI	0.2946	0.9490	0.0260	0.0250
BPGPDI	0.2632	0.9353	0.0030	0.0617
BTGPDI	0.3695	0.9541	0.0400	0.0059
BPPI	0.3239	0.9643	0.0255	0.0102
BTPI	0.3937	0.9560	0.0201	0.0239

		GE0	(Bu2, n=10000)	
	ACIL	CP	LNCP	UNCP
SNI	0.4793	0.9731	0.0267	0.0002
STI	0.4781	0.9693	0.0104	0.0203
BPI	0.3025	0.9520	0.0201	0.0279
PTBPI	0.3154	0.9640	0.0211	0.0149
BTI	0.6456	0.9310	0.0169	0.0521
BCAI	0.3115	0.9560	0.0189	0.0251
PTBCAI	0.3031	0.9650	0.0158	0.0192
BPGPDI	0.2645	0.9567	0.0200	0.0233
BTGPDI	0.4292	0.9504	0.0200	0.0296
BPPI	0.3112	0.9512	0.0286	0.0202
BTPI	0.3754	0.9591	0.0273	0.0136

Table F.3: Properties of Confidence Intervals for GE0 when Samples come from Bu2

		GE0	(Bu3, n=500)	
	ACIL	CP	LNCP	UNCP
SNI	0.1293	0.8902	0.0081	0.1017
STI	0.1292	0.9074	0.0400	0.0526
BPI	0.1783	0.9678	0.0213	0.0109
PTBPI	0.1745	0.9693	0.0112	0.0195
BTI	0.2640	0.9514	0.0217	0.0269
BCAI	0.1655	0.9372	0.0519	0.0109
PTBCAI	0.1672	0.9587	0.0319	0.0094
BPGPDI	0.1775	0.8383	0.0007	0.1610
BTGPDI	0.2785	0.9724	0.0021	0.0255
BPPI	0.2595	0.9652	0.0070	0.0278
BTPI	0.2657	0.9315	0.0334	0.0351

		GE0	(Bu3, n=1000)	
	ACIL	CP	LNCP	UNCP
SNI	0.1148	0.9116	0.0400	0.0484
STI	0.1150	0.9150	0.0550	0.0300
BPI	0.1331	0.9673	0.0146	0.0181
PTBPI	0.1343	0.9625	0.0193	0.0182
BTI	0.1948	0.9495	0.0436	0.0069
BCAI	0.1261	0.9341	0.0576	0.0083
PTBCAI	0.1316	0.9428	0.0491	0.0081
BPGPDI	0.1320	0.9433	0.0060	0.0507
BTGPDI	0.1997	0.9678	0.0043	0.0279
BPPI	0.1581	0.9636	0.0121	0.0243
BTPI	0.1595	0.9513	0.0142	0.0345

		GE0	(Bu3, n=5000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0631	0.9143	0.0102	0.0755
STI	0.0634	0.9157	0.0196	0.0647
BPI	0.0708	0.9699	0.0145	0.0156
PTBPI	0.0762	0.9699	0.0145	0.0156
BTI	0.0992	0.9597	0.0347	0.0056
BCAI	0.0680	0.9441	0.0503	0.0056
PTBCAI	0.0697	0.9315	0.0629	0.0056
BPGPDI	0.0529	0.9433	0.0066	0.0501
BTGPDI	0.0534	0.9519	0.0080	0.0401
BPPI	0.0690	0.9527	0.0237	0.0236
BTPI	0.0613	0.9535	0.0187	0.0278

		GE0	(Bu3, n=10000)	
	ACIL	CP	LNCP	UNCP
SNI	0.0532	0.9200	0.0110	0.0690
STI	0.0532	0.9143	0.0000	0.0857
BPI	0.0698	0.9676	0.0168	0.0156
PTBPI	0.0739	0.9654	0.0190	0.0156
BTI	0.0818	0.9594	0.0350	0.0056
BCAI	0.0670	0.9434	0.0485	0.0081
PTBCAI	0.0687	0.9514	0.0442	0.0044
BPGPDI	0.0523	0.9500	0.0260	0.0240
BTGPDI	0.0538	0.9575	0.0207	0.0218
BPPI	0.0586	0.9596	0.0185	0.0219
BTPI	0.0518	0.9585	0.0155	0.0260

Table F.4: Properties of Confidence Intervals for GE0 when Samples come from Bu3

GE0 (Fr1, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.1552	0.8963	0.0431	0.0606
STI	0.1583	0.9171	0.0297	0.0532
BPI	0.1660	0.9150	0.0040	0.0810
PTBPI	0.1680	0.9330	0.0030	0.0640
BTI	0.2653	0.9250	0.0310	0.0440
BCAI	0.1556	0.9020	0.0320	0.0660
PTBCAI	0.1571	0.9640	0.0180	0.0180
BPGPDI	0.1326	0.9417	0.0230	0.0353
BTGPDI	0.1493	0.9445	0.0300	0.0255
BPPI	0.1763	0.9569	0.0207	0.0224
BTPI	0.1806	0.9267	0.0381	0.0352

GE0 (Fr1, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.2793	0.9576	0.0221	0.0203
STI	0.2784	0.9578	0.0381	0.0041
BPI	0.1311	0.9240	0.0009	0.0751
PTBPI	0.1357	0.9470	0.0027	0.0503
BTI	0.1879	0.9370	0.0269	0.0361
BCAI	0.1266	0.9060	0.0438	0.0502
PTBCAI	0.1282	0.9120	0.0444	0.0436
BPGPDI	0.0845	0.9219	0.0231	0.0550
BTGPDI	0.0862	0.9440	0.0049	0.0511
BPPI	0.0851	0.9581	0.0213	0.0206
BTPI	0.0766	0.9532	0.0205	0.0263

GE0 (Fr1, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.2103	0.9567	0.0031	0.0402
STI	0.2115	0.9517	0.0032	0.0451
BPI	0.0725	0.9652	0.0186	0.0162
PTBPI	0.0735	0.9610	0.0160	0.0230
BTI	0.0943	0.9670	0.0327	0.0003
BCAI	0.0655	0.9220	0.0510	0.0270
PTBCAI	0.0670	0.9156	0.0484	0.0360
BPGPDI	0.0553	0.9217	0.0300	0.0483
BTGPDI	0.0541	0.9440	0.0049	0.0511
BPPI	0.0732	0.9545	0.0253	0.0202
BTPI	0.0665	0.9555	0.0185	0.0260

GE0 (Fr1, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.2097	0.9530	0.0267	0.0203
STI	0.2071	0.9544	0.0250	0.0206
BPI	0.0689	0.9640	0.0250	0.0110
PTBPI	0.0734	0.9670	0.0209	0.0121
BTI	0.1457	0.9645	0.0305	0.0050
BCAI	0.0670	0.9210	0.0417	0.0373
PTBCAI	0.0694	0.9467	0.0520	0.0013
BPGPDI	0.0573	0.9600	0.0017	0.0383
BTGPDI	0.0621	0.9562	0.0023	0.0415
BPPI	0.0739	0.9529	0.0269	0.0202
BTPI	0.0657	0.9565	0.0264	0.0171

Table F.5: Properties of Confidence Intervals for GE0 when Samples come from Fr1

GE0 (Fr2, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.3281	0.9698	0.0200	0.0102
STI	0.3494	0.9686	0.0200	0.0114
BPI	0.3043	0.9656	0.0038	0.0306
PTBPI	0.3034	0.9712	0.0013	0.0275
BTI	0.5964	0.9507	0.0277	0.0216
BCAI	0.2837	0.9470	0.0284	0.0246
PTBCAI	0.2883	0.9342	0.0367	0.0291
BPGPDI	0.3002	0.9600	0.0200	0.0200
BTGPDI	0.4708	0.9645	0.0047	0.0308
BPPI	0.5227	0.9743	0.0069	0.0188
BTPI	0.4968	0.9468	0.0264	0.0268

GE0 (Fr2, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.2244	0.9438	0.0183	0.0379
STI	0.2298	0.9501	0.0085	0.0414
BPI	0.2478	0.9508	0.0200	0.0292
PTBPI	0.2536	0.9529	0.0225	0.0246
BTI	0.4392	0.9551	0.0248	0.0201
BCAI	0.2405	0.9358	0.0412	0.0230
PTBCAI	0.2385	0.9312	0.0470	0.0218
BPGPDI	0.2277	0.9600	0.0100	0.0300
BTGPDI	0.2456	0.9573	0.0207	0.0220
BPPI	0.2596	0.9564	0.0139	0.0297
BTPI	0.2313	0.9519	0.0243	0.0238

GE0 (Fr2, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.2777	0.9686	0.0204	0.0110
STI	0.2770	0.9646	0.0243	0.0111
BPI	0.1449	0.9664	0.0135	0.0201
PTBPI	0.1453	0.9679	0.0147	0.0174
BTI	0.2497	0.9469	0.0369	0.0162
BCAI	0.1351	0.9350	0.0449	0.0201
PTBCAI	0.1383	0.9370	0.0400	0.0230
BPGPDI	0.0977	0.9533	0.0404	0.0063
BTGPDI	0.1589	0.9510	0.0200	0.0290
BPPI	0.1407	0.9524	0.0205	0.0271
BTPI	0.1320	0.9562	0.0160	0.0278

GE0 (Fr2, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.2780	0.9697	0.0193	0.0110
STI	0.2748	0.9676	0.0214	0.0110
BPI	0.1353	0.9661	0.0024	0.0315
PTBPI	0.1448	0.9514	0.0212	0.0274
BTI	0.2390	0.9585	0.0254	0.0161
BCAI	0.1391	0.9423	0.0361	0.0216
PTBCAI	0.1415	0.9461	0.0265	0.0274
BPGPDI	0.0976	0.9533	0.0410	0.0057
BTGPDI	0.1566	0.9505	0.0204	0.0291
BPPI	0.1379	0.9544	0.0285	0.0171
BTPI	0.1252	0.9552	0.0170	0.0278

Table F.6: Properties of Confidence Intervals for GE0 when Samples come from Fr2

		GE0	(T2, n=500)	
	ACIL	CP	LNCP	UNCP
SNI	0.5674	0.9586	0.0323	0.0091
STI	0.5824	0.9514	0.0354	0.0132
BPI	0.2558	0.9050	0.0080	0.0870
PTBPI	0.2485	0.9660	0.0160	0.0180
BTI	0.2805	0.9490	0.0400	0.0110
BCAI	0.2204	0.9020	0.0540	0.0440
PTBCAI	0.2300	0.9240	0.0530	0.0230
BPGPDI	0.2838	0.9667	0.0017	0.0316
BTGPDI	0.3248	0.9818	0.0020	0.0162
BPPI	0.3944	0.9680	0.0253	0.0067
BTPI	0.3581	0.9426	0.0400	0.0174

		GE0	(T2, n=1000)	
	ACIL	CP	LNCP	UNCP
SNI	0.4347	0.9123	0.0625	0.0252
STI	0.4373	0.9382	0.0456	0.0162
BPI	0.1920	0.9510	0.0270	0.0220
PTBPI	0.1946	0.9010	0.0030	0.0960
BTI	0.2247	0.9410	0.0440	0.0150
BCAI	0.1744	0.9440	0.0470	0.0090
PTBCAI	0.1813	0.9280	0.0520	0.0200
BPGPDI	0.2037	0.9600	0.0300	0.0100
BTGPDI	0.2502	0.9653	0.0107	0.0240
BPPI	0.2436	0.9631	0.0268	0.0101
BTPI	0.2286	0.9485	0.0272	0.0243

		GE0	(T2, n=5000)	
	ACIL	CP	LNCP	UNCP
SNI	0.4619	0.9687	0.0112	0.0201
STI	0.4640	0.9610	0.0191	0.0199
BPI	0.1013	0.9113	0.0044	0.0843
PTBPI	0.1011	0.9490	0.0109	0.0401
BTI	0.1185	0.9450	0.0200	0.0350
BCAI	0.0922	0.9331	0.0561	0.0108
PTBCAI	0.0921	0.9520	0.0249	0.0231
BPGPDI	0.0830	0.9517	0.0200	0.0283
BTGPDI	0.0793	0.9565	0.0023	0.0412
BPPI	0.1043	0.9562	0.0237	0.0201
BTPI	0.0914	0.9598	0.0248	0.0154

		GE0	(T2, n=10000)	
	ACIL	CP	LNCP	UNCP
SNI	0.3628	0.9627	0.0072	0.0301
STI	0.3656	0.9610	0.0089	0.0301
BPI	0.1017	0.9202	0.0053	0.0745
PTBPI	0.1016	0.9377	0.0063	0.0560
BTI	0.1213	0.9573	0.0172	0.0255
BCAI	0.0900	0.9364	0.0443	0.0193
PTBCAI	0.0888	0.9487	0.0330	0.0183
BPGPDI	0.0819	0.9500	0.0200	0.0300
BTGPDI	0.0750	0.9556	0.0023	0.0421
BPPI	0.0960	0.9543	0.0355	0.0102
BTPI	0.0922	0.9530	0.0264	0.0206

Table F.7: Properties of Confidence Intervals for GE0 when Samples come from T2

F.2 Generalized Entropy with Parameter 1 (GE1)

GE1 (Pa, n=500)	ACIL	CP	LNCP	UNCP
SNI	1.0376	0.9644	0.0127	0.0229
STI	1.0791	0.9635	0.0224	0.0141
BPI	1.0446	0.9627	0.0200	0.0173
PTBPI	1.0375	0.9780	0.0120	0.0100
BTI	0.9057	0.9056	0.0288	0.0656
BCAI	0.9596	0.9633	0.0224	0.0143
PTBCAI	0.9596	0.9565	0.0152	0.0283
BPGPDI	0.4570	0.9437	0.0050	0.0513
BTGPDI	0.7839	0.9303	0.0109	0.0588
BPPI	0.7919	0.9444	0.0219	0.0337
BTPI	0.6322	0.9319	0.0073	0.0608

GE1 (Pa, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.8708	0.9460	0.0260	0.0280
STI	0.9650	0.9229	0.0497	0.0274
BPI	0.9749	0.9420	0.0400	0.0180
PTBPI	1.0144	0.9417	0.0320	0.0263
BTI	0.8944	0.9492	0.0403	0.0105
BCAI	0.9084	0.9029	0.0090	0.0881
PTBCAI	0.9003	0.9397	0.0226	0.0377
BPGPDI	0.3827	0.9387	0.0448	0.0165
BTGPDI	0.6175	0.9299	0.0351	0.0350
BPPI	0.6696	0.9556	0.0100	0.0344
BTPI	0.4856	0.9493	0.0099	0.0408

GE1 (Pa, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.6449	0.9647	0.0196	0.0157
STI	0.6535	0.9637	0.0139	0.0224
BPI	0.8115	0.9432	0.0200	0.0368
PTBPI	0.8128	0.9514	0.0276	0.0210
BTI	0.9125	0.9360	0.0271	0.0369
BCAI	0.8157	0.9332	0.0141	0.0527
PTBCAI	0.7969	0.9318	0.0180	0.0502
BPGPDI	0.2392	0.9504	0.0178	0.0318
BTGPDI	0.3327	0.9544	0.0248	0.0208
BPPI	0.4040	0.9545	0.0200	0.0255
BTPI	0.2893	0.9498	0.0158	0.0344

GE1 (Pa, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.6247	0.9689	0.0853	0.0458
STI	0.5906	0.9416	0.0304	0.0280
BPI	0.7997	0.9485	0.0300	0.0215
PTBPI	0.8522	0.9307	0.0300	0.0393
BTI	4.4474	0.9486	0.0256	0.0258
BCAI	0.8404	0.9447	0.0177	0.0376
PTBCAI	0.7854	0.9447	0.0117	0.0436
BPGPDI	0.2394	0.9454	0.0179	0.0367
BTGPDI	0.3789	0.9576	0.0131	0.0293
BPPI	0.2969	0.9565	0.0233	0.0202
BTPI	0.2809	0.9491	0.0219	0.0290

Table F.8: Properties of Confidence Intervals for GE1 when Samples come from Pa

GE1 (Bu1, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.9613	0.9408	0.0074	0.0518
STI	0.9673	0.9406	0.0015	0.0579
BPI	0.9619	0.8902	0.0500	0.0598
PTBPI	0.9677	0.9052	0.0400	0.0548
BTI	0.9781	0.8931	0.0556	0.0513
BCAI	0.9603	0.8927	0.0197	0.0876
PTBCAI	0.9049	0.8759	0.0263	0.0978
BPGPDI	0.6881	0.9104	0.0548	0.0348
BTGPDI	0.7746	0.9279	0.0360	0.0361
BPPI	0.6228	0.9496	0.0103	0.0401
BTPI	0.5976	0.9571	0.0048	0.0381

GE1 (Bu1, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.9593	0.9299	0.0122	0.0579
STI	0.9512	0.9315	0.0078	0.0607
BPI	0.9519	0.9323	0.0300	0.0377
PTBPI	0.9661	0.9378	0.0300	0.0322
BTI	0.9624	0.9130	0.0456	0.0414
BCAI	0.9102	0.9095	0.0239	0.0666
PTBCAI	0.9206	0.9069	0.0241	0.0690
BPGPDI	0.5591	0.9337	0.0348	0.0315
BTGPDI	0.6171	0.9418	0.0255	0.0327
BPPI	0.7700	0.9477	0.0273	0.0250
BTPI	0.5690	0.9312	0.0339	0.0349

GE1 (Bu1, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.6767	0.9391	0.0305	0.0304
STI	0.6392	0.9463	0.0233	0.0304
BPI	0.8341	0.9415	0.0205	0.0380
PTBPI	0.7855	0.9442	0.0200	0.0358
BTI	0.9367	0.9404	0.0203	0.0393
BCAI	0.7842	0.9265	0.0221	0.0514
PTBCAI	0.7786	0.9240	0.0224	0.0536
BPGPDI	0.2989	0.9571	0.0179	0.0250
BTGPDI	0.3355	0.9442	0.0248	0.0310
BPPI	0.4833	0.9404	0.0278	0.0318
BTPI	0.4373	0.9500	0.0435	0.0065

GE1 (Bu1, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.6282	0.9412	0.0357	0.0231
STI	0.6637	0.9436	0.0282	0.0282
BPI	0.8137	0.9404	0.0040	0.0556
PTBPI	0.8272	0.9581	0.0215	0.0204
BTI	2.5581	0.9493	0.0229	0.0278
BCAI	0.7554	0.9363	0.0299	0.0338
PTBCAI	0.7606	0.9340	0.0282	0.0378
BPGPDI	0.2941	0.9471	0.0271	0.0258
BTGPDI	0.3694	0.9404	0.0248	0.0348
BPPI	0.4810	0.9551	0.0284	0.0165
BTPI	0.4363	0.9487	0.0449	0.0064

Table F.9: Properties of Confidence Intervals for GE1 when Samples come from Bu1

	GE1 (Bu2, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.9357	0.9452	0.0024	0.0524
STI	0.9015	0.9488	0.0047	0.0465
BPI	0.9656	0.9234	0.0300	0.0466
PTBPI	0.9586	0.9338	0.0300	0.0362
BTI	0.9883	0.9158	0.0483	0.0359
BCAI	0.9601	0.9474	0.0308	0.0218
PTBCAI	0.9391	0.9041	0.0206	0.0753
BPGPDI	0.6689	0.9221	0.0248	0.0531
BTGPDI	0.7527	0.9621	0.0078	0.0301
BPPI	0.6486	0.9227	0.0110	0.0663
BTPI	0.7058	0.9237	0.0348	0.0415

	GE1 (Bu2, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.9237	0.9051	0.0231	0.0718
STI	0.8650	0.9030	0.0278	0.0692
BPI	0.9019	0.9330	0.0320	0.0350
PTBPI	0.8683	0.9372	0.0302	0.0326
BTI	0.9611	0.9449	0.0391	0.0160
BCAI	0.9766	0.9578	0.0067	0.0355
PTBCAI	0.8488	0.9330	0.0212	0.0458
BPGPDI	0.5541	0.9371	0.0081	0.0548
BTGPDI	0.6171	0.9321	0.0128	0.0551
BPPI	0.5331	0.9511	0.0240	0.0249
BTPI	0.5955	0.9461	0.0278	0.0261

	GE1 (Bu2, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.9120	0.9210	0.0384	0.0406
STI	0.8266	0.9309	0.0062	0.0629
BPI	0.8918	0.9457	0.0300	0.0243
PTBPI	0.8109	0.9369	0.0400	0.0231
BTI	0.8562	0.9067	0.0327	0.0606
BCAI	0.8532	0.9569	0.0110	0.0321
PTBCAI	0.8063	0.9397	0.0114	0.0489
BPGPDI	0.3500	0.9257	0.0078	0.0665
BTGPDI	0.4563	0.9445	0.0448	0.0107
BPPI	0.4107	0.9547	0.0303	0.0150
BTPI	0.4805	0.9464	0.0199	0.0337

	GE1 (Bu2, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.9048	0.9460	0.0214	0.0326
STI	0.8215	0.9425	0.0303	0.0272
BPI	0.8720	0.9476	0.0300	0.0224
PTBPI	0.7639	0.9469	0.0320	0.0211
BTI	0.8298	0.9468	0.0300	0.0232
BCAI	0.7830	0.9414	0.0131	0.0455
PTBCAI	0.7151	0.9487	0.0259	0.0254
BPGPDI	0.3513	0.9571	0.0148	0.0281
BTGPDI	0.5160	0.9508	0.0248	0.0244
BPPI	0.3980	0.9516	0.0234	0.0250
BTPI	0.4622	0.9495	0.0321	0.0184

Table F.10: Properties of Confidence Intervals for GE1 when Samples come from Bu2

	GE1 (Bu3, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.3863	0.9570	0.0104	0.0326
STI	0.3629	0.9598	0.0027	0.0375
BPI	0.3688	0.9506	0.0200	0.0294
PTBPI	0.3919	0.9568	0.0200	0.0232
BTI	0.3598	0.9488	0.0205	0.0307
BCAI	0.3760	0.9322	0.0263	0.0415
PTBCAI	0.3458	0.9338	0.0211	0.0451
BPGPDI	0.2643	0.8287	0.0055	0.1658
BTGPDI	0.3653	0.9628	0.0069	0.0303
BPPI	0.3463	0.9556	0.0118	0.0326
BTPI	0.3525	0.9219	0.0382	0.0399

	GE1 (Bu3, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.2999	0.9439	0.0201	0.0360
STI	0.2913	0.9522	0.0119	0.0359
BPI	0.3049	0.9574	0.0014	0.0412
PTBPI	0.3048	0.9611	0.0013	0.0376
BTI	0.3631	0.9474	0.0281	0.0245
BCAI	0.3094	0.9271	0.0371	0.0358
PTBCAI	0.3274	0.9241	0.0345	0.0414
BPGPDI	0.2188	0.9337	0.0108	0.0555
BTGPDI	0.1865	0.9582	0.0091	0.0327
BPPI	0.2449	0.9540	0.0169	0.0291
BTPI	0.2463	0.9417	0.0190	0.0393

	GE1 (Bu3, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.1627	0.9214	0.0494	0.0292
STI	0.1582	0.9207	0.0468	0.0325
BPI	0.2170	0.9429	0.0325	0.0246
PTBPI	0.2100	0.9687	0.0038	0.0275
BTI	0.2692	0.9626	0.0145	0.0229
BCAI	0.2140	0.9323	0.0401	0.0276
PTBCAI	0.2157	0.9275	0.0450	0.0275
BPGPDI	0.1397	0.9337	0.0114	0.0549
BTGPDI	0.1402	0.9423	0.0128	0.0449
BPPI	0.1558	0.9431	0.0285	0.0284
BTPI	0.1481	0.9439	0.0235	0.0326

	GE1 (Bu3, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.1532	0.9281	0.0379	0.0340
STI	0.1614	0.9186	0.0491	0.0323
BPI	0.2136	0.9512	0.0213	0.0275
PTBPI	0.2208	0.9513	0.0225	0.0262
BTI	0.2117	0.9570	0.0227	0.0203
BCAI	0.2036	0.9291	0.0418	0.0291
PTBCAI	0.2059	0.9273	0.0434	0.0293
BPGPDI	0.1391	0.9404	0.0308	0.0288
BTGPDI	0.1406	0.9479	0.0296	0.0225
BPPI	0.1454	0.9500	0.0233	0.0267
BTPI	0.1386	0.9489	0.0303	0.0208

Table F.11: Properties of Confidence Intervals for GE1 when Samples come from Bu3

	GE1 (Fr1, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.3975	0.9641	0.0051	0.0308
STI	0.4015	0.9530	0.0094	0.0376
BPI	0.3854	0.9488	0.0200	0.0312
PTBPI	0.3977	0.9550	0.0200	0.0250
BTI	0.4307	0.9547	0.0176	0.0277
BCAI	0.3733	0.9171	0.0357	0.0472
PTBCAI	0.3613	0.9242	0.0262	0.0496
BPGPDI	0.2194	0.9321	0.0278	0.0401
BTGPDI	0.2361	0.9349	0.0348	0.0303
BPPI	0.2631	0.9473	0.0255	0.0272
BTPI	0.2674	0.9171	0.0429	0.0400

	GE1 (Fr1, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.3139	0.9594	0.0114	0.0292
STI	0.3127	0.9601	0.0090	0.0309
BPI	0.3191	0.9591	0.0014	0.0395
PTBPI	0.3396	0.9556	0.0014	0.0430
BTI	0.3765	0.9430	0.0310	0.0260
BCAI	0.3422	0.9261	0.0363	0.0376
PTBCAI	0.3037	0.9424	0.0201	0.0375
BPGPDI	0.1713	0.9123	0.0279	0.0598
BTGPDI	0.1730	0.9344	0.0097	0.0559
BPPI	0.1719	0.9485	0.0261	0.0254
BTPI	0.1634	0.9436	0.0253	0.0311

	GE1 (Fr1, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.1610	0.9338	0.0320	0.0342
STI	0.1722	0.9145	0.0513	0.0342
BPI	0.2195	0.9692	0.0049	0.0259
PTBPI	0.2166	0.9571	0.0200	0.0229
BTI	0.3592	0.9528	0.0242	0.0230
BCAI	0.2137	0.9398	0.0326	0.0276
PTBCAI	0.2073	0.9391	0.0348	0.0261
BPGPDI	0.1421	0.9121	0.0348	0.0531
BTGPDI	0.1409	0.9344	0.0097	0.0559
BPPI	0.1600	0.9449	0.0301	0.0250
BTPI	0.1533	0.9459	0.0233	0.0308

	GE1 (Fr1, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.1617	0.9382	0.0310	0.0308
STI	0.1649	0.9177	0.0514	0.0309
BPI	0.2156	0.9653	0.0039	0.0308
PTBPI	0.2139	0.9699	0.0025	0.0276
BTI	0.3403	0.9505	0.0278	0.0217
BCAI	0.1966	0.9473	0.0282	0.0245
PTBCAI	0.2277	0.9288	0.0438	0.0274
BPGPDI	0.1441	0.9504	0.0065	0.0431
BTGPDI	0.1489	0.9466	0.0071	0.0463
BPPI	0.1607	0.9533	0.0217	0.0250
BTPI	0.1525	0.9569	0.0212	0.0219

Table F.12: Properties of Confidence Intervals for GE1 when Samples come from Fr1

	GE1 (Fr2, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.6866	0.9279	0.0064	0.0657
STI	0.7093	0.9310	0.0032	0.0658
BPI	0.6846	0.9268	0.0300	0.0432
PTBPI	0.7006	0.9207	0.0300	0.0493
BTI	0.7925	0.9410	0.0141	0.0449
BCAI	0.6758	0.9120	0.0224	0.0656
PTBCAI	0.6600	0.9058	0.0154	0.0788
BPGPDI	0.3870	0.9504	0.0248	0.0248
BTGPDI	0.5576	0.9549	0.0095	0.0356
BPPI	0.6095	0.9647	0.0117	0.0236
BTPI	0.5836	0.9372	0.0312	0.0316

	GE1 (Fr2, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.6006	0.9045	0.0329	0.0626
STI	0.5950	0.9137	0.0179	0.0684
BPI	0.6688	0.9272	0.0300	0.0428
PTBPI	0.6513	0.9343	0.0300	0.0357
BTI	0.5359	0.9255	0.0272	0.0473
BCAI	0.6142	0.9301	0.0122	0.0577
PTBCAI	0.6406	0.9025	0.0346	0.0629
BPGPDI	0.3145	0.9504	0.0148	0.0348
BTGPDI	0.3324	0.9477	0.0255	0.0268
BPPI	0.3464	0.9468	0.0187	0.0345
BTPI	0.3181	0.9423	0.0291	0.0286

	GE1 (Fr2, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.3069	0.9279	0.0340	0.0381
STI	0.3550	0.9301	0.0300	0.0399
BPI	0.5020	0.9470	0.0015	0.0515
PTBPI	0.4894	0.9510	0.0200	0.0290
BTI	0.4606	0.9511	0.0145	0.0344
BCAI	0.4594	0.9331	0.0213	0.0456
PTBCAI	0.4802	0.9269	0.0281	0.0450
BPGPDI	0.1845	0.9437	0.0452	0.0111
BTGPDI	0.2457	0.9414	0.0248	0.0338
BPPI	0.2275	0.9428	0.0253	0.0319
BTPI	0.2188	0.9466	0.0208	0.0326

	GE1 (Fr2, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.3634	0.9455	0.0342	0.0203
STI	0.3437	0.9536	0.0232	0.0232
BPI	0.4721	0.9509	0.0000	0.0491
PTBPI	0.4822	0.9534	0.0014	0.0452
BTI	0.4239	0.9480	0.0195	0.0325
BCAI	0.5059	0.9480	0.0245	0.0275
PTBCAI	0.4804	0.9407	0.0203	0.0390
BPGPDI	0.1844	0.9437	0.0458	0.0105
BTGPDI	0.2434	0.9509	0.0252	0.0239
BTPI	0.2247	0.9548	0.0233	0.0219
BPPI	0.2120	0.9476	0.0211	0.0313

Table F.13: Properties of Confidence Intervals for GE1 when Samples come from Fr2

	GE1	(T2, n=500)		
	ACIL	CP	LNCP	UNCP
SNI	0.5305	0.9418	0.0334	0.0248
STI	0.5436	0.9431	0.0234	0.0335
BPI	0.4557	0.9514	0.0014	0.0472
PTBPI	0.4718	0.9527	0.0200	0.0273
BTI	0.4831	0.9345	0.0314	0.0341
BCAI	0.4424	0.9299	0.0288	0.0413
PTBCAI	0.4788	0.9224	0.0360	0.0416
BPGPDI	0.3706	0.9571	0.0065	0.0364
BTGPDI	0.4116	0.9722	0.0068	0.0210
BPPI	0.4812	0.9584	0.0301	0.0115
BTPI	0.4449	0.9330	0.0448	0.0222

	GE1	(T2, n=1000)		
	ACIL	CP	LNCP	UNCP
SNI	0.4279	0.9434	0.0204	0.0362
STI	0.4219	0.9481	0.0057	0.0462
BPI	0.3872	0.9537	0.0028	0.0435
PTBPI	0.3905	0.9603	0.0100	0.0297
BTI	0.4373	0.9392	0.0317	0.0291
BCAI	0.4044	0.9378	0.0265	0.0357
PTBCAI	0.3813	0.9327	0.0359	0.0314
BPGPDI	0.2905	0.9504	0.0348	0.0148
BTGPDI	0.3370	0.9557	0.0155	0.0288
BPPI	0.3304	0.9535	0.0215	0.0250
BTPI	0.3154	0.9389	0.0320	0.0291

	GE1	(T2, n=5000)		
	ACIL	CP	LNCP	UNCP
SNI	0.2221	0.9460	0.0379	0.0161
STI	0.2146	0.9503	0.0267	0.0230
BPI	0.2609	0.9527	0.0212	0.0261
PTBPI	0.2687	0.9528	0.0213	0.0259
BTI	0.3306	0.9553	0.0218	0.0229
BCAI	0.2670	0.9364	0.0374	0.0262
PTBCAI	0.2571	0.9473	0.0251	0.0276
BPGPDI	0.1698	0.9421	0.0248	0.0331
BTGPDI	0.1661	0.9469	0.0071	0.0460
BPPI	0.1911	0.9466	0.0285	0.0249
BTPI	0.1782	0.9502	0.0296	0.0202

	GE1	(T2, n=10000)		
	ACIL	CP	LNCP	UNCP
SNI	0.2313	0.9474	0.0325	0.0201
STI	0.2129	0.9542	0.0229	0.0229
BPI	0.2710	0.9698	0.0025	0.0277
PTBPI	0.2671	0.9696	0.0013	0.0291
BTI	0.2104	0.9512	0.0244	0.0244
BCAI	0.2668	0.9437	0.0288	0.0275
PTBCAI	0.2526	0.9427	0.0282	0.0291
BPGPDI	0.1687	0.9504	0.0148	0.0348
BTGPDI	0.1618	0.9560	0.0071	0.0369
BPPI	0.1828	0.9547	0.0303	0.0150
BTPI	0.1740	0.9534	0.0212	0.0254

Table F.14: Properties of Confidence Intervals for GE1 when Samples come from T2

F.3 Generalized Entropy with Parameter 1.3 (GE1.3)

GE1.3 (Pa, n=500)	ACIL	CP	LNCP	UNCP
SNI	1.0151	0.9572	0.0163	0.0265
STI	1.0557	0.9563	0.0260	0.0177
BPI	1.0220	0.9523	0.0252	0.0225
PTBPI	1.0150	0.9676	0.0172	0.0152
BTI	0.8870	0.8952	0.0340	0.0708
BCAI	0.9388	0.9529	0.0276	0.0195
PTBCAI	0.9388	0.9561	0.0204	0.0235
BPGPDI	0.4514	0.9333	0.0102	0.0565
BTGPDI	0.7653	0.9099	0.0211	0.0690
BPPI	0.7731	0.9432	0.0225	0.0343
BTPI	0.6172	0.9215	0.0125	0.0660

GE1.3 (Pa, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.8519	0.9388	0.0296	0.0316
STI	0.9441	0.9157	0.0533	0.0310
BPI	0.9538	0.9316	0.0452	0.0232
PTBPI	0.9924	0.9313	0.0372	0.0315
BTI	0.8759	0.9388	0.0455	0.0157
BCAI	0.8887	0.8925	0.0142	0.0933
PTBCAI	0.8808	0.9293	0.0278	0.0429
BPGPDI	0.3780	0.9283	0.0500	0.0217
BTGPDI	0.6029	0.9095	0.0453	0.0452
BPPI	0.6537	0.9544	0.0106	0.0350
BTPI	0.4741	0.9389	0.0151	0.0460

GE1.3 (Pa, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.6309	0.9575	0.0232	0.0193
STI	0.6393	0.9565	0.0175	0.0260
BPI	0.7939	0.9328	0.0252	0.0420
PTBPI	0.7952	0.9410	0.0328	0.0262
BTI	0.8936	0.9256	0.0323	0.0421
BCAI	0.7980	0.9228	0.0193	0.0579
PTBCAI	0.7796	0.9214	0.0231	0.0555
BPGPDI	0.2363	0.9400	0.0230	0.0370
BTGPDI	0.3248	0.9340	0.0350	0.0310
BPPI	0.3944	0.9533	0.0206	0.0261
BTPI	0.2824	0.9394	0.0210	0.0396

GE1.3 (Pa, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.6112	0.9617	0.0189	0.0194
STI	0.5778	0.9344	0.0340	0.0316
BPI	0.7824	0.9381	0.0352	0.0267
PTBPI	0.8337	0.9203	0.0352	0.0445
BTI	4.3554	0.9382	0.0308	0.0310
BCAI	0.8222	0.9343	0.0229	0.0428
PTBCAI	0.7684	0.9343	0.0169	0.0488
BPGPDI	0.2365	0.9350	0.0231	0.0419
BTGPDI	0.3699	0.9372	0.0233	0.0395
BPPI	0.2899	0.9553	0.0239	0.0208
BTPI	0.2742	0.9387	0.0271	0.0342

Table F.15: Properties of Confidence Intervals for GE1.3 when Samples come from Pa

GE1.3 (Bu1, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.9405	0.9336	0.0110	0.0554
STI	0.9463	0.9334	0.0051	0.0615
BPI	0.9410	0.8998	0.0452	0.0550
PTBPI	0.9467	0.8948	0.0552	0.0500
BTI	0.9579	0.8827	0.0608	0.0565
BCAI	0.9395	0.8823	0.0249	0.0928
PTBCAI	0.8853	0.8655	0.0315	0.1030
BPGPDI	0.6797	0.9000	0.0600	0.0400
BTGPDI	0.7563	0.9075	0.0462	0.0463
BPPI	0.6081	0.9484	0.0109	0.0407
BTPI	0.5834	0.9467	0.0100	0.0433

GE1.3 (Bu1, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.9385	0.9227	0.0158	0.0615
STI	0.9306	0.9243	0.0114	0.0643
BPI	0.9313	0.9219	0.0352	0.0429
PTBPI	0.9452	0.9274	0.0352	0.0374
BTI	0.9425	0.9026	0.0508	0.0466
BCAI	0.8905	0.8991	0.0291	0.0718
PTBCAI	0.9006	0.8965	0.0293	0.0742
BPGPDI	0.5523	0.9233	0.0400	0.0367
BTGPDI	0.6025	0.9214	0.0357	0.0429
BPPI	0.7518	0.9465	0.0279	0.0256
BTPI	0.5555	0.9208	0.0391	0.0401

GE1.3 (Bu1, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.6620	0.9319	0.0341	0.0340
STI	0.6253	0.9391	0.0269	0.0340
BPI	0.8160	0.9311	0.0257	0.0432
PTBPI	0.7685	0.9338	0.0252	0.0410
BTI	0.9173	0.9300	0.0255	0.0445
BCAI	0.7672	0.9161	0.0273	0.0566
PTBCAI	0.7617	0.9136	0.0276	0.0588
BPGPDI	0.2953	0.9467	0.0231	0.0302
BTGPDI	0.3276	0.9238	0.0350	0.0412
BPPI	0.4719	0.9392	0.0284	0.0324
BTPI	0.4269	0.9396	0.0487	0.0117

GE1.3 (Bu1, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.6146	0.9340	0.0393	0.0267
STI	0.6493	0.9364	0.0318	0.0318
BPI	0.7961	0.9300	0.0092	0.0608
PTBPI	0.8093	0.9477	0.0267	0.0256
BTI	2.5052	0.9389	0.0281	0.0330
BCAI	0.7390	0.9259	0.0251	0.0490
PTBCAI	0.7441	0.9236	0.0334	0.0430
BPGPDI	0.2905	0.9367	0.0323	0.0310
BTGPDI	0.3607	0.9200	0.0350	0.0450
BPPI	0.4696	0.9539	0.0290	0.0171
BTPI	0.4260	0.9383	0.0501	0.0116

Table F.16: Properties of Confidence Intervals for GE1.3 when Samples come from Bu1

GE1.3 (Bu2, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.9154	0.9380	0.0060	0.0560
STI	0.8820	0.9416	0.0083	0.0501
BPI	0.9447	0.9130	0.0352	0.0518
PTBPI	0.9378	0.9234	0.0352	0.0414
BTI	0.9679	0.9054	0.0535	0.0411
BCAI	0.9393	0.9370	0.0360	0.0270
PTBCAI	0.9187	0.8937	0.0258	0.0805
BPGPDI	0.6608	0.9117	0.0300	0.0583
BTGPDI	0.7349	0.9417	0.0180	0.0403
BPPI	0.6332	0.9215	0.0116	0.0669
BTPI	0.6891	0.9133	0.0400	0.0467

GE1.3 (Bu2, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.9037	0.8979	0.0267	0.0754
STI	0.8462	0.8958	0.0314	0.0728
BPI	0.8823	0.9226	0.0372	0.0402
PTBPI	0.8495	0.9268	0.0354	0.0378
BTI	0.9412	0.9345	0.0343	0.0312
BCAI	0.9554	0.9474	0.0119	0.0407
PTBCAI	0.8304	0.9226	0.0264	0.0510
BPGPDI	0.5474	0.9267	0.0133	0.0600
BTGPDI	0.6025	0.9117	0.0230	0.0653
BPPI	0.5205	0.9499	0.0246	0.0255
BTPI	0.5814	0.9357	0.0330	0.0313

GE1.3 (Bu2, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.8922	0.9138	0.0420	0.0442
STI	0.8087	0.9237	0.0098	0.0665
BPI	0.8725	0.9353	0.0352	0.0295
PTBPI	0.7933	0.9265	0.0452	0.0283
BTI	0.8385	0.8963	0.0379	0.0658
BCAI	0.8347	0.9465	0.0162	0.0373
PTBCAI	0.7888	0.9293	0.0166	0.0541
BPGPDI	0.3457	0.9153	0.0130	0.0717
BTGPDI	0.4455	0.9241	0.0550	0.0209
BPPI	0.4010	0.9535	0.0309	0.0156
BTPI	0.4691	0.9360	0.0251	0.0389

GE1.3 (Bu2, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.8852	0.9388	0.0250	0.0362
STI	0.8037	0.9353	0.0339	0.0308
BPI	0.8531	0.9372	0.0352	0.0276
PTBPI	0.7473	0.9365	0.0372	0.0263
BTI	0.8126	0.9364	0.0352	0.0284
BCAI	0.7660	0.9310	0.0183	0.0507
PTBCAI	0.6996	0.9283	0.0411	0.0306
BPGPDI	0.3470	0.9467	0.0200	0.0333
BTGPDI	0.5038	0.9304	0.0350	0.0346
BPPI	0.3886	0.9504	0.0240	0.0256
BTPI	0.4513	0.9391	0.0373	0.0236

Table F.17: Properties of Confidence Intervals for GE1.3 when Samples come from Bu2

GE1.3 (Bu3, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.3779	0.9498	0.0140	0.0362
STI	0.3550	0.9526	0.0063	0.0411
BPI	0.3608	0.9402	0.0252	0.0346
PTBPI	0.3834	0.9464	0.0252	0.0284
BTI	0.3524	0.9384	0.0257	0.0359
BCAI	0.3678	0.9218	0.0315	0.0467
PTBCAI	0.3383	0.9234	0.0263	0.0503
BPGPDI	0.2611	0.8183	0.0107	0.1710
BTGPDI	0.3566	0.9424	0.0171	0.0405
BPPI	0.3381	0.9544	0.0124	0.0332
BTPI	0.3442	0.9115	0.0434	0.0451

GE1.3 (Bu3, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.2934	0.9367	0.0237	0.0396
STI	0.2850	0.9450	0.0166	0.0384
BPI	0.2983	0.9470	0.0066	0.0464
PTBPI	0.2982	0.9507	0.0065	0.0428
BTI	0.3556	0.9370	0.0333	0.0297
BCAI	0.3027	0.9167	0.0423	0.0410
PTBCAI	0.3203	0.9136	0.0397	0.0467
BPGPDI	0.2161	0.9233	0.0160	0.0607
BTGPDI	0.1821	0.9378	0.0193	0.0429
BPPI	0.2391	0.9528	0.0175	0.0297
BTPI	0.2405	0.9313	0.0242	0.0445

GE1.3 (Bu3, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.1592	0.9142	0.0530	0.0328
STI	0.1548	0.9135	0.0504	0.0361
BPI	0.2123	0.9325	0.0377	0.0298
PTBPI	0.2054	0.9583	0.0090	0.0327
BTI	0.2636	0.9522	0.0197	0.0281
BCAI	0.2094	0.9219	0.0453	0.0328
PTBCAI	0.2110	0.9171	0.0502	0.0327
BPGPDI	0.1380	0.9233	0.0166	0.0601
BTGPDI	0.1369	0.9219	0.0230	0.0551
BPPI	0.1521	0.9419	0.0291	0.0290
BTPI	0.1446	0.9335	0.0287	0.0378

GE1.3 (Bu3, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.1499	0.9209	0.0415	0.0376
STI	0.1579	0.9114	0.0527	0.0359
BPI	0.2090	0.9408	0.0264	0.0328
PTBPI	0.2160	0.9409	0.0277	0.0314
BTI	0.2073	0.9466	0.0279	0.0255
BCAI	0.1992	0.9187	0.0470	0.0343
PTBCAI	0.2014	0.9169	0.0486	0.0345
BPGPDI	0.1374	0.9300	0.0360	0.0340
BTGPDI	0.1373	0.9275	0.0398	0.0327
BPPI	0.1420	0.9488	0.0239	0.0273
BTPI	0.1353	0.9385	0.0355	0.0260

Table F.18: Properties of Confidence Intervals for GE1.3 when Samples come from Bu3

	GE1.3	(Fr1, n=500)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.3889	0.9569	0.0087	0.0344
STI	0.3928	0.9458	0.0130	0.0412
BPI	0.3770	0.9384	0.0252	0.0364
PTBPI	0.3891	0.9446	0.0252	0.0302
BTI	0.4218	0.9443	0.0228	0.0329
BCAI	0.3652	0.9067	0.0409	0.0524
PTBCAI	0.3535	0.9138	0.0314	0.0548
BPGPDI	0.2167	0.9217	0.0330	0.0453
BTGPDI	0.2305	0.9145	0.0450	0.0405
BPPI	0.2569	0.9461	0.0261	0.0278
BTPI	0.2611	0.9067	0.0481	0.0452

	GE1.3	(Fr1, n=1000)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.3071	0.9522	0.0150	0.0328
STI	0.3059	0.9529	0.0126	0.0345
BPI	0.3122	0.9487	0.0065	0.0448
PTBPI	0.3322	0.9452	0.0066	0.0482
BTI	0.3687	0.9326	0.0362	0.0312
BCAI	0.3348	0.9157	0.0415	0.0428
PTBCAI	0.2971	0.9320	0.0253	0.0427
BPGPDI	0.1692	0.9019	0.0331	0.0650
BTGPDI	0.1689	0.9140	0.0199	0.0661
BPPI	0.1678	0.9473	0.0267	0.0260
BTPI	0.1595	0.9332	0.0305	0.0363

	GE1.3	(Fr1, n=5000)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.1575	0.9266	0.0366	0.0368
STI	0.1685	0.9073	0.0549	0.0378
BPI	0.2147	0.9588	0.0101	0.0311
PTBPI	0.2119	0.9467	0.0252	0.0281
BTI	0.3518	0.9424	0.0294	0.0282
BCAI	0.2091	0.9294	0.0378	0.0328
PTBCAI	0.2028	0.9287	0.0400	0.0313
BPGPDI	0.1404	0.9017	0.0400	0.0583
BTGPDI	0.1376	0.9340	0.0199	0.0461
BPPI	0.1562	0.9437	0.0307	0.0256
BTPI	0.1497	0.9455	0.0285	0.0260

	GE1.3	(Fr1, n=10000)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.1582	0.9310	0.0346	0.0344
STI	0.1613	0.9105	0.0550	0.0345
BPI	0.2109	0.9549	0.0090	0.0361
PTBPI	0.2093	0.9595	0.0077	0.0328
BTI	0.3333	0.9401	0.0329	0.0270
BCAI	0.1923	0.9369	0.0334	0.0297
PTBCAI	0.2228	0.9184	0.0490	0.0326
BPGPDI	0.1423	0.9400	0.0117	0.0483
BTGPDI	0.1454	0.9262	0.0173	0.0565
BPPI	0.1569	0.9521	0.0223	0.0256
BTPI	0.1489	0.9465	0.0264	0.0271

Table F.19: Properties of Confidence Intervals for GE1.3 when Samples come from Fr1

	GE1.3	(Fr2, n=500)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.6717	0.9207	0.0100	0.0693
STI	0.6939	0.9238	0.0068	0.0694
BPI	0.6698	0.9164	0.0352	0.0484
PTBPI	0.6854	0.9103	0.0452	0.0445
BTI	0.7761	0.9306	0.0193	0.0501
BCAI	0.6611	0.9016	0.0276	0.0708
PTBCAI	0.6457	0.8954	0.0206	0.0840
BPGPDI	0.3823	0.9400	0.0300	0.0300
BTGPDI	0.5444	0.9345	0.0197	0.0458
BPPI	0.5951	0.9635	0.0123	0.0242
BTPI	0.5698	0.9268	0.0364	0.0368

	GE1.3	(Fr2, n=1000)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.5876	0.8973	0.0365	0.0662
STI	0.5821	0.9065	0.0215	0.0720
BPI	0.6543	0.9168	0.0452	0.0380
PTBPI	0.6372	0.9239	0.0352	0.0409
BTI	0.5248	0.9151	0.0324	0.0525
BCAI	0.6009	0.9197	0.0174	0.0629
PTBCAI	0.6267	0.8921	0.0398	0.0681
BPGPDI	0.3107	0.9400	0.0200	0.0400
BTGPDI	0.3245	0.9273	0.0357	0.0370
BPPI	0.3382	0.9456	0.0193	0.0351
BTPI	0.3106	0.9319	0.0343	0.0338

	GE1.3	(Fr2, n=5000)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.3002	0.9207	0.0376	0.0417
STI	0.3473	0.9230	0.0335	0.0435
BPI	0.4911	0.9366	0.0067	0.0567
PTBPI	0.4788	0.9406	0.0252	0.0342
BTI	0.4511	0.9408	0.0197	0.0395
BCAI	0.4494	0.9227	0.0265	0.0508
PTBCAI	0.4698	0.9165	0.0333	0.0502
BPGPDI	0.1823	0.9333	0.0504	0.0163
BTGPDI	0.2399	0.9210	0.0350	0.0440
BPPI	0.2221	0.9416	0.0259	0.0325
BTPI	0.2136	0.9362	0.0260	0.0378

	GE1.3	(Fr2, n=10000)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.3555	0.9383	0.0378	0.0239
STI	0.3362	0.9464	0.0268	0.0268
BPI	0.4619	0.9405	0.0052	0.0543
PTBPI	0.4717	0.9430	0.0066	0.0504
BTI	0.4151	0.9376	0.0247	0.0377
BCAI	0.4949	0.9376	0.0297	0.0327
PTBCAI	0.4700	0.9303	0.0255	0.0442
BPGPDI	0.1822	0.9333	0.0510	0.0157
BTGPDI	0.2376	0.9305	0.0354	0.0341
BTPI	0.2194	0.9536	0.0239	0.0225
BPPI	0.2070	0.9372	0.0270	0.0358

Table F.20: Properties of Confidence Intervals for GE1.3 when Samples come from Fr2

GE1.3 (T2, n=500)				
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.5190	0.9346	0.0370	0.0284
STI	0.5318	0.9359	0.0270	0.0371
BPI	0.4458	0.9410	0.0066	0.0524
PTBPI	0.4616	0.9423	0.0252	0.0325
BTI	0.4731	0.9241	0.0367	0.0392
BCAI	0.4328	0.9195	0.0340	0.0465
PTBCAI	0.4684	0.9120	0.0412	0.0468
BPGPDI	0.3661	0.9467	0.0117	0.0416
BTGPDI	0.4019	0.9518	0.0170	0.0312
BPPI	0.4698	0.9572	0.0307	0.0121
BTPI	0.4344	0.9226	0.0500	0.0274

GE1.3 (T2, n=1000)				
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.4186	0.9362	0.0240	0.0398
STI	0.4128	0.9409	0.0094	0.0497
BPI	0.3788	0.9433	0.0080	0.0487
PTBPI	0.3820	0.9499	0.0152	0.0349
BTI	0.4283	0.9288	0.0369	0.0343
BCAI	0.3956	0.9274	0.0317	0.0409
PTBCAI	0.3730	0.9223	0.0411	0.0366
BPGPDI	0.2870	0.9400	0.0400	0.0200
BTGPDI	0.3290	0.9353	0.0257	0.0390
BPPI	0.3226	0.9523	0.0221	0.0256
BTPI	0.3079	0.9485	0.0272	0.0243

GE1.3 (T2, n=5000)				
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.2173	0.9388	0.0415	0.0197
STI	0.2099	0.9431	0.0303	0.0266
BPI	0.2552	0.9423	0.0264	0.0313
PTBPI	0.2629	0.9424	0.0265	0.0311
BTI	0.3238	0.9449	0.0270	0.0281
BCAI	0.2612	0.9260	0.0426	0.0314
PTBCAI	0.2515	0.9369	0.0303	0.0328
BPGPDI	0.1677	0.9317	0.0300	0.0383
BTGPDI	0.1622	0.9265	0.0173	0.0562
BPPI	0.1866	0.9454	0.0291	0.0255
BTPI	0.1740	0.9398	0.0348	0.0254

GE1.3 (T2, n=10000)				
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.2263	0.9402	0.0361	0.0237
STI	0.2083	0.9470	0.0265	0.0265
BPI	0.2651	0.9594	0.0077	0.0329
PTBPI	0.2613	0.9592	0.0065	0.0343
BTI	0.2060	0.9408	0.0296	0.0296
BCAI	0.2610	0.9333	0.0340	0.0327
PTBCAI	0.2471	0.9323	0.0334	0.0343
BPGPDI	0.1666	0.9400	0.0200	0.0400
BTGPDI	0.1580	0.9356	0.0173	0.0471
BPPI	0.1785	0.9535	0.0309	0.0156
BTPI	0.1699	0.9430	0.0264	0.0306

Table F.21: Properties of Confidence Intervals for GE1.3 when Samples come from T2

F.4 Atkinson Coefficient with Parameter 1 (A1)

A1 (Pa, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.1097	0.9639	0.0230	0.0131
STI	0.0842	0.9497	0.0283	0.0220
BPI	0.0813	0.9237	0.0312	0.0451
PTBPI	0.0766	0.9287	0.0384	0.0329
BTI	0.0935	0.9233	0.0372	0.0395
BCAI	0.0720	0.9117	0.0478	0.0405
PTBCAI	0.0720	0.9208	0.0391	0.0401
BPGPDI	0.0761	0.9382	0.0215	0.0403
BTGPDI	0.0830	0.9249	0.0204	0.0547
BPPI	0.0909	0.9389	0.0302	0.0309
BTPI	0.0813	0.9265	0.0228	0.0507

A1 (Pa, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.0813	0.9185	0.0257	0.0558
STI	0.0825	0.9313	0.0203	0.0484
BPI	0.0727	0.9258	0.0311	0.0431
PTBPI	0.0704	0.9336	0.0311	0.0353
BTI	0.0794	0.9228	0.0329	0.0443
BCAI	0.0754	0.9227	0.0338	0.0435
PTBCAI	0.0766	0.9139	0.0260	0.0601
BPGPDI	0.0620	0.9332	0.0470	0.0198
BTGPDI	0.0766	0.9443	0.0275	0.0282
BPPI	0.0787	0.9499	0.0126	0.0375
BTPI	0.0649	0.9437	0.0125	0.0438

A1 (Pa, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.0851	0.9351	0.0393	0.0256
STI	0.0862	0.9422	0.0352	0.0226
BPI	0.0532	0.9428	0.0188	0.0384
PTBPI	0.0525	0.9448	0.0223	0.0329
BTI	0.0768	0.9260	0.0324	0.0416
BCAI	0.0543	0.9174	0.0355	0.0471
PTBCAI	0.0623	0.9235	0.0380	0.0385
BPGPDI	0.0585	0.9448	0.0203	0.0349
BTGPDI	0.0521	0.9389	0.0371	0.0240
BPPI	0.0594	0.9587	0.0126	0.0287
BTPI	0.0586	0.9541	0.0183	0.0276

A1 (Pa, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.0711	0.9404	0.0228	0.0368
STI	0.0729	0.9532	0.0242	0.0226
BPI	0.0315	0.9402	0.0224	0.0374
PTBPI	0.0371	0.9560	0.0223	0.0217
BTI	0.0739	0.9475	0.0293	0.0232
BCAI	0.0311	0.9428	0.0327	0.0245
PTBCAI	0.0386	0.9531	0.0207	0.0262
BPGPDI	0.0390	0.9399	0.0204	0.0397
BTGPDI	0.0385	0.9519	0.0156	0.0325
BPPI	0.0464	0.9508	0.0257	0.0235
BTPI	0.0503	0.9534	0.0243	0.0223

Table F.22: Properties of Confidence Intervals for A1 when Samples come from Pa

A1 (Bu1, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.0968	0.8935	0.0543	0.0522
STI	0.0959	0.9291	0.0511	0.0198
BPI	0.1369	0.9484	0.0288	0.0228
PTBPI	0.1374	0.9509	0.0276	0.0215
BTI	0.1421	0.9163	0.0569	0.0268
BCAI	0.1205	0.9099	0.0489	0.0412
PTBCAI	0.1203	0.9176	0.0480	0.0344
BPGPDI	0.0970	0.9043	0.0622	0.0335
BTGPDI	0.0935	0.9216	0.0436	0.0348
BPPI	0.0918	0.9431	0.0182	0.0387
BTPI	0.0965	0.9505	0.0225	0.0270

A1 (Bu1, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.0896	0.8930	0.0107	0.0963
STI	0.0899	0.8990	0.0421	0.0589
BPI	0.1190	0.9533	0.0253	0.0214
PTBPI	0.1194	0.9600	0.0244	0.0156
BTI	0.1256	0.9283	0.0427	0.0290
BCAI	0.1088	0.9177	0.0469	0.0354
PTBCAI	0.1078	0.9158	0.0462	0.0380
BPGPDI	0.0881	0.9274	0.0375	0.0351
BTGPDI	0.0861	0.9354	0.0235	0.0411
BPPI	0.0791	0.9412	0.0301	0.0287
BTPI	0.0880	0.9249	0.0366	0.0385

A1 (Bu1, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.0902	0.9125	0.0481	0.0394
STI	0.0906	0.9244	0.0361	0.0395
BPI	0.0931	0.9619	0.0138	0.0243
PTBPI	0.0924	0.9671	0.0149	0.0180
BTI	0.0985	0.9243	0.0441	0.0316
BCAI	0.0891	0.9213	0.0484	0.0303
PTBCAI	0.0895	0.9258	0.0402	0.0340
BPGPDI	0.0811	0.9505	0.0208	0.0287
BTGPDI	0.0846	0.9477	0.0275	0.0248
BPPI	0.0824	0.9439	0.0306	0.0255
BTPI	0.0863	0.9435	0.0461	0.0104

A1 (Bu1, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.0811	0.9406	0.0301	0.0293
STI	0.0806	0.9365	0.0314	0.0321
BPI	0.0935	0.9421	0.0350	0.0229
PTBPI	0.0835	0.9557	0.0238	0.0205
BTI	0.0982	0.9298	0.0423	0.0279
BCAI	0.0899	0.9333	0.0298	0.0369
PTBCAI	0.0887	0.9572	0.0199	0.0229
BPGPDI	0.0733	0.9406	0.0299	0.0295
BTGPDI	0.0686	0.9439	0.0276	0.0285
BPPI	0.0603	0.9535	0.0212	0.0253
BTPI	0.0656	0.9422	0.0277	0.0301

Table F.23: Properties of Confidence Intervals for A1 when Samples come from Bu1

A1 (Bu2, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.0985	0.9359	0.0346	0.0295
STI	0.0973	0.9428	0.0337	0.0235
BPI	0.1514	0.9261	0.0302	0.0437
PTBPI	0.1493	0.9324	0.0243	0.0433
BTI	0.1904	0.9009	0.0473	0.0518
BCAI	0.1248	0.8913	0.0282	0.0805
PTBCAI	0.1248	0.9027	0.0319	0.0654
BPGPDI	0.0978	0.8961	0.0178	0.0861
BTGPDI	0.0917	0.9555	0.0108	0.0337
BPPI	0.0975	0.9165	0.0140	0.0695
BTPI	0.0947	0.9175	0.0375	0.0450

A1 (Bu2, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.0872	0.9421	0.0235	0.0344
STI	0.0983	0.9411	0.0264	0.0325
BPI	0.1232	0.9198	0.0315	0.0487
PTBPI	0.1220	0.9314	0.0315	0.0371
BTI	0.1645	0.9126	0.0385	0.0489
BCAI	0.1053	0.9065	0.0282	0.0653
PTBCAI	0.1036	0.9361	0.0315	0.0324
BPGPDI	0.0931	0.9307	0.0111	0.0582
BTGPDI	0.0960	0.9258	0.0157	0.0585
BPPI	0.0921	0.9446	0.0268	0.0286
BTPI	0.0944	0.9396	0.0306	0.0298

A1 (Bu2, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.0855	0.9306	0.0325	0.0369
STI	0.0792	0.9390	0.0190	0.0420
BPI	0.0954	0.9307	0.0300	0.0393
PTBPI	0.0941	0.9406	0.0227	0.0367
BTI	0.0938	0.9124	0.0386	0.0490
BCAI	0.0923	0.9269	0.0337	0.0394
PTBCAI	0.0810	0.9316	0.0333	0.0351
BPGPDI	0.0791	0.9392	0.0207	0.0401
BTGPDI	0.0853	0.9381	0.0474	0.0145
BPPI	0.0699	0.9481	0.0331	0.0188
BTPI	0.0796	0.9399	0.0228	0.0373

A1 (Bu2, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.0976	0.9380	0.0266	0.0354
STI	0.0969	0.9395	0.0223	0.0382
BPI	0.0795	0.9312	0.0315	0.0373
PTBPI	0.0772	0.9573	0.0116	0.0311
BTI	0.0886	0.9110	0.0342	0.0548
BCAI	0.0810	0.9238	0.0260	0.0502
PTBCAI	0.0793	0.9471	0.0256	0.0273
BPGPDI	0.0804	0.9505	0.0177	0.0318
BTGPDI	0.0850	0.9443	0.0275	0.0282
BPPI	0.0771	0.9451	0.0262	0.0287
BTPI	0.0813	0.9529	0.0249	0.0222

Table F.24: Properties of Confidence Intervals for A1 when Samples come from Bu2

A1 (Bu3, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.1499	0.9344	0.0244	0.0412
STI	0.1431	0.9361	0.0192	0.0447
BPI	0.1293	0.9606	0.0138	0.0256
PTBPI	0.1275	0.9621	0.0149	0.0230
BTI	0.1553	0.9349	0.0408	0.0243
BCAI	0.1158	0.9042	0.0539	0.0419
PTBCAI	0.1187	0.9131	0.0552	0.0317
BPGPDI	0.1130	0.9224	0.0382	0.0394
BTGPDI	0.1140	0.9562	0.0198	0.0240
BPPI	0.1150	0.9490	0.0148	0.0362
BTPI	0.1113	0.9157	0.0409	0.0434

A1 (Bu3, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.0890	0.9496	0.0109	0.0395
STI	0.0935	0.9619	0.0138	0.0243
BPI	0.0999	0.9659	0.0149	0.0192
PTBPI	0.0895	0.9607	0.0126	0.0267
BTI	0.0995	0.9405	0.0363	0.0232
BCAI	0.0922	0.9166	0.0555	0.0279
PTBCAI	0.0893	0.9231	0.0527	0.0242
BPGPDI	0.0977	0.9274	0.0138	0.0588
BTGPDI	0.0954	0.9516	0.0121	0.0363
BPPI	0.0938	0.9475	0.0197	0.0328
BTPI	0.0952	0.9353	0.0218	0.0429

A1 (Bu3, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.0886	0.9491	0.0203	0.0306
STI	0.0821	0.9448	0.0203	0.0349
BPI	0.0513	0.9502	0.0278	0.0220
PTBPI	0.0520	0.9558	0.0257	0.0185
BTI	0.0677	0.9326	0.0477	0.0197
BCAI	0.0500	0.9272	0.0521	0.0207
PTBCAI	0.0497	0.9396	0.0304	0.0300
BPGPDI	0.0393	0.9472	0.0143	0.0385
BTGPDI	0.0398	0.9557	0.0157	0.0286
BPPI	0.0552	0.9367	0.0312	0.0321
BTPI	0.0476	0.9572	0.0164	0.0264

A1 (Bu3, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.0711	0.9491	0.0204	0.0305
STI	0.0770	0.9548	0.0213	0.0239
BPI	0.0542	0.9526	0.0278	0.0196
PTBPI	0.0528	0.9560	0.0266	0.0174
BTI	0.0631	0.9388	0.0393	0.0219
BCAI	0.0490	0.9457	0.0288	0.0255
PTBCAI	0.0486	0.9529	0.0240	0.0231
BPGPDI	0.0387	0.9439	0.0237	0.0324
BTGPDI	0.0401	0.9513	0.0225	0.0262
BPPI	0.0449	0.9475	0.0242	0.0283
BTPI	0.0481	0.9523	0.0232	0.0245

Table F.25: Properties of Confidence Intervals for A1 when Samples come from Bu3

A1 (Fr1, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.1587	0.9398	0.0113	0.0489
STI	0.1611	0.9399	0.0180	0.0421
BPI	0.1317	0.9595	0.0150	0.0255
PTBPI	0.1323	0.9620	0.0138	0.0242
BTI	0.1622	0.9464	0.0269	0.0267
BCAI	0.1176	0.9136	0.0523	0.0341
PTBCAI	0.1186	0.9194	0.0477	0.0329
BPGPDI	0.0983	0.9258	0.0306	0.0436
BTGPDI	0.0950	0.9286	0.0375	0.0339
BPPI	0.0921	0.9408	0.0283	0.0309
BTPI	0.0963	0.9307	0.0357	0.0336

A1 (Fr1, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.1448	0.9533	0.0075	0.0392
STI	0.1441	0.9538	0.0238	0.0224
BPI	0.1002	0.9669	0.0138	0.0193
PTBPI	0.1035	0.9508	0.0237	0.0255
BTI	0.1241	0.9348	0.0409	0.0243
BCAI	0.0938	0.9323	0.0410	0.0267
PTBCAI	0.0928	0.9334	0.0399	0.0267
BPGPDI	0.0705	0.9062	0.0306	0.0632
BTGPDI	0.0722	0.9281	0.0126	0.0593
BPPI	0.0711	0.9420	0.0289	0.0291
BTPI	0.0627	0.9372	0.0280	0.0348

A1 (Fr1, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.1171	0.9481	0.0304	0.0215
STI	0.1207	0.9470	0.0306	0.0224
BPI	0.0855	0.9421	0.0335	0.0244
PTBPI	0.0842	0.9450	0.0376	0.0174
BTI	0.0886	0.9315	0.0477	0.0208
BCAI	0.0802	0.9448	0.0297	0.0255
PTBCAI	0.0815	0.9407	0.0350	0.0243
BPGPDI	0.0713	0.9258	0.0374	0.0368
BTGPDI	0.0701	0.9281	0.0126	0.0593
BPPI	0.0593	0.9385	0.0328	0.0287
BTPI	0.0626	0.9394	0.0261	0.0345

A1 (Fr1, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.1022	0.9536	0.0243	0.0221
STI	0.1003	0.9679	0.0126	0.0195
BPI	0.0548	0.9657	0.0124	0.0219
PTBPI	0.0560	0.9537	0.0277	0.0186
BTI	0.0614	0.9398	0.0383	0.0219
BCAI	0.0499	0.9328	0.0429	0.0243
PTBCAI	0.0511	0.9555	0.0226	0.0219
BPGPDI	0.0535	0.9439	0.0292	0.0269
BTGPDI	0.0582	0.9401	0.0299	0.0300
BPPI	0.0600	0.9468	0.0245	0.0287
BTPI	0.0519	0.9503	0.0240	0.0257

Table F.26: Properties of Confidence Intervals for A1 when Samples come from Fr1

A1 (Fr2, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.1434	0.9355	0.0270	0.0375
STI	0.1409	0.9337	0.0180	0.0483
BPI	0.1097	0.9172	0.0274	0.0554
PTBPI	0.1107	0.9297	0.0233	0.0470
BTI	0.1140	0.9265	0.0363	0.0372
BCAI	0.0887	0.9294	0.0366	0.0340
PTBCAI	0.0883	0.9294	0.0366	0.0340
BPGPDI	0.1058	0.9439	0.0276	0.0285
BTGPDI	0.1064	0.9483	0.0125	0.0392
BPPI	0.1083	0.9580	0.0147	0.0273
BTPI	0.0827	0.9308	0.0340	0.0352

A1 (Fr2, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.1006	0.9217	0.0246	0.0537
STI	0.0965	0.9506	0.0249	0.0245
BPI	0.0700	0.9332	0.0354	0.0314
PTBPI	0.0779	0.9374	0.0538	0.0088
BTI	0.1000	0.9352	0.0246	0.0402
BCAI	0.0726	0.9401	0.0363	0.0236
PTBCAI	0.0784	0.9481	0.0205	0.0314
BPGPDI	0.0935	0.9439	0.0178	0.0383
BTGPDI	0.0914	0.9412	0.0283	0.0305
BPPI	0.0953	0.9403	0.0216	0.0381
BTPI	0.0772	0.9359	0.0319	0.0322

A1 (Fr2, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.0902	0.9325	0.0289	0.0386
STI	0.0986	0.9499	0.0227	0.0274
BPI	0.0701	0.9349	0.0348	0.0303
PTBPI	0.0744	0.9580	0.0220	0.0200
BTI	0.0964	0.9394	0.0237	0.0369
BCAI	0.0771	0.9477	0.0174	0.0349
PTBCAI	0.0780	0.9495	0.0233	0.0272
BPGPDI	0.0836	0.9373	0.0477	0.0150
BTGPDI	0.0847	0.9350	0.0276	0.0374
BPPI	0.0865	0.9364	0.0281	0.0355
BTPI	0.0779	0.9401	0.0237	0.0362

A1 (Fr2, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.0745	0.9418	0.0252	0.0330
STI	0.0789	0.9556	0.0227	0.0217
BPI	0.0776	0.9454	0.0279	0.0267
PTBPI	0.0695	0.9530	0.0218	0.0252
BTI	0.0752	0.9554	0.0148	0.0298
BCAI	0.0668	0.9369	0.0209	0.0422
PTBCAI	0.0650	0.9417	0.0194	0.0389
BPGPDI	0.0736	0.9373	0.0483	0.0144
BTGPDI	0.0726	0.9544	0.0179	0.0277
BPPI	0.0739	0.9582	0.0161	0.0257
BTPI	0.0712	0.9490	0.0246	0.0264

Table F.27: Properties of Confidence Intervals for A1 when Samples come from Fr2

	A1	(T2, n=500)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.0703	0.8558	0.0534	0.0908
STI	0.0697	0.9296	0.0195	0.0509
BPI	0.1408	0.9618	0.0126	0.0256
PTBPI	0.1448	0.9588	0.0181	0.0231
BTI	0.1089	0.9377	0.0404	0.0219
BCAI	0.1110	0.9214	0.0483	0.0303
PTBCAI	0.1029	0.9147	0.0461	0.0392
BPGPDI	0.0995	0.9505	0.0095	0.0400
BTGPDI	0.1005	0.9556	0.0197	0.0247
BPPI	0.1001	0.9518	0.0329	0.0153
BTPI	0.1037	0.9465	0.0256	0.0279

	A1	(T2, n=1000)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.0708	0.9326	0.0303	0.0371
STI	0.0680	0.9292	0.0305	0.0403
BPI	0.1083	0.9437	0.0264	0.0299
PTBPI	0.1105	0.9512	0.0257	0.0231
BTI	0.1004	0.9478	0.0360	0.0162
BCAI	0.0950	0.9144	0.0614	0.0242
PTBCAI	0.0930	0.9236	0.0544	0.0220
BPGPDI	0.0895	0.9439	0.0375	0.0186
BTGPDI	0.0959	0.9491	0.0184	0.0325
BPPI	0.0894	0.9470	0.0244	0.0286
BTPI	0.0943	0.9325	0.0348	0.0327

	A1	(T2, n=5000)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.0683	0.9472	0.0136	0.0392
STI	0.0686	0.9441	0.0138	0.0421
BPI	0.0872	0.9463	0.0386	0.0151
PTBPI	0.0884	0.9529	0.0208	0.0263
BTI	0.0877	0.9369	0.0469	0.0162
BCAI	0.0791	0.9272	0.0485	0.0243
PTBCAI	0.0800	0.9415	0.0299	0.0286
BPGPDI	0.0690	0.9357	0.0276	0.0367
BTGPDI	0.0654	0.9404	0.0299	0.0297
BPPI	0.0901	0.9401	0.0313	0.0286
BTPI	0.0783	0.9437	0.0324	0.0239

	A1	(T2, n=10000)		
	ACIL	**CP**	**LNCP**	**UNCP**
SNI	0.0686	0.9400	0.0298	0.0302
STI	0.0598	0.9401	0.0277	0.0322
BPI	0.0569	0.9516	0.0299	0.0185
PTBPI	0.0582	0.9583	0.0276	0.0141
BTI	0.0602	0.9397	0.0430	0.0173
BCAI	0.0507	0.9349	0.0333	0.0318
PTBCAI	0.0500	0.9453	0.0263	0.0284
BPGPDI	0.0679	0.9439	0.0178	0.0383
BTGPDI	0.0611	0.9494	0.0200	0.0306
BPPI	0.0720	0.9481	0.0229	0.0290
BTPI	0.0781	0.9568	0.0211	0.0221

Table F.28: Properties of Confidence Intervals for A1 when Samples come from T2

F.5 Atkinson Coefficient with Parameter 1.5 (A1.5)

	A1.5 (Pa, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.1109	0.9646	0.0125	0.0229
STI	0.0851	0.9401	0.0337	0.0262
BPI	0.0822	0.9240	0.0253	0.0507
PTBPI	0.0775	0.9290	0.0237	0.0473
BTI	0.0945	0.9033	0.0472	0.0495
BCAI	0.0728	0.9118	0.0339	0.0543
PTBCAI	0.0728	0.9109	0.0364	0.0527
BPGPDI	0.0770	0.9386	0.0076	0.0538
BTGPDI	0.0839	0.9252	0.0135	0.0613
BPPI	0.0919	0.9393	0.0245	0.0362
BTPI	0.0822	0.9268	0.0099	0.0633

	A1.5 (Pa, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0822	0.9187	0.0259	0.0554
STI	0.0834	0.9317	0.0204	0.0479
BPI	0.0735	0.9261	0.0313	0.0426
PTBPI	0.0712	0.9340	0.0313	0.0347
BTI	0.0803	0.9231	0.0331	0.0438
BCAI	0.0762	0.9230	0.0340	0.0430
PTBCAI	0.0775	0.9141	0.0262	0.0597
BPGPDI	0.0627	0.9336	0.0474	0.0190
BTGPDI	0.0775	0.9448	0.0277	0.0275
BPPI	0.0796	0.9505	0.0126	0.0369
BTPI	0.0656	0.9442	0.0125	0.0433

	A1.5 (Pa, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0861	0.9355	0.0396	0.0249
STI	0.0872	0.9427	0.0355	0.0218
BPI	0.0538	0.9433	0.0189	0.0378
PTBPI	0.0531	0.9453	0.0024	0.0523
BTI	0.0777	0.9263	0.0327	0.0410
BCAI	0.0549	0.9176	0.0358	0.0466
PTBCAI	0.0630	0.9238	0.0383	0.0379
BPGPDI	0.0592	0.9453	0.0204	0.0343
BTGPDI	0.0527	0.9393	0.0373	0.0234
BPPI	0.0601	0.9594	0.0125	0.0281
BTPI	0.0593	0.9547	0.0184	0.0269

	A1.5 (Pa, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0719	0.9409	0.0230	0.0361
STI	0.0737	0.9538	0.0244	0.0218
BPI	0.0319	0.9407	0.0225	0.0368
PTBPI	0.0375	0.9566	0.0224	0.0210
BTI	0.0747	0.9480	0.0295	0.0225
BCAI	0.0314	0.9433	0.0330	0.0237
PTBCAI	0.0390	0.9537	0.0208	0.0255
BPGPDI	0.0394	0.9403	0.0205	0.0392
BTGPDI	0.0389	0.9525	0.0157	0.0318
BPPI	0.0469	0.9514	0.0259	0.0227
BTPI	0.0509	0.9540	0.0245	0.0215

Table F.29: Properties of Confidence Intervals for A1.5 when Samples come from Pa

	A1.5 (Bu1, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.0979	0.8843	0.0898	0.0259
STI	0.0970	0.8798	0.0967	0.0235
BPI	0.1384	0.9499	0.0036	0.0465
PTBPI	0.1389	0.9524	0.0024	0.0452
BTI	0.1437	0.9174	0.0519	0.0307
BCAI	0.1219	0.9109	0.0439	0.0452
PTBCAI	0.1217	0.9187	0.0430	0.0383
BPGPDI	0.0981	0.9053	0.0574	0.0373
BTGPDI	0.0946	0.9228	0.0386	0.0386
BPPI	0.0928	0.9445	0.0129	0.0426
BTPI	0.0976	0.9520	0.0074	0.0406

	A1.5 (Bu1, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0906	0.8938	0.0103	0.0959
STI	0.0909	0.8999	0.0020	0.0981
BPI	0.1203	0.9548	0.0050	0.0402
PTBPI	0.1207	0.9616	0.0141	0.0243
BTI	0.1270	0.9295	0.0426	0.0279
BCAI	0.1100	0.9188	0.0469	0.0343
PTBCAI	0.1090	0.9169	0.0462	0.0369
BPGPDI	0.0891	0.9286	0.0374	0.0340
BTGPDI	0.0871	0.9367	0.0281	0.0352
BPPI	0.0800	0.9426	0.0299	0.0275
BTPI	0.0890	0.9261	0.0365	0.0374

	A1.5 (Bu1, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0912	0.9136	0.0481	0.0383
STI	0.0916	0.9256	0.0359	0.0385
BPI	0.0941	0.9635	0.0034	0.0331
PTBPI	0.0934	0.9688	0.0044	0.0268
BTI	0.0996	0.9255	0.0440	0.0305
BCAI	0.0901	0.9224	0.0484	0.0292
PTBCAI	0.0905	0.9269	0.0401	0.0330
BPGPDI	0.0820	0.9520	0.0205	0.0275
BTGPDI	0.0855	0.9491	0.0274	0.0235
BPPI	0.0833	0.9453	0.0304	0.0243
BTPI	0.0873	0.9449	0.0261	0.0290

	A1.5 (Bu1, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0820	0.9319	0.0096	0.0585
STI	0.0815	0.9378	0.0311	0.0311
BPI	0.0946	0.9637	0.0045	0.0318
PTBPI	0.0844	0.9673	0.0034	0.0293
BTI	0.0993	0.9310	0.0423	0.0267
BCAI	0.0909	0.9346	0.0296	0.0358
PTBCAI	0.0897	0.9486	0.0297	0.0217
BPGPDI	0.0741	0.9420	0.0298	0.0282
BTGPDI	0.0694	0.9453	0.0274	0.0273
BPPI	0.0610	0.9500	0.0310	0.0190
BTPI	0.0663	0.9436	0.0275	0.0289

Table F.30: Properties of Confidence Intervals for A1.5 when Samples come from Bu1

	A1.5	(Bu2, n=500)		
	ACIL	CP	LNCP	UNCP
SNI	0.0996	0.9372	0.0344	0.0284
STI	0.0984	0.9442	0.0335	0.0223
BPI	0.1531	0.9273	0.0300	0.0427
PTBPI	0.1510	0.9337	0.0240	0.0423
BTI	0.1925	0.9018	0.0473	0.0509
BCAI	0.1262	0.8921	0.0280	0.0799
PTBCAI	0.1262	0.9036	0.0317	0.0647
BPGPDI	0.0989	0.9870	0.0074	0.0056
BTGPDI	0.0927	0.9570	0.0104	0.0326
BPPI	0.0986	0.9176	0.0136	0.0688
BTPI	0.0958	0.9186	0.0374	0.0440

	A1.5	(Bu2, n=1000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0882	0.9435	0.0232	0.0333
STI	0.0994	0.9425	0.0261	0.0314
BPI	0.1246	0.9209	0.0213	0.0578
PTBPI	0.1234	0.9327	0.0013	0.0660
BTI	0.1663	0.9137	0.0384	0.0479
BCAI	0.1065	0.9075	0.0279	0.0646
PTBCAI	0.1048	0.9374	0.0313	0.0313
BPGPDI	0.0941	0.9320	0.0107	0.0573
BTGPDI	0.0971	0.9270	0.0154	0.0576
BPPI	0.0931	0.9460	0.0266	0.0274
BTPI	0.0955	0.9410	0.0304	0.0286

	A1.5	(Bu2, n=5000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0865	0.9319	0.0323	0.0358
STI	0.0801	0.9403	0.0187	0.0410
BPI	0.0965	0.9320	0.0340	0.0340
PTBPI	0.0952	0.9420	0.0112	0.0468
BTI	0.0949	0.9134	0.0385	0.0481
BCAI	0.0933	0.9281	0.0335	0.0384
PTBCAI	0.0819	0.9329	0.0331	0.0340
BPGPDI	0.0800	0.9406	0.0204	0.0390
BTGPDI	0.0863	0.9394	0.0474	0.0132
BPPI	0.0707	0.9496	0.0329	0.0175
BTPI	0.0805	0.9413	0.0225	0.0362

	A1.5	(Bu2, n=10000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0987	0.9393	0.0263	0.0344
STI	0.0980	0.9409	0.0220	0.0371
BPI	0.0804	0.9325	0.0013	0.0662
PTBPI	0.0781	0.9589	0.0012	0.0399
BTI	0.0896	0.9120	0.0342	0.0538
BCAI	0.0819	0.9250	0.0257	0.0493
PTBCAI	0.0802	0.9485	0.0254	0.0261
BPGPDI	0.0813	0.9520	0.0173	0.0307
BTGPDI	0.0860	0.9457	0.0274	0.0269
BPPI	0.0780	0.9465	0.0260	0.0275
BTPI	0.0822	0.9544	0.0247	0.0209

Table F.31: Properties of Confidence Intervals for A1.5 when Samples come from Bu2

	A1.5	(Bu3, n=500)		
	ACIL	CP	LNCP	UNCP
SNI	0.1516	0.9357	0.0191	0.0452
STI	0.1447	0.9374	0.0189	0.0437
BPI	0.1308	0.9622	0.0034	0.0344
PTBPI	0.1289	0.9637	0.0045	0.0318
BTI	0.1570	0.9362	0.0407	0.0231
BCAI	0.1171	0.9052	0.0539	0.0409
PTBCAI	0.1200	0.9142	0.0553	0.0305
BPGPDI	0.1143	0.9235	0.0281	0.0484
BTGPDI	0.1153	0.9577	0.0094	0.0329
BPPI	0.1163	0.9505	0.0143	0.0352
BTPI	0.1125	0.9168	0.0407	0.0425

	A1.5	(Bu3, n=1000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0900	0.9511	0.0105	0.0384
STI	0.0945	0.9635	0.0034	0.0331
BPI	0.1010	0.9675	0.0044	0.0281
PTBPI	0.0905	0.9623	0.0122	0.0255
BTI	0.1006	0.9419	0.0362	0.0219
BCAI	0.0932	0.9177	0.0556	0.0267
PTBCAI	0.0903	0.9243	0.0526	0.0231
BPGPDI	0.0988	0.9286	0.0133	0.0581
BTGPDI	0.0965	0.9531	0.0116	0.0353
BPPI	0.0949	0.9489	0.0194	0.0317
BTPI	0.0963	0.9366	0.0215	0.0419

	A1.5	(Bu3, n=5000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0896	0.9506	0.0200	0.0294
STI	0.0830	0.9462	0.0200	0.0338
BPI	0.0519	0.9517	0.0276	0.0207
PTBPI	0.0526	0.9573	0.0254	0.0173
BTI	0.0685	0.9339	0.0477	0.0184
BCAI	0.0506	0.9284	0.0521	0.0195
PTBCAI	0.0503	0.9410	0.0206	0.0384
BPGPDI	0.0397	0.9486	0.0140	0.0374
BTGPDI	0.0402	0.9572	0.0154	0.0274
BPPI	0.0558	0.9380	0.0311	0.0309
BTPI	0.0481	0.9588	0.0161	0.0251

	A1.5	(Bu3, n=10000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0719	0.9506	0.0200	0.0294
STI	0.0779	0.9563	0.0211	0.0226
BPI	0.0548	0.9541	0.0276	0.0183
PTBPI	0.0534	0.9575	0.0264	0.0161
BTI	0.0638	0.9401	0.0392	0.0207
BCAI	0.0495	0.9471	0.0286	0.0243
PTBCAI	0.0491	0.9544	0.0237	0.0219
BPGPDI	0.0391	0.9453	0.0233	0.0314
BTGPDI	0.0406	0.9528	0.0222	0.0250
BPPI	0.0454	0.9489	0.0239	0.0272
BTPI	0.0486	0.9538	0.0228	0.0234

Table F.32: Properties of Confidence Intervals for A1.5 when Samples come from Bu3

	A1.5	(Fr1, n=500)		
	ACIL	CP	LNCP	UNCP
SNI	0.1605	0.9412	0.0108	0.0480
STI	0.1629	0.9413	0.0176	0.0411
BPI	0.1332	0.9611	0.0046	0.0343
PTBPI	0.1338	0.9636	0.0034	0.0330
BTI	0.1640	0.9478	0.0266	0.0256
BCAI	0.1189	0.9147	0.0523	0.0330
PTBCAI	0.1199	0.9205	0.0477	0.0318
BPGPDI	0.0994	0.9270	0.0302	0.0428
BTGPDI	0.0961	0.9298	0.0373	0.0329
BPPI	0.0931	0.9422	0.0280	0.0298
BTPI	0.0974	0.9320	0.0354	0.0326

	A1.5	(Fr1, n=1000)		
	ACIL	CP	LNCP	UNCP
SNI	0.1464	0.9548	0.0070	0.0382
STI	0.1457	0.9553	0.0035	0.0412
BPI	0.1013	0.9686	0.0034	0.0280
PTBPI	0.1047	0.9523	0.0233	0.0244
BTI	0.1255	0.9361	0.0407	0.0232
BCAI	0.0949	0.9336	0.0409	0.0255
PTBCAI	0.0938	0.9347	0.0398	0.0255
BPGPDI	0.0713	0.9072	0.0305	0.0623
BTGPDI	0.0730	0.9293	0.0123	0.0584
BPPI	0.0719	0.9434	0.0287	0.0279
BTPI	0.0634	0.9385	0.0279	0.0336

	A1.5	(Fr1, n=5000)		
	ACIL	CP	LNCP	UNCP
SNI	0.1184	0.9495	0.0302	0.0203
STI	0.1221	0.9484	0.0304	0.0212
BPI	0.0865	0.9435	0.0333	0.0232
PTBPI	0.0851	0.9464	0.0375	0.0161
BTI	0.0896	0.9328	0.0477	0.0195
BCAI	0.0811	0.9462	0.0295	0.0243
PTBCAI	0.0824	0.9421	0.0349	0.0230
BPGPDI	0.0721	0.9270	0.0374	0.0356
BTGPDI	0.0709	0.9293	0.0123	0.0584
BPPI	0.0600	0.9398	0.0327	0.0275
BTPI	0.0633	0.9408	0.0259	0.0333

	A1.5	(Fr1, n=10000)		
	ACIL	CP	LNCP	UNCP
SNI	0.1033	0.9551	0.0240	0.0209
STI	0.1014	0.9696	0.0121	0.0183
BPI	0.0554	0.9673	0.0120	0.0207
PTBPI	0.0566	0.9552	0.0276	0.0172
BTI	0.0621	0.9412	0.0381	0.0207
BCAI	0.0505	0.9341	0.0428	0.0231
PTBCAI	0.0517	0.9570	0.0222	0.0208
BPGPDI	0.0541	0.9453	0.0291	0.0256
BTGPDI	0.0589	0.9415	0.0097	0.0488
BPPI	0.0607	0.9482	0.0243	0.0275
BTPI	0.0525	0.9518	0.0238	0.0244

Table F.33: Properties of Confidence Intervals for A1.5 when Samples come from Fr1

	A1.5	(Fr2, n=500)		
	ACIL	CP	LNCP	UNCP
SNI	0.1450	0.9368	0.0267	0.0365
STI	0.1425	0.9350	0.0176	0.0474
BPI	0.1109	0.9183	0.0272	0.0545
PTBPI	0.1119	0.9309	0.0230	0.0461
BTI	0.1153	0.9277	0.0062	0.0661
BCAI	0.0897	0.9306	0.0365	0.0329
PTBCAI	0.0893	0.9306	0.0365	0.0329
BPGPDI	0.1070	0.9453	0.0274	0.0273
BTGPDI	0.1076	0.9498	0.0121	0.0381
BPPI	0.1095	0.9596	0.0143	0.0261
BTPI	0.0836	0.9321	0.0338	0.0341

	A1.5	(Fr2, n=1000)		
	ACIL	CP	LNCP	UNCP
SNI	0.1017	0.9229	0.0243	0.0528
STI	0.0976	0.9521	0.0246	0.0233
BPI	0.0708	0.9345	0.0352	0.0303
PTBPI	0.0788	0.9387	0.0538	0.0075
BTI	0.1011	0.9365	0.0043	0.0592
BCAI	0.0734	0.9415	0.0362	0.0223
PTBCAI	0.0793	0.9495	0.0202	0.0303
BPGPDI	0.0945	0.9453	0.0174	0.0373
BTGPDI	0.0924	0.9426	0.0281	0.0293
BPPI	0.0964	0.9417	0.0213	0.0370
BTPI	0.0781	0.9372	0.0317	0.0311

	A1.5	(Fr2, n=5000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0912	0.9338	0.0287	0.0375
STI	0.0997	0.9514	0.0024	0.0462
BPI	0.0709	0.9362	0.0346	0.0292
PTBPI	0.0752	0.9596	0.0202	0.0202
BTI	0.0975	0.9408	0.0234	0.0358
BCAI	0.0780	0.9491	0.0170	0.0339
PTBCAI	0.0789	0.9510	0.0230	0.0260
BPGPDI	0.0845	0.9386	0.0478	0.0136
BTGPDI	0.0857	0.9363	0.0274	0.0363
BPPI	0.0875	0.9377	0.0279	0.0344
BTPI	0.0788	0.9415	0.0234	0.0351

	A1.5	(Fr2, n=10000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0753	0.9432	0.0249	0.0319
STI	0.0798	0.9571	0.0224	0.0205
BPI	0.0785	0.9468	0.0277	0.0255
PTBPI	0.0703	0.9545	0.0218	0.0237
BTI	0.0760	0.9569	0.0144	0.0287
BCAI	0.0675	0.9382	0.0206	0.0412
PTBCAI	0.0657	0.9431	0.0190	0.0379
BPGPDI	0.0744	0.9386	0.0484	0.0130
BTGPDI	0.0734	0.9458	0.0278	0.0264
BPPI	0.0747	0.9497	0.0259	0.0244
BTPI	0.0720	0.9505	0.0245	0.0250

Table F.34: Properties of Confidence Intervals for A1.5 when Samples come from Fr2

A1.5 (T2, n=500)	ACIL	CP	LNCP	UNCP
SNI	0.0711	0.8562	0.0534	0.0904
STI	0.0705	0.8803	0.0692	0.0505
BPI	0.1424	0.9734	0.0022	0.0244
PTBPI	0.1464	0.9704	0.0077	0.0219
BTI	0.1101	0.9390	0.0403	0.0207
BCAI	0.1122	0.9226	0.0483	0.0291
PTBCAI	0.1041	0.9158	0.0461	0.0381
BPGPDI	0.1006	0.9520	0.0091	0.0389
BTGPDI	0.1016	0.9671	0.0094	0.0235
BPPI	0.1012	0.9533	0.0327	0.0140
BTPI	0.1049	0.9479	0.0274	0.0247

A1.5 (T2, n=1000)	ACIL	CP	LNCP	UNCP
SNI	0.0716	0.9339	0.0301	0.0360
STI	0.0688	0.9304	0.0303	0.0393
BPI	0.1095	0.9451	0.0254	0.0295
PTBPI	0.1117	0.9527	0.0255	0.0218
BTI	0.1015	0.9492	0.0359	0.0149
BCAI	0.0961	0.9155	0.0614	0.0231
PTBCAI	0.0940	0.9248	0.0545	0.0207
BPGPDI	0.0905	0.9453	0.0374	0.0173
BTGPDI	0.0970	0.9506	0.0181	0.0313
BPPI	0.0904	0.9484	0.0241	0.0275
BTPI	0.0954	0.9338	0.0346	0.0316

A1.5 (T2, n=5000)	ACIL	CP	LNCP	UNCP
SNI	0.0691	0.9486	0.0131	0.0383
STI	0.0694	0.9455	0.0134	0.0411
BPI	0.0882	0.9477	0.0385	0.0138
PTBPI	0.0894	0.9544	0.0307	0.0149
BTI	0.0887	0.9382	0.0469	0.0149
BCAI	0.0800	0.9284	0.0485	0.0231
PTBCAI	0.0809	0.9429	0.0398	0.0173
BPGPDI	0.0698	0.9370	0.0274	0.0356
BTGPDI	0.0661	0.9418	0.0097	0.0485
BPPI	0.0911	0.9415	0.0311	0.0274
BTPI	0.0792	0.9451	0.0322	0.0227

A1.5 (T2, n=10000)	ACIL	CP	LNCP	UNCP
SNI	0.0694	0.9414	0.0204	0.0382
STI	0.0605	0.9415	0.0275	0.0310
BPI	0.0575	0.9531	0.0297	0.0172
PTBPI	0.0589	0.9599	0.0274	0.0127
BTI	0.0609	0.9411	0.0429	0.0160
BCAI	0.0513	0.9362	0.0331	0.0307
PTBCAI	0.0506	0.9467	0.0260	0.0273
BPGPDI	0.0687	0.9453	0.0174	0.0373
BTGPDI	0.0618	0.9509	0.0197	0.0294
BPPI	0.0728	0.9496	0.0329	0.0175
BTPI	0.0790	0.9583	0.0208	0.0209

Table F.35: Properties of Confidence Intervals for A1.5 when Samples come from T2

F.6 Atkinson Coefficient with Parameter 2 (A2)

	A2	(Pa, n=500)		
	ACIL	CP	LNCP	UNCP
SNI	0.0986	0.9677	0.0160	0.0163
STI	0.0856	0.9532	0.0272	0.0196
BPI	0.0831	0.9371	0.0188	0.0441
PTBPI	0.0789	0.9421	0.0172	0.0407
BTI	0.0940	0.9164	0.0407	0.0429
BCAI	0.0747	0.9249	0.0274	0.0477
PTBCAI	0.0747	0.9240	0.0299	0.0461
BPGPDI	0.0570	0.9507	0.0015	0.0478
BTGPDI	0.0839	0.9373	0.0074	0.0553
BPPI	0.0919	0.9514	0.0184	0.0302
BTPI	0.0722	0.9389	0.0038	0.0573

	A2	(Pa, n=1000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0831	0.9318	0.0193	0.0489
STI	0.0841	0.9448	0.0137	0.0415
BPI	0.0687	0.9392	0.0247	0.0361
PTBPI	0.0666	0.9471	0.0247	0.0282
BTI	0.0814	0.9362	0.0265	0.0373
BCAI	0.0511	0.9361	0.0274	0.0365
PTBCAI	0.0622	0.9272	0.0195	0.0533
BPGPDI	0.0827	0.9457	0.0413	0.0130
BTGPDI	0.0775	0.9369	0.0316	0.0315
BPPI	0.0696	0.9426	0.0265	0.0309
BTPI	0.0856	0.9563	0.0064	0.0373

	A2	(Pa, n=5000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0865	0.9486	0.0330	0.0184
STI	0.0875	0.9458	0.0288	0.0254
BPI	0.0603	0.9564	0.0123	0.0313
PTBPI	0.0694	0.9550	0.0230	0.0220
BTI	0.0791	0.9394	0.0262	0.0344
BCAI	0.0610	0.9307	0.0293	0.0400
PTBCAI	0.0693	0.9369	0.0318	0.0313
BPGPDI	0.0692	0.9574	0.0143	0.0283
BTGPDI	0.0627	0.9514	0.0313	0.0173
BPPI	0.0640	0.9615	0.0165	0.0220
BTPI	0.0693	0.9568	0.0123	0.0309

	A2	(Pa, n=10000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0739	0.9431	0.0274	0.0295
STI	0.0755	0.9669	0.0179	0.0152
BPI	0.0684	0.9438	0.0260	0.0302
PTBPI	0.0633	0.9597	0.0159	0.0244
BTI	0.0764	0.9611	0.0230	0.0159
BCAI	0.0679	0.9564	0.0265	0.0171
PTBCAI	0.0647	0.9668	0.0143	0.0189
BPGPDI	0.0694	0.9524	0.0144	0.0332
BTGPDI	0.0789	0.9646	0.0096	0.0258
BPPI	0.0669	0.9635	0.0198	0.0167
BTPI	0.0609	0.9561	0.0184	0.0255

Table F.36: Properties of Confidence Intervals for A2 when Samples come from Pa

	A2	(Bu1, n=500)		
	ACIL	CP	LNCP	UNCP
SNI	0.0847	0.9363	0.0338	0.0299
STI	0.0838	0.9319	0.0307	0.0374
BPI	0.1252	0.9499	0.0236	0.0265
PTBPI	0.1257	0.9524	0.0224	0.0252
BTI	0.1305	0.9295	0.0460	0.0245
BCAI	0.1087	0.9230	0.0379	0.0391
PTBCAI	0.1084	0.9308	0.0370	0.0322
BPGPDI	0.0881	0.9174	0.0513	0.0313
BTGPDI	0.0746	0.9349	0.0325	0.0326
BPPI	0.1028	0.9566	0.0168	0.0266
BTPI	0.1076	0.9641	0.0113	0.0246

	A2	(Bu1, n=1000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0774	0.9359	0.0242	0.0399
STI	0.0777	0.9499	0.0219	0.0282
BPI	0.1071	0.9548	0.0250	0.0202
PTBPI	0.1075	0.9616	0.0150	0.0234
BTI	0.1138	0.9416	0.0366	0.0218
BCAI	0.0968	0.9309	0.0409	0.0282
PTBCAI	0.0958	0.9290	0.0402	0.0308
BPGPDI	0.0691	0.9407	0.0313	0.0280
BTGPDI	0.0671	0.9488	0.0220	0.0292
BPPI	0.0902	0.9547	0.0238	0.0215
BTPI	0.0990	0.9382	0.0304	0.0314

	A2	(Bu1, n=5000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0780	0.9457	0.0221	0.0322
STI	0.0784	0.9456	0.0259	0.0285
BPI	0.0809	0.9635	0.0034	0.0331
PTBPI	0.0802	0.9688	0.0045	0.0267
BTI	0.0864	0.9376	0.0380	0.0244
BCAI	0.0769	0.9345	0.0424	0.0231
PTBCAI	0.0773	0.9391	0.0341	0.0268
BPGPDI	0.0689	0.9641	0.0144	0.0215
BTGPDI	0.0655	0.9512	0.0213	0.0275
BPPI	0.0733	0.9474	0.0243	0.0283
BTPI	0.0773	0.9570	0.0400	0.0030

	A2	(Bu1, n=10000)		
	ACIL	CP	LNCP	UNCP
SNI	0.0688	0.9440	0.0036	0.0524
STI	0.0683	0.9478	0.0211	0.0311
BPI	0.0814	0.9637	0.0045	0.0318
PTBPI	0.0812	0.9673	0.0034	0.0293
BTI	0.0861	0.9431	0.0363	0.0206
BCAI	0.0777	0.9467	0.0236	0.0297
PTBCAI	0.0765	0.9607	0.0237	0.0156
BPGPDI	0.0641	0.9541	0.0236	0.0223
BTGPDI	0.0604	0.9474	0.0213	0.0313
BPPI	0.0710	0.9621	0.0249	0.0130
BTPI	0.0763	0.9557	0.0414	0.0029

Table F.37: Properties of Confidence Intervals for A2 when Samples come from Bu1

	A2 (Bu2, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.0854	0.9325	0.0367	0.0308
STI	0.0843	0.9395	0.0358	0.0247
BPI	0.1312	0.9226	0.0323	0.0451
PTBPI	0.1294	0.9390	0.0263	0.0347
BTI	0.1650	0.8971	0.0496	0.0533
BCAI	0.1082	0.8874	0.0303	0.0823
PTBCAI	0.1082	0.9389	0.0340	0.0271
BPGPDI	0.0989	0.8991	0.0013	0.0996
BTGPDI	0.0927	0.9691	0.0043	0.0266
BPPI	0.0986	0.9297	0.0075	0.0628
BTPI	0.0958	0.9307	0.0313	0.0380

	A2 (Bu2, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0927	0.9388	0.0256	0.0356
STI	0.0938	0.9378	0.0285	0.0337
BPI	0.1068	0.9162	0.0037	0.0801
PTBPI	0.1058	0.9280	0.0035	0.0685
BTI	0.1425	0.9090	0.0406	0.0504
BCAI	0.0913	0.9228	0.0303	0.0469
PTBCAI	0.0898	0.9327	0.0337	0.0336
BPGPDI	0.0841	0.9441	0.0046	0.0513
BTGPDI	0.0871	0.9391	0.0093	0.0516
BPPI	0.0831	0.9481	0.0205	0.0314
BTPI	0.0855	0.9431	0.0243	0.0326

	A2 (Bu2, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0827	0.9272	0.0347	0.0381
STI	0.0772	0.9356	0.0211	0.0433
BPI	0.0827	0.9273	0.0364	0.0363
PTBPI	0.0816	0.9373	0.0036	0.0591
BTI	0.0813	0.9087	0.0409	0.0504
BCAI	0.0800	0.9334	0.0359	0.0307
PTBCAI	0.0888	0.9482	0.0155	0.0363
BPGPDI	0.0800	0.9327	0.0043	0.0630
BTGPDI	0.0863	0.9515	0.0413	0.0072
BPPI	0.0807	0.9617	0.0268	0.0115
BTPI	0.0805	0.9534	0.0164	0.0302

	A2 (Bu2, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0846	0.9446	0.0286	0.0268
STI	0.0740	0.9362	0.0243	0.0395
BPI	0.0689	0.9278	0.0036	0.0686
PTBPI	0.0769	0.9542	0.0035	0.0423
BTI	0.0768	0.9373	0.0366	0.0261
BCAI	0.0616	0.9402	0.0281	0.0317
PTBCAI	0.0602	0.9438	0.0278	0.0284
BPGPDI	0.0813	0.9641	0.0113	0.0246
BTGPDI	0.0760	0.9578	0.0213	0.0209
BPPI	0.0780	0.9586	0.0199	0.0215
BTPI	0.0692	0.9565	0.0286	0.0149

Table F.38: Properties of Confidence Intervals for A2 when Samples come from Bu2

	A2 (Bu3, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.1299	0.9407	0.0166	0.0427
STI	0.1240	0.9424	0.0164	0.0412
BPI	0.1121	0.9572	0.0109	0.0319
PTBPI	0.1105	0.9687	0.0020	0.0293
BTI	0.1346	0.9412	0.0382	0.0206
BCAI	0.1004	0.9102	0.0514	0.0384
PTBCAI	0.1029	0.9192	0.0528	0.0280
BPGPDI	0.0643	0.9357	0.0320	0.0323
BTGPDI	0.1053	0.9698	0.0034	0.0268
BPPI	0.0863	0.9426	0.0283	0.0291
BTPI	0.0825	0.9289	0.0347	0.0364

	A2 (Bu3, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0771	0.9461	0.0180	0.0359
STI	0.0810	0.9485	0.0209	0.0306
BPI	0.0866	0.9525	0.0220	0.0255
PTBPI	0.0776	0.9473	0.0297	0.0230
BTI	0.0862	0.9469	0.0337	0.0194
BCAI	0.0799	0.9327	0.0431	0.0242
PTBCAI	0.0774	0.9393	0.0402	0.0205
BPGPDI	0.0888	0.9407	0.0073	0.0520
BTGPDI	0.0865	0.9452	0.0256	0.0292
BPPI	0.0849	0.9410	0.0334	0.0256
BTPI	0.0863	0.9487	0.0155	0.0358

	A2 (Bu3, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0754	0.9456	0.0275	0.0269
STI	0.0797	0.9512	0.0175	0.0313
BPI	0.0645	0.9567	0.0251	0.0182
PTBPI	0.0651	0.9623	0.0229	0.0148
BTI	0.0536	0.9389	0.0452	0.0159
BCAI	0.0634	0.9334	0.0496	0.0170
PTBCAI	0.0631	0.9460	0.0281	0.0259
BPGPDI	0.0697	0.9407	0.0079	0.0514
BTGPDI	0.0602	0.9493	0.0093	0.0414
BPPI	0.0658	0.9501	0.0250	0.0249
BTPI	0.0681	0.9509	0.0200	0.0291

	A2 (Bu3, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0692	0.9556	0.0175	0.0269
STI	0.0753	0.9613	0.0186	0.0201
BPI	0.0670	0.9591	0.0251	0.0158
PTBPI	0.0758	0.9625	0.0239	0.0136
BTI	0.0647	0.9451	0.0367	0.0182
BCAI	0.0624	0.9521	0.0261	0.0218
PTBCAI	0.0621	0.9594	0.0212	0.0194
BPGPDI	0.0691	0.9474	0.0273	0.0253
BTGPDI	0.0606	0.9549	0.0261	0.0190
BPPI	0.0654	0.9570	0.0198	0.0232
BTPI	0.0636	0.9559	0.0268	0.0173

Table F.39: Properties of Confidence Intervals for A2 when Samples come from Bu3

	A2 (Fr1, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.1376	0.9462	0.0083	0.0455
STI	0.1396	0.9463	0.0151	0.0386
BPI	0.1142	0.9661	0.0021	0.0318
PTBPI	0.1147	0.9686	0.0009	0.0305
BTI	0.1406	0.9528	0.0241	0.0231
BCAI	0.1019	0.9197	0.0498	0.0305
PTBCAI	0.1028	0.9255	0.0452	0.0293
BPGPDI	0.1094	0.9391	0.0243	0.0366
BTGPDI	0.1061	0.9419	0.0313	0.0268
BPPI	0.1031	0.9543	0.0220	0.0237
BTPI	0.1074	0.9241	0.0394	0.0365

	A2 (Fr1, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.1255	0.9498	0.0145	0.0357
STI	0.1249	0.9403	0.0210	0.0387
BPI	0.1068	0.9436	0.0309	0.0255
PTBPI	0.1097	0.9573	0.0208	0.0219
BTI	0.1076	0.9411	0.0382	0.0207
BCAI	0.0813	0.9386	0.0384	0.0230
PTBCAI	0.1004	0.9397	0.0373	0.0230
BPGPDI	0.0913	0.9193	0.0244	0.0563
BTGPDI	0.0930	0.9414	0.0062	0.0524
BPPI	0.1019	0.9555	0.0226	0.0219
BTPI	0.0934	0.9506	0.0218	0.0276

	A2 (Fr1, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.1015	0.9545	0.0275	0.0180
STI	0.1047	0.9634	0.0275	0.0091
BPI	0.0984	0.9485	0.0308	0.0207
PTBPI	0.0972	0.9414	0.0350	0.0236
BTI	0.0911	0.9378	0.0452	0.0170
BCAI	0.0938	0.9512	0.0270	0.0218
PTBCAI	0.0849	0.9471	0.0324	0.0205
BPGPDI	0.0921	0.9191	0.0313	0.0496
BTGPDI	0.0909	0.9414	0.0062	0.0524
BPPI	0.0800	0.9519	0.0266	0.0215
BTPI	0.0833	0.9529	0.0198	0.0273

	A2 (Fr1, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0885	0.9601	0.0215	0.0184
STI	0.0869	0.9546	0.0296	0.0158
BPI	0.0675	0.9523	0.0295	0.0182
PTBPI	0.0685	0.9602	0.0251	0.0147
BTI	0.0632	0.9462	0.0356	0.0182
BCAI	0.0633	0.9391	0.0403	0.0206
PTBCAI	0.0743	0.9520	0.0297	0.0183
BPGPDI	0.0641	0.9574	0.0030	0.0396
BTGPDI	0.0689	0.9536	0.0036	0.0428
BPPI	0.0707	0.9603	0.0182	0.0215
BTPI	0.0725	0.9639	0.0177	0.0184

Table F.40: Properties of Confidence Intervals for A2 when Samples come from Fr1

	A2 (Fr2, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	0.1043	0.9418	0.0242	0.0340
STI	0.1021	0.9400	0.0151	0.0449
BPI	0.1051	0.9233	0.0247	0.0520
PTBPI	0.1059	0.9359	0.0205	0.0436
BTI	0.1088	0.9327	0.0037	0.0636
BCAI	0.1069	0.9356	0.0340	0.0304
PTBCAI	0.1065	0.9356	0.0340	0.0304
BPGPDI	0.1070	0.9574	0.0213	0.0213
BTGPDI	0.1076	0.9619	0.0060	0.0321
BPPI	0.1095	0.9717	0.0082	0.0201
BTPI	0.1036	0.9442	0.0277	0.0281

	A2 (Fr2, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0872	0.9279	0.0218	0.0503
STI	0.0987	0.9571	0.0221	0.0208
BPI	0.0607	0.9395	0.0327	0.0278
PTBPI	0.0675	0.9437	0.0513	0.0050
BTI	0.0867	0.9415	0.0219	0.0366
BCAI	0.0843	0.9465	0.0337	0.0198
PTBCAI	0.0808	0.9545	0.0177	0.0278
BPGPDI	0.0845	0.9574	0.0113	0.0313
BTGPDI	0.0824	0.9547	0.0220	0.0233
BPPI	0.0864	0.9538	0.0152	0.0310
BTPI	0.0881	0.9493	0.0256	0.0251

	A2 (Fr2, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0867	0.9388	0.0262	0.0350
STI	0.0940	0.9564	0.0200	0.0236
BPI	0.0651	0.9412	0.0321	0.0267
PTBPI	0.0687	0.9574	0.0209	0.0217
BTI	0.0864	0.9458	0.0209	0.0333
BCAI	0.0811	0.9541	0.0145	0.0314
PTBCAI	0.0819	0.9560	0.0205	0.0235
BPGPDI	0.0845	0.9507	0.0417	0.0076
BTGPDI	0.0857	0.9484	0.0213	0.0303
BPPI	0.0875	0.9498	0.0218	0.0284
BTPI	0.0808	0.9536	0.0173	0.0291

	A2 (Fr2, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	0.0717	0.9482	0.0224	0.0294
STI	0.0755	0.9621	0.0199	0.0180
BPI	0.0690	0.9518	0.0252	0.0230
PTBPI	0.0688	0.9595	0.0193	0.0212
BTI	0.0751	0.9619	0.0119	0.0262
BCAI	0.0721	0.9432	0.0181	0.0387
PTBCAI	0.0706	0.9581	0.0165	0.0254
BPGPDI	0.0844	0.9507	0.0423	0.0070
BTGPDI	0.0734	0.9579	0.0217	0.0204
BPPI	0.0747	0.9518	0.0248	0.0234
BTPI	0.0720	0.9526	0.0223	0.0251

Table F.41: Properties of Confidence Intervals for A2 when Samples come from Fr2

F.7 Quintile Share Ratio (QSR)

	QSR (Pa, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	8.8209	0.9406	0.0198	0.0396
STI	11.0811	0.9606	0.0196	0.0198
BPI	8.0263	0.9634	0.0027	0.0339
PTBPI	8.3234	0.9607	0.0100	0.0293
BTI	40.6495	0.9547	0.0208	0.0245
BCAI	7.2385	0.9359	0.0301	0.0340
PTBCAI	7.2735	0.9261	0.0415	0.0324
BPGPDI	6.4342	0.9259	0.0351	0.0390
BTGPDI	6.4653	0.9161	0.0465	0.0374
BPPI	6.3337	0.9459	0.0151	0.0390
BTPI	6.3643	0.9461	0.0165	0.0374

	QSR (Pa, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	6.4016	0.9675	0.0132	0.0193
STI	9.1298	0.9783	0.0145	0.0072
BPI	5.8249	0.9680	0.0013	0.0307
PTBPI	5.7462	0.9624	0.0100	0.0276
BTI	33.4915	0.9558	0.0227	0.0215
BCAI	6.9420	0.9442	0.0298	0.0260
PTBCAI	7.1069	0.9410	0.0314	0.0276
BPGPDI	6.1707	0.9342	0.0348	0.0310
BTGPDI	6.3172	0.9310	0.0364	0.0326
BPPI	6.0743	0.9542	0.0148	0.0310
BTPI	6.2185	0.9410	0.0264	0.0326

	QSR (Pa, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	4.0250	0.9616	0.0154	0.0230
STI	6.4970	0.9618	0.0157	0.0225
BPI	3.6624	0.9726	0.0012	0.0262
PTBPI	3.8315	0.9758	0.0012	0.0230
BTI	12.8283	0.9473	0.0282	0.0245
BCAI	3.5082	0.9407	0.0303	0.0290
PTBCAI	3.5392	0.9310	0.0414	0.0276
BPGPDI	3.1184	0.9307	0.0353	0.0340
BTGPDI	3.1460	0.9210	0.0464	0.0326
BPPI	3.0697	0.9507	0.0253	0.0240
BTPI	3.0968	0.9410	0.0264	0.0326

	QSR (Pa, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	4.0072	0.9613	0.0258	0.0129
STI	5.1543	0.9543	0.0227	0.0230
BPI	3.6917	0.9688	0.0037	0.0275
PTBPI	3.6846	0.9719	0.0037	0.0244
BTI	10.9062	0.9425	0.0330	0.0245
BCAI	4.4132	0.9439	0.0287	0.0274
PTBCAI	3.8409	0.9452	0.0286	0.0262
BPGPDI	3.9228	0.9339	0.0337	0.0324
BTGPDI	3.4141	0.9352	0.0336	0.0312
BPPI	3.8616	0.9539	0.0137	0.0324
BTPI	3.3608	0.9552	0.0136	0.0312

Table F.42: Properties of Confidence Intervals for QSR when Samples come from Pa

	QSR (Bu1, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	168.7268	0.9624	0.0178	0.0198
STI	167.8737	0.9472	0.0185	0.0343
BPI	153.5276	0.9831	0.0034	0.0135
PTBPI	198.4662	0.9867	0.0022	0.0111
BTI	248.9747	0.9570	0.0322	0.0108
BCAI	167.5650	0.9498	0.0354	0.0148
PTBCAI	163.8727	0.9429	0.0423	0.0148
BPGPDI	148.9467	0.9398	0.0404	0.0198
BTGPDI	145.6646	0.9329	0.0473	0.0198
BPPI	146.6194	0.9398	0.0404	0.0198
BTPI	143.3886	0.9329	0.0473	0.0198

	QSR (Bu1, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	132.1916	0.9647	0.0013	0.0340
STI	141.6704	0.9368	0.0310	0.0322
BPI	120.2835	0.9854	0.0022	0.0124
PTBPI	129.8761	0.9819	0.0045	0.0136
BTI	299.7681	0.9420	0.0457	0.0123
BCAI	117.0695	0.9454	0.0423	0.0123
PTBCAI	125.5116	0.9383	0.0469	0.0148
BPGPDI	104.0618	0.9354	0.0473	0.0173
BTGPDI	111.5659	0.9283	0.0519	0.0198
BPPI	102.4358	0.9354	0.0473	0.0173
BTPI	109.8227	0.9283	0.0519	0.0198

	QSR (Bu1, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	76.4444	0.9524	0.0184	0.0292
STI	85.8296	0.9472	0.0185	0.0343
BPI	69.5581	0.9833	0.0045	0.0122
PTBPI	82.1352	0.9844	0.0033	0.0123
BTI	180.9759	0.9471	0.0419	0.0110
BCAI	61.0239	0.9455	0.0371	0.0174
PTBCAI	65.4182	0.9429	0.0448	0.0123
BPGPDI	54.2435	0.9355	0.0421	0.0224
BTGPDI	58.1495	0.9329	0.0498	0.0173
BPPI	53.3959	0.9455	0.0321	0.0224
BTPI	57.2409	0.9529	0.0281	0.0190

	QSR (Bu1, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	73.5756	0.9504	0.0204	0.0292
STI	57.6630	0.9478	0.0181	0.0341
BPI	66.9478	0.9810	0.0067	0.0123
PTBPI	76.3426	0.9825	0.0077	0.0098
BTI	122.0122	0.9479	0.0374	0.0147
BCAI	65.0971	0.9228	0.0585	0.0187
PTBCAI	74.4138	0.9316	0.0536	0.0148
BPGPDI	57.8641	0.9128	0.0635	0.0237
BTGPDI	66.1456	0.9216	0.0586	0.0198
BPPI	56.9600	0.9528	0.0235	0.0237
BTPI	65.1121	0.9516	0.0286	0.0198

Table F.43: Properties of Confidence Intervals for QSR when Samples come from Bu1

	QSR (Bu2, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	73.9271	0.9408	0.0074	0.0518
STI	70.5813	0.9546	0.0215	0.0239
BPI	67.2676	0.9627	0.0013	0.0360
PTBPI	71.4198	0.9661	0.0013	0.0326
BTI	91.3555	0.9510	0.0181	0.0309
BCAI	60.9455	0.9357	0.0268	0.0375
PTBCAI	61.9027	0.9287	0.0370	0.0343
BPGPDI	54.1738	0.9257	0.0318	0.0425
BTGPDI	55.0246	0.9187	0.0420	0.0393
BPPI	53.3273	0.9257	0.0318	0.0425
BTPI	54.1649	0.9387	0.0220	0.0393

	QSR (Bu2, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	69.3233	0.9299	0.0122	0.0579
STI	54.2805	0.9315	0.0378	0.0307
BPI	63.0785	0.9639	0.0000	0.0361
PTBPI	50.2629	0.9693	0.0000	0.0307
BTI	94.7966	0.9384	0.0308	0.0308
BCAI	56.0954	0.9419	0.0238	0.0343
PTBCAI	65.9010	0.9512	0.0145	0.0343
BPGPDI	49.8626	0.9319	0.0288	0.0393
BTGPDI	58.5787	0.9412	0.0195	0.0393
BPPI	49.0835	0.9319	0.0288	0.0393
BTPI	57.6634	0.9412	0.0195	0.0393

	QSR (Bu2, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	41.7088	0.9341	0.0325	0.0334
STI	53.3954	0.9463	0.0233	0.0304
BPI	37.9516	0.9651	0.0026	0.0323
PTBPI	35.3638	0.9661	0.0013	0.0326
BTI	92.3380	0.9394	0.0297	0.0309
BCAI	31.9875	0.9529	0.0094	0.0377
PTBCAI	35.3955	0.9241	0.0399	0.0360
BPGPDI	28.4333	0.9429	0.0144	0.0427
BTGPDI	31.4627	0.9141	0.0449	0.0410
BPPI	27.9891	0.9429	0.0144	0.0427
BTPI	30.9711	0.9141	0.0449	0.0410

	QSR (Bu2, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	34.7407	0.9412	0.0357	0.0231
STI	38.7494	0.9436	0.0282	0.0282
BPI	31.6112	0.9647	0.0013	0.0340
PTBPI	30.8295	0.9678	0.0013	0.0309
BTI	81.9920	0.9368	0.0310	0.0322
BCAI	37.8220	0.9479	0.0195	0.0326
PTBCAI	38.8824	0.9336	0.0339	0.0325
BPGPDI	33.6196	0.9579	0.0145	0.0276
BTGPDI	34.5621	0.9236	0.0389	0.0375
BPPI	33.0943	0.9479	0.0145	0.0376
BTPI	34.0221	0.9436	0.0289	0.0275

Table F.44: Properties of Confidence Intervals for QSR when Samples come from Bu2

	QSR (Bu3, n=500)			
	ACIL	CP	LNCP	UNCP
SNI	3.9271	0.9562	0.0234	0.0204
STI	4.5813	0.9303	0.0392	0.0305
BPI	3.0232	0.9833	0.0045	0.0122
PTBPI	3.4264	0.9869	0.0033	0.0098
BTI	4.2876	0.9438	0.0452	0.0110
BCAI	3.1500	0.9207	0.0657	0.0136
PTBCAI	3.0046	0.9263	0.0614	0.0123
BPGPDI	2.8000	0.9407	0.0407	0.0186
BTGPDI	2.6708	0.9463	0.0364	0.0173
BPPI	2.7563	0.9407	0.0407	0.0186
BTPI	2.6290	0.9463	0.0364	0.0173

	QSR (Bu3, n=1000)			
	ACIL	CP	LNCP	UNCP
SNI	3.3233	0.9339	0.0301	0.0360
STI	3.2805	0.9304	0.0303	0.0393
BPI	2.4279	0.9834	0.0044	0.0122
PTBPI	2.4193	0.9817	0.0097	0.0086
BTI	3.0335	0.9440	0.0462	0.0098
BCAI	2.2596	0.9414	0.0499	0.0087
PTBCAI	2.4302	0.9321	0.0557	0.0122
BPGPDI	2.0085	0.9314	0.0549	0.0137
BTGPDI	2.1602	0.9221	0.0607	0.0172
BPPI	1.9772	0.9314	0.0549	0.0137
BTPI	2.1264	0.9221	0.0607	0.0172

	QSR (Bu3, n=5000)			
	ACIL	CP	LNCP	UNCP
SNI	1.7088	0.9286	0.0331	0.0383
STI	1.3954	0.9455	0.0134	0.0411
BPI	1.3562	0.9816	0.0098	0.0086
PTBPI	1.2445	0.9841	0.0095	0.0064
BTI	1.7045	0.9454	0.0448	0.0098
BCAI	1.1352	0.9233	0.0668	0.0099
PTBCAI	1.2244	0.9376	0.0526	0.0098
BPGPDI	1.0091	0.9133	0.0718	0.0149
BTGPDI	1.0884	0.9276	0.0576	0.0148
BPPI	0.9933	0.9533	0.0318	0.0149
BTPI	1.0714	0.9276	0.0576	0.0148

	QSR (Bu3, n=10000)			
	ACIL	CP	LNCP	UNCP
SNI	1.7407	0.9314	0.0304	0.0382
STI	1.7494	0.9415	0.0275	0.0310
BPI	1.2198	0.9838	0.0086	0.0076
PTBPI	1.2163	0.9873	0.0074	0.0053
BTI	1.9285	0.9478	0.0435	0.0087
BCAI	1.1266	0.9254	0.0648	0.0098
PTBCAI	1.1350	0.9322	0.0580	0.0098
BPGPDI	1.0014	0.9154	0.0697	0.0149
BTGPDI	1.0089	0.9222	0.0630	0.0148
BPPI	0.9858	0.9453	0.0398	0.0149
BTPI	0.9931	0.9422	0.0430	0.0148

Table F.45: Properties of Confidence Intervals for QSR when Samples come from Bu3

QSR (Fr1, n=500)	ACIL	CP	LNCP	UNCP
SNI	3.7046	0.9478	0.0113	0.0410
STI	3.0084	0.9221	0.0384	0.0396
BPI	3.3709	0.9820	0.0045	0.0135
PTBPI	3.3615	0.9834	0.0055	0.0111
BTI	5.2539	0.9497	0.0380	0.0123
BCAI	3.2922	0.9258	0.0607	0.0135
PTBCAI	3.3769	0.9240	0.0612	0.0148
BPGPDI	2.9264	0.9158	0.0657	0.0185
BTGPDI	3.0017	0.9140	0.0662	0.0198
BPPI	2.8807	0.9158	0.0657	0.0185
BTPI	2.9548	0.9140	0.0662	0.0198

QSR (Fr1, n=1000)	ACIL	CP	LNCP	UNCP
SNI	2.8208	0.9651	0.0026	0.0323
STI	2.5269	0.9394	0.0297	0.0309
BPI	2.5667	0.9779	0.0110	0.0111
PTBPI	2.5298	0.9827	0.0086	0.0087
BTI	3.2707	0.9439	0.0451	0.0110
BCAI	2.2101	0.9369	0.0496	0.0135
PTBCAI	2.6337	0.9393	0.0472	0.0135
BPGPDI	1.9645	0.9269	0.0545	0.0186
BTGPDI	2.3411	0.9293	0.0522	0.0185
BPPI	1.9338	0.9269	0.0545	0.0186
BTPI	2.3045	0.9293	0.0522	0.0185

QSR (Fr1, n=5000)	ACIL	CP	LNCP	UNCP
SNI	1.3412	0.9438	0.0283	0.0279
STI	1.2767	0.9401	0.0185	0.0414
BPI	1.2204	0.9813	0.0077	0.0110
PTBPI	1.2628	0.9799	0.0138	0.0063
BTI	1.4721	0.9414	0.0499	0.0087
BCAI	1.1292	0.9277	0.0624	0.0099
PTBCAI	1.1493	0.9238	0.0652	0.0110
BPGPDI	1.0037	0.9477	0.0374	0.0149
BTGPDI	1.0216	0.9438	0.0402	0.0160
BPPI	0.9881	0.9477	0.0374	0.0149
BTPI	1.0056	0.9438	0.0402	0.0160

QSR (Fr1, n=10000)	ACIL	CP	LNCP	UNCP
SNI	1.3365	0.9660	0.0033	0.0307
STI	1.1698	0.9458	0.0227	0.0315
BPI	1.2161	0.9630	0.0106	0.0264
PTBPI	1.2896	0.9682	0.0143	0.0175
BTI	2.0282	0.9374	0.0540	0.0086
BCAI	1.1035	0.9357	0.0545	0.0098
PTBCAI	1.1912	0.9286	0.0604	0.0110
BPGPDI	0.9809	0.9457	0.0395	0.0148
BTGPDI	1.0588	0.9486	0.0354	0.0160
BPPI	0.9656	0.9457	0.0395	0.0148
BTPI	1.0423	0.9586	0.0254	0.0160

Table F.46: Properties of Confidence Intervals for QSR when Samples come from Fr1

QSR (Fr2, n=500)	ACIL	CP	LNCP	UNCP
SNI	24.9043	0.9670	0.0104	0.0226
STI	27.0390	0.9698	0.0027	0.0275
BPI	12.4584	0.8655	0.0897	0.0448
PTBPI	19.3910	0.8786	0.0809	0.0405
BTI	98.5890	0.9113	0.0600	0.0287
BCAI	10.6698	0.8721	0.0853	0.0426
PTBCAI	13.3991	0.8742	0.0839	0.0419
BPGPDI	9.4843	0.8621	0.0903	0.0476
BTGPDI	11.9103	0.8642	0.0889	0.0469
BPPI	9.3361	0.8621	0.0903	0.0476
BTPI	11.7242	0.8642	0.0889	0.0469

QSR (Fr2, n=1000)	ACIL	CP	LNCP	UNCP
SNI	19.3480	0.9439	0.0201	0.0360
STI	23.5191	0.9522	0.0119	0.0359
BPI	8.4489	0.8864	0.0424	0.0712
PTBPI	13.3428	0.9176	0.0216	0.0608
BTI	86.2771	0.8665	0.0890	0.0445
BCAI	9.0468	0.9392	0.0072	0.0536
PTBCAI	12.5742	0.9399	0.0068	0.0534
BPGPDI	8.0416	0.9292	0.0122	0.0586
BTGPDI	11.1771	0.9299	0.0118	0.0584
BPPI	7.9160	0.9292	0.0122	0.0586
BTPI	11.0024	0.9299	0.0118	0.0584

QSR (Fr2, n=5000)	ACIL	CP	LNCP	UNCP
SNI	11.4507	0.9414	0.0294	0.0292
STI	14.8710	0.9407	0.0268	0.0325
BPI	5.7282	0.9631	0.0246	0.0123
PTBPI	7.1736	0.9533	0.0311	0.0156
BTI	25.8959	0.9711	0.0459	0.0730
BCAI	5.7455	0.9803	0.0132	0.0066
PTBCAI	5.8312	0.9057	0.0462	0.0481
BPGPDI	5.1071	0.9703	0.0182	0.0116
BTGPDI	5.1833	0.9457	0.0212	0.0331
BPPI	5.0273	0.9703	0.0182	0.0116
BTPI	5.1023	0.9157	0.0412	0.0431

QSR (Fr2, n=10000)	ACIL	CP	LNCP	UNCP
SNI	6.4983	0.9481	0.0279	0.0240
STI	8.1901	0.9486	0.0191	0.0323
BPI	5.9129	0.9524	0.0184	0.0292
PTBPI	5.5714	0.9631	0.0246	0.0123
BTI	17.3299	0.9472	0.0185	0.0343
BCAI	6.2187	0.9446	0.0103	0.0452
PTBCAI	6.5288	0.9212	0.0359	0.0429
BPGPDI	5.5277	0.9746	0.0153	0.0102
BTGPDI	5.8034	0.9412	0.0209	0.0379
BPPI	5.4414	0.9746	0.0153	0.0102
BTPI	5.7127	0.9312	0.0309	0.0379

Table F.47: Properties of Confidence Intervals for QSR when Samples come from Fr2

Appendix G

Standard Errors for the CPs and ACILs

The following tables display the standard errors for the CPs and ACILs given in Appendix F.

G.1 Generalized Entropy with Parameter 0 (GE0)

GE0				Pa Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0198	0.0204	0.0221	0.0221	0.0128	0.0148	0.0194	0.0105	0.0175	0.0481	0.0443	0.0229
1000	0.0207	0.0213	0.0221	0.0215	0.0122	0.0175	0.0148	0.0107	0.0153	0.0201	0.0241	0.0182
5000	0.0161	0.0167	0.0152	0.0179	0.0121	0.0130	0.0129	0.0257	0.0137	0.0144	0.0113	0.0154
10000	0.0153	0.0168	0.0151	0.0152	0.0119	0.0124	0.0119	0.0124	0.0170	0.0123	0.0167	0.0143

GE0				Bu1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0126	0.0134	0.0220	0.0223	0.0123	0.0159	0.0191	0.0218	0.0291	0.0726	0.0785	0.0291
1000	0.0132	0.0137	0.0221	0.0205	0.0117	0.0124	0.0110	0.0250	0.0226	0.0682	0.0806	0.0274
5000	0.0157	0.0147	0.0158	0.0158	0.0118	0.0120	0.0121	0.0176	0.0138	0.0218	0.0227	0.0158
10000	0.0159	0.0151	0.0156	0.0159	0.0117	0.0127	0.0128	0.0128	0.0139	0.0125	0.0147	0.0140

GE0				Bu2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0276	0.0384	0.0299	0.0316	0.0389	0.0347	0.0278	0.0204	0.0198	0.0411	0.0488	0.0326
1000	0.0289	0.0378	0.0277	0.0304	0.0163	0.0200	0.0216	0.0107	0.0231	0.0370	0.0360	0.0263
5000	0.0208	0.0210	0.0233	0.0221	0.0183	0.0199	0.0184	0.0103	0.0188	0.0147	0.0145	0.0184
10000	0.0196	0.0208	0.0125	0.0125	0.0179	0.0195	0.0184	0.0107	0.0169	0.0173	0.0115	0.0162

GE0				Bu3 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0272	0.0270	0.0268	0.0271	0.0182	0.0216	0.0224	0.0170	0.0125	0.0241	0.0254	0.0227
1000	0.0275	0.0275	0.0216	0.0218	0.0188	0.0185	0.0210	0.0189	0.0185	0.0312	0.0315	0.0233
5000	0.0220	0.0212	0.0216	0.0211	0.0175	0.0178	0.0162	0.0129	0.0156	0.0216	0.0181	0.0187
10000	0.0221	0.0217	0.0217	0.0209	0.0185	0.0165	0.0157	0.0199	0.0165	0.0221	0.0172	0.0193

GE0				Fr1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0270	0.0267	0.0274	0.0270	0.0243	0.0233	0.0233	0.0188	0.0216	0.0221	0.0150	0.0233
1000	0.0214	0.0218	0.0270	0.0215	0.0185	0.0242	0.0232	0.0156	0.0179	0.0149	0.0220	0.0207
5000	0.0218	0.0214	0.0212	0.0213	0.0183	0.0158	0.0158	0.0198	0.0036	0.0155	0.0155	0.0173
10000	0.0219	0.0215	0.0219	0.0215	0.0175	0.0182	0.0162	0.0129	0.0083	0.0085	0.0067	0.0159

GE0				Fr2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0265	0.0264	0.0269	0.0268	0.0234	0.0247	0.0217	0.0202	0.0101	0.0269	0.0277	0.0237
1000	0.0264	0.0271	0.0273	0.0275	0.0246	0.0242	0.0241	0.0163	0.0201	0.0276	0.0255	0.0246
5000	0.0269	0.0272	0.0276	0.0270	0.0240	0.0229	0.0225	0.0181	0.0214	0.0139	0.0227	0.0231
10000	0.0272	0.0269	0.0274	0.0268	0.0236	0.0210	0.0226	0.0165	0.0090	0.0142	0.0020	0.0197

GE0				T2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0268	0.0219	0.0270	0.0271	0.0244	0.0225	0.0222	0.0129	0.0136	0.0209	0.0319	0.0228
1000	0.0213	0.0213	0.0265	0.0213	0.0191	0.0226	0.0235	0.0137	0.0150	0.0152	0.0171	0.0197
5000	0.0212	0.0219	0.0215	0.0221	0.0165	0.0175	0.0161	0.0125	0.0145	0.0217	0.0188	0.0186
10000	0.0216	0.0219	0.0216	0.0216	0.0174	0.0219	0.0180	0.0136	0.0167	0.0187	0.0150	0.0189

Table G.1: Standard Errors for the CPs for GE0

GE0				Pa Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0238	0.0239	0.0233	0.0238	0.0390	0.0208	0.0202	0.0219	0.0937	0.0251	0.0691	0.0350
1000	0.0190	0.0183	0.0198	0.0188	0.0357	0.0182	0.0177	0.0178	0.1253	0.0130	0.0097	0.0285
5000	0.0126	0.0123	0.0116	0.0117	0.0266	0.0110	0.0111	0.0093	0.0480	0.0040	0.0034	0.0147
10000	0.0108	0.0108	0.0101	0.0104	0.0200	0.0089	0.0093	0.0079	0.0125	0.0035	0.0028	0.0097
GE0				Bu1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0118	0.0120	0.0129	0.0131	0.0177	0.0110	0.0101	0.0125	0.3100	0.0262	0.0231	0.0419
1000	0.0100	0.0098	0.0105	0.0096	0.0145	0.0087	0.0094	0.0104	0.0239	0.0266	0.0213	0.0141
5000	0.0063	0.0061	0.0059	0.0060	0.0088	0.0055	0.0055	0.0049	0.0116	0.0118	0.0106	0.0075
10000	0.0054	0.0054	0.0052	0.0052	0.0079	0.0044	0.0045	0.0042	0.0092	0.0104	0.0099	0.0065
GE0				Bu2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0174	0.0179	0.0194	0.0184	0.0312	0.0154	0.0148	0.0170	0.0570	0.0180	0.0132	0.0218
1000	0.0155	0.0151	0.0155	0.0149	0.0264	0.0136	0.0137	0.0144	0.2766	0.0161	0.0134	0.0396
5000	0.0109	0.0109	0.0102	0.0103	0.0190	0.0095	0.0095	0.0082	0.0532	0.0082	0.0069	0.0142
10000	0.0093	0.0095	0.0091	0.0090	0.0158	0.0079	0.0079	0.0067	0.0339	0.0071	0.0064	0.0111
GE0				Bu3 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0115	0.0117	0.0113	0.0115	0.0156	0.0102	0.0102	0.0668	0.0630	0.0084	0.0074	0.0207
1000	0.0085	0.0084	0.0087	0.0085	0.0132	0.0081	0.0083	0.0432	0.0653	0.0060	0.0059	0.0168
5000	0.0045	0.0045	0.0045	0.0045	0.0068	0.0040	0.0041	0.0035	0.0040	0.0036	0.0032	0.0043
10000	0.0040	0.0039	0.0038	0.0038	0.0047	0.0033	0.0033	0.0032	0.0037	0.0030	0.0027	0.0036
GE0				Fr1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0119	0.0120	0.0114	0.0114	0.0183	0.0107	0.0109	0.0117	0.0636	0.0069	0.0060	0.0159
1000	0.0090	0.0088	0.0083	0.0082	0.0098	0.0081	0.0081	0.0087	0.0151	0.0052	0.0045	0.0085
5000	0.0044	0.0048	0.0044	0.0044	0.0052	0.0041	0.0041	0.0035	0.0123	0.0027	0.0023	0.0047
10000	0.0040	0.0041	0.0038	0.0038	0.0047	0.0035	0.0037	0.0031	0.0033	0.0023	0.0020	0.0035
GE0				Fr2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0161	0.0158	0.0148	0.0148	0.0274	0.0136	0.0137	0.0150	0.2457	0.0103	0.0075	0.0359
1000	0.0131	0.0126	0.0119	0.0119	0.0168	0.0109	0.0109	0.0120	0.0268	0.0058	0.0048	0.0125
5000	0.0071	0.0077	0.0070	0.0068	0.0103	0.0064	0.0065	0.0052	0.0086	0.0026	0.0022	0.0064
10000	0.0064	0.0065	0.0060	0.0059	0.0094	0.0056	0.0059	0.0045	0.0050	0.0022	0.0019	0.0054
GE0				T2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0105	0.0100	0.0103	0.0104	0.0127	0.0090	0.0093	0.0111	0.0152	0.0067	0.0051	0.0100
1000	0.0088	0.0077	0.0078	0.0077	0.0093	0.0072	0.0070	0.0080	0.0171	0.0043	0.0033	0.0080
5000	0.0041	0.0043	0.0040	0.0041	0.0101	0.0037	0.0036	0.0032	0.0029	0.0020	0.0018	0.0040
10000	0.0036	0.0037	0.0036	0.0036	0.0048	0.0033	0.0031	0.0028	0.0024	0.0017	0.0015	0.0031

Table G.2: Standard Errors for the ACILs for GE0

G.2 Generalized Entropy with Parameter 1 (GE1)

GE1				Pa Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0226	0.0233	0.0252	0.0252	0.0146	0.0169	0.0222	0.0120	0.0200	0.0550	0.0506	0.0262
1000	0.0236	0.0243	0.0252	0.0245	0.0140	0.0200	0.0169	0.0123	0.0174	0.0230	0.0276	0.0208
5000	0.0185	0.0168	0.0173	0.0170	0.0138	0.0149	0.0147	0.0294	0.0145	0.0164	0.0129	0.0169
10000	0.0174	0.0192	0.0185	0.0173	0.0136	0.0142	0.0125	0.0142	0.0195	0.0141	0.0191	0.0163

GE1				Bu1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0144	0.0153	0.0251	0.0255	0.0141	0.0182	0.0218	0.0249	0.0333	0.0830	0.0897	0.0332
1000	0.0151	0.0156	0.0252	0.0234	0.0134	0.0142	0.0126	0.0286	0.0259	0.0780	0.0921	0.0313
5000	0.0179	0.0168	0.0181	0.0181	0.0135	0.0137	0.0138	0.0201	0.0158	0.0249	0.0260	0.0181
10000	0.0182	0.0172	0.0178	0.0182	0.0134	0.0145	0.0146	0.0135	0.0125	0.0223	0.0168	0.0163

GE1				Bu2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0315	0.0439	0.0342	0.0361	0.0444	0.0397	0.0317	0.0233	0.0226	0.0470	0.0558	0.0373
1000	0.0331	0.0432	0.0316	0.0348	0.0187	0.0228	0.0246	0.0123	0.0264	0.0423	0.0412	0.0301
5000	0.0237	0.0240	0.0267	0.0252	0.0209	0.0227	0.0210	0.0118	0.0215	0.0168	0.0165	0.0210
10000	0.0224	0.0237	0.0143	0.0143	0.0205	0.0223	0.0210	0.0123	0.0194	0.0198	0.0132	0.0185

GE1				Bu3 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0311	0.0308	0.0306	0.0309	0.0208	0.0246	0.0257	0.0195	0.0143	0.0276	0.0290	0.0259
1000	0.0314	0.0314	0.0246	0.0249	0.0215	0.0212	0.0240	0.0216	0.0212	0.0357	0.0360	0.0267
5000	0.0251	0.0242	0.0246	0.0241	0.0200	0.0204	0.0186	0.0147	0.0178	0.0246	0.0207	0.0213
10000	0.0252	0.0248	0.0248	0.0239	0.0212	0.0189	0.0179	0.0227	0.0189	0.0253	0.0197	0.0221

GE1				Fr1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0308	0.0305	0.0313	0.0308	0.0278	0.0267	0.0267	0.0100	0.0246	0.0252	0.0171	0.0256
1000	0.0244	0.0249	0.0308	0.0245	0.0212	0.0277	0.0266	0.0064	0.0205	0.0170	0.0251	0.0226
5000	0.0249	0.0244	0.0242	0.0243	0.0209	0.0181	0.0180	0.0113	0.0042	0.0177	0.0177	0.0187
10000	0.0250	0.0245	0.0250	0.0245	0.0200	0.0208	0.0186	0.0033	0.0095	0.0097	0.0077	0.0171

GE1				Fr2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0303	0.0302	0.0307	0.0306	0.0268	0.0282	0.0248	0.0231	0.0116	0.0307	0.0316	0.0271
1000	0.0302	0.0309	0.0312	0.0314	0.0281	0.0277	0.0276	0.0187	0.0230	0.0315	0.0291	0.0281
5000	0.0307	0.0311	0.0315	0.0308	0.0275	0.0262	0.0258	0.0207	0.0244	0.0159	0.0260	0.0264
10000	0.0311	0.0307	0.0313	0.0306	0.0270	0.0275	0.0259	0.0189	0.0102	0.0162	0.0023	0.0229

GE1				T2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0306	0.0250	0.0308	0.0309	0.0279	0.0258	0.0254	0.0033	0.0155	0.0239	0.0365	0.0250
1000	0.0243	0.0243	0.0303	0.0243	0.0218	0.0259	0.0269	0.0156	0.0171	0.0173	0.0196	0.0225
5000	0.0242	0.0250	0.0245	0.0252	0.0189	0.0200	0.0185	0.0143	0.0165	0.0248	0.0215	0.0212
10000	0.0246	0.0250	0.0246	0.0246	0.0199	0.0250	0.0206	0.0144	0.0077	0.0214	0.0171	0.0204

Table G.3: Standard Errors for the CPs for GE1

GE1				Pa Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0214	0.0269	0.0223	0.0214	0.0351	0.0187	0.0177	0.0192	0.0820	0.0231	0.0635	0.0319
1000	0.0171	0.0206	0.0189	0.0169	0.0321	0.0164	0.0155	0.0156	0.1096	0.0120	0.0089	0.0258
5000	0.0114	0.0138	0.0111	0.0105	0.0239	0.0099	0.0097	0.0081	0.0420	0.0037	0.0031	0.0134
10000	0.0097	0.0122	0.0097	0.0093	0.0180	0.0080	0.0081	0.0069	0.0136	0.0032	0.0026	0.0092
GE1				Bu1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0106	0.0135	0.0123	0.0118	0.0159	0.0099	0.0088	0.0110	0.2712	0.0240	0.0212	0.0373
1000	0.0090	0.0111	0.0100	0.0086	0.0130	0.0078	0.0082	0.0091	0.0209	0.0245	0.0196	0.0129
5000	0.0057	0.0069	0.0057	0.0054	0.0079	0.0049	0.0048	0.0043	0.0101	0.0108	0.0097	0.0069
10000	0.0049	0.0061	0.0050	0.0047	0.0071	0.0039	0.0040	0.0037	0.0081	0.0096	0.0091	0.0060
GE1				Bu2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0157	0.0201	0.0186	0.0165	0.0281	0.0139	0.0130	0.0149	0.0498	0.0165	0.0121	0.0199
1000	0.0139	0.0170	0.0148	0.0134	0.0238	0.0123	0.0120	0.0126	0.2420	0.0147	0.0123	0.0354
5000	0.0099	0.0122	0.0098	0.0093	0.0171	0.0085	0.0083	0.0071	0.0465	0.0075	0.0063	0.0130
10000	0.0084	0.0106	0.0087	0.0081	0.0142	0.0071	0.0069	0.0058	0.0296	0.0065	0.0058	0.0102
GE1				Bu3 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0104	0.0132	0.0108	0.0104	0.0140	0.0092	0.0089	0.0585	0.0551	0.0077	0.0068	0.0186
1000	0.0077	0.0095	0.0084	0.0077	0.0119	0.0073	0.0072	0.0378	0.0572	0.0055	0.0055	0.0150
5000	0.0041	0.0051	0.0043	0.0040	0.0061	0.0036	0.0036	0.0031	0.0035	0.0033	0.0030	0.0040
10000	0.0036	0.0044	0.0037	0.0034	0.0042	0.0029	0.0029	0.0028	0.0033	0.0028	0.0025	0.0033
GE1				Fr1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0107	0.0135	0.0109	0.0103	0.0165	0.0096	0.0095	0.0102	0.0557	0.0063	0.0055	0.0144
1000	0.0081	0.0099	0.0079	0.0073	0.0089	0.0073	0.0071	0.0076	0.0132	0.0048	0.0042	0.0078
5000	0.0039	0.0054	0.0042	0.0039	0.0047	0.0037	0.0036	0.0031	0.0108	0.0025	0.0021	0.0044
10000	0.0036	0.0046	0.0037	0.0034	0.0042	0.0031	0.0032	0.0027	0.0029	0.0021	0.0018	0.0032
GE1				Fr2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0144	0.0177	0.0142	0.0134	0.0246	0.0123	0.0120	0.0132	0.2150	0.0095	0.0069	0.0321
1000	0.0118	0.0142	0.0114	0.0107	0.0151	0.0098	0.0095	0.0105	0.0235	0.0054	0.0044	0.0115
5000	0.0063	0.0087	0.0067	0.0061	0.0093	0.0058	0.0057	0.0045	0.0076	0.0024	0.0020	0.0059
10000	0.0058	0.0073	0.0058	0.0053	0.0084	0.0051	0.0051	0.0040	0.0044	0.0020	0.0018	0.0050
GE1				T2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0094	0.0113	0.0099	0.0094	0.0114	0.0081	0.0081	0.0097	0.0133	0.0061	0.0047	0.0092
1000	0.0079	0.0087	0.0075	0.0069	0.0084	0.0065	0.0061	0.0070	0.0149	0.0039	0.0031	0.0073
5000	0.0037	0.0048	0.0038	0.0037	0.0091	0.0033	0.0032	0.0028	0.0025	0.0019	0.0016	0.0037
10000	0.0032	0.0041	0.0034	0.0032	0.0043	0.0029	0.0027	0.0024	0.0021	0.0016	0.0014	0.0029

Table G.4: Standard Errors for the ACILs for GE1

G.3 Generalized Entropy with Parameter 1.3 (GE1.3)

GE1.3				Pa Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0297	0.0306	0.0331	0.0331	0.0192	0.0221	0.0291	0.0158	0.0263	0.0722	0.0664	0.0343
1000	0.0310	0.0319	0.0331	0.0322	0.0183	0.0263	0.0221	0.0161	0.0229	0.0301	0.0362	0.0273
5000	0.0242	0.0250	0.0227	0.0268	0.0182	0.0195	0.0193	0.0385	0.0205	0.0216	0.0170	0.0230
10000	0.0229	0.0252	0.0242	0.0227	0.0179	0.0186	0.0179	0.0186	0.0255	0.0185	0.0251	0.0216
GE1.3				Bu1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0189	0.0201	0.0329	0.0335	0.0185	0.0239	0.0286	0.0326	0.0437	0.1090	0.1177	0.0436
1000	0.0198	0.0205	0.0331	0.0307	0.0176	0.0186	0.0165	0.0375	0.0340	0.1023	0.1209	0.0410
5000	0.0235	0.0220	0.0238	0.0238	0.0177	0.0180	0.0182	0.0264	0.0207	0.0326	0.0341	0.0237
10000	0.0239	0.0226	0.0233	0.0239	0.0176	0.0190	0.0192	0.0192	0.0209	0.0187	0.0220	0.0209
GE1.3				Bu2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0413	0.0576	0.0449	0.0474	0.0583	0.0521	0.0416	0.0306	0.0297	0.0617	0.0732	0.0490
1000	0.0434	0.0567	0.0415	0.0456	0.0245	0.0300	0.0323	0.0161	0.0347	0.0555	0.0540	0.0395
5000	0.0312	0.0315	0.0350	0.0331	0.0275	0.0298	0.0276	0.0155	0.0282	0.0220	0.0217	0.0275
10000	0.0294	0.0312	0.0188	0.0188	0.0269	0.0292	0.0276	0.0161	0.0254	0.0260	0.0173	0.0242
GE1.3				Bu3 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0408	0.0405	0.0402	0.0406	0.0273	0.0323	0.0337	0.0255	0.0188	0.0362	0.0381	0.0340
1000	0.0412	0.0412	0.0323	0.0326	0.0282	0.0278	0.0315	0.0284	0.0278	0.0468	0.0473	0.0350
5000	0.0329	0.0317	0.0323	0.0316	0.0263	0.0267	0.0244	0.0193	0.0233	0.0323	0.0272	0.0280
10000	0.0331	0.0325	0.0325	0.0313	0.0278	0.0248	0.0235	0.0298	0.0248	0.0332	0.0258	0.0290
GE1.3				Fr1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0405	0.0400	0.0410	0.0405	0.0365	0.0350	0.0350	0.0281	0.0323	0.0331	0.0224	0.0349
1000	0.0320	0.0326	0.0405	0.0322	0.0278	0.0363	0.0348	0.0234	0.0269	0.0223	0.0329	0.0311
5000	0.0326	0.0320	0.0317	0.0319	0.0275	0.0238	0.0236	0.0298	0.0055	0.0232	0.0232	0.0259
10000	0.0328	0.0322	0.0328	0.0322	0.0263	0.0273	0.0244	0.0193	0.0125	0.0127	0.0100	0.0239
GE1.3				Fr2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0397	0.0396	0.0403	0.0402	0.0351	0.0371	0.0325	0.0303	0.0152	0.0403	0.0415	0.0356
1000	0.0396	0.0406	0.0409	0.0412	0.0369	0.0363	0.0362	0.0245	0.0301	0.0413	0.0382	0.0369
5000	0.0403	0.0408	0.0413	0.0405	0.0360	0.0344	0.0338	0.0272	0.0320	0.0208	0.0341	0.0347
10000	0.0408	0.0403	0.0410	0.0402	0.0354	0.0360	0.0340	0.0248	0.0134	0.0213	0.0030	0.0300
GE1.3				T2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0402	0.0328	0.0405	0.0406	0.0366	0.0338	0.0334	0.0193	0.0204	0.0313	0.0478	0.0342
1000	0.0319	0.0319	0.0397	0.0319	0.0286	0.0340	0.0353	0.0205	0.0224	0.0227	0.0257	0.0295
5000	0.0317	0.0328	0.0322	0.0331	0.0248	0.0263	0.0242	0.0188	0.0217	0.0325	0.0282	0.0278
10000	0.0323	0.0328	0.0323	0.0323	0.0261	0.0328	0.0270	0.0204	0.0250	0.0281	0.0224	0.0283

Table G.5: Standard Errors for the CPs for GE1.3

GE1.3				Pa Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0271	0.0340	0.0282	0.0271	0.0444	0.0237	0.0223	0.0243	0.1038	0.0292	0.0803	0.0404
1000	0.0217	0.0261	0.0239	0.0214	0.0407	0.0207	0.0196	0.0197	0.1387	0.0152	0.0113	0.0326
5000	0.0144	0.0175	0.0140	0.0133	0.0303	0.0125	0.0123	0.0103	0.0532	0.0047	0.0040	0.0169
10000	0.0123	0.0154	0.0122	0.0118	0.0227	0.0101	0.0103	0.0087	0.0172	0.0040	0.0033	0.0117

GE1.3				Bu1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0134	0.0170	0.0156	0.0149	0.0202	0.0126	0.0112	0.0139	0.3433	0.0304	0.0269	0.0472
1000	0.0114	0.0140	0.0127	0.0109	0.0165	0.0099	0.0104	0.0115	0.0265	0.0310	0.0248	0.0163
5000	0.0106	0.0087	0.0072	0.0069	0.0100	0.0062	0.0061	0.0054	0.0128	0.0137	0.0123	0.0091
10000	0.0095	0.0077	0.0064	0.0060	0.0090	0.0050	0.0050	0.0047	0.0102	0.0131	0.0116	0.0080

GE1.3				Bu2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0199	0.0255	0.0235	0.0209	0.0355	0.0175	0.0164	0.0188	0.0631	0.0209	0.0153	0.0252
1000	0.0176	0.0215	0.0188	0.0170	0.0301	0.0155	0.0152	0.0159	0.3063	0.0187	0.0155	0.0448
5000	0.0125	0.0155	0.0124	0.0117	0.0217	0.0108	0.0105	0.0090	0.0589	0.0095	0.0080	0.0164
10000	0.0106	0.0135	0.0110	0.0102	0.0180	0.0090	0.0087	0.0074	0.0375	0.0082	0.0074	0.0129

GE1.3				Bu3 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0131	0.0166	0.0137	0.0131	0.0178	0.0116	0.0113	0.0740	0.0698	0.0097	0.0086	0.0236
1000	0.0097	0.0120	0.0106	0.0097	0.0150	0.0092	0.0091	0.0479	0.0723	0.0070	0.0069	0.0190
5000	0.0052	0.0065	0.0054	0.0051	0.0077	0.0045	0.0045	0.0039	0.0044	0.0042	0.0038	0.0050
10000	0.0045	0.0055	0.0046	0.0043	0.0053	0.0037	0.0037	0.0035	0.0041	0.0037	0.0031	0.0042

GE1.3				Fr1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0135	0.0170	0.0138	0.0130	0.0208	0.0122	0.0120	0.0129	0.0705	0.0080	0.0070	0.0183
1000	0.0103	0.0126	0.0100	0.0093	0.0112	0.0092	0.0089	0.0097	0.0167	0.0060	0.0053	0.0099
5000	0.0050	0.0069	0.0053	0.0050	0.0059	0.0047	0.0045	0.0039	0.0137	0.0031	0.0027	0.0055
10000	0.0045	0.0058	0.0046	0.0043	0.0053	0.0043	0.0041	0.0034	0.0036	0.0027	0.0023	0.0041

GE1.3				Fr2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0183	0.0225	0.0180	0.0169	0.0312	0.0155	0.0152	0.0166	0.2721	0.0120	0.0087	0.0406
1000	0.0149	0.0180	0.0144	0.0135	0.0191	0.0124	0.0120	0.0133	0.0297	0.0068	0.0056	0.0145
5000	0.0080	0.0110	0.0084	0.0077	0.0117	0.0073	0.0072	0.0058	0.0096	0.0030	0.0026	0.0075
10000	0.0073	0.0093	0.0073	0.0067	0.0107	0.0064	0.0065	0.0050	0.0055	0.0026	0.0023	0.0063

GE1.3				T2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0119	0.0143	0.0125	0.0118	0.0145	0.0103	0.0103	0.0123	0.0169	0.0078	0.0059	0.0117
1000	0.0100	0.0110	0.0094	0.0088	0.0106	0.0082	0.0077	0.0088	0.0189	0.0050	0.0039	0.0093
5000	0.0047	0.0061	0.0048	0.0047	0.0115	0.0042	0.0040	0.0036	0.0032	0.0024	0.0020	0.0047
10000	0.0041	0.0057	0.0043	0.0044	0.0054	0.0037	0.0034	0.0033	0.0026	0.0020	0.0018	0.0037

Table G.6: Standard Errors for the ACILs for GE1.3

G.4 Atkinson Coefficient with Parameter 1 (A1)

A1				Pa Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0247	0.0254	0.0275	0.0275	0.0160	0.0184	0.0242	0.0131	0.0219	0.0600	0.0552	0.0285
1000	0.0258	0.0265	0.0275	0.0268	0.0152	0.0219	0.0184	0.0134	0.0190	0.0250	0.0301	0.0227
5000	0.0201	0.0183	0.0189	0.0185	0.0151	0.0162	0.0161	0.0320	0.0158	0.0179	0.0141	0.0185
10000	0.0190	0.0210	0.0201	0.0189	0.0149	0.0155	0.0136	0.0155	0.0212	0.0153	0.0209	0.0178

A1				Bu1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0157	0.0167	0.0274	0.0279	0.0153	0.0199	0.0238	0.0271	0.0363	0.0906	0.0978	0.0362
1000	0.0165	0.0171	0.0275	0.0255	0.0146	0.0155	0.0137	0.0312	0.0282	0.0851	0.1005	0.0341
5000	0.0195	0.0183	0.0198	0.0198	0.0147	0.0150	0.0151	0.0220	0.0172	0.0271	0.0284	0.0197
10000	0.0199	0.0188	0.0194	0.0199	0.0146	0.0158	0.0160	0.0147	0.0136	0.0243	0.0183	0.0178

A1				Bu2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0344	0.0479	0.0373	0.0394	0.0485	0.0433	0.0346	0.0254	0.0247	0.0513	0.0609	0.0407
1000	0.0361	0.0471	0.0345	0.0379	0.0204	0.0249	0.0269	0.0134	0.0288	0.0462	0.0449	0.0328
5000	0.0259	0.0261	0.0291	0.0275	0.0228	0.0248	0.0230	0.0129	0.0234	0.0183	0.0180	0.0229
10000	0.0244	0.0259	0.0156	0.0156	0.0223	0.0243	0.0230	0.0134	0.0211	0.0216	0.0144	0.0201

A1				Bu3 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0339	0.0336	0.0334	0.0338	0.0227	0.0269	0.0280	0.0212	0.0156	0.0301	0.0317	0.0283
1000	0.0343	0.0343	0.0269	0.0271	0.0234	0.0231	0.0261	0.0236	0.0231	0.0389	0.0393	0.0291
5000	0.0274	0.0264	0.0269	0.0263	0.0219	0.0222	0.0203	0.0161	0.0194	0.0269	0.0226	0.0233
10000	0.0275	0.0270	0.0270	0.0260	0.0231	0.0206	0.0195	0.0248	0.0206	0.0276	0.0215	0.0241

A1				Fr1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0336	0.0333	0.0341	0.0336	0.0303	0.0291	0.0291	0.0109	0.0269	0.0275	0.0187	0.0279
1000	0.0266	0.0271	0.0336	0.0268	0.0231	0.0302	0.0290	0.0070	0.0223	0.0185	0.0274	0.0247
5000	0.0271	0.0266	0.0264	0.0265	0.0228	0.0198	0.0196	0.0123	0.0045	0.0193	0.0193	0.0204
10000	0.0273	0.0268	0.0273	0.0268	0.0219	0.0227	0.0203	0.0036	0.0104	0.0106	0.0083	0.0187

A1				Fr2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0330	0.0329	0.0335	0.0334	0.0292	0.0308	0.0270	0.0252	0.0126	0.0335	0.0345	0.0296
1000	0.0329	0.0338	0.0340	0.0343	0.0307	0.0302	0.0301	0.0204	0.0250	0.0344	0.0318	0.0307
5000	0.0335	0.0339	0.0344	0.0336	0.0300	0.0286	0.0281	0.0226	0.0266	0.0173	0.0284	0.0288
10000	0.0339	0.0335	0.0341	0.0334	0.0295	0.0300	0.0282	0.0206	0.0112	0.0177	0.0025	0.0250

A1				T2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0334	0.0273	0.0336	0.0338	0.0304	0.0281	0.0277	0.0036	0.0169	0.0260	0.0398	0.0273
1000	0.0265	0.0265	0.0330	0.0265	0.0238	0.0282	0.0293	0.0171	0.0187	0.0189	0.0214	0.0245
5000	0.0264	0.0273	0.0268	0.0275	0.0206	0.0219	0.0201	0.0156	0.0180	0.0270	0.0234	0.0231
10000	0.0269	0.0273	0.0269	0.0269	0.0217	0.0273	0.0225	0.0157	0.0083	0.0233	0.0187	0.0223

Table G.7: Standard Errors for the CPs for A1

A1				Pa Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0239	0.0240	0.0234	0.0239	0.0392	0.0209	0.0203	0.0220	0.0941	0.0252	0.0694	0.0351
1000	0.0191	0.0184	0.0198	0.0189	0.0358	0.0183	0.0178	0.0179	0.1258	0.0131	0.0098	0.0286
5000	0.0127	0.0124	0.0116	0.0117	0.0267	0.0110	0.0111	0.0094	0.0482	0.0041	0.0034	0.0148
10000	0.0109	0.0109	0.0102	0.0115	0.0200	0.0089	0.0094	0.0079	0.0156	0.0035	0.0028	0.0101

A1				Bu1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0132	0.0134	0.0144	0.0146	0.0198	0.0124	0.0113	0.0140	0.3471	0.0293	0.0259	0.0469
1000	0.0112	0.0110	0.0117	0.0107	0.0162	0.0098	0.0105	0.0116	0.0268	0.0298	0.0239	0.0158
5000	0.0071	0.0069	0.0066	0.0068	0.0099	0.0061	0.0061	0.0055	0.0130	0.0132	0.0118	0.0085
10000	0.0061	0.0061	0.0059	0.0059	0.0088	0.0049	0.0051	0.0047	0.0103	0.0117	0.0111	0.0073

A1				Bu2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0195	0.0201	0.0217	0.0206	0.0349	0.0172	0.0166	0.0190	0.0638	0.0202	0.0148	0.0244
1000	0.0174	0.0169	0.0174	0.0167	0.0296	0.0153	0.0154	0.0161	0.3097	0.0180	0.0150	0.0443
5000	0.0123	0.0122	0.0114	0.0115	0.0213	0.0106	0.0106	0.0091	0.0595	0.0091	0.0077	0.0159
10000	0.0104	0.0106	0.0102	0.0101	0.0177	0.0088	0.0088	0.0075	0.0379	0.0079	0.0071	0.0125

A1				Bu3 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0129	0.0131	0.0127	0.0129	0.0175	0.0114	0.0114	0.0748	0.0705	0.0094	0.0083	0.0232
1000	0.0096	0.0095	0.0098	0.0096	0.0148	0.0090	0.0092	0.0484	0.0731	0.0068	0.0066	0.0188
5000	0.0051	0.0051	0.0050	0.0050	0.0076	0.0045	0.0046	0.0039	0.0045	0.0041	0.0036	0.0048
10000	0.0045	0.0044	0.0043	0.0043	0.0053	0.0037	0.0037	0.0036	0.0042	0.0034	0.0030	0.0040

A1				Fr1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0133	0.0134	0.0128	0.0128	0.0205	0.0119	0.0122	0.0131	0.0713	0.0077	0.0068	0.0178
1000	0.0101	0.0099	0.0092	0.0091	0.0110	0.0090	0.0090	0.0098	0.0169	0.0058	0.0051	0.0095
5000	0.0049	0.0054	0.0049	0.0049	0.0058	0.0046	0.0046	0.0039	0.0138	0.0030	0.0026	0.0053
10000	0.0045	0.0045	0.0043	0.0043	0.0053	0.0039	0.0041	0.0035	0.0037	0.0026	0.0022	0.0039

A1				Fr2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0180	0.0177	0.0166	0.0166	0.0306	0.0153	0.0154	0.0168	0.2751	0.0115	0.0084	0.0402
1000	0.0146	0.0141	0.0133	0.0133	0.0188	0.0122	0.0122	0.0134	0.0300	0.0065	0.0054	0.0140
5000	0.0079	0.0086	0.0078	0.0076	0.0115	0.0072	0.0073	0.0058	0.0097	0.0029	0.0025	0.0072
10000	0.0072	0.0073	0.0068	0.0066	0.0105	0.0063	0.0066	0.0051	0.0056	0.0025	0.0021	0.0061

A1				T2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0117	0.0112	0.0115	0.0116	0.0142	0.0101	0.0104	0.0125	0.0170	0.0075	0.0057	0.0112
1000	0.0099	0.0086	0.0087	0.0086	0.0104	0.0081	0.0078	0.0089	0.0191	0.0048	0.0037	0.0090
5000	0.0046	0.0048	0.0045	0.0046	0.0113	0.0042	0.0041	0.0036	0.0032	0.0023	0.0020	0.0045
10000	0.0040	0.0041	0.0040	0.0040	0.0053	0.0037	0.0035	0.0031	0.0027	0.0020	0.0017	0.0035

Table G.8: Standard Errors for the ACILs for A1

G.5 Atkinson Coefficient with Parameter 1.5 (A1.5)

A1.5				Pa Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0212	0.0218	0.0236	0.0236	0.0137	0.0158	0.0207	0.0113	0.0187	0.0515	0.0474	0.0245
1000	0.0221	0.0227	0.0236	0.0229	0.0130	0.0187	0.0158	0.0115	0.0163	0.0215	0.0258	0.0194
5000	0.0173	0.0157	0.0162	0.0159	0.0129	0.0139	0.0138	0.0275	0.0136	0.0154	0.0121	0.0158
10000	0.0163	0.0180	0.0173	0.0162	0.0127	0.0133	0.0117	0.0133	0.0182	0.0132	0.0179	0.0153
A1.5				Bu1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0135	0.0143	0.0235	0.0239	0.0132	0.0170	0.0204	0.0233	0.0311	0.0777	0.0839	0.0311
1000	0.0141	0.0146	0.0236	0.0219	0.0125	0.0133	0.0118	0.0267	0.0242	0.0729	0.0862	0.0293
5000	0.0167	0.0157	0.0169	0.0169	0.0126	0.0128	0.0129	0.0188	0.0147	0.0233	0.0243	0.0169
10000	0.0170	0.0161	0.0166	0.0170	0.0125	0.0136	0.0137	0.0126	0.0117	0.0208	0.0157	0.0152
A1.5				Bu2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0295	0.0410	0.0320	0.0338	0.0416	0.0371	0.0297	0.0218	0.0212	0.0440	0.0522	0.0349
1000	0.0309	0.0404	0.0296	0.0325	0.0175	0.0214	0.0230	0.0115	0.0247	0.0396	0.0385	0.0281
5000	0.0222	0.0224	0.0249	0.0236	0.0196	0.0213	0.0197	0.0110	0.0201	0.0157	0.0155	0.0196
10000	0.0209	0.0222	0.0134	0.0134	0.0192	0.0208	0.0197	0.0115	0.0181	0.0185	0.0123	0.0173
A1.5				Bu3 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0290	0.0288	0.0286	0.0289	0.0195	0.0230	0.0240	0.0182	0.0134	0.0258	0.0271	0.0242
1000	0.0294	0.0294	0.0230	0.0233	0.0201	0.0198	0.0224	0.0202	0.0198	0.0334	0.0337	0.0249
5000	0.0235	0.0226	0.0230	0.0225	0.0187	0.0190	0.0174	0.0138	0.0166	0.0230	0.0194	0.0200
10000	0.0236	0.0232	0.0232	0.0223	0.0198	0.0177	0.0167	0.0213	0.0177	0.0237	0.0184	0.0207
A1.5				Fr1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0288	0.0285	0.0293	0.0288	0.0260	0.0249	0.0249	0.0094	0.0230	0.0236	0.0160	0.0239
1000	0.0228	0.0233	0.0288	0.0229	0.0198	0.0259	0.0248	0.0060	0.0192	0.0159	0.0235	0.0212
5000	0.0233	0.0228	0.0226	0.0227	0.0196	0.0169	0.0168	0.0105	0.0039	0.0165	0.0165	0.0175
10000	0.0234	0.0229	0.0234	0.0229	0.0187	0.0195	0.0174	0.0031	0.0089	0.0090	0.0072	0.0160
A1.5				Fr2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0283	0.0282	0.0287	0.0286	0.0250	0.0264	0.0232	0.0216	0.0108	0.0287	0.0296	0.0254
1000	0.0282	0.0289	0.0291	0.0294	0.0263	0.0259	0.0258	0.0175	0.0215	0.0295	0.0273	0.0263
5000	0.0287	0.0290	0.0295	0.0288	0.0257	0.0245	0.0241	0.0194	0.0228	0.0148	0.0243	0.0247
10000	0.0290	0.0287	0.0293	0.0286	0.0253	0.0257	0.0242	0.0177	0.0096	0.0152	0.0021	0.0214
A1.5				T2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0286	0.0234	0.0288	0.0289	0.0261	0.0241	0.0238	0.0031	0.0145	0.0223	0.0341	0.0234
1000	0.0227	0.0227	0.0283	0.0227	0.0204	0.0242	0.0251	0.0146	0.0160	0.0162	0.0183	0.0210
5000	0.0226	0.0234	0.0229	0.0236	0.0177	0.0187	0.0173	0.0134	0.0155	0.0232	0.0201	0.0198
10000	0.0230	0.0234	0.0230	0.0230	0.0186	0.0234	0.0193	0.0135	0.0072	0.0200	0.0160	0.0191

Table G.9: Standard Errors for the CPs for A1.5

A1.5				Pa Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0269	0.0270	0.0263	0.0215	0.0353	0.0239	0.0215	0.0234	0.1000	0.0227	0.0625	0.0355
1000	0.0215	0.0207	0.0223	0.0170	0.0323	0.0209	0.0189	0.0190	0.1337	0.0118	0.0088	0.0297
5000	0.0143	0.0139	0.0131	0.0106	0.0240	0.0126	0.0118	0.0099	0.0512	0.0036	0.0031	0.0153
10000	0.0122	0.0122	0.0114	0.0103	0.0180	0.0102	0.0099	0.0084	0.0166	0.0031	0.0026	0.0105

A1.5				Bu1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0148	0.0151	0.0162	0.0132	0.0179	0.0141	0.0120	0.0149	0.3688	0.0264	0.0233	0.0488
1000	0.0126	0.0124	0.0132	0.0096	0.0146	0.0112	0.0111	0.0124	0.0285	0.0268	0.0215	0.0158
5000	0.0079	0.0077	0.0075	0.0061	0.0089	0.0070	0.0065	0.0059	0.0138	0.0119	0.0107	0.0085
10000	0.0068	0.0068	0.0066	0.0053	0.0079	0.0056	0.0054	0.0050	0.0110	0.0105	0.0100	0.0074

A1.5				Bu2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0220	0.0226	0.0244	0.0185	0.0314	0.0197	0.0177	0.0202	0.0678	0.0181	0.0133	0.0251
1000	0.0195	0.0191	0.0195	0.0151	0.0266	0.0175	0.0163	0.0171	0.3291	0.0162	0.0135	0.0463
5000	0.0138	0.0137	0.0129	0.0104	0.0192	0.0121	0.0113	0.0097	0.0633	0.0082	0.0069	0.0165
10000	0.0117	0.0119	0.0114	0.0091	0.0159	0.0101	0.0094	0.0079	0.0403	0.0071	0.0064	0.0128

A1.5				Bu3 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0145	0.0147	0.0143	0.0116	0.0157	0.0131	0.0121	0.0795	0.0750	0.0084	0.0075	0.0242
1000	0.0108	0.0106	0.0110	0.0086	0.0133	0.0103	0.0098	0.0514	0.0777	0.0061	0.0060	0.0196
5000	0.0057	0.0057	0.0056	0.0045	0.0068	0.0051	0.0049	0.0042	0.0047	0.0036	0.0033	0.0049
10000	0.0050	0.0049	0.0048	0.0038	0.0047	0.0042	0.0040	0.0038	0.0044	0.0030	0.0027	0.0041

A1.5				Fr1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0150	0.0151	0.0144	0.0115	0.0184	0.0137	0.0129	0.0139	0.0757	0.0069	0.0061	0.0185
1000	0.0113	0.0111	0.0104	0.0082	0.0099	0.0103	0.0096	0.0104	0.0180	0.0052	0.0046	0.0099
5000	0.0055	0.0061	0.0055	0.0044	0.0052	0.0052	0.0049	0.0042	0.0147	0.0027	0.0023	0.0055
10000	0.0050	0.0051	0.0048	0.0038	0.0047	0.0045	0.0044	0.0037	0.0039	0.0023	0.0020	0.0040

A1.5				Fr2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0202	0.0199	0.0187	0.0150	0.0276	0.0175	0.0163	0.0179	0.2923	0.0104	0.0076	0.0421
1000	0.0165	0.0159	0.0150	0.0120	0.0169	0.0139	0.0129	0.0142	0.0319	0.0059	0.0049	0.0145
5000	0.0089	0.0097	0.0088	0.0068	0.0104	0.0082	0.0077	0.0062	0.0103	0.0026	0.0022	0.0074
10000	0.0081	0.0082	0.0076	0.0059	0.0095	0.0072	0.0070	0.0054	0.0060	0.0022	0.0019	0.0063

A1.5				T2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0132	0.0126	0.0130	0.0105	0.0128	0.0115	0.0110	0.0132	0.0181	0.0067	0.0051	0.0116
1000	0.0111	0.0097	0.0098	0.0078	0.0094	0.0093	0.0083	0.0095	0.0203	0.0043	0.0034	0.0093
5000	0.0051	0.0054	0.0050	0.0041	0.0102	0.0047	0.0043	0.0039	0.0034	0.0021	0.0018	0.0045
10000	0.0045	0.0046	0.0045	0.0036	0.0048	0.0042	0.0037	0.0033	0.0028	0.0018	0.0015	0.0036

Table G.10: Standard Errors for the ACILs for A1.5

G.6 Atkinson Coefficient with Parameter 2 (A2)

A2				Pa Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0196	0.0202	0.0219	0.0219	0.0127	0.0146	0.0192	0.0104	0.0174	0.0477	0.0439	0.0227
1000	0.0205	0.0211	0.0219	0.0213	0.0121	0.0174	0.0146	0.0106	0.0151	0.0199	0.0239	0.0180
5000	0.0160	0.0145	0.0150	0.0147	0.0120	0.0129	0.0128	0.0255	0.0126	0.0143	0.0112	0.0147
10000	0.0151	0.0167	0.0160	0.0150	0.0118	0.0123	0.0108	0.0123	0.0169	0.0122	0.0166	0.0142
A2				Bu1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0125	0.0133	0.0218	0.0222	0.0122	0.0158	0.0189	0.0216	0.0289	0.0721	0.0778	0.0288
1000	0.0131	0.0136	0.0219	0.0203	0.0116	0.0123	0.0109	0.0248	0.0225	0.0677	0.0800	0.0271
5000	0.0155	0.0145	0.0157	0.0157	0.0117	0.0119	0.0120	0.0175	0.0137	0.0216	0.0226	0.0157
10000	0.0158	0.0149	0.0154	0.0158	0.0116	0.0126	0.0127	0.0117	0.0108	0.0193	0.0145	0.0141
A2				Bu2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0273	0.0381	0.0297	0.0313	0.0386	0.0345	0.0275	0.0202	0.0196	0.0408	0.0484	0.0324
1000	0.0287	0.0375	0.0274	0.0302	0.0162	0.0198	0.0214	0.0106	0.0229	0.0367	0.0357	0.0261
5000	0.0206	0.0208	0.0231	0.0219	0.0182	0.0197	0.0183	0.0103	0.0186	0.0145	0.0144	0.0182
10000	0.0194	0.0206	0.0124	0.0124	0.0178	0.0193	0.0183	0.0106	0.0168	0.0172	0.0114	0.0160
A2				Bu3 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0269	0.0268	0.0266	0.0268	0.0181	0.0214	0.0223	0.0169	0.0124	0.0239	0.0252	0.0225
1000	0.0272	0.0272	0.0214	0.0216	0.0186	0.0184	0.0208	0.0187	0.0184	0.0309	0.0312	0.0231
5000	0.0218	0.0210	0.0214	0.0209	0.0174	0.0177	0.0161	0.0128	0.0154	0.0214	0.0180	0.0185
10000	0.0219	0.0215	0.0215	0.0207	0.0184	0.0164	0.0155	0.0197	0.0164	0.0220	0.0171	0.0192
A2				Fr1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0268	0.0265	0.0271	0.0268	0.0241	0.0231	0.0231	0.0087	0.0214	0.0219	0.0148	0.0222
1000	0.0212	0.0216	0.0268	0.0213	0.0184	0.0240	0.0230	0.0056	0.0178	0.0147	0.0218	0.0196
5000	0.0216	0.0212	0.0210	0.0211	0.0182	0.0157	0.0156	0.0098	0.0036	0.0153	0.0153	0.0162
10000	0.0217	0.0213	0.0217	0.0213	0.0174	0.0181	0.0161	0.0028	0.0083	0.0084	0.0066	0.0149
A2				Fr2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0263	0.0262	0.0267	0.0266	0.0232	0.0245	0.0215	0.0200	0.0101	0.0267	0.0274	0.0235
1000	0.0262	0.0268	0.0270	0.0272	0.0244	0.0240	0.0239	0.0162	0.0199	0.0273	0.0253	0.0244
5000	0.0267	0.0269	0.0273	0.0268	0.0238	0.0227	0.0224	0.0180	0.0212	0.0138	0.0226	0.0229
10000	0.0269	0.0267	0.0271	0.0266	0.0234	0.0238	0.0225	0.0164	0.0089	0.0141	0.0020	0.0198

Table G.11: Standard Errors for the CPs for A2

A2				Pa Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0266	0.0248	0.0260	0.0213	0.0349	0.0234	0.0200	0.0232	0.0980	0.0211	0.0590	0.0344
1000	0.0213	0.0190	0.0221	0.0168	0.0319	0.0205	0.0175	0.0188	0.1310	0.0110	0.0083	0.0289
5000	0.0141	0.0128	0.0130	0.0105	0.0238	0.0123	0.0110	0.0098	0.0502	0.0034	0.0029	0.0149
10000	0.0121	0.0112	0.0113	0.0102	0.0178	0.0100	0.0092	0.0083	0.0162	0.0029	0.0024	0.0102

A2				Bu1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0147	0.0139	0.0161	0.0130	0.0177	0.0139	0.0112	0.0147	0.3613	0.0245	0.0220	0.0475
1000	0.0125	0.0114	0.0131	0.0095	0.0144	0.0109	0.0154	0.0122	0.0279	0.0249	0.0203	0.0157
5000	0.0089	0.0071	0.0074	0.0060	0.0088	0.0069	0.0069	0.0058	0.0135	0.0110	0.0101	0.0084
10000	0.0067	0.0063	0.0065	0.0052	0.0079	0.0055	0.0057	0.0050	0.0108	0.0098	0.0095	0.0072

A2				Bu2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0217	0.0207	0.0242	0.0183	0.0311	0.0193	0.0164	0.0200	0.0664	0.0169	0.0125	0.0243
1000	0.0193	0.0175	0.0193	0.0149	0.0264	0.0171	0.0152	0.0169	0.3224	0.0150	0.0127	0.0452
5000	0.0137	0.0126	0.0127	0.0103	0.0190	0.0119	0.0105	0.0096	0.0620	0.0077	0.0065	0.0160
10000	0.0116	0.0110	0.0113	0.0090	0.0158	0.0099	0.0087	0.0079	0.0395	0.0066	0.0061	0.0125

A2				Bu3 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0143	0.0135	0.0141	0.0115	0.0155	0.0128	0.0113	0.0787	0.0734	0.0078	0.0071	0.0236
1000	0.0106	0.0098	0.0109	0.0085	0.0131	0.0101	0.0091	0.0509	0.0761	0.0057	0.0057	0.0191
5000	0.0057	0.0053	0.0056	0.0044	0.0068	0.0050	0.0045	0.0042	0.0047	0.0034	0.0031	0.0048
10000	0.0050	0.0045	0.0048	0.0038	0.0047	0.0041	0.0037	0.0037	0.0044	0.0028	0.0026	0.0040

A2				Fr1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0148	0.0139	0.0142	0.0114	0.0182	0.0134	0.0120	0.0138	0.0742	0.0064	0.0057	0.0180
1000	0.0112	0.0102	0.0103	0.0081	0.0098	0.0101	0.0089	0.0103	0.0176	0.0049	0.0043	0.0096
5000	0.0054	0.0056	0.0054	0.0043	0.0052	0.0051	0.0045	0.0042	0.0144	0.0025	0.0022	0.0054
10000	0.0050	0.0047	0.0048	0.0038	0.0047	0.0044	0.0040	0.0037	0.0038	0.0022	0.0019	0.0039

A2				Fr2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0200	0.0183	0.0185	0.0148	0.0273	0.0171	0.0152	0.0177	0.2864	0.0096	0.0072	0.0411
1000	0.0163	0.0146	0.0148	0.0118	0.0168	0.0136	0.0120	0.0141	0.0313	0.0055	0.0046	0.0141
5000	0.0088	0.0089	0.0087	0.0068	0.0103	0.0080	0.0072	0.0061	0.0101	0.0024	0.0021	0.0072
10000	0.0080	0.0076	0.0075	0.0059	0.0094	0.0071	0.0065	0.0053	0.0058	0.0021	0.0018	0.0061

Table G.12: Standard Errors for the ACILs for A2

G.7 Quintile Share Ratio (QSR)

QSR				Pa Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0584	0.0367	0.0532	0.0532	0.0309	0.0356	0.0468	0.0254	0.0422	0.0860	0.0879	0.0506
1000	0.0584	0.0350	0.0532	0.0517	0.0294	0.0422	0.0356	0.0259	0.0368	0.0484	0.0581	0.0432
5000	0.0402	0.0347	0.0365	0.0358	0.0292	0.0313	0.0311	0.0619	0.0306	0.0346	0.0273	0.0358
10000	0.0428	0.0341	0.0389	0.0365	0.0287	0.0299	0.0263	0.0299	0.0411	0.0297	0.0403	0.0344
QSR				Bu1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0582	0.0353	0.0529	0.0539	0.0297	0.0384	0.0460	0.0524	0.0702	0.0751	0.0914	0.0549
1000	0.0584	0.0336	0.0532	0.0494	0.0282	0.0299	0.0266	0.0603	0.0546	0.0645	0.0944	0.0503
5000	0.0420	0.0339	0.0382	0.0382	0.0285	0.0290	0.0292	0.0425	0.0332	0.0524	0.0548	0.0384
10000	0.0412	0.0336	0.0375	0.0384	0.0282	0.0306	0.0309	0.0285	0.0263	0.0470	0.0354	0.0343
QSR				Bu2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0793	0.1115	0.0721	0.0762	0.0937	0.0838	0.0669	0.0491	0.0477	0.0992	0.0977	0.0797
1000	0.0733	0.0468	0.0667	0.0733	0.0394	0.0482	0.0520	0.0259	0.0558	0.0892	0.0869	0.0598
5000	0.0618	0.0525	0.0562	0.0532	0.0441	0.0479	0.0444	0.0249	0.0453	0.0354	0.0349	0.0455
10000	0.0331	0.0514	0.0301	0.0301	0.0432	0.0470	0.0444	0.0259	0.0408	0.0418	0.0278	0.0378
QSR				Bu3 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0709	0.0522	0.0646	0.0653	0.0439	0.0520	0.0541	0.0411	0.0301	0.0581	0.0612	0.0540
1000	0.0571	0.0539	0.0520	0.0524	0.0453	0.0446	0.0505	0.0456	0.0446	0.0752	0.0759	0.0543
5000	0.0571	0.0502	0.0520	0.0508	0.0422	0.0430	0.0392	0.0311	0.0375	0.0520	0.0437	0.0453
10000	0.0574	0.0530	0.0522	0.0503	0.0446	0.0399	0.0377	0.0479	0.0399	0.0534	0.0415	0.0471
QSR				Fr1 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0725	0.0697	0.0660	0.0650	0.0586	0.0562	0.0562	0.0211	0.0520	0.0532	0.0361	0.0551
1000	0.0715	0.0530	0.0650	0.0517	0.0446	0.0584	0.0560	0.0135	0.0432	0.0358	0.0529	0.0496
5000	0.0561	0.0525	0.0510	0.0513	0.0441	0.0382	0.0380	0.0237	0.0088	0.0373	0.0373	0.0398
10000	0.0579	0.0502	0.0527	0.0517	0.0422	0.0439	0.0392	0.0069	0.0201	0.0204	0.0161	0.0365
QSR				Fr2 Samples		SE.CP						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0712	0.0672	0.0648	0.0646	0.0565	0.0596	0.0522	0.0487	0.0244	0.0648	0.0667	0.0582
1000	0.0722	0.0705	0.0657	0.0662	0.0593	0.0584	0.0581	0.0394	0.0484	0.0664	0.0415	0.0588
5000	0.0730	0.0688	0.0664	0.0650	0.0579	0.0553	0.0543	0.0437	0.0515	0.0335	0.0548	0.0568
10000	0.0505	0.0677	0.0460	0.0646	0.0570	0.0579	0.0546	0.0399	0.0216	0.0342	0.0347	0.0481

Table G.13: Standard Errors for the CPs for QSR

QSR				Pa Samples		SE.ACIL					
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0391	0.0464	0.0435	0.0374	0.0549	0.0401	0.0357	0.0398	0.1360	0.0372	0.0858	0.0542
1000	0.0346	0.0432	0.0384	0.0317	0.0510	0.0364	0.0326	0.0342	0.1784	0.0241	0.0207	0.0477
5000	0.0240	0.0343	0.0267	0.0234	0.0406	0.0259	0.0241	0.0226	0.0745	0.0144	0.0137	0.0295
10000	0.0221	0.0279	0.0245	0.0231	0.0329	0.0228	0.0219	0.0207	0.0309	0.0137	0.0131	0.0231

QSR				Bu1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0276	0.0277	0.0307	0.0268	0.0327	0.0278	0.0244	0.0290	0.4746	0.0415	0.0383	0.0710
1000	0.0241	0.0242	0.0268	0.0223	0.0286	0.0241	0.0233	0.0257	0.0459	0.0421	0.0361	0.0294
5000	0.0176	0.0180	0.0195	0.0177	0.0213	0.0188	0.0178	0.0174	0.0274	0.0242	0.0229	0.0202
10000	0.0166	0.0170	0.0184	0.0167	0.0201	0.0171	0.0164	0.0164	0.0238	0.0226	0.0222	0.0188

QSR				Bu2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0370	0.0423	0.0411	0.0336	0.0500	0.0349	0.0311	0.0357	0.0954	0.0317	0.0261	0.0417
1000	0.0314	0.0372	0.0348	0.0292	0.0439	0.0320	0.0295	0.0318	0.4245	0.0293	0.0263	0.0682
5000	0.0237	0.0291	0.0264	0.0232	0.0344	0.0253	0.0235	0.0224	0.0897	0.0198	0.0184	0.0305
10000	0.0221	0.0256	0.0245	0.0215	0.0303	0.0227	0.0212	0.0201	0.0608	0.0185	0.0178	0.0259

QSR				Bu3 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0253	0.0254	0.0281	0.0248	0.0300	0.0265	0.0245	0.1111	0.1044	0.0201	0.0191	0.0399
1000	0.0216	0.0228	0.0240	0.0209	0.0269	0.0230	0.0217	0.0755	0.1079	0.0173	0.0173	0.0344
5000	0.0154	0.0158	0.0171	0.0157	0.0187	0.0164	0.0158	0.0153	0.0160	0.0144	0.0140	0.0159
10000	0.0145	0.0136	0.0161	0.0149	0.0160	0.0153	0.0148	0.0148	0.0156	0.0136	0.0133	0.0148

QSR				Fr1 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0255	0.0283	0.0283	0.0246	0.0334	0.0272	0.0254	0.0277	0.1054	0.0183	0.0174	0.0329
1000	0.0209	0.0191	0.0232	0.0205	0.0226	0.0230	0.0215	0.0232	0.0327	0.0163	0.0156	0.0217
5000	0.0153	0.0141	0.0170	0.0156	0.0167	0.0166	0.0158	0.0153	0.0285	0.0132	0.0128	0.0164
10000	0.0145	0.0136	0.0161	0.0149	0.0160	0.0156	0.0152	0.0147	0.0149	0.0128	0.0124	0.0146

QSR				Fr2 Samples		SE.ACIL						
n	**SNI**	**STI**	**BPI**	**PTBPI**	**BTI**	**BCAI**	**PTBCAI**	**BPGPDI**	**BTGPDI**	**BPPI**	**BTPI**	**Average**
500	0.0304	0.0382	0.0338	0.0290	0.0451	0.0320	0.0295	0.0328	0.3782	0.0224	0.0192	0.0628
1000	0.0261	0.0267	0.0290	0.0252	0.0315	0.0275	0.0254	0.0281	0.0502	0.0170	0.0159	0.0275
5000	0.0190	0.0196	0.0212	0.0187	0.0232	0.0203	0.0192	0.0179	0.0229	0.0131	0.0127	0.0189
10000	0.0177	0.0186	0.0197	0.0175	0.0220	0.0191	0.0184	0.0169	0.0175	0.0127	0.0123	0.0175

Table G.14: Standard Errors for the ACILs for QSR