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Vector Refinable Splines

and Subdivision

by

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Declaration

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Summary

In this thesis we study a standard example of refinable functions, that is, functions which can be reproduced by the integer shifts of their own dilations. Using the cardinal B-spline as an introductory example, we prove some of its properties, thereby building a basis for a later extension to the vector setting. Defining a subdivision scheme associated to the B-spline refinement mask, we then present the proof of a well-known convergence result.

Subdivision is a powerful tool used in computer-aided geometric design (CAGD) for the generation of curves and surfaces. The basic step of a subdivision algorithm consists of starting with a given set of points, called the initial control points, and creating new points as a linear combination of the previous ones, thereby generating new control points. Under certain conditions, repeated applications of this procedure yields a continuous limit curve. One important goal of this thesis is to study a particular extension of scalar subdivision to matrix subdivision.

More precisely, we introduce a new class of refinable spline vectors with one degree of smoothness less than the scalar case, by using a simple generalization of the scalar B-splines. We then analyze the properties satisfied by this class of refinable functions, proving also that the components of the refinable vectors thus obtained present some symmetry with respect to each other, thereby extending the symmetry within the scalar B-splines to the vector case. A comparative study between the newly-created vector splines and some vector B-splines from the literature is also given.

Besides, we extend the recursive formula for the scalar B-spline mask, providing a very simple and efficient way to compute the matrix mask corresponding to the vector refinable splines. We then give an explicit formula for the matrix refinement mask which generalizes the explicit formulation of the scalar B-spline mask.

Based on the explicit formula for the matrix refinement mask, we define the associated matrix subdivision scheme, and prove that it is convergent under a specific condition, yielding an efficient way to evaluate the corresponding refinable vector spline. The whole convergence proof uses purely algebraic arguments. The theory is then illustrated by means of numerical examples. We compare the results obtained with the ones yielded by the scalar B-spline masks and another B-spline matrix mask from the literature.

Opsomming

In hierdie tesis bestudeer ons 'n standaard voorbeeld van verfynbare funksies, oftewel funksies wat gereproduseer word deur die heelgetal skuiwe van hulle eie dilasies. Deur die kardinale B-latfunksie ("B-spline") as 'n inleidende voorbeeld te gebruik, bewys ons sommige van die eienskappe daarvan, en bou ons daardeur 'n fondament vir 'n latere uitbreiding na die vektor konteks. Ons beskou 'n subdivisieskema geassosieer met die B-latfunksie verfyningsmasker, en ons gee die bewys van 'n bekende konvergensieresultaat.

Subdivisie is 'n kragtige stuk gereedskap wat gebruik word in rekenaar-gesteunde geometriese ontwerp (CAGD) vir die generering van krommes en oppervlakke. Die basiese stap van 'n subdivisie algoritme bestaan daaruit om te begin met 'n gegewe stel punte, die sogenaamde aanvanklike kontrolepunte, en dan nuwe punte te skep as lineêre kombinasies van die vorige punte, en sodoende 'n stel nuwe kontrolepunte te genereer. Onderhewig aan sekere voorwaardes lewer herhaaldelike toepassings van hierdie prosedure dan 'n kontinue limietkromme. Een belangrike doelwit van hierdie tesis is om 'n spesifieke uitbreiding van skalare subdivisie na vektor subdivisie te bestudeer.

Meer presies: ons stel bekend 'n nuwe klas verfynbare latfunksie vektore met een gladheidsgraad minder as die skalare geval, deur 'n eenvoudige veralgemening van die skalare B-latfunksies te gebruik. Ons analiseer dan die eienskappe wat bevredig word deur hierdie klas verfynbare funksies; onder andere word bewys dat die komponente van die verfynbare vektore sodoende verkry 'n sekere tipe simmetrie ten opsigte van mekaar het, wat toon dat die simmetrie teenwoordig binne skalare B-latfunksies deur middel van

ons konstruksie na die vektor geval uitgebrei word. 'n Vergelykende studie tussen ons nuwe konstruksie en bestaande vektor B-latfunksies in die literatuur word ook gegee.

Daarbenewens brei ons die rekursiewe formule vir die skalare B-latfunksie maskers uit na ons vektor geval, wat gevolglik 'n baie eenvoudige en effektiewe manier is om die matriks masker wat ooreenstem met die vektor verfynbare latfunksie te bereken. Ons gee verder 'n eksplisiete formule vir die matriksverfyningsmasker wat die eksplisiete formulering van die skalare B-latfunksies maskers veralgemeen.

Gebaseer op die eksplisiete formule vir die matriks verfyningsmasker, definieer ons die ooreenkomstige matriks subdivisieskema, en bewys dat dit konvergent is onderhewig aan 'n spesifieke voorwaarde, en wat dan 'n effektiewe manier verteenwoordig vir die evaluering van die ooreenstemmende verfynbare vektor latfunksie. Die konvergensie bewys berus geheel en al op suiwer algebraïese argumente. Die teorie word dan geïllustreer met behulp van numeriese voorbeelde. Ons vergelyk die resultate sodoende verkry met die resultate gelewer deur die skalare B-latfunksie maskers, asook 'n ander B-latfunksie matriks masker in die literatuur.

Acknowledgements

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Contents

List of symbols

Symbol	Definition
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of non-negative integers
$\mathbb{N}$	the set of positive integers
$\mathbb{R}^2$	the set of real pairs
$\mathbb{R}^3$	the set of real triplets
$\lfloor x \rfloor$	the integer part of $x \in \mathbb{R}$
$\sum_j$	the sum over all the integers $j \in \mathbb{Z}$
$\binom{m}{j}$	for $j \in \mathbb{Z}_+$, the binomial coefficient given by $\dfrac{m!}{j!(m-j)!}$ if $j \in \{0, \ldots, m\}$, and 0 if $j \in \mathbb{Z} \setminus \{0, \ldots, m\}$
$M(\mathbb{R})$	the set of functions $f : \mathbb{R} \to \mathbb{R}$
$M(\mathbb{Z})$	the set of sequences $c : \mathbb{Z} \to \mathbb{R}$
$C^l(\mathbb{R})$	the set of l times continuously differentiable functions on $\mathbb{R}$
$C^0(\mathbb{R}) = C(\mathbb{R})$	the set of continuous functions on $\mathbb{R}$
$C^{-1}(\mathbb{R})$	the set of piecewise continuous functions on $\mathbb{R}$
π_l	the set of polynomials of degree at most l
$(\cdot)_+^m$	the truncated power of degree m

Δ	the backwards difference operator defined by $\Delta f := f - f(\cdot - 1)$
$S_{m,r}(\mathbb{Z})$	the space of splines of order m with respect to the knots $t_j^{(r)} := \left\lfloor \frac{j}{r} \right\rfloor,\ j \in \mathbb{Z}$
$S_m(\mathbb{Z})$	the space $S_{m,1}(\mathbb{Z})$ of cardinal splines
$f\,[\alpha_0, \ldots, \alpha_m]$	the divided difference of f with respect to $\alpha_0, \ldots, \alpha_m \in \mathbb{R}$
$f\vert_{[a,b)}$	the restricton of the function f to the interval $[a, b) \subset \mathbb{R}$
δ	the delta sequence defined by $\delta_0 = 1$ and $\delta_j = 0$ if $j \in \mathbb{Z} \setminus \{0\}$
$\operatorname{supp}(f)$	the support of a function f, given by the closure of the set $\{x \in \mathbb{R} : \ f(x) \neq 0\}$

Preface

Subdivision is a strong tool used in computer graphics for the generation of smooth curves or surfaces [FvDFH97]. Subdivision methods consist of generating denser and denser set of points, starting with a given set of points called the initial control points. More precisely, at each step of a subdivision algorithm, new control points are computed by means of linear combinations of the previous control points. If the algorithm is appropriately chosen, then repeated applications of the procedure lead to some limit curve, in which case the subdivision scheme is termed convergent.

Vector refinable functions, and in particular the corresponding matrix masks, arise by means of subdivision in geometric modeling where the components of the curve or surface can be generated by different linear combinations of the initial control points. They also appear in the study of shift-invariant spaces generated by more than one function and the construction of corresponding multi-wavelets (see [GLT93], [GL94], and [Plo96]) which have a great variety of applications in signal processing.

A lot of work has been done on refinable vectors $\mathbf{\Phi} = (\phi_1, \ldots, \phi_r)^{\mathrm{T}}$ satisfying vector refinement equations. The case $r = 2$ is studied in [Goo97] where refinable pairs of spline functions are fully characterized in terms of Fourier transforms. For the more general setting $r \in \mathbb{N}$, $r \geq 2$, we refer to [GL98] for spline vectors with general knots. In [CDP97], the existence and regularity of compactly supported vector functions are analyzed by means of particular factorization properties of the (matrix) mask.

As an introduction to refinable functions, we proceed in Chapter 2 to define and study the well-known cardinal B-splines (see [Chu92]), properties of which will be extended to the vector setting later in Chapter 4. Most of the results are quoted from [dV07b] and [dV07a].

Chapter 3 highlights the strong link between cardinal B-splines and subdivision, by means of the Lane-Riesenfeld algorithm [Rie75] which provides us with a first example of scalar subdivision schemes. A convergence result for the Lane-Riesenfeld subdivision algorithm is given in Theorem 3.3. The proof given here follows a similar pattern as in [Goo00]. Extending the Lane-Riesenfeld algorithm, we proceed to define a general scalar subdivision scheme.

Motivated by the results of Chapter 2, we then, in Chapter 4, focus on the generalization of the scalar B-splines to the vector setting with $r = 2$, that is, refinable pairs of splines. For an extension of the B-splines to the vector setting by means of divided differences, we refer to [Plo95a]. The vector splines presented here are constructed from an alternative generalization based on a simple recursion formula.

Next, in Chapter 5, we define a matrix subdivision scheme based on the results of Chapter 4, and prove that subdivision convergence occurs under certain conditions on the control points. Matrix subdivision convergence has been studied in a lot of papers, amongst them [JRZ98] by means of the joint spectral radius of two finite matrices derived from the mask, and in [CDL96] in terms of a new stability notion for the limit function (see also [CDL95]). Our approach has the significant advantage of providing an innovative extension of the scalar B-splines to the vector setting. Furthermore, apart from the symmetry properties presented in Theorem 4.4 and Theorem 4.8, an explicit and very simple formula is given for the refinement mask. Also, all our results are proved in terms of pure algebraic arguments.

Throughout the thesis, analytical results are illustrated by means of numerical graphs.

Chapter 1

Preliminaries

This chapter is intended to introduce all the basic notations which will be used throughout this thesis, as well as the theory and conventions which will be useful for the understanding of the text.

1.1 Definitions and notations

We will respectively denote by $\mathbb{C}, \mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_+$ the set of complex numbers, real numbers, integers, positive integers and non-negative integers. The symbol $\mathbb{R}^2$ will refer to the set $\{(x,y): \ x \in \mathbb{R}, \ y \in \mathbb{R}\}$. The notations $M(\mathbb{R})$ and $M(\mathbb{Z})$ will respectively denote the set of real-valued functions on $\mathbb{R}$ and the set of real-valued sequences on $\mathbb{Z}$. We define the Kronecker delta sequence $\{\delta_j: \ j \in \mathbb{Z}\} \in M(\mathbb{Z})$ by

$$\delta_j = \begin{cases} 1, & j = 0, \\ 0, & j \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{1.1}$$

For $l \in \mathbb{N}$, we will use the symbol $C^l(\mathbb{R})$ for the set of functions defined on $\mathbb{R}$ which are l times continuously differentiable. If $l = 0$, then $C^0(\mathbb{R}) = C(\mathbb{R})$, the set of continuous functions on $\mathbb{R}$, whereas $C^{-1}(\mathbb{R})$ will denote the space of piecewise continuous functions on $\mathbb{R}$. Boldface symbols will be used exclusively for vectors, with the exception that the notation $\mathbf{0}$ will refer either to the zero vector or to the null matrix, depending on the context. For $n \in \mathbb{N}$, the symbol $\mathbb{R}^n$ will denote the set $\{(\eta_1, \ldots, \eta_n) : \eta_j \in \mathbb{R},\ j = 1, \ldots, n\}$. In particular, we have that $\mathbb{R}^1 = \mathbb{R}$. For $l \in \mathbb{Z}_+$, we will write π_l for the space of polynomials of degree at most l. For $m \in \mathbb{Z}_+$, the symbol $(\cdot)_+^m$ denotes the truncated power of degree m, as defined by

$$x_+^m = \begin{cases} x^m, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{1.2}$$

with the convention that

$$(0)_+^0 = 1. \tag{1.3}$$

Observe that $(\cdot)_+^m \in C^{m-1}(\mathbb{R})$.

The operator $\Delta : M(\mathbb{R}) \to M(\mathbb{R})$ is given by

$$\Delta f := f - f(\cdot - 1),\ f \in M(\mathbb{R}), \tag{1.4}$$

in terms of which we define

$$\Delta^{l+1} f := \Delta \left(\Delta^l f \right),\ l \in \mathbb{Z}_+, \tag{1.5}$$

with $\Delta^0 f = f$. Depending on the context, the operator Δ may also be applied to sequences in $M(\mathbb{Z})$. The following well-known results will be needed:

$$\Delta^m f = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(\cdot - k),\ m \in \mathbb{Z}_+,\ f \in M(\mathbb{R}), \tag{1.6}$$

and

$$\Delta^m(fg) = \sum_{k=0}^{m} \binom{m}{j} \Delta^k f \Delta^{m-k} g(\cdot - k), \; m \in \mathbb{Z}_+, \; f \in M(\mathbb{R}), \; g \in M(\mathbb{R}), \tag{1.7}$$

where the binomial coefficients are defined by

$$\binom{k}{j} = \begin{cases} \dfrac{k!}{j!(k-j)!} &, \quad 0 \leq j \leq k, \\ & \qquad\qquad k \in \mathbb{Z}_+, \; j \in \mathbb{Z}, \\ 0 &, \quad j \notin \{0, \ldots, k\}, \end{cases} \tag{1.8}$$

with the convention that

$$0! = 1, \tag{1.9}$$

and satisfy the identity

$$\binom{k}{j} + \binom{k}{j+1} = \binom{k+1}{j+1}, \; k \in \mathbb{Z}_+, \; j \in \mathbb{Z}. \tag{1.10}$$

For $m \in \mathbb{Z}_+$ and $f \in C^m(\mathbb{R})$, $f^{(m)}$ is the m-th derivative of the function f with $f^{(0)} := f$, whereas $f[\alpha_0, \ldots, \alpha_m]$ denotes the divided difference of f with respect to the points $\alpha_0, \ldots, \alpha_m \in \mathbb{R}$. In particular, if $\{\alpha_j : \; j \in \mathbb{Z}\}$ is a non-decreasing sequence, we shall rely on the recursive formula [dB05]

$$\begin{cases} (\cdot - x)_+^m[\alpha_j, \ldots, \alpha_{j+l}] &= \dfrac{(\cdot - x)_+^m[\alpha_{j+1} \ldots, \alpha_{j+l}] - (\cdot - x)_+^m[\alpha_j, \ldots, \alpha_{j+l-1}]}{\alpha_{j+l} - \alpha_j}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha_j \neq \alpha_{j+l}, \\ (\cdot - x)_+^m[\alpha_j, \ldots, \alpha_{j+l}] &= \left\{(\cdot - x)_+^m\right\}^{(l)}(\alpha_j), \; \text{if } \alpha_j = \ldots = \alpha_{j+l}, \end{cases} \tag{1.11}$$

for $x \in \mathbb{R},\ j \in \mathbb{Z},\ l \in \mathbb{N}$ and $m \in \mathbb{N}_0$, with the convention that

$$(\cdot - x)_+(0,0) = 0. \tag{1.12}$$

Let the sequences $\{\xi_j :\ j \in \mathbb{Z}\}$, with $\xi_j < \xi_{j+1},\ j \in \mathbb{Z}$, and $\{r_j :\ j \in \mathbb{Z}\} \subseteq \mathbb{N}$ be given, and define the sequence $\{t_j :\ j \in \mathbb{Z}\}$ by

$$\begin{aligned} \ldots < \underbrace{t_{-r_{-1}-r_{-2}} = \ldots = t_{-r_{-1}-1}}_{\xi_{-2}} &< \underbrace{t_{-r_{-1}} = \ldots = t_{-1}}_{\xi_{-1}} < \underbrace{t_0 = \ldots = t_{r_0-1}}_{\xi_0} \\ < \underbrace{t_{r_0} = \ldots = t_{r_0+r_1-1}}_{\xi_1} &< \underbrace{t_{r_0+r_1} = \ldots = t_{r_0+r_1+r_2-1}}_{\xi_2} < \ldots \end{aligned} \tag{1.13}$$

Hence, $\{t_j :\ j \in \mathbb{Z}\}$ is a non-decreasing sequence with, for $j \in \mathbb{Z}$, r_j denoting the number of times (or multiplicity) that the number t_{k_j} appears in the sequence where

$$k_j = \begin{cases} 0, & \text{if } j = 0, \\ r_0 + r_1 + \ldots + r_{j-1}, & \text{if } j \geq 1, \\ -r_{-1} - r_{-2} - \ldots - r_j, & \text{if } j \leq -1. \end{cases}$$

For $m \in \mathbb{N}$, a function $s \in M(\mathbb{R})$ is then said to be a spline of order m with respect to the (knot) sequence $\{t_j :\ j \in \mathbb{Z}\}$ if

$$s|_{[\xi_j, \xi_{j+1})} = p_j \in \pi_{m-1},\ j \in \mathbb{Z},$$

and such that, for $j \in \mathbb{Z}$, the function s is $m - r_j - 1$ times continuously differentiable at ξ_j. For simplicity, we shall assume that

$$m - r_j \geq 0,\ j \in \mathbb{Z}, \tag{1.14}$$

which means that the multiplicity of any point is at most equal to the order of the spline.

In the definition (1.13), if $r_j = r \in \mathbb{N}$ for $j \in \mathbb{Z}$, that is, if every ξ_j appears r times in the sequence t, then the function s is called a spline of order m and multiplicity r with respect to the knots $\{t_j : \; j \in \mathbb{Z}\} \subseteq \mathbb{R}$. In other words, s is such that

$$s|_{[\xi_j,\xi_{j+1})} = p_j \in \pi_{m-1}, \; j \in \mathbb{Z},$$

with s being $m-r-1$ times continuously differentiable at each point $\xi_j, \; j \in \mathbb{Z}$.

In the particular case [Plo95b]

$$t_j^{(r)} := \left\lfloor \frac{j}{r} \right\rfloor, \; j \in \mathbb{Z}, \tag{1.15}$$

with $\lfloor x \rfloor$ denoting the integer part of $x \in \mathbb{R}$, the symbol $S_{m,r}(\mathbb{Z})$ shall refer to the linear space of splines of order m and multiplicity r with respect to the knots $t_j^{(r)}$ defined by (1.15). We write $S_m(\mathbb{Z})$ for $S_{m,1}(\mathbb{Z})$ which is the space of cardinal spline functions with respect to the knot sequence $\left\{t_j^{(1)} = j, \; j \in \mathbb{Z}\right\}$.

Next, for $d = 1, 2$ or 3, let $f := (f_0, \ldots, f_{d-1})^{\mathrm{T}} : \mathbb{R} \to \mathbb{R}^d$ denote a given function. The function f is said to be refinable if there exists a sequence a such that

$$f = \sum_j a_j \; f(2 \cdot -j), \tag{1.16}$$

where a is either a real- or matrix-valued sequence on $\mathbb{Z}$. Equation (1.16) is called the refinement equation, the sequence a is the refinement mask, and (a, f) is called a refinement pair. The polynomial A defined by

$$A(z) := \sum_j a_j \; z^j, \; z \in \mathbb{C}, \tag{1.17}$$

is called the refinement mask symbol.

If, moreover, the function $f = (f_0, \ldots, f_{d-1})^{\mathrm{T}}$ satisfies

$$\sum_{k=0}^{d-1} \int_{-\infty}^{+\infty} f_k(x)\ \mathrm{d}x = 1, \tag{1.18}$$

then (a, f) is said to be a normalized refinement pair.

The following results from standard algebra will also be needed.

1.2 Matrices, vectors and norms

In the sequel, let

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix}$$

denote a real square matrix, for which we define the norms

$$\|A\|_S := \sum_{i=0}^{1} \sum_{j=0}^{1} |a_{i,j}|, \tag{1.19}$$

$$\|A\|_1 := \max\left\{|a_{0,0}| + |a_{0,1}|,\ |a_{1,0}| + |a_{1,1}|\right\}. \tag{1.20}$$

Similarly, for a vector $\mathbf{v} = (v_0, v_1)^{\mathrm{T}} \in \mathbb{R}^2$, we define the norms

$$\|\mathbf{v}\|_S := |v_0| + |v_1|, \tag{1.21}$$

$$\|\mathbf{v}\|_1 := \max\left\{|v_0|,\ |v_1|\right\}. \tag{1.22}$$

Note that if the term inside $\|\ \|_S$ is either a matrix or a vector, we shall respectively use (1.19) or (1.21). Similarly, depending on the term inside $\|\ \|_1$, we shall implicitly refer to (1.20) or (1.22).

Remark 1.1. Note from equations (1.19), (1.20), (1.21) and (1.22) that we have

$$\left.\begin{aligned} \|A\|_1 \le \|A\|_S, \quad & \|\mathbf{v}\|_1 \le \|\mathbf{v}\|_S, \\ \\ \|A\|_S \le 2\ \|A\|_1, \quad & \|\mathbf{v}\|_S \le 2\ \|\mathbf{v}\|_1. \end{aligned}\right\} \tag{1.23}$$

We have the following result.

Lemma 1.2. *Let A be a real square matrix of dimension 2 and $\mathbf{v} = (v_0, v_1)^T$ a vector in $\mathbb{R}^2$. We then have*

$$\|A\mathbf{v}\|_1 \le \|A\|_1\ \|\mathbf{v}\|_1. \tag{1.24}$$

In particular, we have that

$$\|A\mathbf{v}\|_1 \le \|A\|_S\ \|\mathbf{v}\|_1. \tag{1.25}$$

Proof. Equations (1.20) and (1.22) imply

$$\begin{aligned} \|A\mathbf{v}\|_1 &= \left\| \begin{pmatrix} a_{0,0}\ v_0 + a_{0,1}\ v_1 \\ a_{1,0}\ v_0 + a_{1,1}\ v_1 \end{pmatrix} \right\|_1 \\ \\ &= \max\left\{|a_{0,0}\ v_0 + a_{0,1}\ v_1|,\ |a_{1,0}\ v_0 + a_{1,1}\ v_1|\right\} \\ \\ &\le \max\left\{|a_{0,0}|\ |v_0| + |a_{0,1}|\ |v_1|,\ |a_{1,0}|\ |v_0| + |a_{1,1}|\ |v_1|\right\} \\ \\ &\le \max\left\{(|a_{0,0}| + |a_{0,1}|)\ \max(|v_0|, |v_1|),\ (|a_{1,0}| + |a_{1,1}|)\ \max(|v_0|, |v_1|)\right\} \\ \\ &= \max\left\{|a_{0,0}| + |a_{0,1}|,\ |a_{1,0}| + |a_{1,1}|\right\}\ \max(|v_0|, |v_1|) \\ \\ &= \|A\|_1\ \|\mathbf{v}\|_1. \end{aligned}$$

Next, the fact that

$$\|A\|_1 \leq \|A\|_S,$$

together with (1.24), yield (1.25). $\square$

Next, let $p = \{p_j : \ j \in \mathbb{Z}\}$ denote a sequence of real square matrices. We define $\|\cdot\|_{S,\infty}$ and $\|\cdot\|_{1,\infty}$ by

$$\|p\|_{S,\infty} := \sup_j \|p_j\|_S, \tag{1.26}$$

$$\|p\|_{1,\infty} := \sup_j \|p_j\|_1. \tag{1.27}$$

The definition of $\|\cdot\|_{S,\infty}$ and $\|\cdot\|_{1,\infty}$ corresponding to a sequence of vectors is similar to (1.26) and (1.27).

Chapter 2

Scalar B-spline functions

2.1 Definition and refinability

Throughout this thesis, we will refer to the well-known class of scalar B-spline functions defined by

$$N_m := \int_0^1 N_{m-1}(\cdot - t)\,\mathrm{d}t, \text{ if } m \geq 2, \tag{2.1}$$

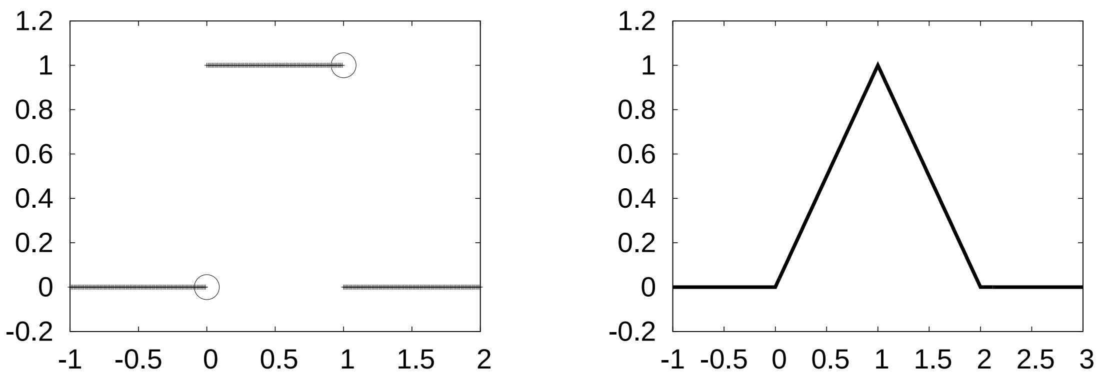
with

$$N_1(x) := \begin{cases} 1 & , \quad x \in [0,1), \\ 0 & , \quad x \in \mathbb{R} \setminus [0,1). \end{cases} \tag{2.2}$$

As an example, equations (2.1) and (2.2) give

$$N_2(x) = \begin{cases} x & , \quad x \in [0,1), \\ 2-x & , \quad x \in [1,2), \\ 0 & , \quad x \in \mathbb{R} \setminus [0,2). \end{cases} \tag{2.3}$$

The B-splines N_1 and N_2 are drawn in Figure 2.1.



(a) $m = 1$ (b) $m = 2$

Figure 2.1: Graph of N_m, with $m = 1, 2$.

Remark 2.1. An equivalent definition (see [dB76]) of the B-splines uses divided differences by means of

$$N_m(x) := m\ (\cdot - x)_+^{m-1}[0, \ldots, m],\ x \in \mathbb{R},\ m \in \mathbb{N}. \tag{2.4}$$

However, in this thesis, we will use the definition (2.1), (2.2) as of B-splines.

The following refinability property can be proved.

Theorem 2.2. *For $m \in \mathbb{N}$, we have that*

$$N_m = \sum_j p_{m,j}\ N_m(2 \cdot -j), \tag{2.5}$$

where

$$\begin{cases} p_{1,0} := 1, \quad p_{1,1} := 1, \quad p_{1,j} := 0 \text{ otherwise}, \\ \\ p_{m,j} := \dfrac{1}{2}\left\{p_{m-1,j} + p_{m-1,j-1}\right\}, \quad j \in \mathbb{Z},\ m \geq 2. \end{cases} \tag{2.6}$$

Proof. Before proving that Theorem 2.2 holds for any integer $m \in \mathbb{N}$, we first prove that the result holds for $m = 1$, and then use an induction on m.

From (2.2), we see that

$$N_1(2x) = \begin{cases} 1, & x \in [0, 1/2), \\ 0, & x \in \mathbb{R} \setminus [0, 1/2), \end{cases}$$

and

$$N_1(2x-1) = \begin{cases} 1, & x \in [1/2, 1), \\ 0, & x \in \mathbb{R} \setminus [1/2, 1), \end{cases}$$

and thus

$$N_1(2x) + N_1(2x-1) = \begin{cases} 1, & x \in [0, 1), \\ 0, & x \in \mathbb{R} \setminus [0, 1). \end{cases} \tag{2.7}$$

It follows from (2.7) and (2.2) that

$$N_1(x) = N_1(2x) + N_1(2x-1), \quad x \in \mathbb{R},$$

which proves that (2.5) and (2.6) hold for $m = 1$.

Proceeding inductively, we now suppose that (p_m, N_m) is a refinement pair for a fixed integer $m \in \mathbb{N}$, and show that (p_{m+1}, N_{m+1}) also forms a refinement pair. Equations (2.1), (2.5) and (2.6) imply that

$$\begin{aligned} N_{m+1}\left(\frac{\cdot}{2}\right) &= \int_0^1 N_m\left(\frac{\cdot}{2} - t\right) \, \mathrm{d}t \\ &= \int_0^1 \sum_j p_{m,j} \, N_m(\cdot - 2t - j) \, \mathrm{d}t \\ &= \sum_j p_{m,j} \, \int_0^1 N_m(\cdot - 2t - j) \, \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \sum_j p_{m,j} \left[\int_0^{\frac{1}{2}} N_m(\cdot - 2t - j)\ \mathrm{d}t + \int_{\frac{1}{2}}^1 N_m(\cdot - 2t - j)\ \mathrm{d}t \right] \\
&= \sum_j p_{m,j} \left[\frac{1}{2} \int_0^1 N_m(\cdot - j - t)\ \mathrm{d}t + \frac{1}{2} \int_0^1 N_m(\cdot - j - 1 - t)\ \mathrm{d}t \right] \\
&= \frac{1}{2} \sum_j p_{m,j} \left[N_{m+1}(\cdot - j) + N_{m+1}(\cdot - j - 1) \right] \\
&= \frac{1}{2} \left[\sum_j p_{m,j}\ N_{m+1}(\cdot - j) + \sum_j p_{m,j}\ N_{m+1}(\cdot - j - 1) \right] \\
&= \frac{1}{2} \left[\sum_j p_{m,j}\ N_{m+1}(\cdot - j) + \sum_j p_{m,j-1}\ N_{m+1}(\cdot - j) \right] \\
&= \sum_j \left[\frac{1}{2} (p_{m,j} + p_{m,j-1}) \right]\ N_{m+1}(\cdot - j) \\
&= \sum_j p_{m+1,j}\ N_{m+1}(\cdot - j),
\end{aligned}
$$

and thus

$$
N_{m+1} = \sum_j p_{m+1,j}\ N_{m+1}(2 \cdot -j), \tag{2.8}
$$

i.e (p_{m+1}, N_{m+1}) is a refinement pair, and thereby completing our inductive proof. □

2.2 Properties of the scalar B-spline functions

We proceed in this section to investigate further properties of the B-splines N_m, for $m \in \mathbb{N}$. We will need the following intermediate result [Theorem 5.2, [dV07b]], in which we use the notation $\mathrm{supp}(f)$ for the support of a function $f : \mathbb{R} \to \mathbb{R}$ which is defined as the closure of the set $\{x \in \mathbb{R} : \ f(x) \neq 0\}$.

Lemma 2.3. *Let ϕ denote a function satisfying the conditions*

$$
\begin{cases}
\phi \in C^l(\mathbb{R}), \\
\mathrm{supp}(\phi) = [0, p],
\end{cases} \tag{2.9}
$$

with $l \in \mathbb{Z}_+ \cup \{-1\}$ and $p \in \mathbb{N}$. Let us define the sequence of functions $(\phi_j)_{j\in\mathbb{Z}_+}$ as

$$\begin{cases} \phi_{j+1} := & \int_0^1 \phi_j(\cdot - t)\ dt,\ j \geq 0, \\ \phi_0 := & \phi. \end{cases} \tag{2.10}$$

We then have

$$\begin{cases} \phi_j \in C^{l+j}(\mathbb{R}), \\ \operatorname{supp}(\phi_j) = [0, p+j], \end{cases} ,\ j \in \mathbb{Z}_+. \tag{2.11}$$

Proof. The proof of Lemma 2.3 is an extended version of the one used in [dV07b]. We shall proceed by induction on j. We know from the assumptions (2.9) that (2.11) holds for $j = 0$. Hence, let us suppose that the first line of (2.11) is true for an arbitrary integer $j \in \mathbb{Z}_+$, and proceed to prove that (2.11) also holds with j replaced by $j+1$. Using equation (2.10), we derive

$$\begin{aligned} \phi_{j+1}(x) &= \int_{x-1}^{x} \phi_j(t)\mathrm{d}t \\ &= \int_0^x \phi_j(t)\ \mathrm{d}t - \int_0^{x-1} \phi_j(t)\ \mathrm{d}t, \quad x \in \mathbb{R}, \end{aligned}$$

which, together with the induction assumption, implies that

$$\phi_{j+1}^{(l+j+1)} = \phi_j^{(l+j)} - \phi_j^{(l+j)}(\cdot - 1).$$

Therefore,

$$\phi_{j+1} \in C^{l+[j+1]}(\mathbb{R}). \tag{2.12}$$

Similarly, suppose that the second line of (2.11) holds for a given $j \in \mathbb{Z}_+$. Now note from (2.12) that ϕ_{j+1} is continuous on $\mathbb{R}$. Moreover, we have by inductive assumption that

$\phi_j(x) = 0,\ x \notin [0, l+j]$. As a consequence,

$$\phi_j(x-t) = 0,\ t \in [0,1],\ x \notin [0, l+j+1]. \tag{2.13}$$

Combining (2.13) with (2.10) gives

$$\phi_{j+1}(x) = 0,\ x \notin [0, l+j+1],$$

which, together with the fact that ϕ_{j+1} is continuous, implies that

$$\phi_{j+1}(x) = 0,\ x \notin (0, l+j+1).$$

It follows that

$$\text{supp}\,(\phi_{j+1}) = [0, l+(j+1)].$$

□

The following properties can be proved.

Theorem 2.4. *For $m \in \mathbb{N}$, we have*

(i) $\text{supp}\ (N_m) = [0, m];$ (2.14)

(ii) $N_m(x) > 0,\ x \in (0, m);$ (2.15)

(iii) $N_m(m - \cdot) = N_m,\ m \geq 2;$ (2.16)

(iv) $\sum_j N_m(x-j) = 1,\ x \in \mathbb{R};$ (2.17)

(v) $\int_{-\infty}^{+\infty} N_m(x)\ dx = 1;$ (2.18)

(vi) $N_m \in C^{m-2}(\mathbb{R});$ (2.19)

(vii) $N'_{m+1} = N_m - N_m(\cdot - 1), \ m \geq 2;$ (2.20)

(viii) $N_m = \dfrac{1}{(m-1)!} \displaystyle\sum_{j=0}^{m} (-1)^j \binom{m}{j} (\cdot - j)_+^{m-1};$ (2.21)

(ix) $N_m(\cdot - j) \in S_m(\mathbb{Z}), \ j \in \mathbb{Z};$ (2.22)

(x) $N_{m+1} = \dfrac{1}{m} \ \{(\cdot)N_m + (m+1-\cdot) \ N_m(\cdot - 1)\}.$ (2.23)

Proof. Most of the following proofs use the same pattern as in [Theorem 1.1, [dV07b]].

We prove (i) by induction on m. First, observe from (2.2) that (i) is true for $m = 1$. Suppose next that (i) holds for a fixed integer $m \geq 1$, that is,

$$\text{supp } (N_m) = [0, m],$$

which, together with (2.1) and Lemma 2.3, yields

$$\text{supp } (N_{m+1}) = [0, m+1].$$

Hence, equation (i) holds for $m + 1$.

To prove (ii), we also use an induction on m, noting from (2.2) that (ii) holds for $m = 1$. Let us now suppose that (ii) holds for a fixed integer $m \geq 1$. We aim to prove that

$$N_{m+1}(x) > 0, \ x \in (0, m+1). \tag{2.24}$$

If $x \in (1, m)$, it follows from (2.1) together with the inductive assumption that $N_{m+1}(x) > 0$.

If $x \in (0, 1]$, then

$$-1 < x - 1 \leq 0 < x \leq 1,$$

which, together with (2.1), gives

$$\begin{aligned}
N_{m+1}(x) &= \int_0^1 N_m(x-t)\ \mathrm{d}t \\
&= \int_{x-1}^{x} N_m(t)\ \mathrm{d}t \\
&= \int_{x-1}^{0} N_m(t)\ \mathrm{d}t + \int_0^x N_m(t)\ \mathrm{d}t \\
&= \int_0^x N_m(t)\ \mathrm{d}t > 0,
\end{aligned}$$

having also used equation (2.14) and the inductive assumption.

Using a similar argument, we can show that, for $x \in [m, m+1)$, we have that $N_{m+1}(x) > 0$. We have therefore proved that (2.24) is true.

The proof of (iii) is by induction on m. Note first from (2.3) that (iii) holds for $m = 2$. Let us assume that (iii) holds for a fixed integer $m \geq 2$, and prove that it still holds for $m+1$. Using the definition (2.1) together with the inductive assumption, we get, for $x \in \mathbb{R}$,

$$\begin{aligned}
N_{m+1}(x) &= \int_0^1 N_m(x-t)\ \mathrm{d}t \\
&= \int_0^1 N_m(m-[x-t])\ \mathrm{d}t \\
&= \int_0^1 N_m(m-x+t)\ \mathrm{d}t \\
&= \int_0^1 N_m(m+1-x+t-1)\ \mathrm{d}t \\
&= \int_0^1 N_m(m+1-x-[1-t])\ \mathrm{d}t \\
&= \int_0^1 N_m(m+1-x-t)\ \mathrm{d}t \\
&= N_{m+1}(m+1-x).
\end{aligned}$$

Our inductive proof is therefore complete.

Again relying on an inductive proof, we first note from (2.2) that (iv) holds for $m = 1$.

Let us then suppose that (iv) holds for a fixed integer $m \in \mathbb{N}$ and prove that it also holds for $m+1$. It follows from (2.1) that, for $x \in \mathbb{R}$, we have

$$\begin{aligned}\sum_j N_{m+1}(x-j) &= \sum_j \int_0^1 N_m(x-j-t)\ \mathrm{d}t \\ &= \int_0^1 \left\{ \sum_j N_m(x-j-t) \right\} \mathrm{d}t \\ &= \int_0^1 1\ \mathrm{d}t = 1,\end{aligned}$$

having also used the inductive assumption. Hence, our inductive proof is complete.

We prove (v) inductively by first observing from (2.2) that (v) is true for $m = 1$. Next, suppose that (v) holds for a fixed integer $m \geq 1$. Our inductive proof will be complete if we can show that

$$\int_{-\infty}^{+\infty} N_{m+1}(x)\ \mathrm{d}x = 1. \tag{2.25}$$

From equations (2.14), (2.1), and the inductive assumption, we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} N_{m+1}(x)\ \mathrm{d}x &= \int_0^{m+1} N_{m+1}(x)\ \mathrm{d}x \\ &= \int_0^{m+1} \left\{ \int_0^1 N_m(x-t)\ \mathrm{d}t \right\}\ \mathrm{d}x \\ &= \int_0^1 \left\{ \int_0^{m+1} N_m(x-t)\ \mathrm{d}x \right\}\ \mathrm{d}t \\ &= \int_0^1 \left\{ \int_{-t}^{m+1-t} N_m(x)\ \mathrm{d}x \right\}\ \mathrm{d}t \\ &= \int_0^1 \left\{ \int_0^{m} N_m(x)\ \mathrm{d}x \right\}\ \mathrm{d}t \\ &= \int_0^1 \left\{ \int_{-\infty}^{+\infty} N_m(x)\ \mathrm{d}x \right\}\ \mathrm{d}t \\ &= \int_0^1 1\ \mathrm{d}t = 1.\end{aligned}$$

We have therefore proved that (2.25) holds.

Observe from (2.2) that (vi) holds for $m = 1$, which, together with (2.1) and Lemma 2.3, implies that (vi) is true for every $m \in \mathbb{N}$.

To prove (vii), observe from (2.1) that

$$\begin{aligned} N_{m+1}(x) &= \int_0^1 N_m(x-t)\ \mathrm{d}t \\ &= \int_{x-1}^x N_m(t)\ \mathrm{d}t \\ &= \int_0^x N_m(t)\ \mathrm{d}t - \int_0^{x-1} N_m(t)\ \mathrm{d}t, \end{aligned}$$

from which we get, for $m \geq 2$,

$$N'_{m+1} = N_m - N_m(\cdot - 1),$$

after having noted also that (2.19) implies $N_m \in C^1(\mathbb{R})$ for $m \geq 3$.

To prove (viii), we first note from (2.2), (1.2) and (1.3) that

$$N_1(x) = x_+^0 - (x-1)_+^0, \quad x \in \mathbb{R},$$

i.e (viii) holds for $m = 1$. Next, let us suppose that (viii) holds for a fixed integer $m \in \mathbb{N}$. Using (2.1), together with the inductive hypothesis, we get

$$\begin{aligned} N_{m+1} &= \int_0^1 N_m(\cdot - t)\ \mathrm{d}t \\ &= \int_0^1 \frac{1}{(m-1)!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (\cdot - j - t)_+^{m-1}\ \mathrm{d}t \\ &= \frac{1}{(m-1)!} \sum_j (-1)^j \binom{m}{j} \int_0^1 (\cdot - j - t)_+^{m-1}\ \mathrm{d}t. \end{aligned} \tag{2.26}$$

Noting that

$$\int_\alpha^\beta t_+^k\ \mathrm{d}t = \frac{\beta_+^{k+1} - \alpha_+^{k+1}}{k+1}, \quad \alpha \in \mathbb{R},\ \beta \in \mathbb{R},\ k \in \mathbb{Z}_+,$$

equations (2.26) and (1.4) yield

$$\begin{aligned} N_{m+1} &= \frac{1}{(m-1)!}\sum_j (-1)^j \binom{m}{j} \frac{(\cdot - j)_+^m - (\cdot - j - 1)_+^m}{m} \\ &= \frac{1}{m!}\left\{\sum_j (-1)^j \binom{m}{j} (\cdot - j)_+^m - \sum_j (-1)^j \binom{m}{j} (\cdot - j - 1)_+^m\right\} \\ &= \frac{1}{m!}\,\Delta\left\{\sum_j (-1)^j \binom{m}{j} (\cdot - j)_+^m\right\}, \end{aligned}$$

which, together with (1.6) and (1.5), gives

$$\begin{aligned} N_{m+1} &= \frac{1}{m!}\,\Delta\left\{\Delta^m (\cdot)_+^m\right\} \\ &= \frac{1}{m!}\,\Delta^{m+1}(\cdot)_+^m \\ &= \frac{1}{m!}\sum_j (-1)^j \binom{m+1}{j} (\cdot - j)_+^m. \end{aligned} \tag{2.27}$$

Therefore, the result (viii) also holds for $m+1$.

We prove (ix) by induction on m, by noting from (2.2) that (ix) holds for $m = 1$. Next, suppose that (ix) holds for a fixed integer $m \geq 1$. From the inductive assumption, we get

$$N_m(\cdot - j)|_{[l,l+1)} \in \pi_{m-1},\ l \in \mathbb{Z},\ j \in \mathbb{Z},$$

which, together with (2.1), implies that

$$N_{m+1}(\cdot - j)|_{[l,l+1)} \in \pi_m,\ l \in \mathbb{Z},\ j \in \mathbb{Z}. \tag{2.28}$$

It follows from (2.28) and (2.19) that $N_{m+1} \in S_{m+1}(\mathbb{Z})$, thereby concluding our inductive proof.

In order to prove the result (x), we shall rely on the fact that

$$x_+^{k+1} = x\ x_+^k,\ x \in \mathbb{R},\ k \in \mathbb{Z}_+. \tag{2.29}$$

Recall from (2.21) and (1.6) that, for $m \geq 1$,

$$N_m = \frac{1}{(m-1)!}\ \Delta^m (\cdot)_+^{m-1}, \tag{2.30}$$

which, together with (2.29) and the definitions

$$\left.\begin{aligned} u(x) &:= x, \\ v(x) &:= x_+^{m-1}, \end{aligned}\right\}\ x \in \mathbb{R}, \tag{2.31}$$

leads to

$$\begin{aligned} N_{m+1} &= \frac{1}{m!}\Delta^{m+1}(uv) \\ &= \frac{1}{m!}\sum_j \binom{m+1}{j}\ \Delta^j u\ \left(\Delta^{m+1-j}\ v\right)(\cdot - j), \end{aligned} \tag{2.32}$$

by virtue of (1.7). Next, noting from (1.4) and (1.5) that

$$\Delta^k u = \begin{cases} u,\ \text{if } k = 0, \\ 1,\ \text{if } k = 1, \\ 0\ \text{ otherwise}, \end{cases}$$

it follows from (2.32) that

$$N_{m+1} = \frac{1}{m!}\left\{u\Delta^{m+1}\ v + (m+1)(\Delta^m\ v)(\cdot - 1)\right\}. \tag{2.33}$$

Observe now from (1.5) and (1.4) that we have

$$
\begin{aligned}
u(\Delta^{m+1}\ v) &= u(\Delta\ \Delta^m\ v) \\
&= u\left\{\Delta^m\ v - (\Delta^m\ v)\,(\cdot - 1)\right\} \\
&= u\Delta^m\ v - u\,(\Delta^m\ v)\,(\cdot - 1).
\end{aligned} \tag{2.34}
$$

Equations (2.33) and (2.34) then give

$$
\begin{aligned}
N_{m+1} &= \frac{1}{m!}\left\{u\Delta^m\ v + (m+1-u)\,(\Delta^m\ v)\,(\cdot - 1)\right\} \\
&= \frac{1}{m}\left\{uN_m + (m+1-u)\ N_m(\cdot - 1)\right\},
\end{aligned}
$$

having also used (2.30) and the second line of (2.31). □

Theorem 2.5. *For $m \in \mathbb{N}$, the set*

$$
\{N_m(\cdot - j):\ \ j \in \mathbb{Z}\} \tag{2.35}
$$

is a basis for $S_m(\mathbb{Z})$. More precisely, for every $s \in S_m(\mathbb{Z})$, there exists a unique sequence $c \in M(\mathbb{Z})$ such that

$$
s = \sum_j c_j\ N_m(\cdot - j). \tag{2.36}
$$

Proof. The following proof is from [Mic95] (see also [Theorem 1.2, [dV07b]]).

To prove Theorem 2.5, we shall prove that the result is true for $m = 1$ and $m = 2$, and then prove inductively that the result holds for $m \in \mathbb{N}$.

Hence, suppose that $m = 1$ and let $s \in S_1(\mathbb{Z})$. Observe that (2.2) yields

$$
N_1(j) = \delta_j,\ \ j \in \mathbb{Z}, \tag{2.37}
$$

where δ is the Kronecker delta sequence defined by (1.1). Introducing the cardinal spline

$s_1 \in S_1(\mathbb{Z})$ by

$$s_1 := \sum_j s(j)\ N_1(\cdot - j),$$

we see that

$$s_1(l) = \sum_j s(j)\ N_1(l-j) = s(l), \quad l \in \mathbb{Z}, \tag{2.38}$$

having also used equation (2.37). It follows from (2.38) that $s_1 = s$. As a consequence,

$$s = \sum_j s(j)\ N_1(\cdot - j), \quad s \in S_1(\mathbb{Z}),$$

and (2.36) is true for $m = 1$, with

$$c_j = s(j),\ j \in \mathbb{Z}. \tag{2.39}$$

To prove that the sequence defined by (2.39) is unique, suppose that $\tilde{c} \in M(\mathbb{Z})$ is an arbitrary sequence for which (2.36) holds, that is,

$$s = \sum_j \tilde{c}_j N_1(\cdot - j).$$

Our proof will be complete if we can show that $\tilde{c} = c$, with c given by (2.39). Using (2.37), we get

$$s(l) = \sum_j \tilde{c}_j\ N_1(l-j) = \tilde{c}_l, \quad l \in \mathbb{Z},$$

thereby proving the uniqueness of the sequence c in (2.39). Therefore, Theorem 2.5 is true for $m = 1$.

In a similar fashion, let us suppose that $m = 2$, and let $s \in S_2(\mathbb{Z})$ be given. We define the cardinal spline $s_2 \in S_2(\mathbb{Z})$ by

$$s_2 := \sum_j s(j+1)\ N_2(\cdot - j). \tag{2.40}$$

Now observe from (2.3) that

$$N_2(j+1) = \delta_j,\ \ j \in \mathbb{Z}, \tag{2.41}$$

which, together with (2.40), implies that

$$s_2(l+1) = \sum_j s(j+1)\ N_2(l+1-j) = s(l+1), \quad l \in \mathbb{Z}. \tag{2.42}$$

For s and s_2 both belong to $S_2(\mathbb{Z})$, they are piecewise linear functions on $\mathbb{R}$, which, together with (2.42), yields $s_2 = s$. As a result, equation (2.36) holds for $m = 2$, with c given by

$$c_j = s(j+1),\ \ j \in \mathbb{Z}. \tag{2.43}$$

To prove that the sequence defined by (2.43) is unique, let $\tilde{c} \in \mathbb{Z}$ be any sequence such that

$$s = \sum_j \tilde{c}_j\ N_2(\cdot - j),$$

and let us show that $\tilde{c} = c$. To this end, note from (2.41) that

$$s(l+1) = \sum_j \tilde{c}_j\ N_2(l+1-j) = \tilde{c}_l, \quad l \in \mathbb{Z},$$

thereby proving the uniqueness of the sequence c in (2.43). It follows that Theorem 2.5 holds for $m = 2$.

Using an induction on m, let us now suppose that the theorem holds for a fixed integer

$m \geq 2$, and let $s \in S_{m+1}(\mathbb{Z})$. Since $m+1 \geq 3$, equations (2.22) and (2.20) give

$$s' \in S_m(\mathbb{Z}). \tag{2.44}$$

It follows from (2.44) and the inductive hypothesis that there exists a unique sequence $q \in M(\mathbb{Z})$ such that

$$s' = \sum_j q_j \; N_m(\cdot - j). \tag{2.45}$$

Let us now define the sequence $r \in M(\mathbb{Z})$ by

$$r_j := \begin{cases} \sum\limits_{k=1}^{j} q_k & , \quad j \geq 1, \\ 0 & , \quad j = 0, \\ -\sum\limits_{k=j+1}^{0} q_k & , \quad j \leq -1, \end{cases} \tag{2.46}$$

so that

$$r_j - r_{j-1} = q_j, \;\; j \in \mathbb{Z}. \tag{2.47}$$

Inserting (2.47) into (2.45), we find that

$$\begin{aligned} s' &= \sum_j \{r_j - r_{j-1}\} \;\; N_m(\cdot - j) \\ &= \sum_j r_j \; N_m(\cdot - j) - \sum_j r_{j-1} \; N_m(\cdot - j) \\ &= \sum_j r_j \; N_m(\cdot - j) - \sum_j r_j \; N_m(\cdot - j - 1) \\ &= \sum_j r_j \;\; \{N_m(\cdot - j) - N_m(\cdot - j - 1)\}, \end{aligned}$$

which, together with (2.20), gives

$$s' = \sum_j r_j \; N'_{m+1}(\cdot - j). \tag{2.48}$$

We deduce from (2.48) that, for $x \in \mathbb{R}$, we have

$$\begin{aligned}\int_0^x s'(t) \; \mathrm{d}t &= \int_0^x \sum_j r_j \; N'_{m+1}(t-j) \; \mathrm{d}t \\ &= \sum_j r_j \int_0^x N'_{m+1}(t-j) \; \mathrm{d}t,\end{aligned}$$

from which it follows that

$$\begin{aligned}s(x) - s(0) &= \sum_j r_j \left\{N_{m+1}(x-j) - N_{m+1}(-j)\right\} \\ &= \sum_j r_j \; N_{m+1}(x-j) - \sum_k r_k \; N_{m+1}(-k), \;\; x \in \mathbb{R}.\end{aligned} \tag{2.49}$$

Using (2.49) and (2.17), we obtain

$$s = \sum_j \left\{ s(0) - \sum_k r_k \; N_{m+1}(-k) + r_j \right\} N_{m+1}(\cdot - j). \tag{2.50}$$

As a result, with the sequence $c \in M(\mathbb{Z})$ defined by

$$c_j := s(0) - \sum_k r_k \; N_{m+1}(-k) + r_j, \;\; j \in \mathbb{Z}, \tag{2.51}$$

we have that

$$s = \sum_j c_j \; N_{m+1}(\cdot - j). \tag{2.52}$$

Hence, it remains to prove that the sequence c in (2.51) is the unique sequence for which

(2.52) holds. To this end, let us suppose that $\tilde{c} \in M(\mathbb{Z})$ is such that

$$s = \sum_j \tilde{c}_j \; N_{m+1}(\cdot - j). \tag{2.53}$$

We want to show that $\tilde{c} = c$. Observe from (2.53) and (2.20) that

$$\begin{aligned} s' &= \sum_j \tilde{c}_j \left\{ N_m(\cdot - j) - N_m(\cdot - j - 1) \right\} \\ &= \sum_j \tilde{c}_j \; N_m(\cdot - j) - \sum_j \tilde{c}_j \; N_m(\cdot - j - 1) \\ &= \sum_j \tilde{c}_j \; N_m(\cdot - j) - \sum_j \tilde{c}_{j-1} \; N_m(\cdot - j) \\ &= \sum_j \left\{ \tilde{c}_j - \tilde{c}_{j-1} \right\} N_m(\cdot - j). \end{aligned} \tag{2.54}$$

According to the inductive assumption, the sequence $q \in M(\mathbb{Z})$ is the unique sequence for which (2.45) is true, which, together with (2.54), yields

$$\tilde{c}_j - \tilde{c}_{j-1} = q_j, \quad j \in \mathbb{Z},$$

and thus,

$$\tilde{c}_j - \tilde{c}_{j-1} = r_j - r_{j-1}, \quad j \in \mathbb{Z}, \tag{2.55}$$

by virtue of (2.47). We claim that

$$\tilde{c}_j = \tilde{c}_0 + r_j, \quad j \in \mathbb{Z}. \tag{2.56}$$

We prove (2.56) by induction by first observing from the middle line of (2.46) that (2.56) holds for $j = 0$. Rewriting (2.55) in the form

$$\tilde{c}_{j+1} = \tilde{c}_j - r_j + r_{j+1}, \; j \in \mathbb{Z},$$

it follows inductively that (2.56) holds for $j \in \mathbb{Z}_+$. In a similar fashion, if we rewrite (2.55) in the form

$$\tilde{c}_{j-1} = \tilde{c}_j - r_j + r_{j-1}, \ j \in \mathbb{Z},$$

it follows inductively that (2.56) holds for every integer $j \leq 0$. We have therefore shown that equation (2.56) is true for $j \in \mathbb{Z}$.

Inserting (2.56) into (2.53), we obtain

$$s = \sum_j \{\tilde{c}_0 + r_j\} N_{m+1}(\cdot - j),$$

which, together with (2.17), yields

$$s = \tilde{c}_0 + \sum_j r_j \ N_{m+1}(\cdot - j), \tag{2.57}$$

and thus,

$$\tilde{c}_0 = s(0) - \sum_k r_k \ N_{m+1}(-k). \tag{2.58}$$

Substituting (2.58) into (2.56), we finally get

$$\tilde{c}_j = s(0) - \sum_k r_k \ N_{m+1}(-k) + r_j = c_j, \ j \in \mathbb{Z}, \tag{2.59}$$

having also used (2.51). In conclusion, $\tilde{c} = c$, and our inductive proof is therefore complete.

$\square$

For a given function $f \in M_0(\mathbb{R})$, we say that f has linearly independent integer shifts on $\mathbb{R}$ if it holds, for a sequence $c \in M(\mathbb{Z})$, that

$$\sum_j c_j \, f(\cdot - j) = 0,$$

implies then

$$c_j = 0, \quad j \in \mathbb{Z}.$$

The following result is then an immediate consequence of Theorem 2.5.

Corollary 2.6. *The m^{th} order cardinal B-spline N_m has linearly independent integer shifts on $\mathbb{R}$.*

Remark 2.7. In Theorem 2.2, the sequence p_m defined by (2.6) is the unique mask corresponding to the scalar B-spline N_m, for $m \in \mathbb{N}$. Indeed, for a fixed $m \in \mathbb{N}$, if we suppose that $q_m \in M(\mathbb{Z})$ is a sequence such that (q_m, N_m) satisfies the refinement equation, then we have

$$\sum_j \{q_{m,j} - p_{m,j}\} \;\; N_m(2x - j) = 0, \quad x \in \mathbb{R},$$

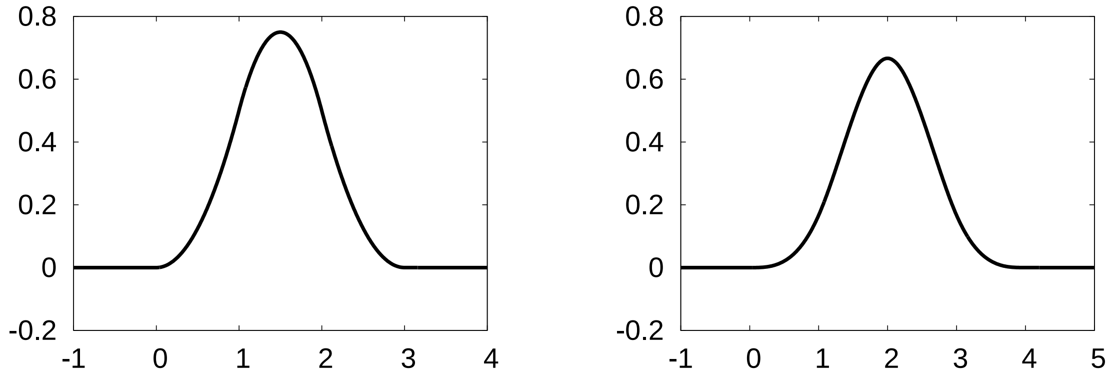
which, together with Theorem 2.5, yields

$$q_{m,j} = p_{m,j}, \;\; j \in \mathbb{Z}.$$

Some examples of cardinal B-spline functions are drawn in Fig. 2.2 below.

2.3 The scalar B-spline masks

The following result gives an explicit formula for the B-spline mask $p_m \in M(\mathbb{Z})$, for $m \in \mathbb{N}$.



(a) $m = 3$ (b) $m = 4$

Figure 2.2: Graph of N_m, with $m = 3, 4$.

Theorem 2.8. *For $m \in \mathbb{N}$, we have for the sequence $\{p_{m,j} : j \in \mathbb{Z}\}$ of Theorem 2.2 the explicit formulation*

$$p_{m,j} = \frac{1}{2^{m-1}} \binom{m}{j}, \quad j \in \mathbb{Z}. \tag{2.60}$$

In particular, the mask symbol P_m is given by

$$P_m(z) = \frac{1}{2^{m-1}} (1+z)^m, \quad z \in \mathbb{C}. \tag{2.61}$$

Proof. Observe that (2.61) is a direct consequence of (2.60) and (1.17).

We prove (2.60) inductively by first noting from (2.6) that (2.60) holds for $m = 1$. Next, suppose that (2.60) holds for a fixed integer $m \in \mathbb{N}$. From (2.6) and the inductive hypothesis, we find that

$$\begin{aligned} p_{m+1,j} &= \frac{1}{2} \left\{ \frac{1}{2^{m-1}} \binom{m}{j} + \frac{1}{2^{m-1}} \binom{m}{j-1} \right\} \\ &= \frac{1}{2^m} \left\{ \binom{m}{j} + \binom{m}{j-1} \right\} \\ &= \frac{1}{2^m} \binom{m+1}{j}, \ j \in \mathbb{Z}, \end{aligned}$$

thereby concluding our inductive proof. $\square$

Chapter 3

Lane-Riesenfeld subdivision

An important theme in this thesis will be that of subdivision, which we proceed to introduce in this chapter. Starting with the de Rham-Chaikin algorithm as an introductory example, we end this chapter by defining the more general Lane-Riesenfeld subdivision algorithm.

3.1 The de Rham-Chaikin algorithm

For $d = 1, 2$ or 3, let $\mathbf{c} : \mathbb{Z} \to \mathbb{R}^d$ be a given vector-valued sequence. We then connect the successive points of $\mathbf{c}$ by straight line segments, thereby forming the initial control polygon generated by the initial control points $\mathbf{c}$. On every segment of the control polygon, we add a new point at each quarter of the length from the endpoints of the segment. The new points thus obtained form a new control polygon. The de Rham-Chaikin algorithm consists in iterating this procedure, generating more and more points which eventually converge to a smooth curve, as analytically proved in Theorem 3.3 below, also noting in Remark 3.2 that the de Rham-Chaikin algorithm is a particular case of the Lane-Riesenfeld subdivision algorithm. More formally, the de Rham-Chaikin algorithm can be described as follows:

$$\mathbf{c}^{(0)} = \mathbf{c}, \qquad \left.\begin{aligned} \mathbf{c}_{2j}^{(r+1)} &= \frac{1}{4}\mathbf{c}_j^{(r)} + \frac{3}{4}\mathbf{c}_{j-1}^{(r)}, \\ \mathbf{c}_{2j+1}^{(r+1)} &= \frac{3}{4}\mathbf{c}_j^{(r)} + \frac{1}{4}\mathbf{c}_{j-1}^{(r)}, \end{aligned}\right\} \quad j \in \mathbb{Z},\ r \in \mathbb{Z}_+, \tag{3.1}$$

where the points $\left\{\mathbf{c}_j^{(r)} :\ j \in \mathbb{Z}\right\}$ are the new points computed at each level $r \in \mathbb{Z}_+$ of the algorithm.

In Figure 3.1, the de Rham-Chaikin algorithm is applied to the sequence $\mathbf{c}$ defined by

$$\begin{cases} \mathbf{c}_0 = (2,0)^T, \quad \mathbf{c}_1 = (1,1)^T, \quad \mathbf{c}_2 = (1,2)^T, \\ \mathbf{c}_3 = (2,3)^T, \quad \mathbf{c}_4 = (3,2)^T, \quad \mathbf{c}_5 = (3,1)^T, \\ \mathbf{c}_6 = (2,0)^T. \end{cases} \tag{3.2}$$

Remark 3.1. In practice, working with a finitely supported sequence often leads to "edge" artifacts near the endpoints, should we define $\mathbf{c}_j = \mathbf{0}$ for $j \notin \{0, \ldots, 6\}$. To generate Figure 3.1, we used instead the periodic extension given by

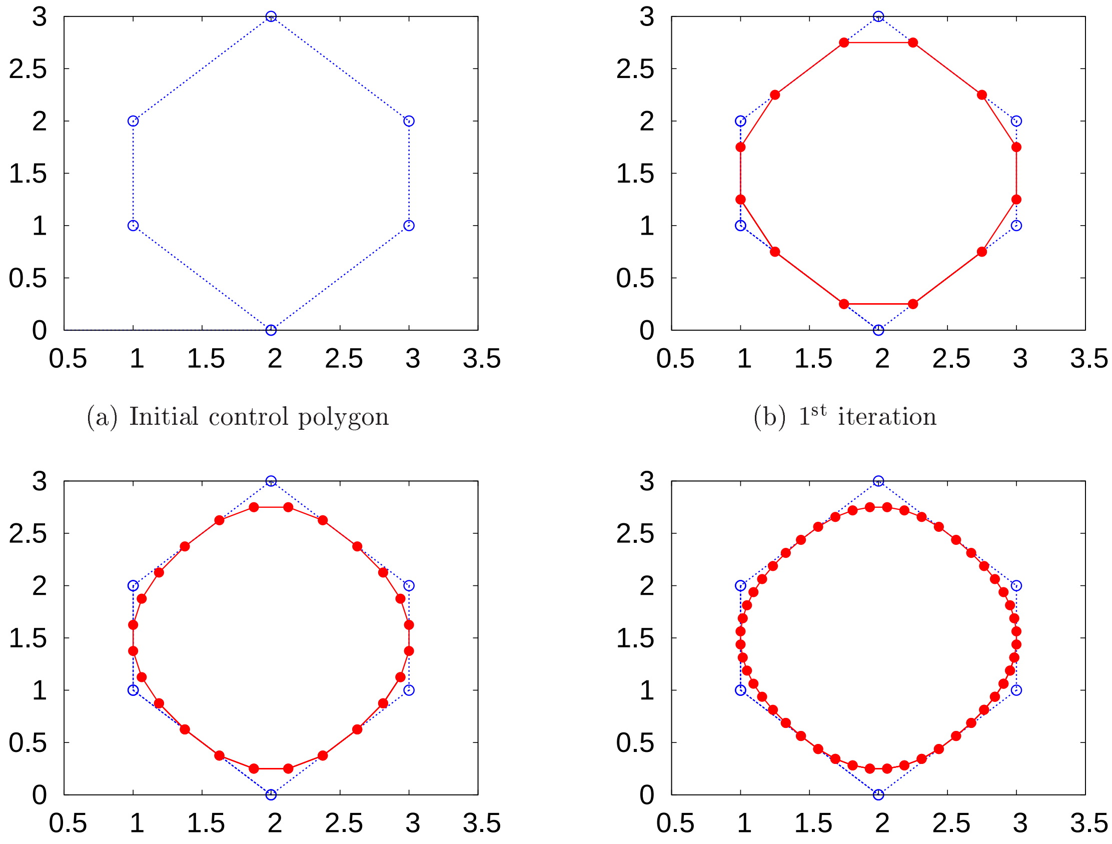
$$\mathbf{c}_j = \mathbf{c}_{j+6}, \quad j \in \mathbb{Z}. \tag{3.3}$$

3.2 The Lane-Riesenfeld algorithm

Definition

We use the same notations as in Section 3.1. For any integer $m \geq 2$, the Lane-Riesenfeld algorithm is defined by

$$\begin{cases} \mathbf{c}^{(0)} &= \mathbf{c}, \\ \mathbf{c}_j^{(r+1)} &= \sum_k p_{m,j-2k}\, \mathbf{c}_k^{(r)},\ j \in \mathbb{Z},\ r \in \mathbb{Z}_+, \end{cases} \tag{3.4}$$



(a) Initial control polygon

(b) 1^{st} iteration

(c) 2^{nd} iteration

(d) 3^{rd} iteration

Figure 3.1: The de Rham-Chaikin subdivision algorithm

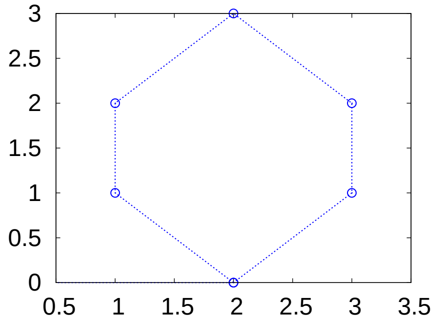
where the sequence p_m is given by (2.6), or, equivalently, (2.60).

Remark 3.2. Observe from (3.1) and (2.60) that an equivalent formulation of the de Rham-Chaikin algorithm is

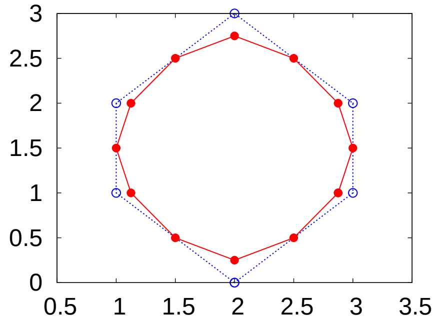
$$\begin{cases} \mathbf{c}^{(0)} & = \mathbf{c}, \\ \mathbf{c}_j^{(r+1)} & = \sum_k p_{3,j-2k}\, \mathbf{c}_k^{(r)}, \;\; j \in \mathbb{Z}, \;\; r \in \mathbb{Z}_+, \end{cases}$$

which, together with (3.4), implies that the de Rham-Chaikin algorithm is the special case $m = 3$ of the Lane-Riesenfeld algorithm.

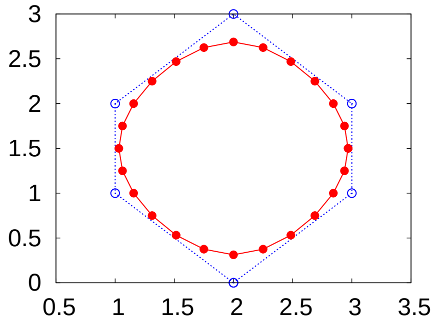
In Figures 3.2 and 3.3, we apply the Lane-Riesenfeld algorithm to the sequence $\mathbf{c}$ defined by (3.2), (3.3), with $m = 4$ and $m = 5$.



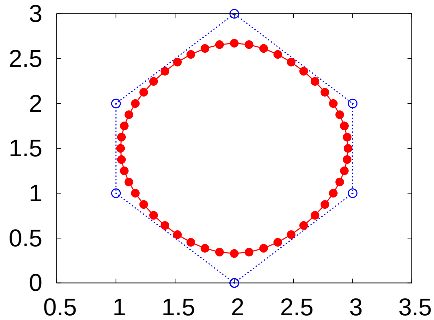
(a) Initial control polygon



(b) 1st iteration

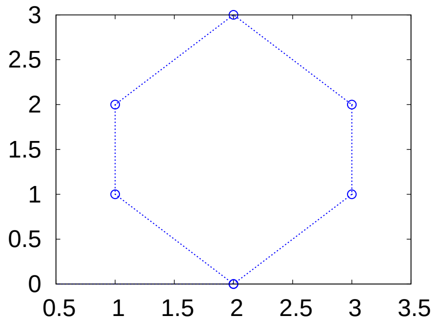


(c) 2nd iteration

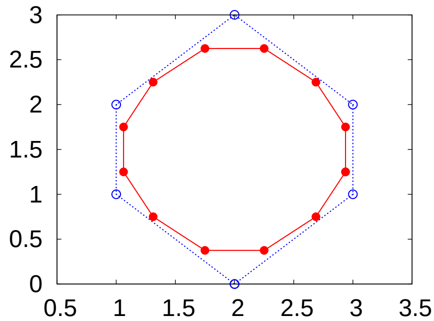


(d) 3rd iteration

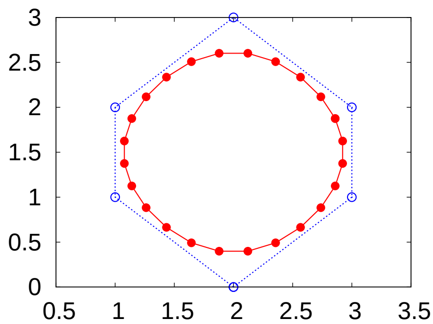
Figure 3.2: The Lane-Riesenfeld subdivision algorithm with $m = 4$



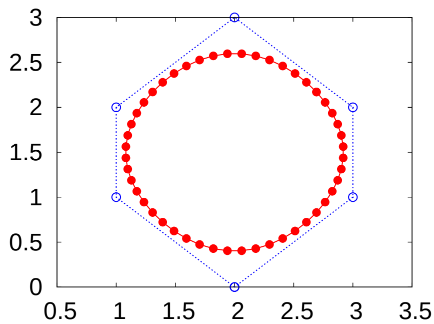
(a) Initial control polygon



(b) 1st iteration



(c) 2nd iteration



(d) 3rd iteration

Figure 3.3: The Lane-Riesenfeld subdivision algorithm with $m = 5$

Convergence result

Let $\mathbf{v} = (v_0, \ldots, v_d)^T$ denote a vector in $\mathbb{R}^d$, for $d = 1, 2$ or 3. We define

$$\|\mathbf{v}\|_{E,d} := \sqrt{\sum_{l=0}^{d-1} v_l^2}. \tag{3.5}$$

For a given sequence $\mathbf{c} : \mathbb{Z} \to \mathbb{R}^d$, we define $\|\mathbf{c}\|_{d,\infty}$ by

$$\|\mathbf{c}\|_{d,\infty} := \sup_j \|\mathbf{c_j}\|_{E,d}. \tag{3.6}$$

The following result can be proved.

Theorem 3.3. *Let $\mathbf{c} : \mathbb{Z} \to \mathbb{R}^d$ be a sequence such that*

$$\|\Delta\mathbf{c}\|_{d,\infty} < \infty. \tag{3.7}$$

Then, for any integer $m \geq 2$, the function $\mathbf{\Phi_m} = \mathbf{\Phi_{m,c}}$ defined by

$$\mathbf{\Phi_m} := \sum_j \mathbf{c}_j \; N_m(\cdot - j), \tag{3.8}$$

is such that

$$\left\| \mathbf{\Phi_m}\left(\frac{\cdot}{2^r}\right) - \mathbf{c}^{(r)}_{(\cdot - 1)} \right\|_{d,\infty} \leq K_m^{(r)} \; \|\Delta\mathbf{c}\|_{d,\infty}, \quad r \in \mathbb{Z}_+, \tag{3.9}$$

with

$$K_m^{(r)} = \frac{m-2}{2^r}, \quad r \in \mathbb{Z}_+. \tag{3.10}$$

Remark 3.4. Observe from (2.19) that the function $\mathbf{\Phi_m}$ given by (3.8) is in $C^{m-2}(\mathbb{R})$, for $m \geq 2$.

Proof of Theorem 3.3. The following proof is based on [Theorem 3.2, [Goo00]].

In the sequel, we fix the integer $m \geq 2$. From equations (3.8), (3.4) and (2.5), we get

$$\begin{aligned}
\mathbf{\Phi_m}\left(\frac{j}{2^r}\right) &= \sum_k \mathbf{c}_k^{(0)}\ N_m\left(\frac{j}{2^r}-k\right) \\
&= \sum_k \mathbf{c}_k^{(0)}\ \sum_l p_{m,l}\ N_m\left(\frac{j}{2^{r-1}}-2k-l\right) \\
&= \sum_k \mathbf{c}_k^{(0)}\ \sum_l p_{m,l-2k}\ N_m\left(\frac{j}{2^{r-1}}-l\right) \\
&= \sum_l \left\{\sum_k p_{m,l-2k}\ \mathbf{c}_k^{(0)}\right\} N_m\left(\frac{j}{2^{r-1}}-l\right) \\
&= \sum_l \mathbf{c}_l^{(1)}\ N_m\left(\frac{j}{2^{r-1}}-l\right) \\
&= \qquad \ldots \\
&= \sum_l \mathbf{c}_l^{(r)}\ N_m\left(j-l\right) \\
&= \sum \mathbf{c}_{j-l}^{(r)}\ N_m(l), \quad j \in \mathbb{Z},\ r \in \mathbb{Z}_+,
\end{aligned}$$

which yields

$$\mathbf{\Phi_m}\left(\frac{j}{2^r}\right) - \mathbf{c}_{j-1}^{(r)} = \sum_k \left\{\mathbf{c}_{j-k}^{(r)} - \mathbf{c}_{j-1}^{(r)}\right\} N_m(k), \quad j \in \mathbb{Z},\ r \in \mathbb{Z}_+, \tag{3.11}$$

by virtue of (2.17). Combining (3.11) and (2.14), together with the fact that $m \geq 2$, we find that

$$\begin{aligned}
\left\|\mathbf{\Phi_m}\left(\frac{j}{2^r}\right) - \mathbf{c}_{j-1}^{(r)}\right\|_{E,d} &= \left\|\sum_{k=1}^{m-1} \left\{\mathbf{c}_{j-k}^{(r)} - \mathbf{c}_{j-1}^{(r)}\right\} N_m(k)\right\|_{E,d} \\
&\leq \sum_{k=1}^{m-1} \left\|\mathbf{c}_{j-k}^{(r)} - \mathbf{c}_{j-1}^{(r)}\right\|_{E,d} N_m(k), \quad j \in \mathbb{Z},\ r \in \mathbb{Z}_+,
\end{aligned} \tag{3.12}$$

having also used the positivity property (2.15). Now observe that, for $r \in \mathbb{Z}_+$,

$$\mathbf{c}_{j-k}^{(r)} - \mathbf{c}_{j-1}^{(r)} = \sum_{l=j-k}^{j-2} \left(\mathbf{c}_l^{(r)} - \mathbf{c}_{l+1}^{(r)}\right), \quad j \in \mathbb{Z},\ k = 1, \ldots, m-1,$$

which, together with (1.4), yields, for $r \in \mathbb{Z}_+$,

$$\begin{aligned} \left\| \mathbf{c}^{(r)}_{j-k} - \mathbf{c}^{(r)}_{j-1} \right\|_{E,d} &\leq (k-1) \left\| \Delta \mathbf{c}^{(r)} \right\|_{d,\infty} \\ &\leq (m-2) \left\| \Delta \mathbf{c}^{(r)} \right\|_{d,\infty}, \quad j \in \mathbb{Z},\ k = 1, \ldots, m-1. \end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain, for $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$,

$$\begin{aligned} \left\| \mathbf{\Phi_m}\left(\frac{j}{2^r}\right) - \mathbf{c}^{(r)}_j \right\|_{E,d} &\leq (m-2) \left\| \Delta \mathbf{c}^{(r)} \right\|_{d,\infty} \sum_{k=1}^{m-1} N_m(k) \\ &= (m-2) \left\| \Delta \mathbf{c}^{(r)} \right\|_{d,\infty}, \end{aligned} \tag{3.14}$$

again using also equation (2.17). We claim that

$$\left\| \Delta \mathbf{c}^{(r)} \right\|_{d,\infty} \leq \frac{1}{2^r} \ \| \Delta \mathbf{c} \|_{d,\infty}, \ r \in \mathbb{Z}_+, \tag{3.15}$$

which, together with (3.10) and (3.14), yields the desired result (3.9).

We prove (3.15) inductively by first noting from (3.4) that (3.15) trivially holds for $r = 0$. Suppose next that (3.15) holds for a fixed integer $r \geq 0$. From (1.4), (3.4) and (2.6), we find that

$$\begin{aligned} \Delta \mathbf{c}^{(r+1)}_j &= \mathbf{c}^{(r+1)}_j - \mathbf{c}^{(r+1)}_{j-1} \\ &= \sum_k p_{m,j-2k} \ \mathbf{c}^{(r)}_k - \sum_k p_{m,(j-1)-2k} \ \mathbf{c}^{(r)}_k \\ &= \sum_k \frac{1}{2} \left\{ p_{m-1,j-2k} + p_{m-1,j-2k-1} \right\} \ \mathbf{c}^{(r)}_k - \sum_k \frac{1}{2} \left\{ p_{m-1,j-1-2k} + p_{m-1,j-2-2k} \right\} \ \mathbf{c}^{(r)}_k \\ &= \sum_k \frac{1}{2} \ p_{m-1,j-2k} \ \mathbf{c}^{(r)}_k - \sum_k \frac{1}{2} \ p_{m-1,j-2(1+k)} \ \mathbf{c}^{(r)}_k \\ &= \sum \frac{1}{2} \ p_{m-1,j-2k} \ \mathbf{c}^{(r)}_k - \sum_k \frac{1}{2} \ p_{m-1,j-2k} \ \mathbf{c}^{(r)}_{k-1} \end{aligned}$$

$$= \sum_k \frac{1}{2}\, p_{m-1,j-2k} \, \left\{ \mathbf{c}_k^{(r)} - \mathbf{c}_{k-1}^{(r)} \right\}$$
$$= \sum_k \frac{1}{2}\, p_{m-1,j-2k} \, \Delta\mathbf{c}_k^{(r)}, \ j \in \mathbb{Z}, \ r \in \mathbb{Z}_+,$$

which implies that

$$\left\| \Delta\mathbf{c}_j^{(r+1)} \right\|_{E,d} \leq \frac{1}{2} \left\| \Delta\mathbf{c}^{(r)} \right\|_{d,\infty} \sum_k p_{m-1,j-2k}, \quad j \in \mathbb{Z}, \ r \in \mathbb{Z}_+, \tag{3.16}$$

since (2.60) shows that $p_{m,j} \geq 0, \ j \in \mathbb{Z}$. Observe now from (3.16) that our inductive proof of (3.15) will be complete if we can show that

$$\sum_k p_{m-1,j-2k} = 1, \ j \in \mathbb{Z}. \tag{3.17}$$

To this end, note that, for every fixed $j \in \mathbb{Z}$, the number $j - 2k$ is either odd or even for every $k \in \mathbb{Z}$. More precisely,

$$\sum_k p_{m-1,j-2k} = \begin{cases} \sum_l p_{m-1,2l}, \text{ if } j \text{ is even}, \\ \sum_l p_{m-1,2l+1}, \quad \text{if } j \text{ is odd}. \end{cases}$$

Therefore, for (3.17) to be true, it is sufficient to have that

$$\sum_l p_{m-1,2l} = \sum_l p_{m-1,2l+1} = 1. \tag{3.18}$$

To see that (3.18) holds, observe first from (2.60) that

$$\begin{aligned} \sum_l p_{m-1,l} &= \frac{1}{2^{m-2}} \sum_l \binom{m-1}{l} \\ &= \frac{1}{2^{m-2}} \sum_{l=0}^{m-1} \binom{m-1}{l} \\ &= \frac{1}{2^{m-2}} (1+1)^{m-1} = 2. \end{aligned} \tag{3.19}$$

Similarly, we get

$$
\begin{aligned}
\sum_l p_{m-1,2l} - \sum_l p_{m-1,2l+1} &= \sum_l (-1)^l \; p_{m-1,l} \\
&= \frac{1}{2^{m-2}} \sum_l (-1)^l \binom{m-1}{l} \\
&= \frac{1}{2^{m-2}} \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} \\
&= \frac{1}{2^{m-2}} (1-1)^{m-1} = 0. \qquad (3.20)
\end{aligned}
$$

Equation (3.18) is now a direct consequence of (3.19) and (3.20), together with the fact that

$$\sum_l p_{m-1,l} = \sum_l p_{m-1,2l} + \sum_l p_{m-1,2l+1}.$$

□

Chapter 4

Vector refinable splines

In this chapter, we aim to extend the scalar definition of the B-splines to the vector setting, and investigate to which extent the properties of the scalar B-splines can be generalized.

4.1 Definition and refinability

Constructing our vector splines analogously to (2.1), we recursively define, for an integer $m \geq 2$, the vector sequence $\left\{\tilde{\mathbf{N}}_{\mathbf{m}} : \ m \in \mathbb{N}\right\}$ by

$$\tilde{\mathbf{N}}_{\mathbf{m}} := \int_0^1 \tilde{\mathbf{N}}_{\mathbf{m-1}}(\cdot - t)\ \mathrm{d}t,\ m \geq 3, \tag{4.1}$$

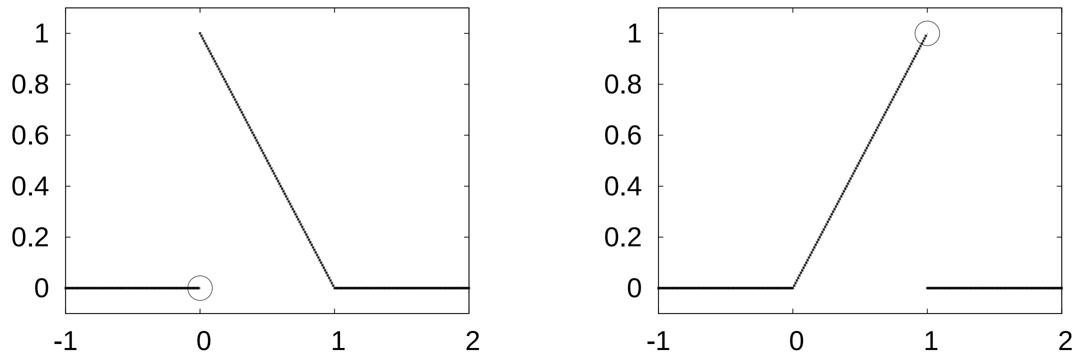
where $\tilde{\mathbf{N}}_{\mathbf{2}}$ is given by

$$\tilde{\mathbf{N}}_{\mathbf{2}}(x) := \begin{cases} \begin{bmatrix} 1-x \\ x \end{bmatrix}, & 0 \leq x < 1, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{4.2}$$

Notation 4.1. For the vector splines defined by (4.1), we introduce the notation

$$\tilde{\mathbf{N}}_{\mathbf{m}} = (\tilde{N}_{m,0}, \tilde{N}_{m,1})^{\mathrm{T}}, \; m \geq 2. \tag{4.3}$$

The graph of $\tilde{\mathbf{N}}_{\mathbf{2}}$ is drawn in Figure 4.1.



(a) $\tilde{N}_{2,0}$ (b) $\tilde{N}_{2,1}$

Figure 4.1: Graph of $\tilde{\mathbf{N}}_{\mathbf{2}}$

Example 4.2. An equivalent formulation of the recursion formula (4.1) is

$$\tilde{\mathbf{N}}_{\mathbf{m}}(x) := \int_{x-1}^{x} \tilde{\mathbf{N}}_{\mathbf{m-1}}(t) \, \mathrm{d}t, \; x \in \mathbb{R}, \; m \geq 3. \tag{4.4}$$

Let us now compute $\tilde{N}_{3,0}$. Observe from (4.2) that

$$\tilde{\mathbf{N}}_{\mathbf{2}}(x) = \mathbf{0}, \; x \in \mathbb{R} \setminus [0, 1),$$

which, together with (4.4), yields

$$\tilde{\mathbf{N}}_{\mathbf{3}}(x) = \mathbf{0}, \; x \in \mathbb{R} \setminus [0, 2), \tag{4.5}$$

which implies that

$$\tilde{N}_{3,0}(x) = 0, \; x \in \mathbb{R} \setminus [0, 2). \tag{4.6}$$

Therefore, it remains to compute $N_{3,0}(x)$, for $x \in [0,2)$. To this end, we successively consider the cases $x \in [0,1)$ and $x \in [1,2)$.

Let us then fix $x \in [0,1)$, so that $-1 \leq x-1 < 0$. Using (4.4), we then get

$$\begin{aligned}\tilde{N}_{3,0}(x) &= \int_{x-1}^{x} \tilde{N}_{2,0}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{0} \tilde{N}_{2,0}(t)\ \mathrm{d}t + \int_{0}^{x} \tilde{N}_{2,0}(t)\ \mathrm{d}t,\end{aligned}$$

which, together with (4.2) and the fact that $0 \leq x < 1$, gives

$$\begin{aligned}\tilde{N}_{3,0}(x) &= \int_{0}^{x} \tilde{N}_{2,0}(t)\ \mathrm{d}t \\ &= \int_{0}^{x} (1-t)\ \mathrm{d}t \\ &= \frac{1}{2}\ x\ (2-x).\end{aligned} \tag{4.7}$$

Next, let us fix $x \in [1,2)$, so that $0 \leq x-1 < 1$. Using (4.4), we then obtain

$$\begin{aligned}\tilde{N}_{3,0}(x) &= \int_{x-1}^{x} \tilde{N}_{2,0}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{1} \tilde{N}_{2,0}(t)\ \mathrm{d}t + \int_{1}^{x} \tilde{N}_{2,0}(t)\ \mathrm{d}t,\end{aligned}$$

which, together with (4.2) and the fact that $0 \leq x-1 < 1$, implies that

$$\begin{aligned}\tilde{N}_{3,0}(x) &= \int_{x-1}^{1} \tilde{N}_{2,0}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{1} (1-t)\ \mathrm{d}t \\ &= \frac{1}{2}\ (x-2)^2.\end{aligned} \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) yields

$$\tilde{N}_{3,0}(x) = \begin{cases} x\,(2-x)/2 & , \quad x \in [0,1), \\ (x-2)^2/2 & , \quad x \in [1,2), \\ 0 & , \quad x \in \mathbb{R} \setminus [0,2), \end{cases} \tag{4.9}$$

Similarly, it follows from (4.5) that

$$\tilde{N}_{3,1}(x) = 0, \;\; x \in \mathbb{R} \setminus [0,2). \tag{4.10}$$

Therefore, it remains to compute $N_{3,1}(x)$, for $x \in [0,2)$. To this end, we successively consider the cases $x \in [0,1)$ and $x \in [1,2)$. Let us then fix $x \in [0,1)$, so that

$$-1 \leq x - 1 < 0.$$

Using (4.4), we then get

$$\begin{aligned} \tilde{N}_{3,1}(x) &= \int_{x-1}^{x} \tilde{N}_{2,1}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{0} \tilde{N}_{2,1}(t)\ \mathrm{d}t + \int_{0}^{x} \tilde{N}_{2,1}(t)\ \mathrm{d}t, \end{aligned}$$

which, together with (4.2) and the fact that $0 \leq x < 1$, gives

$$\begin{aligned} \tilde{N}_{3,1}(x) &= \int_{0}^{x} \tilde{N}_{2,1}(t)\ \mathrm{d}t \\ &= \int_{0}^{x} t\ \mathrm{d}t \\ &= \frac{1}{2}\ x^2. \end{aligned} \tag{4.11}$$

Next, let us fix $x \in [1,2)$, so that $0 \leq x-1 < 1$. Using (4.4), we then obtain

$$\begin{aligned}\tilde{N}_{3,1}(x) &= \int_{x-1}^{x} \tilde{N}_{2,1}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{1} \tilde{N}_{2,1}(t)\ \mathrm{d}t + \int_{1}^{x} \tilde{N}_{2,1}(t)\ \mathrm{d}t,\end{aligned}$$

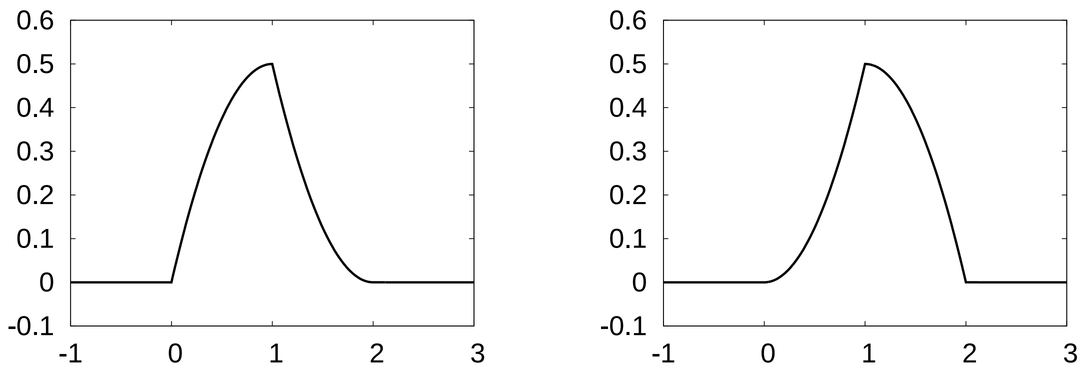
which, together with (4.2) and the fact that $0 \leq x-1 < 1$, implies that

$$\begin{aligned}\tilde{N}_{3,1}(x) &= \int_{x-1}^{1} \tilde{N}_{2,1}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{1} t\ \mathrm{d}t \\ &= \frac{1}{2}\ x\ (2-x).\end{aligned} \tag{4.12}$$

Combining (4.10), (4.11) and (4.12) yields

$$\tilde{N}_{3,1}(x) = \begin{cases} x^2/2 & ,\quad x \in [0,1), \\ x(2-x)/2 & ,\quad x \in [1,2), \\ 0 & ,\quad x \in \mathbb{R} \setminus [0,2), \end{cases} \tag{4.13}$$

The graph of $\tilde{\mathbf{N}}_\mathbf{3}$ is drawn in Figure 4.2.



(a) $\tilde{N}_{3,0}$ (b) $\tilde{N}_{3,1}$

Figure 4.2: Graph of $\tilde{\mathbf{N}}_\mathbf{3}$

In a similar fashion, let us compute $\tilde{N}_{4,0}$. According to (4.9) and (4.13), we have

$$\tilde{\mathbf{N}}_{\mathbf{3}}(x) = \mathbf{0},\ x \in \mathbb{R} \setminus [0,2),$$

which, by virtue of (4.1), yields

$$\tilde{\mathbf{N}}_{\mathbf{4}}(x) = \mathbf{0},\ x \in \mathbb{R} \setminus [0,3). \tag{4.14}$$

It follows from (4.14) that

$$\tilde{N}_{4,0}(x) = 0,\ x \in \mathbb{R} \setminus [0,3). \tag{4.15}$$

Therefore, it remains to compute $\tilde{N}_{4,0}(x)$, for $x \in [0,3)$. To this end, we successively consider the cases $x \in [0,1)$, $x \in [1,2)$ and $x \in [2,3)$.

Hence, let us fix $x \in [0,1)$, so that $-1 \leq x-1 < 0$. According to (4.4), we then have

$$\begin{aligned}\tilde{N}_{4,0}(x) &= \int_{x-1}^{x} \tilde{N}_{3,0}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{0} \tilde{N}_{3,0}(t)\ \mathrm{d}t + \int_{0}^{x} \tilde{N}_{3,0}(t)\ \mathrm{d}t,\end{aligned}$$

which, together with (4.9) and the fact that $0 \leq x < 1$, gives

$$\begin{aligned}\tilde{N}_{4,0}(x) &= \int_{0}^{x} \tilde{N}_{3,0}(t)\ \mathrm{d}t \\ &= \int_{0}^{x} \frac{1}{2}\ t\ (2-t)\ \mathrm{d}t \\ &= -\frac{1}{6}\ x^2\ (x-3).\end{aligned} \tag{4.16}$$

Let us now fix $x \in [1,2)$, so that $0 \leq x-1 < 1$. According to (4.4), we then get

$$\begin{aligned}\tilde{N}_{4,0}(x) &= \int_{x-1}^{x} \tilde{N}_{3,0}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{1} \tilde{N}_{3,0}(t)\ \mathrm{d}t + \int_{1}^{x} \tilde{N}_{3,0}(t)\ \mathrm{d}t,\end{aligned}$$

which, together with (4.9), yields

$$\begin{aligned}\tilde{N}_{4,0}(x) &= \int_{x-1}^{1} \tilde{N}_{3,0}(t)\ \mathrm{d}t + \int_{1}^{x} \tilde{N}_{3,0}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{1} \frac{1}{2}\ t\ (2-t)\ \mathrm{d}t + \int_{1}^{x} \frac{1}{2}\ (t-2)^2\ \mathrm{d}t \\ &= \frac{1}{6}\ (x-3)\ (2x^2-6x+3),\end{aligned} \tag{4.17}$$

by virtue of the fact that $1 \leq x < 2$ and $0 \leq x-1 < 1$.

Next, let us fix $x \in [2,3)$, so that $1 \leq x-1 < 2$. According to (4.4), we then have

$$\begin{aligned}\tilde{N}_{4,0}(x) &= \int_{x-1}^{x} \tilde{N}_{3,0}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{2} \tilde{N}_{3,0}(t)\ \mathrm{d}t + \int_{2}^{x} \tilde{N}_{3,0}(t)\ \mathrm{d}t,\end{aligned}$$

which, together with (4.9), gives

$$\begin{aligned}\tilde{N}_{4,0}(x) &= \int_{x-1}^{2} \tilde{N}_{3,0}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{2} \frac{1}{2}\ (t-2)^2\ \mathrm{d}t \\ &= -\frac{1}{6}\ (x-3)^3,\end{aligned} \tag{4.18}$$

by virtue of the fact that $1 \leq x-1 < 2$.

Combining (4.15), (4.16), (4.17) and (4.18), we find that

$$\tilde{N}_{4,0}(x) = \begin{cases} -x^2(x-3)/6 & , \quad x \in [0,1), \\ (x-3)\ (2x^2-6x+3)/6 & , \quad x \in [1,2), \\ -(x-3)^3/6 & , \quad x \in [2,3), \\ 0 & , \quad x \in \mathbb{R} \setminus [0,3). \end{cases} \tag{4.19}$$

Let us now compute $\tilde{N}_{4,1}(x)$, for $x \in \mathbb{R}$. It follows from (4.14) that

$$\tilde{N}_{4,1}(x) = 0, \ \ x \in \mathbb{R} \setminus [0,3). \tag{4.20}$$

Therefore, it remains to compute $\tilde{N}_{4,1}(x)$, for $x \in [0,3)$. To this end, we successively consider the cases $x \in [0,1)$, $x \in [1,2)$ and $x \in [2,3)$.

Hence, let us fix $x \in [0,1)$, so that $-1 \leq x-1 < 0$. According to (4.4), we then have

$$\begin{aligned} \tilde{N}_{4,1}(x) &= \int_{x-1}^{x} \tilde{N}_{3,1}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{0} \tilde{N}_{3,1}(t)\ \mathrm{d}t + \int_{0}^{x} \tilde{N}_{3,1}(t)\ \mathrm{d}t, \end{aligned}$$

which, together with (4.13) and the fact that $0 \leq x < 1$, gives

$$\begin{aligned} \tilde{N}_{4,1}(x) &= \int_{0}^{x} \tilde{N}_{3,1}(t)\ \mathrm{d}t \\ &= \int_{0}^{x} \frac{1}{2}\ t^2\ \mathrm{d}t \\ &= \frac{1}{6}\ x^3. \end{aligned} \tag{4.21}$$

Let us now fix $x \in [1,2)$, so that $0 \leq x-1 < 1$. According to (4.4), we then get

$$\begin{aligned} \tilde{N}_{4,1}(x) &= \int_{x-1}^{x} \tilde{N}_{3,1}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{1} \tilde{N}_{3,1}(t)\ \mathrm{d}t + \int_{1}^{x} \tilde{N}_{3,1}(t)\ \mathrm{d}t, \end{aligned}$$

which, together with (4.13), yields

$$\begin{aligned}\tilde{N}_{4,1}(x) &= \int_{x-1}^{1} \tilde{N}_{3,1}(t)\ \mathrm{d}t + \int_{1}^{x} \tilde{N}_{3,1}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{1} \frac{1}{2}\ t^2\ \mathrm{d}t + \int_{1}^{x} \frac{1}{2}\ t\ (2-t)\ \mathrm{d}t \\ &= -\frac{1}{6}\ x\ (2x^2 - 6x + 3), \end{aligned} \tag{4.22}$$

by virtue of the fact that $1 \leq x < 2$ and $0 \leq x - 1 < 1$.

Next, let us fix $x \in [2, 3)$, so that $1 \leq x - 1 < 2$. According to (4.4), we then have

$$\begin{aligned}\tilde{N}_{4,1}(x) &= \int_{x-1}^{x} \tilde{N}_{3,1}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{2} \tilde{N}_{3,1}(t)\ \mathrm{d}t + \int_{2}^{x} \tilde{N}_{3,1}(t)\ \mathrm{d}t,\end{aligned}$$

which, together with (4.9), gives

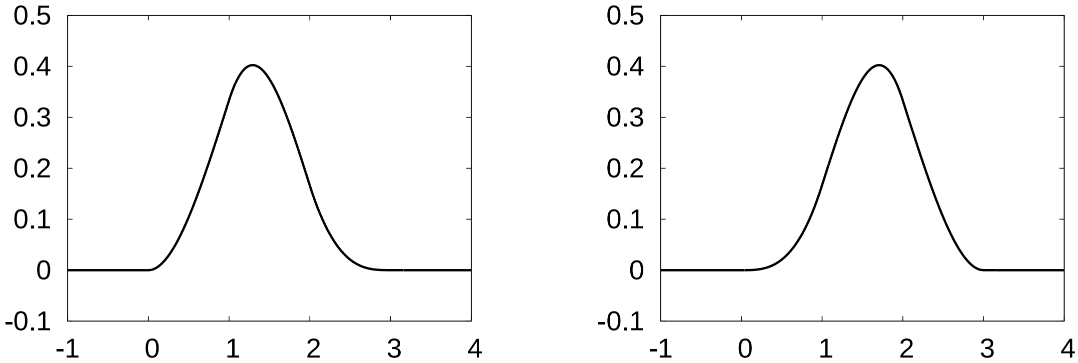
$$\begin{aligned}\tilde{N}_{4,1}(x) &= \int_{x-1}^{2} \tilde{N}_{3,1}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{2} \frac{1}{2}\ t\ (2-t)\ \mathrm{d}t \\ &= \frac{1}{6}\ x\ (x-3)^2,\end{aligned} \tag{4.23}$$

by virtue of the fact that $1 \leq x - 1 < 2$.

Combining (4.20), (4.21), (4.22) and (4.23), we find that

$$\tilde{N}_{4,1}(x) = \begin{cases} x^3/6 & ,\quad x \in [0,1), \\ -x\ (2x^2 - 6x + 3)/6 & ,\quad x \in [1,2), \\ x\ (x-3)^2/6 & ,\quad x \in [2,3), \\ 0 & ,\quad x \in \mathbb{R} \setminus [0,3). \end{cases} \tag{4.24}$$

The graph of $\tilde{\mathbf{N}}_{\mathbf{4}}$ is drawn in Figure 4.3.



(a) $\tilde{N}_{4,0}$ (b) $\tilde{N}_{4,1}$

Figure 4.3: Graph of $\tilde{\mathbf{N}}_\mathbf{4}$

We claim the following result.

Theorem 4.3. *For any integer $m \geq 2$, we have the refinement equation*

$$\tilde{\mathbf{N}}_\mathbf{m} = \sum_j \tilde{p}_{m,j}\ \tilde{\mathbf{N}}_\mathbf{m}(2\cdot -j), \tag{4.25}$$

where

$$\begin{cases} \tilde{p}_{2,0} := \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}, \ \tilde{p}_{2,1} := \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}, \ \tilde{p}_{2,j} := \mathbf{0} \ \textit{otherwise}, \\ \tilde{p}_{m,j} := \frac{1}{2}\left(\tilde{p}_{m-1,j} + \tilde{p}_{m-1,j-1}\right), \quad j \in \mathbb{Z}, \quad m \geq 3. \end{cases} \tag{4.26}$$

Proof. In order to prove the theorem, we shall first show that the result holds for $m = 2$, and then proceed by induction on m. To show that the result holds for $m = 2$, we will prove that, with the matrix sequence $\tilde{p}_2 := \{\tilde{p}_{2,j} = \ j \in \mathbb{Z}\}$ given by (4.26), we have that $(\tilde{p}_2, \tilde{\mathbf{N}}_\mathbf{2})$ is a refinement pair. Observe first from (4.26) that

$$\text{supp}\,(\tilde{p}_2) = \{0, 1\},$$

which means that

$$\sum_{j} \tilde{p}_{2,j}\ \tilde{\mathbf{N}}_{\mathbf{2}}(2\cdot -j) = \sum_{j=0}^{1} \tilde{p}_{2,j}\ \tilde{\mathbf{N}}_{\mathbf{2}}(2\cdot -j). \tag{4.27}$$

Next, note from (4.25) that it will suffice to prove that the right-hand side of (4.27) is equal to $\tilde{\mathbf{N}}_{\mathbf{2}}$. To this end, we successively consider the cases $x \in [0,1/2)$, $x \in [1/2,1)$ and $x \in \mathbb{R} \setminus [0,1)$.

Hence, let us first fix $x \in [0,1/2)$, so that, according to (4.2),

$$\tilde{\mathbf{N}}_{\mathbf{2}}(x) = \begin{bmatrix} 1-x \\ x \end{bmatrix}. \tag{4.28}$$

Besides, the second line of (4.2) implies that

$$\tilde{\mathbf{N}}_{\mathbf{2}}(2x-1) = \mathbf{0},$$

which, together with (4.27) and (4.3), yields

$$\begin{aligned} \sum_{j=0}^{1} \tilde{p}_{2,j}\ \tilde{\mathbf{N}}_{\mathbf{2}}(2x-j) &= \tilde{p}_{2,0}\ \tilde{\mathbf{N}}_{\mathbf{2}}(2x) \\ &= \tilde{p}_{2,0} \begin{bmatrix} \tilde{N}_{2,0}(2x) \\ \tilde{N}_{2,1}(2x) \end{bmatrix}. \end{aligned} \tag{4.29}$$

Substituting (4.2) and (4.26) into (4.29), we find that

$$\begin{aligned} \tilde{p}_{2,0} \begin{bmatrix} \tilde{N}_{2,0}(2x) \\ \tilde{N}_{2,1}(2x) \end{bmatrix} &= \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1-2x \\ 2x \end{bmatrix} \\ &= \begin{bmatrix} 1-x \\ x \end{bmatrix}, \end{aligned}$$

which, together with (4.29) and (4.28), gives

$$\sum_{j=0}^{1} \tilde{p}_{2,j} \ \tilde{\mathbf{N}}_{\mathbf{2}}(2x-j) = \tilde{\mathbf{N}}_{\mathbf{2}}(x). \tag{4.30}$$

Let us now fix $x \in [1/2, 1)$, so that, according to (4.2),

$$\tilde{\mathbf{N}}_{\mathbf{2}}(x) = \begin{bmatrix} 1-x \\ x \end{bmatrix}. \tag{4.31}$$

Besides, the second line of (4.2) implies that

$$\tilde{\mathbf{N}}_{\mathbf{2}}(2x) = \mathbf{0}.$$

which, together with (4.27) and (4.3), yields

$$\begin{aligned} \sum_{j=0}^{1} \tilde{p}_{2,j} \ \tilde{\mathbf{N}}_{\mathbf{2}}(2x-j) &= \tilde{p}_{2,1} \ \tilde{\mathbf{N}}_{\mathbf{2}}(2x-1) \\ &= \tilde{p}_{2,1} \begin{bmatrix} \tilde{N}_{2,0}(2x-1) \\ \tilde{N}_{2,1}(2x-1) \end{bmatrix}. \end{aligned} \tag{4.32}$$

Substituting (4.2) and (4.26) into (4.32), we find that

$$\begin{aligned} \tilde{p}_{2,1} \begin{bmatrix} \tilde{N}_{2,0}(2x-1) \\ \tilde{N}_{2,1}(2x-1) \end{bmatrix} &= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2-2x \\ 2x-1 \end{bmatrix} \\ &= \begin{bmatrix} 1-x \\ x \end{bmatrix}, \end{aligned}$$

which, together with (4.32) and (4.31), gives

$$\sum_{j=0}^{1} \tilde{p}_{2,j}\ \tilde{\mathbf{N}}_{\mathbf{2}}(2x-j) = \tilde{\mathbf{N}}_{\mathbf{2}}(x). \tag{4.33}$$

Next, let us fix $x \in \mathbb{R} \setminus [0,1)$. But then the second line of (4.2) yields

$$\tilde{\mathbf{N}}_{\mathbf{2}}(x) = \mathbf{0}$$

and

$$\tilde{\mathbf{N}}_{\mathbf{2}}(2x) = \tilde{\mathbf{N}}_{\mathbf{2}}(2x-1) = \mathbf{0},$$

which, together with (4.27), leads to

$$\sum_{j=0}^{1} \tilde{p}_{2,j}\ \tilde{\mathbf{N}}_{\mathbf{2}}(2x-j) = \mathbf{0} = \tilde{\mathbf{N}}_{\mathbf{2}}(x). \tag{4.34}$$

It follows from (4.30), (4.33), (4.34) that

$$\sum_{j=0}^{1} \tilde{p}_{2,j}\ \tilde{\mathbf{N}}_{\mathbf{2}}(2x-j) = \tilde{\mathbf{N}}_{\mathbf{2}}(x), \quad x \in \mathbb{R}. \tag{4.35}$$

Therefore, (4.25) holds for $m = 2$.

Proceeding inductively, let us suppose that equation (4.25) holds for a fixed integer $m \geq 2$. Using (4.1), we get

$$\tilde{\mathbf{N}}_{\mathbf{m+1}}\left(\frac{\cdot}{2}\right) = \int_0^1 \tilde{\mathbf{N}}_{\mathbf{m}}\left(\frac{\cdot}{2} - t\right) \mathrm{d}t,$$

which, together with the inductive hypothesis, yields

$$
\begin{aligned}
\tilde{\mathbf{N}}_{\mathbf{m+1}}\left(\frac{\cdot}{2}\right) &= \int_0^1 \sum_j \tilde{p}_{m,j}\ \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - 2t - j)\ dt \\
&= \sum_j \tilde{p}_{m,j} \int_0^1 \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - 2t - j)\ dt
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_j \tilde{p}_{m,j} \int_0^{\frac{1}{2}} \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - 2t - j)\ dt + \sum_j \tilde{p}_{m,j} \int_{\frac{1}{2}}^1 \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - 2t - j)\ dt \\
&= \sum_j \tilde{p}_{m,j} \int_0^{\frac{1}{2}} \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - 2t - j)\ dt + \sum_j \tilde{p}_{m,j-1} \int_{\frac{1}{2}}^1 \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - 2t + 1 - j)\ dt \\
&= \sum_j \tilde{p}_{m,j}\ \frac{1}{2} \int_0^1 \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - t - j)\ dt + \sum_j \tilde{p}_{m,j-1}\ \frac{1}{2} \int_0^1 \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - t - j)\ dt \\
&= \sum_j \frac{1}{2} \left[\tilde{p}_{m,j} + \tilde{p}_{m,j-1}\right] \int_0^1 \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - t - j)\ dt. \qquad (4.36)
\end{aligned}
$$

Inserting the second line of (4.26) into (4.36), we obtain

$$\tilde{\mathbf{N}}_{\mathbf{m+1}}\left(\frac{\cdot}{2}\right) = \sum_j \tilde{p}_{m+1,j}\ \tilde{\mathbf{N}}_{\mathbf{m+1}}(\cdot - j),$$

which is equivalent to

$$\tilde{\mathbf{N}}_{\mathbf{m+1}} = \sum_j \tilde{p}_{m+1,j}\ \tilde{\mathbf{N}}_{\mathbf{m+1}}(2 \cdot - j). \qquad (4.37)$$

It follows from (4.37) that $\left(\tilde{p}_{m+1}, \tilde{\mathbf{N}}_{\mathbf{m+1}}\right)$ forms a refinement pair, which completes our inductive proof. □

4.2 Properties of the vector spline $\tilde{\mathbf{N}}_{\mathbf{m}}$

Analogously to Theorem 2.4 for the scalar case, we proceed to prove the following properties for the vector spline $\tilde{\mathbf{N}}_{\mathbf{m}}$.

Theorem 4.4. *For an integer $m \geq 2$, let the vector spline $\tilde{\mathbf{N}}_{\mathbf{m}}$ be defined as in (4.1), (4.2). Then:*

$$\textit{(i)} \quad \text{supp}\left(\tilde{N}_{m,k}\right) = [0, m-1], \quad k = 0, 1; \tag{4.38}$$

$$\textit{(ii)} \quad \tilde{N}_{m,k}(x) > 0, \ x \in (0, m-1), \quad k = 0, 1; \tag{4.39}$$

$$\textit{(iii)} \quad \tilde{N}_{m,0} = \tilde{N}_{m,1}(m-1-\cdot); \tag{4.40}$$

$$\textit{(iv)} \quad \sum_j \left\{\tilde{N}_{m,0}(x-j) + \tilde{N}_{m,1}(x-j)\right\} = 1, \quad x \in \mathbb{R}; \tag{4.41}$$

$$\textit{(v)} \quad \int_{-\infty}^{+\infty} \left\{\tilde{N}_{m,0}(x) + \tilde{N}_{m,1}(x)\right\} dx = 1; \tag{4.42}$$

$$\textit{(vi)} \quad \tilde{N}_{m,k} \in C^{m-3}(\mathbb{R}), \quad k = 0, 1; \tag{4.43}$$

$$\textit{(vii)} \quad \tilde{\mathbf{N}}_{\mathbf{m}} \in C^{m-3}(\mathbb{R}); \tag{4.44}$$

$$\textit{(viii)} \quad \tilde{\mathbf{N}}'_{m+1} = \tilde{\mathbf{N}}_{\mathbf{m}} - \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - 1), \quad m \geq 3; \tag{4.45}$$

$$\textit{(ix)} \quad \tilde{N}_{m,k}\big|_{[j,j+1)} \in \pi_{m-1}, \ j \in \mathbb{Z}, \quad k = 0, 1. \tag{4.46}$$

Proof. We prove (i) inductively, by first observing from (4.2) that (i) holds for $m = 2$. Let us next assume that (i) holds for a fixed integer $m \geq 2$. But then (4.1) implies that

$$\begin{bmatrix} \tilde{N}_{m+1,0} \\ \tilde{N}_{m+1,1} \end{bmatrix} = \begin{bmatrix} \int_0^1 \tilde{N}_{m,0}(\cdot - t) \ \mathrm{d}t \\ \int_0^1 \tilde{N}_{m,1}(\cdot - t) \ \mathrm{d}t \end{bmatrix} \tag{4.47}$$

It follows from (4.47) and Lemma 2.3, together with the inductive hypothesis, that

$$\text{supp}\left(\tilde{N}_{m+1,0}\right) = \text{supp}\left(\tilde{N}_{m+1,1}\right) = [0, m],$$

thereby completing our inductive proof of (i).

To prove (ii), again relying on an inductive proof, note from (4.2) that (ii) holds for $m = 2$. Let us now suppose that (ii) holds for a fixed integer $m \geq 2$. Observe from (4.47) that

$$\begin{bmatrix} \tilde{N}_{m+1,0}(x) \\ \tilde{N}_{m+1,1}(x) \end{bmatrix} = \begin{bmatrix} \int_{x-1}^{x} \tilde{N}_{m,0}(t)\ \mathrm{d}t \\ \int_{x-1}^{x} \tilde{N}_{m,1}(t)\ \mathrm{d}t \end{bmatrix}, \quad x \in \mathbb{R}.$$

Using a similar argument as the one leading to (2.24) in the proof of Theorem 2.4, we proceed to show that (ii) holds for $m+1$, as follows:

If $x \in (1, m-1)$, then (4.1) and the inductive assumption imply that $\tilde{N}_{m+1,k}(x) > 0$, for $k = 0, 1$. If $x \in (0, 1]$, then

$$-1 < x - 1 \leq 0 < x \leq 1,$$

which, together with (4.1), gives, for $k = 0, 1$,

$$\begin{aligned} \tilde{N}_{m+1,k}(x) &= \int_0^1 \tilde{N}_{m,k}(x-t)\ \mathrm{d}t \\ &= \int_{x-1}^{x} \tilde{N}_{m,k}(t)\ \mathrm{d}t \\ &= \int_{x-1}^{0} \tilde{N}_{m,k}(t)\ \mathrm{d}t + \int_0^x \tilde{N}_{m,k}(t)\ \mathrm{d}t \\ &= \int_0^x \tilde{N}_{m,k}(t)\ \mathrm{d}t > 0, \end{aligned} \tag{4.48}$$

having also used (4.38) and the inductive assumption. Using a similar argument as the one leading to (4.48), we can show that, for $x \in [m-1, m)$, we have that $\tilde{N}_{m+1,k}(x) > 0$, for $k = 0, 1$. Therefore,

$$\tilde{N}_{m+1,k}(x) > 0, \quad x \in (0, m), \ \ k = 0, 1, \tag{4.49}$$

thereby completing our inductive proof.

In order to prove the symmetry result (iii), we use an induction on the integer m. It is trivial from (4.2) that (4.40) holds for $m = 2$. Let us next assume that (iii) holds for a fixed integer $m \geq 2$. We then have, for $x \in \mathbb{R}$,

$$\begin{aligned}
\tilde{N}_{m+1,0}(x) &= \int_0^1 \tilde{N}_{m,0}(x-t)\ \mathrm{d}t \\
&= \int_0^1 \tilde{N}_{m,1}(m-1-[x-t])\ \mathrm{d}t \\
&= \int_0^1 \tilde{N}_{m,1}(m-x-1+t)\ \mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \tilde{N}_{m,1}(m-x-[1-t])\ \mathrm{d}t \\
&= \int_0^1 \tilde{N}_{m,1}(m-x-t)\ \mathrm{d}t \\
&= \tilde{N}_{m+1,1}(m-x),
\end{aligned}$$

thereby concluding our inductive proof.

To prove (iv), we first show that (iv) holds for $m = 2$, and then use an induction on the integer $m \geq 2$. Hence, observe that for any fixed $x \in \mathbb{R}$, there exist a unique $k \in \mathbb{Z}$ such that $x - k \in [0, 1)$, so that, from (4.38) with $m = 2$,

$$\begin{aligned}
\sum_j \left\{\tilde{N}_{2,0}(x-j) + \tilde{N}_{2,1}(x-j)\right\} &= \tilde{N}_{2,0}(x-k) + \tilde{N}_{2,1}(x-k) \\
&= [1-(x-k)] + [x-k] = 1,
\end{aligned}$$

by virtue of (4.2). It follows that (iv) is true for $m = 2$.

Let us next suppose that (iv) holds for a fixed integer $m \geq 2$. Our inductive proof will be complete if we can show that (iv) holds with m replaced by $m+1$. According to the first line of (4.1), we have that

$$\begin{aligned}
&\sum_j \left\{ \tilde{N}_{m+1,0}(x-j) + \tilde{N}_{m+1,1}(x-j) \right\} \\
&\quad = \sum_j \left\{ \int_0^1 \tilde{N}_{m,0}(x-j-t)\mathrm{d}t + \int_0^1 \tilde{N}_{m,1}(x-j-t)\mathrm{d}t \right\} \\
&\quad = \sum_j \int_0^1 \left\{ \tilde{N}_{m,0}(x-j-t) + \tilde{N}_{m,1}(x-j-t) \right\} \mathrm{d}t \\
&\quad = \int_0^1 \sum_j \left\{ \tilde{N}_{m,0}(x-j-t) + \tilde{N}_{m,1}(x-j-t) \right\} \mathrm{d}t \\
&\quad = \int_0^1 1 \; \mathrm{d}t = 1.
\end{aligned}$$

Similarly, we use an induction on m to prove (v). The key of the proof is to show that

$$\int_{-\infty}^{+\infty} \tilde{N}_{m,0}(x) \; \mathrm{d}x = \int_{-\infty}^{+\infty} \tilde{N}_{m,1}(x) \; \mathrm{d}x = \frac{1}{2}, \quad m \geq 2, \tag{4.50}$$

from which the desired result (v) follows directly, since also (4.2) shows that (4.50) trivially holds for $m = 2$. Let us next suppose that (4.50) holds for a fixed integer $m \geq 2$. Using (4.38) together with (4.1), we obtain

$$\begin{aligned}
\int_{-\infty}^{+\infty} \tilde{N}_{m+1,0}(x) \; \mathrm{d}x &= \int_0^m \tilde{N}_{m+1,0}(x) \; \mathrm{d}x \\
&= \int_0^m \int_0^1 \tilde{N}_{m,0}(x-t) \; \mathrm{d}t \; \mathrm{d}x \\
&= \int_0^1 \left[\int_0^m \tilde{N}_{m,0}(x-t) \; \mathrm{d}x \right] \mathrm{d}t \\
&= \int_0^1 \left[\int_{-t}^{m-t} \tilde{N}_{m,0}(x) \; \mathrm{d}x \right] \mathrm{d}t.
\end{aligned} \tag{4.51}$$

Since $0 \leq t \leq 1$, we have that

$$\begin{cases} -t & \leq 0, \\ m-t & \geq m-1, \end{cases}$$

which, together with (4.38), (4.51), and the inductive hypothesis (4.50), yields

$$\begin{aligned}
\int_{-\infty}^{+\infty} \tilde{N}_{m+1,0}(x)\ \mathrm{d}x &= \int_0^1 \left[\int_0^{m-1} \tilde{N}_{m,0}(x)\ \mathrm{d}x\right] \mathrm{d}t \\
&= \int_0^1 \left[\int_{-\infty}^{+\infty} \tilde{N}_{m,0}(x)\ \mathrm{d}x\right] \mathrm{d}t \\
&= \int_0^1 \frac{1}{2}\ \mathrm{d}t = \frac{1}{2}.
\end{aligned}$$

Therefore,

$$\int_{-\infty}^{+\infty} \tilde{N}_{m+1,0}(x)\ \mathrm{d}x = \frac{1}{2}, \quad m \geq 2.$$

Using a similar argument, we show that

$$\int_{-\infty}^{+\infty} \tilde{N}_{m+1,1}(x)\ \mathrm{d}x = \frac{1}{2}, \quad m \geq 2,$$

thereby completing our inductive proof of (v).

Next, we note from (4.2) that (vi) holds for $m = 2$. Now, suppose that (vi) holds for a fixed integer $m \geq 2$. But then (4.1), Lemma 2.3 and the inductive hypothesis yield

$$\tilde{\mathbf{N}}_{\mathbf{m+1}} \in C^{m-2}(\mathbb{R}),$$

which means that (vi) holds for $m + 1$, which completes our inductive proof of (vi).

Observe that (vii) is a direct consequence of (vi).

In order to prove (viii), observe from (4.1) that

$$\begin{aligned}
\tilde{\mathbf{N}}_{m+1} &= \int_{x-1}^{x} \tilde{\mathbf{N}}_{\mathbf{m}}(t)\ \mathrm{d}t \\
&= \int_0^x \tilde{\mathbf{N}}_{\mathbf{m}}(t)\ \mathrm{d}t - \int_0^{x-1} \tilde{\mathbf{N}}_{\mathbf{m}}(t)\ \mathrm{d}t.
\end{aligned}$$

Again relying on an inductive proof, observe from (4.2) that (ix) holds for $m = 2$. Hence, let us suppose that (ix) holds for a fixed integer $m \geq 2$. Using (4.1) together with the inductive assumption, we see that

$$\tilde{N}_{m+1,k}\big|_{[j,j+1)} \in \pi_m, \quad j \in \mathbb{Z}, \quad k = 0, 1,$$

thereby concluding our inductive proof of (ix). $\square$

Observe in particular that Figures 4.1, 4.2 and 4.3 are consistent with the result (4.38) as well as with the symmetry property (4.40). Also, as stated by (4.44), it is clear from the figures that, as m increases, the vector $\tilde{\mathbf{N}}_{\mathbf{m}}$ gets smoother.

Definition 4.5. We say that a vector function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ has linearly independent integer shifts on $\mathbb{R}$ if the only sequence $c \in M(\mathbb{Z})$ for which it holds that

$$\sum_j c_j \, \mathbf{f}(\cdot - j) = \mathbf{0},$$

is the zero sequence $c_j = 0, \ j \in \mathbb{Z}$.

Proposition 4.6. *The set* $\left\{\tilde{\mathbf{N}}_{\mathbf{2}}(\cdot - l); \ l \in \mathbb{Z}\right\}$ *is linearly independent on* $\mathbb{R}$.

Proof. Suppose that $\{c_j : \ j \in \mathbb{Z}\}$ is such that

$$\sum_j c_j \, \tilde{\mathbf{N}}_{\mathbf{2}}(x - j) = \mathbf{0}, \quad x \in \mathbb{R},$$

so that, from (4.3),

$$\left.\begin{aligned} \sum_j c_j \, \tilde{N}_{2,0}(x-j) = 0, \\ \sum_j c_j \, \tilde{N}_{2,1}(x-j) = 0, \end{aligned}\right\} \ x \in \mathbb{R},$$

and thus

$$\left.\begin{aligned} \sum_j c_j \, \tilde{N}_{2,0}(k-j) = 0, \\ \sum_j c_j \, \tilde{N}_{2,1}(k-j) = 0, \end{aligned}\right\} \ k \in \mathbb{Z}. \tag{4.52}$$

Since (4.2) gives

$$\tilde{N}_{2,0}(x) = \begin{cases} 1 - x, & x \in [0,1), \\ 0, & x \in \mathbb{R} \setminus [0,1), \end{cases}$$

we have, moreover, that

$$\sum_j c_j \, \tilde{N}_{2,0}(k-j) = c_k, \quad k \in \mathbb{Z}. \tag{4.53}$$

It follows from (4.53) and the first line of (4.52) that the integer shift sequence

$$\left\{ \tilde{\mathbf{N}}_{\mathbf{2}}(\cdot - j) : \; j \in \mathbb{Z} \right\}$$

is linearly independent on $\mathbb{R}$.

□

Theorem 4.7. *Let the integer $m \geq 2$ be given. Then $\tilde{\mathbf{N}}_{\mathbf{m}}$ has linearly independent integer shifts on $\mathbb{R}$.*

Proof. The case $m = 2$ has already been proved in Propostion 4.6. In order to prove Theorem 4.7, we proceed to show that the result holds for $m = 3$, and then use induction on the integer m.

Hence, suppose that $c \in M(\mathbb{Z})$ is such that

$$\sum_j c_j \, \tilde{\mathbf{N}}_{\mathbf{3}}(x-j) = \mathbf{0}, \quad x \in \mathbb{R},$$

so that, from (4.3),

$$\left. \begin{aligned} \sum_j c_j \, \tilde{N}_{3,0}(x-j) = 0, \\ \sum_j c_j \, \tilde{N}_{3,1}(x-j) = 0, \end{aligned} \right\} \; x \in \mathbb{R},$$

and therefore,

$$\left.\begin{aligned}\sum_j c_j\ \tilde{N}_{3,0}(k-j) = 0,\\ \sum_j c_j\ \tilde{N}_{3,1}(k-j) = 0,\end{aligned}\right\}\ k \in \mathbb{Z}. \tag{4.54}$$

But (4.9) gives

$$\sum_j c_j\ \tilde{N}_{3,0}(k-j) = \frac{1}{2}\ c_{k-1}, \quad k \in \mathbb{Z},$$

which, together with the first line of (4.54), implies that $c_{k-1} = 0,\ k \in \mathbb{Z}$, and hence

$$c_j = 0, \quad j \in \mathbb{Z}.$$

Therefore, $\mathbf{\tilde{N}_3}$ has linearly independent integer shifts.

Proceeding inductively, let $m \geq 3$ and suppose that $\mathbf{\tilde{N}_m}$ has linearly independent integer shifts. Moreover, let $c \in M(\mathbb{Z})$ be such that

$$\sum_j c_j\ \mathbf{\tilde{N}_{m+1}}(x-j) = \mathbf{0}, \quad x \in \mathbb{R}. \tag{4.55}$$

Since $m+1 \geq 4$, it follows from (4.44) and (4.55) that

$$\sum_j c_j\ \mathbf{\tilde{N}'_{m+1}}(x-j) = \mathbf{0}, \quad x \in \mathbb{R}. \tag{4.56}$$

Equations (4.56) and (4.45) yield

$$\begin{aligned}
\mathbf{0} &= \sum_j c_j \left[\tilde{\mathbf{N}}_{\mathbf{m}}(x-j) - \tilde{\mathbf{N}}_{\mathbf{m}}(x-j-1)\right] \\
&= \sum_j c_j \; \tilde{\mathbf{N}}_{\mathbf{m}}(x-j) - \sum_j c_j \; \tilde{\mathbf{N}}_{\mathbf{m}}(x-j-1) \\
&= \sum_j c_j \; \tilde{\mathbf{N}}_{\mathbf{m}}(x-j) - \sum_j c_{j-1} \; \tilde{\mathbf{N}}_{\mathbf{m}}(x-j) \\
&= \sum_j (c_j - c_{j-1}) \; \tilde{\mathbf{N}}_{\mathbf{m}}(x-j),
\end{aligned}$$

which, together with the inductive hypothesis, gives

$$c_j - c_{j-1} = 0, \quad j \in \mathbb{Z},$$

yielding

$$c_j = c, \quad j \in \mathbb{Z}, \tag{4.57}$$

with c denoting a constant. It follows from (4.55) and (4.57) that

$$c \sum_j \tilde{\mathbf{N}}_{\mathbf{m+1}}(x-j) = \mathbf{0}, \quad x \in \mathbb{R},$$

and thus,

$$\left.\begin{aligned}
c \sum_j \tilde{N}_{m+1,0}(x-j) = 0, \\
c \sum_j \tilde{N}_{m+1,1}(x-j) = 0,
\end{aligned}\right\} \; x \in \mathbb{R},$$

which, together with (4.41), gives

$$0 = c \left[\sum_j \tilde{N}_{m+1,0}(x-j) + \sum_j \tilde{N}_{m+1,1}(x-j)\right] = c.$$

Therefore, $c = 0$, which, by virtue of (4.57), yields

$$c_j = 0, \quad j \in \mathbb{Z}.$$

As a consequence, $\tilde{\mathbf{N}}_{\mathbf{m+1}}$ has linearly independent integer shifts, which concludes our inductive proof. □

4.3 The matrix mask $\tilde{p}_m$

We aim in this section to discuss the properties of the matrix mask $\tilde{p}_m,\ m \geq 2$, as given recursively by (4.26). Defining the corresponding matrix mask symbol $\tilde{P}_m$ as in (1.17), we let

$$\tilde{P}_m(z) := \sum_j \tilde{p}_{m,j}\ z^j, \quad z \in \mathbb{C}. \tag{4.58}$$

We proceed to prove the following formulations of $\tilde{P}_m$.

Theorem 4.8. *For $m \geq 2$, the refinement mask symbol $\tilde{P}_m$ satisfies*

$$\left.\begin{aligned} \tilde{P}_2(z) &= \frac{1}{2}\begin{bmatrix} z+2 & 1 \\ z & 2z+1 \end{bmatrix}, \\ \tilde{P}_m(z) &= \frac{1+z}{2}\ \tilde{P}_{m-1}(z),\ m \geq 3, \end{aligned}\right\} z \in \mathbb{C}, \tag{4.59}$$

and is given explicitly by the formula

$$\tilde{P}_m(z) = \frac{1}{2^{m-1}}\ (1+z)^{m-2} \begin{bmatrix} z+2 & 1 \\ z & 2z+1 \end{bmatrix}, \quad z \in \mathbb{C}. \tag{4.60}$$

Proof. First, to prove the formula in the first line of (4.59), we use (4.58) and the first line of (4.26) to obtain, for $z \in \mathbb{C}$,

$$\tilde{P}_2(z) = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} z = \begin{bmatrix} 1+z/2 & 1/2 \\ z/2 & 1/2+z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} z+2 & 1 \\ z & 2z+1 \end{bmatrix}.$$

Next, to prove the recursion formula in the second line of (4.59), we use (4.58) and the second line of (4.26) to obtain, for $z \in \mathbb{C}$,

$$\begin{aligned} \frac{1+z}{2}\, \tilde{P}_{m-1}(z) =& \frac{1}{2}\,(1+z) \sum_j \tilde{p}_{m-1,j}\, z^j \\ =& \frac{1}{2} \left[\sum_j \tilde{p}_{m-1,j}\, z^j + \sum_j \tilde{p}_{m-1,j}\, z^{j+1} \right] \\ =& \frac{1}{2} \left[\sum_j \tilde{p}_{m-1,j}\, z^j + \sum_j \tilde{p}_{m-1,j-1}\, z^j \right] \\ =& \sum_j \frac{1}{2}\, (\tilde{p}_{m-1,j} + \tilde{p}_{m-1,j-1})\, z^j \\ =& \sum \tilde{p}_{m,j}\, z^j \\ =& \tilde{P}_m(z). \end{aligned}$$

Next, we prove (4.60) inductively, by first noting from the first line of (4.59) that (4.60) holds for $m = 2$. Suppose now that (4.60) holds for a fixed integer $m \geq 2$. Using (4.59), together with the inductive hypothesis, we obtain

$$\begin{aligned} \tilde{P}_{m+1}(z) &= \frac{1+z}{2}\, P_m(z) \\ &= \frac{1+z}{2}\, \frac{1}{2^{m-1}}\, (1+z)^{m-2} \begin{bmatrix} z+2 & 1 \\ z & 2z+1 \end{bmatrix} \\ &= \frac{1}{2^m}\, (1+z)^{m-1} \begin{bmatrix} z+2 & 1 \\ z & 2z+1 \end{bmatrix}, \quad z \in \mathbb{C}, \end{aligned}$$

which shows that (4.60) holds with m replaced by $m+1$, and thereby completing our inductive proof of (4.60). □

The explicit formulation (4.60) of the mask symbol now enables us to explicitly compute the matrix masks $\tilde{p}_m = \{\tilde{p}_{m,j} : \ j \in \mathbb{Z}\}$, as follows.

Theorem 4.9. *For any integer $m \geq 2$, we have:*

$$\tilde{p}_{m,j} = \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}_{m,j} & \tilde{b}_{m,j} \\ \tilde{b}_{m,m-1-j} & \tilde{a}_{m,m-1-j} \end{bmatrix}, \ j \in \mathbb{Z}, \tag{4.61}$$

with

$$\left.\begin{aligned} \tilde{a}_{m,j} &= \binom{m-2}{j} + \binom{m-1}{j}, \\ \tilde{b}_{m,j} &= \binom{m-2}{j}, \end{aligned}\right\} j \in \mathbb{Z}. \tag{4.62}$$

Proof. Let the integer $m \geq 2$ be fixed. Since, according to (4.58),

$$\tilde{P}_m(z) = \sum_j \tilde{p}_{m,j}\, z^j, \quad z \in \mathbb{C},$$

we see from (4.60) that

$$\tilde{p}_{m,j} = \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}_{m,j} & \tilde{b}_{m,j} \\ \tilde{c}_{m,j} & \tilde{d}_{m,j} \end{bmatrix}, \quad j \in \mathbb{Z}, \tag{4.63}$$

where, for $z \in \mathbb{C}$, we have

$$
\begin{aligned}
\sum_j \tilde{a}_{m,j} z^j &= (1+z)^{m-2}(z+2) \\
&= z\ (1+z)^{m-2} + 2\ (1+z)^{m-2} \\
&= z \sum_{j=0}^{m-2} \binom{m-2}{j} z^j + 2 \sum_{j=0}^{m-2} \binom{m-2}{j} z^j \\
&= \sum_{j=0}^{m-2} \binom{m-2}{j} z^{j+1} + \sum_{j=0}^{m-2} 2 \binom{m-2}{j} z^j \\
&= \sum_{j=1}^{m-1} \binom{m-2}{j-1} z^j + \sum_{j=0}^{m-2} 2 \binom{m-2}{j} z^j \\
&= \sum_{j=0}^{m-1} \binom{m-2}{j-1} z^j + \sum_{j=0}^{m-1} 2 \binom{m-2}{j} z^j \\
&= \sum_{j=0}^{m-1} \left[\binom{m-2}{j-1} + 2 \binom{m-2}{j} \right] z^j \\
&= \sum_{j=0}^{m-1} \left[\binom{m-1}{j} + \binom{m-2}{j} \right] z^j, \quad z \in \mathbb{C},
\end{aligned}
$$

and thus

$$
\tilde{a}_{m,j} = \binom{m-1}{j} + \binom{m-2}{j}, \quad j \in \mathbb{Z}; \tag{4.64}
$$

whereas

$$
\sum_j \tilde{b}_{m,j}\ z^j = (1+z)^{m-2} = \sum_{j=0}^{m-2} \binom{m-2}{j} z^j,
$$

so that

$$
\tilde{b}_{m,j} = \binom{m-2}{j}, \quad j \in \mathbb{Z}; \tag{4.65}
$$

also,

$$\begin{aligned}\sum_j \tilde{c}_{m,j}\ z^j =& z\ (1+z)^{m-2}\\ =& z\sum_j \binom{m-2}{j}\ z^j\\ =& \sum_j \binom{m-2}{j}\ z^{j+1}\\ =& \sum_j \binom{m-2}{j-1}\ z^j,\end{aligned}$$

and thus

$$\tilde{c}_{m,j} = \binom{m-2}{j-1}, \quad j\in\mathbb{Z}; \tag{4.66}$$

whereas

$$\begin{aligned}\sum_j \tilde{d}_{m,j}\ z^j =& (2z+1)\ (1+z)^{m-2}\\ =& 2z\ (1+z)^{m-2} + (1+z)^{m-2}\\ =& 2\sum_j \binom{m-2}{j}\ z^{j+1} + \sum_j \binom{m-2}{j}\ z^j\\ =& 2\sum_j \binom{m-2}{j-1}\ z^j + \sum_j \binom{m-2}{j}\ z^j\\ =& \sum_j \left[2\binom{m-2}{j-1} + \binom{m-2}{j}\right]\ z^j\\ =& \sum_j \left[\binom{m-2}{j-1} + \left\{\binom{m-2}{j-1} + \binom{m-2}{j}\right\}\right]\ z^j\\ =& \sum_j \left[\binom{m-2}{j-1} + \binom{m-1}{j}\right]\ z^j,\end{aligned}$$

so that

$$\tilde{d}_{m,j} = \binom{m-2}{j-1} + \binom{m-1}{j}, \quad j\in\mathbb{Z}. \tag{4.67}$$

Observing from (4.64), (4.65), (4.66) and (4.67) that, for $j \in \mathbb{Z}$,

$$\tilde{b}_{m,m-1-j} = \binom{m-2}{m-1-j} = \binom{m-2}{m-2-(j-1)} = \binom{m-2}{j-1} = \tilde{c}_{m,j}, \tag{4.68}$$

and

$$\tilde{a}_{m,m-1-j} = \binom{m-1}{m-1-j} + \binom{m-2}{m-1-j} = \binom{m-1}{j} + \binom{m-2}{m-2-(j-1)} = \binom{m-1}{j} + \binom{m-2}{j-1} = \tilde{c}_{m,j}, \tag{4.69}$$

the result (4.61), (4.62), follows by combining (4.63), (4.64), (4.65), (4.68) and (4.69). □

Observe in particular, for an integer j, the cross symmetry between the first and the second rows of the matrix $\tilde{p}_{m,j}$, for $j \in \mathbb{Z}$. Moreover, we have the following intermediate consequence of Theorem 4.9.

Corollary 4.10. *For any integer $m \geq 2$, it holds that*

$$\text{supp }\ (\tilde{p}_m) = \{0, \ldots, m-1\}. \tag{4.70}$$

Example 4.11. Using (4.61) and (4.62), for $m = 3$, we find that

$$\tilde{p}_{3,0} = \begin{bmatrix} 1/2 & 1/4 \\ 0 & 1/4 \end{bmatrix}, \quad \tilde{p}_{3,1} = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}, \quad \tilde{p}_{3,2} = \begin{bmatrix} 1/4 & 0 \\ 1/4 & 1/2 \end{bmatrix}, \tag{4.71}$$

with

$$\tilde{p}_{3,j} = \mathbf{0}, \ \ j \in \mathbb{Z} \setminus \{0, 1, 2\}. \tag{4.72}$$

Similarly, for $m = 4$, we obtain the matrices

$$\left.\begin{aligned} &\tilde{p}_{4,0} = \begin{bmatrix} 1/4 & 1/8 \\ 0 & 1/8 \end{bmatrix}, \quad \tilde{p}_{4,1} = \begin{bmatrix} 5/8 & 1/4 \\ 1/8 & 1/2 \end{bmatrix}, \quad \tilde{p}_{4,2} = \begin{bmatrix} 1/2 & 1/8 \\ 1/4 & 5/8 \end{bmatrix}, \\ &\tilde{p}_{4,3} = \begin{bmatrix} 1/8 & 0 \\ 1/8 & 1/4 \end{bmatrix}, \end{aligned}\right\} \tag{4.73}$$

with

$$\tilde{p}_{4,j} = \mathbf{0}, \; j \in \mathbb{Z} \setminus \{0, \ldots, 3\}. \tag{4.74}$$

4.4 Other constructions in the literature

In [Plo95b], Plonka defines a class of refinable vector splines which is different from those obtained from our definition (4.1). A particular case of the splines considered by Plonka is the case where the knot sequence $t = \{t_l : \; l \in \mathbb{Z}\}$ is given by

$$t_l := \left\lfloor \frac{l}{2} \right\rfloor, \; l \in \mathbb{Z}. \tag{4.75}$$

Note that (4.76) implies that each knot is repeated twice, which means that we obtain some splines of multiplicity 2. In order to have a refinable vector, Plonka first extends the definition (2.4) by defining the cardinal spline of order $m \geq 2$, denoted by $N_{m,k}$, with respect to the knots $t_k, \ldots, t_{k+m}$, by means of

$$N_{m,k}(x) := (t_{k+m} - t_k) \; (\cdot - x)_+^{m-1}[t_k, \ldots, t_{k+m}], \; x \in \mathbb{R}, \; k \in \mathbb{Z}, \tag{4.76}$$

with t_l given by (4.75) for $l \in \{k, \ldots, k+m\}$.

The vector cardinal B-spline, denoted as $\mathbf{N_m}$, is then defined as

$$\mathbf{N_m} := \begin{pmatrix} N_{m,0} \\ N_{m,1} \end{pmatrix}, \ m \geq 2. \tag{4.77}$$

The following result is proved in [Plo95b].

Theorem 4.12. *For an integer $m \geq 2$, let $\mathbf{N}_m$ be the vector spline defined by (4.76), (4.77), and let t_l be given by (4.75) for $l \in \mathbb{Z}$. Then:*

$$\text{(i)} \quad \left.\begin{aligned} &\operatorname{supp}(N_{m,k}) = [0, \lfloor (m+k)/2 \rfloor], \\ &N_{k,m}(x) > 0, \ x \in (0, \lfloor (m+k)/2 \rfloor), \end{aligned}\right\}, \ k = 0, 1; \tag{4.78}$$

$$\text{(ii)} \quad \operatorname{supp}(\mathbf{N_m}) = [0, \lfloor (m+1)/2 \rfloor]; \tag{4.79}$$

$$\text{(iii)} \quad \sum_j \{N_{m,0}(x-j) + N_{m,1}(x-j)\} = 1, \ x \in \mathbb{R}; \tag{4.80}$$

$$\text{(iv)} \quad \int_{-\infty}^{+\infty} N_{m,k}(x) \ dx = \frac{\lfloor (k+m)/2 \rfloor}{m}, \ k = 0, 1; \tag{4.81}$$

$$\text{(v)} \quad \int_{-\infty}^{+\infty} \{N_{m,0}(x) + N_{m,1}(x)\} \ dx = 1; \tag{4.82}$$

$$\text{(vi)} \quad N_{m,k} = \frac{\cdot - t_k}{t_{k+m-1} - t_k} N_{m-1,k} + \frac{t_{k+m} - \cdot}{t_{k+m} - t_{k+1}} N_{m-1,k+1}, \ k \in \mathbb{Z}, \ m \geq 3; \tag{4.83}$$

$$\text{(vii)} \quad N_{m,k} \in C^{m-3}(\mathbb{R}), \ k \in \mathbb{Z}; \tag{4.84}$$

$$\text{(viii)} \quad \mathbf{N_m} \in C^{m-3}(\mathbb{R}); \tag{4.85}$$

$$\text{(ix)} \quad N_{m,k}|_{[j,j+1)} \in \pi_{m-1}, j \in \mathbb{Z}, \ k \in \mathbb{Z}. \tag{4.86}$$

Example 4.13. From equations (4.77), (4.76) together with (4.75), we find that

$$\mathbf{N_2}(x) = \begin{bmatrix} N_{2,0}(x) \\ N_{2,1}(x) \end{bmatrix} := \begin{bmatrix} (\cdot - x)_+[0,0,1] \\ (\cdot - x)_+[0,1,1] \end{bmatrix}, \quad x \in \mathbb{R}. \tag{4.87}$$

Using the recursive formula (1.11) for divided differences, we then obtain

$$N_{2,0}(x) := (\cdot - x)_+[0,0,1] = (\cdot - x)_+[0,1] - (\cdot - x)_+[0,0], \quad x \in \mathbb{R}. \tag{4.88}$$

Again using (1.11), we get, for $x \in \mathbb{R}$,

$$\begin{aligned}(\cdot - x)_+[0,1] &= (\cdot - x)_+[1] - (\cdot - x)_+[0] \\ &= \begin{cases} 1-x &, \quad x \in [0,1), \\ 0 &, \quad x \in \mathbb{R} \setminus [0,1). \end{cases}\end{aligned} \tag{4.89}$$

Besides, observe from (1.12) that

$$(\cdot - x)_+[0,0] = 0, \ x \in \mathbb{R}. \tag{4.90}$$

Combining (4.88), (4.89) and (4.90), we deduce that

$$N_{2,0}(x) = \begin{cases} 1-x &, \quad x \in [0,1), \\ 0 &, \quad x \in \mathbb{R} \setminus [0,1). \end{cases} \tag{4.91}$$

In a similar fashion, let us compute $N_{2,1}(x)$, for $x \in \mathbb{R}$. Using the recursive formula (1.11) for divided differences, we then obtain

$$N_{2,0}(x) := (\cdot - x)_+[0,1,1] = (\cdot - x)_+[1,1] - (\cdot - x)_+[0,1], \quad x \in \mathbb{R}. \tag{4.92}$$

From the second line of (1.11), we get, for $x \in \mathbb{R}$,

$$(\cdot - x)_+[1,1] = (\cdot - x)'_+(1) = \begin{cases} 1 &, \quad x \in [0,1), \\ 0 &, \quad x \in \mathbb{R} \setminus [0,1), \end{cases} \tag{4.93}$$

since also the cardinal B-splines are chosen to be right-continuous. Combining (4.92), (4.93) and (4.89), we find that

$$N_{2,1}(x) = \begin{cases} x &, \quad x \in [0,1), \\ 0 &, \quad x \in \mathbb{R} \setminus [0,1). \end{cases} \tag{4.94}$$

Observe from (4.91), (4.94) and (4.2) that

$$\mathbf{N_2} = \tilde{\mathbf{N}}_\mathbf{2}, \tag{4.95}$$

the graph of which is drawn in Figure 4.1.

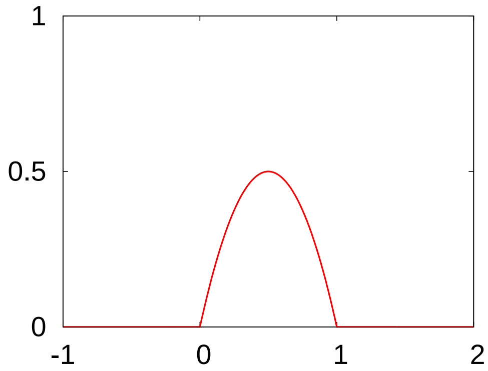
Using a similar argument as the one leading to (4.91) and (4.94), one can show that

$$\text{for even } k \in \mathbb{Z},\ N_{2,k}(x) = \begin{cases} \dfrac{x - t_{k+2}}{t_k - t_{k+2}} & , \quad x \in [t_k, t_{k+2}), \\ 0 & , \quad x \in \mathbb{R} \setminus [t_k, t_{k+2}), \end{cases} \tag{4.96}$$

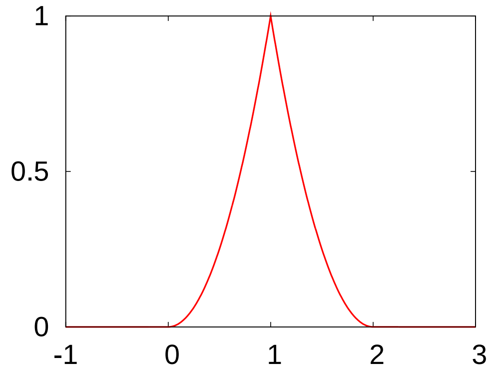
whereas

$$\text{for odd } k \in \mathbb{Z},\ N_{2,k}(x) = \begin{cases} \dfrac{x - t_k}{t_{k+2} - t_k} & , \quad x \in [t_k, t_{k+2}), \\ 0 & , \quad x \in \mathbb{R} \setminus [t_k, t_{k+2}). \end{cases} \tag{4.97}$$

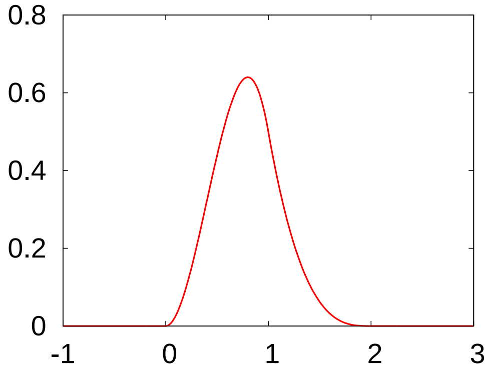
Using (4.96), (4.97), together with the recursive formula (4.83) and the definition (4.77), we proceed to draw the graphs of $\mathbf{N_3}$ and $\mathbf{N_4}$ in Figure 4.4.



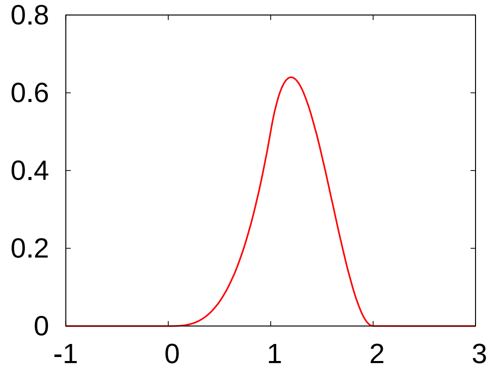
(a) $N_{3,0}$



(b) $N_{3,1}$



(c) $N_{4,0}$



(d) $N_{4,1}$

Figure 4.4: Graph of $\mathbf{N_m}$ for $m = 3, 4$.

Observe from Figure 4.4 that, as implied by (4.38) and (4.79), the vector spline $\mathbf{N_m}$ has a shorter support than $\tilde{\mathbf{N}}_\mathbf{m}$. Comparing Figure 4.2 with Figures 4.4(a) and 4.4(b) (respectively Figure 4.3 with Figures 4.3(a) and 4.3(b)), we see that $\mathbf{N}_m$ and $\tilde{\mathbf{N}}_\mathbf{m}$ have the same smoothness degree, as implied by (4.44) and (4.85). It should also be pointed out that, for $m = 3$, the components of the vector $\mathbf{N_3}$ present no symmetry with respect to each other. It turns out that there is no symmetry between the two components of the vector $\mathbf{N_m}$ for any odd value of m. Indeed, whenever m is odd, the position of the knots (4.75) corresponding to $N_{m,0}$ and $N_{m,1}$ are not symmetric, resulting in an absence of symmetry between $N_{m,0}$ and $N_{m,1}$. Besides, note from (4.79) that, for odd values of m, the components $N_{m,0}$ and $N_{m,1}$ do not even have the same support. Note, however, that, for odd m, both $N_{m,0}$ and $N_{m,1}$ are symmetric with respect to the midpoints of their supports.

As proved in [Plo95b], the vector $\mathbf{N}_m$ is refinable. For any integer $m \geq 2$, let us denote by r_m the mask corresponding to the refinable vector $\mathbf{N_m}$. Plonka then defines the corresponding refinement mask symbol R_m as

$$R_m(z) := \frac{1}{2}\sum_j r_{m,j}\ z^j,\ \ z \in \mathbb{C}. \tag{4.98}$$

It is shown in [Plo95b] that the mask symbol R_m thus defined satisfies the following formulation.

Theorem 4.14. *Let $m > 2$ and let $z \in \mathbb{C} \setminus \{1\}$. The two-scale symbol matrix R_m is uniquely determined by*

$$R_m(z) = \frac{1}{2(1-z)} \begin{bmatrix} 1/t_{m-1} & -1/t_m \\ -z^2/t_{m-1} & 1/t_m \end{bmatrix} R_{m-1}(z) \begin{bmatrix} t_{m-1} & t_{m-1} \\ z\ t_m & t_m \end{bmatrix}. \tag{4.99}$$

with

$$R_2(z) = \frac{1}{4}\begin{bmatrix} z+2 & 1 \\ z & 2z+1 \end{bmatrix}, \tag{4.100}$$

and where $t_{m-k},\ k = 0, 1$, are given by (4.75).

Furthermore, we have that

$$R_m(1) = \lim_{u\to 0} \frac{1}{2(1-e^{iu})} \begin{bmatrix} 1/t_{m-1} & -1/t_m \\ -e^{2iu}/t_{m-1} & 1/t_m \end{bmatrix} R_{m-1}(e^{iu}) \begin{bmatrix} t_{m-1} & t_{m-1} \\ e^{iu}t_m & t_m \end{bmatrix}. \tag{4.101}$$

Example 4.15. Using (4.75), (4.100) and (4.99) with $m = 3$, we get, for $z \in \mathbb{C}$,

$$\begin{aligned} R_3(z) =& \frac{1}{2(1-z)} \begin{bmatrix} 1 & -1 \\ -z^2 & 1 \end{bmatrix} \frac{1}{4} \begin{bmatrix} z+2 & 1 \\ z & 2z+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ z & 1 \end{bmatrix} \\ =& \frac{1}{8} \begin{bmatrix} 2z+2 & 2 \\ 2z^2+2z & z^2+4z+1 \end{bmatrix}, \end{aligned} \tag{4.102}$$

which, together with (4.98), yields

$$r_{3,0} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/4 \end{bmatrix}, \quad r_{3,1} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}, \quad r_{3,2} = \begin{bmatrix} 0 & 0 \\ 1/2 & 1/4 \end{bmatrix}, \tag{4.103}$$

with

$$r_{3,j} = \mathbf{0}, \quad j \in \mathbb{Z} \setminus \{0, 1, 2\}. \tag{4.104}$$

In a similar fashion, let us compute $R_4(z)$, for $z \in \mathbb{C}$. Using (4.75), (4.102) and (4.99) with $m = 4$, we find that, for $z \in \mathbb{C}$,

$$\begin{aligned} R_4(z) =& \frac{1}{2(1-z)} \begin{bmatrix} 1 & -1/2 \\ -z^2 & 1/2 \end{bmatrix} \frac{1}{8} \begin{bmatrix} 2z+2 & 2 \\ 2z^2+2z & z^2+4z+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2z & 2 \end{bmatrix} \\ =& \frac{1}{16} \begin{bmatrix} z^2+6z+2 & 2z+5 \\ 5z^2+2z & 2z^2+6z+1 \end{bmatrix}, \end{aligned} \tag{4.105}$$

from which we deduce that

$$r_{4,0} = \begin{bmatrix} 1/4 & 5/8 \\ 0 & 1/8 \end{bmatrix}, \quad r_{4,1} = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}, \quad r_{4,2} = \begin{bmatrix} 1/8 & 0 \\ 5/8 & 1/4 \end{bmatrix}, \tag{4.106}$$

with

$$r_{4,j} = 0, \quad j \in \mathbb{Z} \setminus \{0, 1, 2\}. \tag{4.107}$$

Remark 4.16. Observe from (4.72), (4.74), (4.104) and (4.107) that p_m seems to have a shorter support than $\tilde{p}_m$. However, as illustrated in Example 4.15, the recursive formula (4.83), involving multiplications of matrices with polynomial entries, is not as simple as the recursive formula in the second line of (4.26) which only involves the addition of two real-valued matrices. Moreover, it should be noted that, in contrast with the mask p_m corresponding to the vector $\mathbf{N_m}$, the mask $\tilde{p}_m$ associated with $\tilde{\mathbf{N}}_\mathbf{m}$ can be directly computed by means of the explicit formulae (4.61) and (4.62).

Definition 4.17. Let fix the integer $m \geq 2$, and let $\mathcal{B} \subset S_{m,2}(\mathbb{Z})$. We say that $\mathcal{B}$ is a basis for $S_{m,2}(\mathbb{Z})$ if any spline $f \in S_{m,2}(\mathbb{Z})$ can be expressed as a unique linear combination of the elements of $\mathcal{B}$.

The following result is presented in [Plo95b].

Theorem 4.18. *Let the integer $m \geq 2$ be given. Then, the set*

$$\{N_{m,k}(\cdot - j);\ k = 0, 1;\ j \in \mathbb{Z}\} \tag{4.108}$$

is a basis for $S_{m,2}(\mathbb{Z})$.

Remark 4.19. Theorem 4.18 implies that $S_{m,2}(\mathbb{Z})$ is generated by linear combinations of $N_{m,1}$ and $N_{m,2}$.

In [Goo98], Goodman defines a polynomial $B_{m,1}$ which, combined with the scalar B-spline N_m, forms the refinable vector spline $\mathbf{G_m} := (N_m, B_{m,1})^{\mathrm{T}}$. It is proved that every function in $S_{m,2}(\mathbb{Z})$ can be expressed as a linear combination of the integer shifts of N_m and $B_{m,1}$. However, since the Fourier transform is used to define $B_{m,1}$, it is not very easy to visualize the vector $\mathbf{G_m}$.

Chapter 5

Matrix subdivision schemes

We aim in this chapter to extend the scalar subdivision schemes and the corresponding convergence analysis of Chapter 3 to the vector setting.

5.1 Definition

Let (p, ϕ) denote a refinement pair such that $\phi : \mathbb{R} \to \mathbb{R}^2$ and such that $p = \{p_j : \ j \in \mathbb{Z}\}$ is a finitely supported sequence of 2×2 matrices with real entries. Motivated by the definition (3.4), we define the corresponding matrix subdivision scheme $(p, \mathbf{c})$ as was done by [CDL95], by

$$\begin{cases} \mathbf{c}^{(0)} & := \mathbf{c}, \\ \mathbf{c}_j^{(r+1)} & := \sum_k (p_{j-2k})^{\mathrm{T}} \ \mathbf{c}_k^{(r)}, \ r \in \mathbb{Z}_+, \end{cases} \tag{5.1}$$

where $\mathbf{c} : \mathbb{Z} \to \mathbb{R}^2$ denotes a given control point sequence.

5.2 Convergence proof

Theoretical result

Extending the result (3.3) of Theorem 3.3, we shall prove that, under certain conditions on the control point sequence $\mathbf{c}$, the subdivision scheme $(\tilde{p}_m, \mathbf{c})$, as defined by (5.1) with p replaced by $\tilde{p}_m$, as given in Theorem 4.8, converges to the limit curve $\tilde{\boldsymbol{\Phi}}_{\mathbf{m}}$ given by

$$\tilde{\boldsymbol{\Phi}}_{\mathbf{m}} = \sum_j \begin{bmatrix} \mathbf{c}_j^{\mathrm{T}} \\ \mathbf{c}_j^{\mathrm{T}} \end{bmatrix} \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - j). \tag{5.2}$$

We shall rely on the following results.

Lemma 5.1. *Let $m \in \mathbb{N}$. We then have*

$$\sum_k \binom{m}{2k} = \sum_k \binom{m}{2k+1} = 2^{m-1}. \tag{5.3}$$

Proof. Let us fix the integer $m \in \mathbb{N}$. Using the binomial theorem, we know that

$$\sum_k \binom{m}{k} = \sum_k \binom{m}{k} 1^k \, 1^{m-k} = (1+1)^m = 2^m. \tag{5.4}$$

Similarly, we have that

$$\begin{aligned} \sum_k \binom{m}{2k} - \sum_k \binom{m}{2k+1} &= \sum_k \left\{ \binom{m}{2k} - \binom{m}{2k+1} \right\} \\ &= \sum_k \binom{m}{k} (-1)^k \, 1^{m-k} \\ &= (1-1)^m = 0, \end{aligned}$$

which gives

$$\sum_k \binom{m}{2k} = \sum_k \binom{m}{2k+1}. \tag{5.5}$$

Since also

$$\sum_k \binom{m}{k} = \sum_k \binom{m}{2k} + \sum_k \binom{m}{2k+1},$$

we find that (5.3) is a direct consequence of (5.4) and (5.5). $\square$

Lemma 5.2. *Let $m \in \mathbb{N}$, and let the sequences $\tilde{a}^{(m)}$ and $\tilde{b}^{(m)}$ be defined as*

$$\left.\begin{array}{rcl} \tilde{a}_j^{(m)} &:=& \binom{m-2}{j} + \binom{m-1}{j}, \\ \tilde{b}_j^{(m)} &:=& \binom{m-2}{j}, \end{array}\right\} \quad j \in \mathbb{Z}, \tag{5.6}$$

with the conventions

$$\binom{l}{k} = 0 \quad , \quad k \in \mathbb{Z},\ l < 0, \tag{5.7}$$

$$\binom{0}{k} = \delta_k \quad , \quad k \in \mathbb{Z}. \tag{5.8}$$

Then, for any integer $m \geq 2$, it holds that

$$\left.\begin{array}{rcl} \tilde{a}_{j-1}^{(m-1)} + \tilde{a}_j^{(m-1)} &=& \tilde{a}_j^{(m)}, \\ \tilde{b}_{j-1}^{(m-1)} + \tilde{b}_j^{(m-1)} &=& \tilde{b}_j^{(m)}, \end{array}\right\} \quad j \in \mathbb{Z}. \tag{5.9}$$

Proof. Let the integer $m \geq 2$ be fixed. According to the first line of (5.6), we have

$$\begin{aligned} \tilde{a}_{j-1}^{(m-1)} + \tilde{a}_j^{(m-1)} &= \left[\binom{m-3}{j-1} + \binom{m-2}{j-1}\right] + \left[\binom{m-3}{j} + \binom{m-2}{j}\right] \\ &= \left[\binom{m-3}{j-1} + \binom{m-3}{j}\right] + \left[\binom{m-2}{j-1} + \binom{m-2}{j}\right] \\ &= \binom{m-2}{j} + \binom{m-1}{j} \\ &= \tilde{a}_j^{(m)}, \quad j \in \mathbb{Z}, \end{aligned}$$

yielding the first line of (5.9).

Similarly, the second line of (5.6) gives

$$\begin{aligned}\tilde{b}_{j-1}^{(m-1)} + \tilde{b}_j^{(m-1)} &= \binom{m-3}{j-1} + \binom{m-3}{j} \\ &= \binom{m-2}{j} \\ &= \tilde{b}_j^{(m)}, \quad j \in \mathbb{Z},\end{aligned}$$

from which the second line of (5.9) immediately follows. □

Lemma 5.3. *Let* $\mathbf{c} : \mathbb{Z} \to \mathbb{R}^2$ *be a sequence such that* $\|\mathbf{c}\|_{S,\infty} < \infty$*. Then, it holds that*

$$\|\Delta \mathbf{c}\|_{1,\infty} < \infty.$$

Proof. Suppose that $\mathbf{c} : \mathbb{Z} \to \mathbb{R}^2$ is a sequence satisfying

$$\|\mathbf{c}\|_{S,\infty} < \infty. \tag{5.10}$$

Then, from (1.4), we obtain

$$\begin{aligned}\|\Delta \mathbf{c}_j\|_1 &= \|\mathbf{c}_j - \mathbf{c}_{j-1}\|_1 \\ &\leq \|\mathbf{c}_j\|_1 + \|\mathbf{c}_{j-1}\|_1 \\ &\leq \|\mathbf{c}\|_{S,\infty} + \|\mathbf{c}\|_{S,\infty} \\ &\leq 2\,\|\mathbf{c}\|_{S,\infty},\end{aligned} \tag{5.11}$$

by virtue of the first line of (1.23) and the definition (1.26). It follows from (5.10), (5.11) and (1.27) that $\|\Delta \mathbf{c}\|_{1,\infty} < \infty$. □

We are now able to prove our main result which is stated as follows.

Theorem 5.4. *Let* $\mathbf{c} : \mathbb{Z} \to \mathbb{R}^2$ *be a sequence such that*

$$\|\mathbf{c}\|_{S,\infty} < \infty, \tag{5.12}$$

where the norm $\|\cdot\|_{S,\infty}$ *is given by (1.26). Then, for any integer* $m \geq 3$, *the subdivision scheme* $(\tilde{p}_m, \mathbf{c})$ *converges at a geometric rate to the limit function* $\tilde{\mathbf{\Phi}}_{\mathbf{m},\mathbf{c}} := \tilde{\mathbf{\Phi}}_{\mathbf{m}}$ *given by*

$$\tilde{\mathbf{\Phi}}_{\mathbf{m}} = \sum_j \begin{bmatrix} \mathbf{c}_j^T \\ \mathbf{c}_j^T \end{bmatrix} \tilde{\mathbf{N}}_{\mathbf{m}}(\cdot - j), \tag{5.13}$$

by means of

$$\left\| \tilde{\mathbf{\Phi}}_{\mathbf{m}}\left(\frac{\cdot}{2^r}\right) - \mathbf{c}^{(r)}_{(\cdot-1)} \right\|_{1,\infty} \leq \tilde{K}_m^{(r)}, \quad r \in \mathbb{N}, \tag{5.14}$$

with

$$\tilde{K}_m^{(r)} = \left(\frac{1}{2}\right)^{r-1} (4m - 11)\; \|\mathbf{c}\|_{S,\infty}, \quad r \in \mathbb{N}. \tag{5.15}$$

Proof. Let fix the integer $m \geq 3$. Note first from Lemma 5.3 and equation (5.12) that $\|\Delta \mathbf{c}\|_{1,\infty} < \infty$. To prove that we have subdivision convergence, we shall show that, for $r \in \mathbb{N}$, with the operator Δ and the norm $\|\cdot\|_{1,\infty}$ respectively given by (1.4) and (1.27), it holds that

$$\left\| \tilde{\mathbf{\Phi}}_{\mathbf{m}}\left(\frac{\cdot}{2^r}\right) - \mathbf{c}^{(r)}_{(\cdot-1)} \right\|_{1,\infty} \leq \left(\frac{1}{2}\right)^{r-1} (2\;(m-3)\|\Delta \mathbf{c}\|_{1,\infty} + \|\mathbf{c}\|_{S,\infty}\;) \sum_{k=1}^{m-2} \left\| \tilde{\mathbf{N}}_{\mathbf{m}}(k) \right\|_1, \tag{5.16}$$

so that, from (5.11), we get

$$\begin{aligned} \left\| \tilde{\mathbf{\Phi}}_{\mathbf{m}}\left(\frac{\cdot}{2^r}\right) - \mathbf{c}^{(r)}_{(\cdot-1)} \right\|_{1,\infty} &\leq \left(\frac{1}{2}\right)^{r-1} (2\;(m-3)\;2\|\mathbf{c}\|_{S,\infty} + \|\mathbf{c}\|_{S,\infty}\;) \sum_{k=1}^{m-2} \left\| \tilde{\mathbf{N}}_{\mathbf{m}}(k) \right\|_1 \\ &\leq \left(\frac{1}{2}\right)^{r-1} (4m-11)\;\|\mathbf{c}\|_{S,\infty} \sum_{k=1}^{m-2} \left\| \tilde{\mathbf{N}}_{\mathbf{m}}(k) \right\|_1. \end{aligned} \tag{5.17}$$

Observing from (1.22), (4.38), (4.39), and (4.41) that

$$\sum_k \left\|\tilde{\mathbf{N}}_{\mathbf{m}}(k)\right\|_1 = \sum_k \sup\left\{\tilde{N}_{m,0}(k),\ \tilde{N}_{m,1}(k)\right\} \leq \sum_k \left\{\tilde{N}_{m,0}(k) + \tilde{N}_{m,1}(k)\right\} = 1, \quad (5.18)$$

the desired result (5.14), (5.15) follows directly from (5.17) and (5.18).

Hence, it remains to prove (5.16). To this end, we first compute

$$\tilde{\mathbf{\Phi}}_{\mathbf{m}}\left(\frac{j}{2^r}\right) - \mathbf{c}_{j-1}^{(r)}, \quad j \in \mathbb{Z}.$$

Using equation (5.13) and the refinement equation (4.25), we obtain, for $j \in \mathbb{Z}$, and $r \in \mathbb{Z}_+$,

$$\tilde{\mathbf{\Phi}}_{\mathbf{m}}\left(\frac{j}{2^r}\right) = \sum_k \begin{bmatrix} \left(\mathbf{c}_{\mathbf{k}}^{(\mathbf{0})}\right)^{\mathrm{T}} \\ \left(\mathbf{c}_{\mathbf{k}}^{(\mathbf{0})}\right)^{\mathrm{T}} \end{bmatrix} \tilde{\mathbf{N}}_{\mathbf{m}}\left(\frac{j}{2^r} - k\right)$$

$$= \sum_k \begin{bmatrix} \left(\mathbf{c}_{\mathbf{k}}^{(\mathbf{0})}\right)^{\mathrm{T}} \\ \left(\mathbf{c}_{\mathbf{k}}^{(\mathbf{0})}\right)^{\mathrm{T}} \end{bmatrix} \sum_l \tilde{p}_{m,l}\, \tilde{\mathbf{N}}_{\mathbf{m}}\left(\frac{j}{2^{r-1}} - 2k - l\right)$$

$$= \sum_k \begin{bmatrix} \left(\mathbf{c}_{\mathbf{k}}^{(\mathbf{0})}\right)^{\mathrm{T}} \\ \left(\mathbf{c}_{\mathbf{k}}^{(\mathbf{0})}\right)^{\mathrm{T}} \end{bmatrix} \sum_l \tilde{p}_{m,l-2k}\, \tilde{\mathbf{N}}_{\mathbf{m}}\left(\frac{j}{2^{r-1}} - l\right)$$

$$= \sum_l \sum_k \begin{bmatrix} \left(\mathbf{c}_{\mathbf{k}}^{(\mathbf{0})}\right)^{\mathrm{T}} \\ \left(\mathbf{c}_{\mathbf{k}}^{(\mathbf{0})}\right)^{\mathrm{T}} \end{bmatrix} \tilde{p}_{m,l-2k}\, \tilde{\mathbf{N}}_{\mathbf{m}}\left(\frac{j}{2^{r-1}} - l\right)$$

$$= \sum_l \sum_k \left[\left(\tilde{p}_{m,l-2k}\right)^{\mathrm{T}} \begin{pmatrix} \mathbf{c}_{\mathbf{k}}^{(\mathbf{0})} & \mathbf{c}_{\mathbf{k}}^{(\mathbf{0})} \end{pmatrix}\right]^{\mathrm{T}} \tilde{\mathbf{N}}_{\mathbf{m}}\left(\frac{j}{2^{r-1}} - l\right)$$

$$= \sum_l \left[\sum_k (\tilde{p}_{m,l-2k})^{\mathrm{T}} \begin{pmatrix} \mathbf{c}_{\mathbf{k}}^{(\mathbf{0})} & \mathbf{c}_{\mathbf{k}}^{(\mathbf{0})} \end{pmatrix} \right]^{\mathrm{T}} \tilde{\mathbf{N}}_{\mathbf{m}} \left(\frac{j}{2^{r-1}} - l \right)$$

$$= \sum_l \left[\begin{matrix} \sum_k [\tilde{p}_{m,l-2k}]^{\mathrm{T}} \mathbf{c}_{\mathbf{k}}^{(\mathbf{0})} & [\tilde{p}_{m,l-2k}]^{\mathrm{T}} \mathbf{c}_{\mathbf{k}}^{(\mathbf{0})} \end{matrix} \right]^{\mathrm{T}} \tilde{\mathbf{N}}_{\mathbf{m}} \left(\frac{j}{2^{r-1}} - l \right),$$

which, together with (5.1), gives, for $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$,

$$\tilde{\mathbf{\Phi}}_{\mathbf{m}} \left(\frac{j}{2^r} \right) = \sum_l \left[\begin{matrix} \mathbf{c}_{\mathbf{l}}^{(\mathbf{1})} & \mathbf{c}_{\mathbf{l}}^{(\mathbf{1})} \end{matrix} \right]^{\mathrm{T}} \tilde{\mathbf{N}}_{\mathbf{m}} \left(\frac{j}{2^{r-1}} - l \right)$$

$$= \sum_l \left[\begin{matrix} \left(\mathbf{c}_{\mathbf{l}}^{(\mathbf{1})} \right)^{\mathrm{T}} \\ \left(\mathbf{c}_{\mathbf{l}}^{(\mathbf{1})} \right)^{\mathrm{T}} \end{matrix} \right] \tilde{\mathbf{N}}_{\mathbf{m}} \left(\frac{j}{2^{r-1}} - l \right).$$

Repeated applications of this procedure yield

$$\begin{aligned} \tilde{\mathbf{\Phi}}_{\mathbf{m}} \left(\frac{j}{2^r} \right) &= \sum_l \left[\begin{matrix} \left(\mathbf{c}_{l}^{(r)} \right)^{\mathrm{T}} \\ \left(\mathbf{c}_{l}^{(r)} \right)^{\mathrm{T}} \end{matrix} \right] \tilde{\mathbf{N}}_{\mathbf{m}}(j-l) \\ &= \sum_l \left[\begin{matrix} \left(\mathbf{c}_{j-l}^{(r)} \right)^{\mathrm{T}} \\ \left(\mathbf{c}_{j-l}^{(r)} \right)^{\mathrm{T}} \end{matrix} \right] \tilde{\mathbf{N}}_{\mathbf{m}}(l), \quad j \in \mathbb{Z}, \ r \in \mathbb{Z}_+. \end{aligned} \tag{5.19}$$

It follows from (4.41) and (5.19) that, for $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$,

$$\tilde{\mathbf{\Phi}}_{\mathbf{m}} \left(\frac{j}{2^r} \right) - \mathbf{c}_{j-1}^{(r)} = \sum_k \left[\begin{matrix} \left(\mathbf{c}_{j-k}^{(r)} \right)^{\mathrm{T}} \\ \left(\mathbf{c}_{j-k}^{(r)} \right)^{\mathrm{T}} \end{matrix} \right] \tilde{\mathbf{N}}_{\mathbf{m}}(k) - \left[\begin{matrix} \mathbf{c}_{j-1}^{(r)} & \mathbf{c}_{j-1}^{(r)} \end{matrix} \right] \sum_k \left[\begin{matrix} \tilde{N}_{m,0}(k) \\ \tilde{N}_{m,1}(k) \end{matrix} \right]$$

$$= \sum_k \left[\begin{matrix} \left(\mathbf{c}_{j-k}^{(r)} \right)^{\mathrm{T}} \\ \left(\mathbf{c}_{j-k}^{(r)} \right)^{\mathrm{T}} \end{matrix} \right] \tilde{\mathbf{N}}_{\mathbf{m}}(k) - \sum_k \left[\begin{matrix} \mathbf{c}_{j-1}^{(r)} & \mathbf{c}_{j-1}^{(r)} \end{matrix} \right] \left[\begin{matrix} \tilde{N}_{m,0}(k) \\ \tilde{N}_{m,1}(k) \end{matrix} \right]$$

$$= \sum_k \begin{bmatrix} \left(\mathbf{c}_{j-k}^{(r)}\right)^{\mathrm{T}} \\ \left(\mathbf{c}_{j-k}^{(r)}\right)^{\mathrm{T}} \end{bmatrix} \tilde{\mathbf{N}}_{\mathbf{m}}(k) - \sum_k \begin{bmatrix} \mathbf{c}_{j-1}^{(r)} & \mathbf{c}_{j-1}^{(r)} \end{bmatrix} \tilde{\mathbf{N}}_{\mathbf{m}}(k)$$

$$= \sum_k \left\{ \begin{bmatrix} \left(\mathbf{c}_{j-k}^{(r)}\right)^{\mathrm{T}} \\ \left(\mathbf{c}_{j-k}^{(r)}\right)^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} \mathbf{c}_{j-1}^{(r)} & \mathbf{c}_{j-1}^{(r)} \end{bmatrix} \right\} \tilde{\mathbf{N}}_{\mathbf{m}}(k)$$

$$= \sum_k \begin{bmatrix} c_{j-k,0}^{(r)} - c_{j-1,0}^{(r)} & c_{j-k,1}^{(r)} - c_{j-1,0}^{(r)} \\ \\ c_{j-k,0}^{(r)} - c_{j-1,1}^{(r)} & c_{j-k,1}^{(r)} - c_{j-1,1}^{(r)} \end{bmatrix} \tilde{\mathbf{N}}_{\mathbf{m}}(k)$$

$$= \sum_{k=1}^{m-2} \begin{bmatrix} c_{j-k,0}^{(r)} - c_{j-1,0}^{(r)} & c_{j-k,1}^{(r)} - c_{j-1,0}^{(r)} \\ \\ c_{j-k,0}^{(r)} - c_{j-1,1}^{(r)} & c_{j-k,1}^{(r)} - c_{j-1,1}^{(r)} \end{bmatrix} \tilde{\mathbf{N}}_{\mathbf{m}}(k), \tag{5.20}$$

by virtue also of (4.38). Hence,

$$\tilde{\boldsymbol{\Phi}}_{\mathbf{m}}\left(\frac{j}{2^r}\right) - \mathbf{c}_{j-1}^{(r)} = \sum_{k=1}^{m-2} B_{j-k,j-1}^{(r)} \, \tilde{\mathbf{N}}_{\mathbf{m}}(k), \quad j \in \mathbb{Z}, \; r \in \mathbb{Z}_+, \tag{5.21}$$

where

$$B_{d,p}^{(r)} := \begin{bmatrix} c_{d,0}^{(r)} - c_{p,0}^{(r)} & c_{d,1}^{(r)} - c_{p,0}^{(r)} \\ \\ c_{d,0}^{(r)} - c_{p,1}^{(r)} & c_{d,1}^{(r)} - c_{p,1}^{(r)} \end{bmatrix}, \quad d \in \mathbb{Z}, \; p \in \mathbb{Z}, \; r \in \mathbb{Z}_+. \tag{5.22}$$

It follows from (5.21) and (1.25) that, for $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$,

$$\begin{aligned}\left\|\tilde{\mathbf{\Phi}}_{\mathbf{m}}\left(\frac{j}{2^r}\right) - \mathbf{c}_{j-1}^{(r)}\right\|_1 &\leq \sum_{k=1}^{m-2} \left\|B_{j-k,j-1}^{(r)}\right\|_S \ \left\|\tilde{\mathbf{N}}_{\mathbf{m}}(k)\right\|_1 \\ &\leq \sup\left\{\left\|B_{d,p}^{(r)}\right\|_S : \ d,p \in \mathbb{Z}, \ |d-p| \leq m-3\right\} \sum_{k=1}^{m-2} \left\|\tilde{\mathbf{N}}_{\mathbf{m}}(k)\right\|_1,\end{aligned} \tag{5.23}$$

by virtue of the fact that

$$|(j-1)-(j-k)| \leq k-1 \leq m-3, \quad j \in \mathbb{Z}, \ k = 1, \ldots, m-2.$$

Comparing (5.23), (5.14) and (5.15), we see that it will suffice to prove that, for $r \in \mathbb{N}$,

$$\sup\left\{\left\|B_{l,k}^{(r)}\right\|_S : \ d,p \in \mathbb{Z}, \ |l-k| \leq m-3\right\} \leq \left(\frac{1}{2}\right)^{r-1} \Big(2\ (m-3)\ \|\Delta \mathbf{c}\|_{1,\infty} + \|\mathbf{c}\|_{S,\infty}\Big). \tag{5.24}$$

Observe from (5.22) that

$$\left\|B_{l,k}^{(r)}\right\|_S = \left\|\begin{bmatrix} c_{l,0}^{(r)} - c_{k,0}^{(r)} & c_{l,1}^{(r)} - c_{k,0}^{(r)} \\ \\ c_{l,0}^{(r)} - c_{k,1}^{(r)} & c_{l,1}^{(r)} - c_{k,1}^{(r)} \end{bmatrix}\right\|_S, \quad l \in \mathbb{Z}, \ k \in \mathbb{Z}, \ r \in \mathbb{N}. \tag{5.25}$$

Substituting (1.19) and (1.21) into (5.25), we get

$$\left\|B_{l,k}^{(r)}\right\|_S = \left\|\begin{bmatrix} c_{l,0}^{(r)} \\ \\ c_{l,1}^{(r)} \end{bmatrix} - \begin{bmatrix} c_{k,0}^{(r)} \\ \\ c_{k,1}^{(r)} \end{bmatrix}\right\|_S + \left\|\begin{bmatrix} c_{l,0}^{(r)} \\ \\ c_{l,1}^{(r)} \end{bmatrix} - \begin{bmatrix} c_{k,1}^{(r)} \\ \\ c_{k,0}^{(r)} \end{bmatrix}\right\|_S$$

$$\leq \left\| \mathbf{c}_l^{(r)} - \mathbf{c}_k^{(r)} \right\|_S + \left\| \begin{bmatrix} c_{l,0}^{(r)} \\ \\ c_{l,1}^{(r)} \end{bmatrix} - \begin{bmatrix} c_{k,0}^{(r)} \\ \\ c_{k,1}^{(r)} \end{bmatrix} \right\|_S + \left\| \begin{bmatrix} c_{k,0}^{(r)} \\ \\ c_{k,1}^{(r)} \end{bmatrix} - \begin{bmatrix} c_{k,1}^{(r)} \\ \\ c_{k,0}^{(r)} \end{bmatrix} \right\|_S$$

$$= 2 \left\| \mathbf{c}_l^{(r)} - \mathbf{c}_k^{(r)} \right\|_S + \left\| \begin{bmatrix} c_{k,0}^{(r)} \\ \\ c_{k,1}^{(r)} \end{bmatrix} - \begin{bmatrix} c_{k,1}^{(r)} \\ \\ c_{k,0}^{(r)} \end{bmatrix} \right\|_S, \quad l \in \mathbb{Z},\ k \in \mathbb{Z},\ r \in \mathbb{Z}_+,$$

which, together with the second line of (1.23), implies that, for $r \in \mathbb{Z}_+$,

$$\left\| B_{l,k}^{(r)} \right\|_S \leq 4 \left\| \mathbf{c}_l^{(r)} - \mathbf{c}_k^{(r)} \right\|_1 + 2 \left\| \begin{bmatrix} c_{k,0}^{(r)} \\ c_{k,1}^{(r)} \end{bmatrix} - \begin{bmatrix} c_{k,1}^{(r)} \\ c_{k,0}^{(r)} \end{bmatrix} \right\|_1, \quad l \in \mathbb{Z},\ k \in \mathbb{Z}. \tag{5.26}$$

Let us now consider two integers u and v satisfying

$$0 \leq u - v \leq m - 3, \tag{5.27}$$

and let us bound the right-hand side of (5.26). Using the definition (1.4), we obtain

$$\mathbf{c}_u^{(r)} - \mathbf{c}_v^{(r)} = \sum_{j=v+1}^{u} \Delta \mathbf{c}_j^{(r)}, \quad r \in \mathbb{Z}_+,$$

which means that

$$\begin{aligned} \left\| \mathbf{c}_u^{(r)} - \mathbf{c}_v^{(r)} \right\|_1 &\leq \sum_{j=v+1}^{u} \left\| \Delta \mathbf{c}^{(r)} \right\|_{1,\infty} \\ &= (u-v) \left\| \Delta \mathbf{c}^{(r)} \right\|_{1,\infty} \\ &\leq (m-3) \left\| \Delta \mathbf{c}^{(r)} \right\|_{1,\infty}, \quad r \in \mathbb{Z}_+, \end{aligned} \tag{5.28}$$

having also used equation (5.27). We claim that

$$\left\|\Delta \mathbf{c}^{(r)}\right\|_{1,\infty} \leq \frac{1}{2}\left\|\mathbf{c}^{(r-1)}\right\|_{1,\infty}, \quad r \in \mathbb{N}. \tag{5.29}$$

Using (1.4) and (5.1), we obtain

$$\begin{aligned} \Delta \mathbf{c}_j^{(r)} &= \mathbf{c}_j^{(r-1)} - \mathbf{c}_{j-1}^{(r-1)} \\ &= \sum_k p_{j-2k}^{(m)}\, \mathbf{c}_k^{(r-1)} - \sum_k p_{(j-1)-2k}^{(m)}\, \mathbf{c}_k^{(r-1)}, \quad j \in \mathbb{Z},\ r \in \mathbb{N}. \end{aligned} \tag{5.30}$$

Next, using the explicit formulae (4.61) and (4.62) for the refinement mask, observe that

$$\tilde{p}_{m,j} = \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}_j^{(m)} & \tilde{b}_j^{(m)} \\ \tilde{b}_{(m-1)-j}^{(m)} & \tilde{a}_{(m-1)-j}^{(m)} \end{bmatrix}, \quad j \in \mathbb{Z}, \tag{5.31}$$

where the sequences $\tilde{a}^{(m)}$ and $\tilde{b}^{(m)}$ are defined by (5.6).

It follows from (5.30), (5.31), (5.9), together with the fact that

$$(m-1) - [(j-1) - 2k] = m - (j-2k),$$

that we have, for any integers $j \in \mathbb{Z}$ and $r \in \mathbb{N}$,

$$\begin{aligned} \Delta \mathbf{c}_j^{(r)} &= \sum_k \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}_{j-2k}^{(m)} & \tilde{b}_{j-2k}^{(m)} \\ \tilde{b}_{(m-1)-(j-2k)}^{(m)} & \tilde{a}_{(m-1)-(j-2k)}^{(m)} \end{bmatrix}^{\mathrm{T}} \mathbf{c}_k^{(r-1)} \\ &\quad - \sum_k \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}_{(j-1)-2k}^{(m)} & \tilde{b}_{(j-1)-2k}^{(m)} \\ \tilde{b}_{m-(j-2k)}^{(m)} & \tilde{a}_{m-(j-2k)}^{(m)} \end{bmatrix}^{\mathrm{T}} \mathbf{c}_k^{(r-1)} \end{aligned}$$

$$= \sum_k \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}^{(m-1)}_{(j-1)-2k} + \tilde{a}^{(m-1)}_{j-2k} & \tilde{b}^{(m-1)}_{(j-1)-2k} + \tilde{b}^{(m-1)}_{j-2k} \\ \\ \tilde{b}^{(m-1)}_{(m-2)-(j-2k)} + \tilde{b}^{(m-1)}_{(m-1)-(j-2k)} & \tilde{a}^{(m-1)}_{(m-2)-(j-2k)} + \tilde{a}^{(m-1)}_{(m-1)-(j-2k)} \end{bmatrix}^{\mathrm{T}} \mathbf{c}^{(r-1)}_k$$

$$- \sum_k \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}^{(m-1)}_{(j-2)-2k} + \tilde{a}^{(m-1)}_{(j-1)-2k} & \tilde{b}^{(m-1)}_{(j-2)-2k} + \tilde{b}^{(m-1)}_{(j-1)-2k} \\ \\ \tilde{b}^{(m-1)}_{(m-1)-(j-2k)} + \tilde{b}^{(m-1)}_{m-(j-2k)} & \tilde{a}^{(m-1)}_{(m-1)-(j-2k)} + \tilde{a}^{(m-1)}_{m-(j-2k)} \end{bmatrix}^{\mathrm{T}} \mathbf{c}^{(r-1)}_k.$$

After simplifications and observing that

$$m - (j - 2k) = (m - 2) - [j - 2(1 + k)],$$

we get, for $j \in \mathbb{Z}$ and $r \in \mathbb{N}$,

$$\Delta \mathbf{c}^{(r)}_j = \sum_k \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}^{(m-1)}_{j-2k} & \tilde{b}^{(m-1)}_{j-2k} \\ \\ \tilde{b}^{(m-1)}_{(m-2)-(j-2k)} & \tilde{a}^{(m-1)}_{(m-2)-(j-2k)} \end{bmatrix}^{\mathrm{T}} \mathbf{c}^{(r-1)}_k$$

$$- \sum_k \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}^{(m-1)}_{j-2(1+k)} & \tilde{b}^{(m-1)}_{j-2(1+k)} \\ \\ \tilde{b}^{(m-1)}_{(m-2)-[j-2(1+k)]} & \tilde{a}^{(m-1)}_{(m-2)-[j-2(1+k)]} \end{bmatrix}^{\mathrm{T}} \mathbf{c}^{(r-1)}_k$$

$$= \sum_k \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}^{(m-1)}_{j-2k} & \tilde{b}^{(m-1)}_{j-2k} \\ \\ \tilde{b}^{(m-1)}_{(m-2)-(j-2k)} & \tilde{a}^{(m-1)}_{(m-2)-(j-2k)} \end{bmatrix}^{\mathrm{T}} \left[\mathbf{c}^{(r-1)}_k - \mathbf{c}^{(r-1)}_{k-1} \right],$$

which, together with (1.4) and (5.31), yields

$$\Delta\mathbf{c}_j^{(r)} = \sum_k \frac{1}{2}\left(\tilde{p}_{m-1,j-2k}\right)^{\mathrm{T}} \; \Delta\mathbf{c}_k^{(r-1)}, \quad j \in \mathbb{Z}, \; r \in \mathbb{N}, \tag{5.32}$$

which in turn implies that

$$\left\|\Delta\mathbf{c}^{(r)}\right\|_{1,\infty} \leq \left[\frac{1}{2} \sup_j \left\{\sum_k \left\|\tilde{p}_{m-1,j-2k}^{\mathrm{T}}\right\|_1\right\}\right] \; \left\|\Delta\mathbf{c}^{(r-1)}\right\|_{1,\infty}, \quad r \in \mathbb{N}, \tag{5.33}$$

by virtue of (1.24) and (1.27).

Next, let us compute $\sum_k \left\|\tilde{p}_{m-1,j-2k}\right\|_1$, for any fixed $j \in \mathbb{Z}$. Using equations (1.20) and (5.31), together with the fact that

$$\left.\begin{array}{l} \tilde{a}_j^{(m)} \geq 0, \\ \tilde{b}_j^{(m)} \geq 0, \end{array}\right\} \; j \in \mathbb{Z}, \; m \in \mathbb{N},$$

we get, for any $j \in \mathbb{Z}$,

$$\begin{aligned} \sum_k \left\|\tilde{p}_{m-1,j-2k}\right\|_1 = & \sum_k \sup\left\{\frac{1}{2^{m-2}}\left(\tilde{a}_{j-2k}^{(m-1)} + \tilde{b}_{(m-2)-(j-2k)}^{(m-1)}\right),\right. \\ & \left.\frac{1}{2^{m-2}}\left(\tilde{b}_{j-2k}^{(m-1)} + \tilde{a}_{(m-2)-(j-2k)}^{(m-1)}\right)\right\} \\ = & \frac{1}{2^{m-2}} \sum_k \sup\left\{\tilde{a}_{j-2k}^{(m-1)} + \tilde{b}_{(m-2)-(j-2k)}^{(m-1)},\right. \\ & \left.\tilde{b}_{j-2k}^{(m-1)} + \tilde{a}_{(m-2)-(j-2k)}^{(m-1)}\right\} \\ = & \frac{1}{2^{m-2}} \sup\left\{\sum_k \tilde{a}_{j-2k}^{(m-1)} + \sum_k \tilde{b}_{(m-2)-(j-2k)}^{(m-1)},\right. \\ & \left.\sum_k \tilde{b}_{j-2k}^{(m-1)} + \sum_k \tilde{a}_{(m-2)-(j-2k)}^{(m-1)}\right\}. \end{aligned} \tag{5.34}$$

Note that for any fixed integers m and j, the indices appearing in the right-hand side of

(5.34) are such that

$$\left.\begin{array}{l} j-2k \text{ are either all odd or all even,} \\ (m-2)-(j-2k) \text{ are either all odd or all even,} \end{array}\right\}, \quad k \in \mathbb{Z}.$$

We claim that

$$\left.\begin{array}{l} \sum_l \tilde{a}_{2l}^{(m)} = \sum_l \tilde{a}_{2l+1}^{(m)}, \\ \\ \sum_l \tilde{b}_{2l}^{(m)} = \sum_l \tilde{b}_{2l+1}^{(m)}, \end{array}\right\} m \geq 3, \tag{5.35}$$

which, together with (5.34), implies

$$\sum_k \|\tilde{p}_{m-1,j-2k}\|_1 = \frac{1}{2^{m-2}}\left(\sum_l \tilde{a}_{2l}^{(m-1)} + \sum_l \tilde{b}_{2l}^{(m-1)}\right), \quad j \in \mathbb{Z},\ m \geq 4. \tag{5.36}$$

Let us then prove that (5.35) indeed holds. According to the first line of (5.6) and the result (5.3) in Lemma 5.1, we have that

$$\begin{aligned} \sum_l \tilde{a}_{2l}^{(m)} &= \sum_l \binom{m-2}{2l} + \binom{m-1}{2l} \\ &= \sum_l \binom{m-2}{2l} + \sum_l \binom{m-1}{2l} \\ &= \sum_l \binom{m-2}{2l+1} + \sum_l \binom{m-2}{2l+1} \\ &= \sum_l \binom{m-2}{2l+1} + \binom{m-2}{2l+1} \\ &= \sum_l \tilde{a}_{2l+1}^{(m)}, \end{aligned}$$

whereas

$$\sum_l \tilde{a}_{2l}^{(m)} = \sum_l \tilde{a}_{2l+1}^{(m)} = 2^{m-3} + 2^{m-2}, \tag{5.37}$$

having again used Lemma 5.1 together with (5.8) and (5.7). Similarly, we obtain

$$\sum_l \tilde{b}_{2l}^{(m)} = \sum_l \tilde{b}_{2l+1}^{(m)} = 2^{m-3}. \tag{5.38}$$

In other words, equation (5.35), and hence also (5.36), hold. Using (5.36), (5.37) and (5.38), we get, for $m \geq 4$ and $j \in \mathbb{Z}$,

$$\begin{aligned}\sum_k \|\tilde{p}_{m-1,j-2k}\|_1 &= \frac{1}{2^{m-2}}\left[\sum_l \tilde{a}_{2l}^{(m-1)} + \sum_l \tilde{b}_{2l}^{(m-1)}\right] \\ &= \frac{1}{2^{m-2}}\left[\left(2^{m-4}+2^{m-3}\right)+2^{m-4}\right] \\ &= 1. \end{aligned}\tag{5.39}$$

Similarly, we compute $\sum_k \|\tilde{p}_{m-1,j-2k}\|_1$, for $j \in \mathbb{Z}$ and $m = 3$. Using the first line of (5.6), (5.7) and (5.8), we obtain

$$\begin{cases} \sum_l \tilde{a}_{2l}^{(2)} = 2, & \sum_l \tilde{a}_{2l+1}^{(2)} = 1, \\ \sum_l \tilde{b}_{2l}^{(2)} = 1, & \sum_l \tilde{b}_{2l+1}^{(2)} = 0. \end{cases}\tag{5.40}$$

Let us now successively consider the cases where $j \in \mathbb{Z}$ is odd, and where $j \in \mathbb{Z}$ is even.

Hence, let us fix an even integer $j \in \mathbb{Z}$. But then, for every $k \in \mathbb{Z}$, we have that $j-2k$ is even, whereas $(3-2)-(j-2k)$ is odd, which, inserted into (5.34), gives

$$\begin{aligned}\sum_k \|\tilde{p}_{3-1,j-2k}\|_1 =&\frac{1}{2^{3-2}}\ \sup\left\{\sum_k \tilde{a}_{j-2k}^{(3-1)} + \sum_k \tilde{b}_{(3-2)-(j-2k)}^{(3-1)},\ \sum_k \tilde{b}_{j-2k}^{(3-1)} + \sum_k \tilde{a}_{(3-2)-(j-2k)}^{(3-1)}\right\} \\ =&\frac{1}{2}\ \sup\left\{\sum_l \tilde{a}_{2l}^{(2)} + \sum_l \tilde{b}_{2l+1}^{(2)},\ \sum_l \tilde{b}_{2l}^{(2)} + \sum_k \tilde{a}_{2l+1}^{(2)}\right\}. \end{aligned}\tag{5.41}$$

Substituting (5.40) into (5.41), we get

$$\sum_k \|\tilde{p}_{3-1,j-2k}\|_1 = 1, \quad j \text{ is even.} \tag{5.42}$$

Next, let us fix an odd integer $j \in \mathbb{Z}$. But then, for every $k \in \mathbb{Z}$, we have that $j - 2k$ is odd, whereas $(3-2)-(j-2k)$ is even. Using a similar argument as in the case where j is even, we obtain

$$\sum_k \|\tilde{p}_{3-1,j-2k}\|_1 = 1, \quad j \text{ is odd.} \tag{5.43}$$

Combining (5.39), (5.42) and (5.43), we find that

$$\sum_k \|\tilde{p}_{m-1,j-2k}\|_1 = 1, \quad j \in \mathbb{Z}. \tag{5.44}$$

The claim (5.29) is now a direct consequence of (5.44) and (5.33).

Next, equations (5.1) and (5.29) recursively yield

$$\left\|\Delta \mathbf{c}^{(r)}\right\|_{1,\infty} \leq \left(\frac{1}{2}\right)^r \|\Delta \mathbf{c}\|_{1,\infty}, \quad r \in \mathbb{N}. \tag{5.45}$$

Combining (5.28), (5.27) and (5.45), we find that, for $r \in \mathbb{N}$,

$$\left\|\mathbf{c}_l^{(r)} - \mathbf{c}_k^{(r)}\right\|_1 \leq (m-3)\left(\frac{1}{2}\right)^r \|\Delta \mathbf{c}\|_{1,\infty}, \quad l \in \mathbb{Z},\ k \in \mathbb{Z},\ |l-k| \leq m-3. \tag{5.46}$$

For $r \in \mathbb{N}$ and $k \in \mathbb{Z}$, let us now find an upper bound for the last term on the right-hand side of (5.26), that is,

$$\left\| \begin{bmatrix} c_{k,0}^{(r)} \\ c_{k,1}^{(r)} \end{bmatrix} - \begin{bmatrix} c_{k,1}^{(r)} \\ c_{k,0}^{(r)} \end{bmatrix} \right\|_1 .$$

To this end, note from the definition (1.20) that

$$\left\| \begin{bmatrix} c_{k,0}^{(r)} \\ c_{k,1}^{(r)} \end{bmatrix} - \begin{bmatrix} c_{k,1}^{(r)} \\ c_{k,0}^{(r)} \end{bmatrix} \right\|_1 = \left| c_{k,0}^{(r)} - c_{k,1}^{(r)} \right|, \quad k \in \mathbb{Z}, \ r \in \mathbb{N}. \tag{5.47}$$

Besides, it follows from (5.1) and (5.31) that

$$\begin{aligned}
\begin{bmatrix} c_{k,0}^{(r)} \\ c_{k,1}^{(r)} \end{bmatrix} &= \sum_j (\tilde{p}_{m,k-2j})^{\mathrm{T}} \begin{bmatrix} c_{j,0}^{(r-1)} \\ c_{j,1}^{(r-1)} \end{bmatrix} \\
&= \sum_j \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}_{k-2j}^{(m)} & \tilde{b}_{k-2j}^{(m)} \\ \tilde{b}_{(m-1)-(k-2j)}^{(m)} & \tilde{a}_{(m-1)-(k-2j)}^{(m)} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} c_{j,0}^{(r-1)} \\ c_{j,1}^{(r-1)} \end{bmatrix} \\
&= \sum_j \frac{1}{2^{m-1}} \begin{bmatrix} \tilde{a}_{k-2j}^{(m)} & \tilde{b}_{(m-1)-(k-2j)}^{(m)} \\ \tilde{b}_{k-2j}^{(m)} & \tilde{a}_{(m-1)-(k-2j)}^{(m)} \end{bmatrix} \begin{bmatrix} c_{j,0}^{(r-1)} \\ c_{j,1}^{(r-1)} \end{bmatrix}, \quad k \in \mathbb{Z}, \ r \in \mathbb{N}.
\end{aligned} \tag{5.48}$$

Therefore, for $k \in \mathbb{Z}$ and $r \in \mathbb{N}$,

$$\begin{aligned}
c_{k,0}^{(r)} - c_{k,1}^{(r)} = &\frac{1}{2^{m-1}} \left\{ \sum_j \left[\tilde{a}_{k-2j}^{(m)} - \tilde{b}_{k-2j}^{(m)} \right] c_{j,0}^{(r-1)} \right. \\
&\left. + \sum_j \left[\tilde{b}_{(m-1)-(k-2j)}^{(m)} - \tilde{a}_{(m-1)-(k-2j)}^{(m)} \right] c_{j,1}^{(r-1)} \right\},
\end{aligned}$$

from which we get

$$\begin{aligned}
\left| c_{k,0}^{(r)} - c_{k,1}^{(r)} \right| \leq \frac{1}{2^{m-1}} \Bigg\{ &\sum_j \left| \tilde{a}_{k-2j}^{(m)} - \tilde{b}_{k-2j}^{(m)} \right| \left| c_{j,0}^{(r-1)} \right| \\
&+ \sum_j \left| \tilde{b}_{(m-1)-(k-2j)}^{(m)} - \tilde{a}_{(m-1)-(k-2j)}^{(m)} \right| \left| c_{j,1}^{(r-1)} \right| \Bigg\}
\end{aligned}$$

$$\leq \frac{1}{2^{m-1}} \left\{ \sup_l \left| c_{l,0}^{(r-1)} \right| \sum_j \left| \tilde{a}_{k-2j}^{(m)} - \tilde{b}_{k-2j}^{(m)} \right| \right.$$

$$\left. + \sup_l \left| c_{l,1}^{(r-1)} \right| \sum_j \left| \tilde{a}_{(m-1)-(k-2j)}^{(m)} - \tilde{b}_{(m-1)-(k-2j)}^{(m)} \right| \right\}. \tag{5.49}$$

Next, observe from (5.6) that

$$\left| \tilde{a}_j^{(m)} - \tilde{b}_j^{(m)} \right| = \tilde{a}_j^{(m)} - \tilde{b}_j^{(m)} = \binom{m-1}{j}, \quad j \in \mathbb{Z}.$$

Applying a similar argument to the one leading to (5.35), we then find

$$\sum_l \left| \tilde{a}_{2l}^{(m)} - \tilde{b}_{2l}^{(m)} \right| = \sum_l \left| \tilde{a}_{2l+1}^{(m)} - \tilde{b}_{2l+1}^{(m)} \right| = 2^{m-2}. \tag{5.50}$$

Since each of the indices $k-2j$ and $(m-1)-(k-2j)$ are either all odd or all even for any fixed integers m and k, we deduce from (5.50) that

$$\sum_j \left| \tilde{a}_{k-2j}^{(m)} - \tilde{b}_{k-2j}^{(m)} \right| = \sum_j \left| \tilde{a}_{(m-1)-(k-2j)}^{(m)} - \tilde{b}_{(m-1)-(k-2j)}^{(m)} \right| = 2^{m-2}, \quad k \in \mathbb{Z},$$

which, inserted into (5.49), gives

$$\left| c_{k,0}^{(r)} - c_{k,1}^{(r)} \right| \leq \frac{1}{2^{m-1}} \left[2^{m-2} \left(\sup_l \left| c_{l,0}^{(r-1)} \right| + \sup_l \left| c_{l,1}^{(r-1)} \right| \right) \right]$$

$$= \frac{1}{2} \left(\sup_l \left| c_{l,0}^{(r-1)} \right| + \sup_l \left| c_{l,1}^{(r-1)} \right| \right)$$

$$= \frac{1}{2} \sup_l \left(\left| c_{l,0}^{(r-1)} \right| + \left| c_{l,1}^{(r-1)} \right| \right), \quad k \in \mathbb{Z}, \; r \in \mathbb{N}. \tag{5.51}$$

Substituting (1.21) into (5.51), we get

$$\left| c_{k,0}^{(r)} - c_{k,1}^{(r)} \right| \leq \frac{1}{2} \left\| \mathbf{c}^{(r-1)} \right\|_{S,\infty}, \quad k \in \mathbb{Z}, \; r \in \mathbb{N},$$

which, together with the first line of (5.1), recursively yields

$$\left| c_{k,0}^{(r)} - c_{k,1}^{(r)} \right| \leq \left(\frac{1}{2}\right)^r \|\mathbf{c}\|_{S,\infty}, \quad k \in \mathbb{Z},\ r \in \mathbb{N}. \tag{5.52}$$

It follows from (5.47) and (5.52) that

$$\left\| \begin{bmatrix} c_{k,0}^{(r)} \\ c_{k,1}^{(r)} \end{bmatrix} - \begin{bmatrix} c_{k,1}^{(r)} \\ c_{k,0}^{(r)} \end{bmatrix} \right\|_1 \leq \left(\frac{1}{2}\right)^r \|\mathbf{c}\|_{S,\infty}, \quad k \in \mathbb{Z},\ r \in \mathbb{N}. \tag{5.53}$$

Combining equations (5.26), (5.46) and (5.53), we find that, for $l \in \mathbb{Z}$ and $k \in \mathbb{Z}$ such that

$$|l-k| \leq m-3,$$

it holds that

$$\begin{aligned} \left\| B_{l,k}^{(r)} \right\|_S &\leq 4\,(m-3)\left(\frac{1}{2}\right)^r \|\Delta \mathbf{c}\|_{1,\infty} + 2\left(\frac{1}{2}\right)^r \|\mathbf{c}\|_{S,\infty} \\ &= \left(\frac{1}{2}\right)^{r-1} \left(2\,(m-3)\,\|\Delta \mathbf{c}\|_{1,\infty} + \|\mathbf{c}\|_{S,\infty}\right), \quad r \in \mathbb{N}. \end{aligned} \tag{5.54}$$

The desired result (5.16) follows directly from (5.23) and (5.54). □

Remark 5.5. As opposed to the vector spline $\tilde{\mathbf{N}}_{\mathbf{m}}$ recursively given by (4.1), the recursive formula (4.83) yields a stable and efficient method for the computation of the vector B-spline $\mathbf{N}_{\mathbf{m}}$, for $m \geq 3$. However, as implied by Theorem 5.4, with the definitions

$$\boldsymbol{\Delta}_{\mathbf{j}}^{(\mathbf{0})} := \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & j = 0, \\ \mathbf{0}, & j \in \mathbb{Z} \setminus \{0\}, \end{cases} \tag{5.55}$$

and

$$\mathbf{\Delta_j^{(1)}} := \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & , \quad j = 0, \\ \mathbf{0} & , \quad j \in \mathbb{Z} \setminus \{0\}, \end{cases} \tag{5.56}$$

the subdivision schemes $\left(\tilde{p}_m, \mathbf{\Delta^{(0)}}\right)$ and $\left(\tilde{p}_m, \mathbf{\Delta^{(1)}}\right)$ converge for $m \geq 3$, providing a very stable way to evaluate respectively the functions $\tilde{N}_{m,0}$ and $\tilde{N}_{m,1}$.

Comparative study

We proceed in this section to discuss the similarities and the differences between the vector splines $\mathbf{\tilde{N}_m}$ and $\mathbf{N_m}$, as respectively studied in Section 4.2 and Section 4.4.

Equations (4.43) (4.46), (4.84) and (4.86), imply that, for $m \geq 2$ and $k = 0, 1$, the functions $N_{m,k}$ and $\tilde{N}_{m,k}$ both belong to $S_{m,2}(\mathbb{Z})$. Besides, (4.38), (4.39) and (4.78) show that, inside the open interval of their respective support, the functions $N_{m,k}$ and $\tilde{N}_{m,k}$ are strictly positive. Similarly, observe that $\mathbf{N_m}$ and $\mathbf{\tilde{N}_m}$ are normalized by means of (4.42), (4.82), and that they both form a partition of unity in the sense of (4.41) and (4.80).

Next, it follows from (4.38) and (4.79) that $\mathbf{N_m}$ has a shorter support than $\mathbf{\tilde{N}_m}$, for $m \geq 3$. What is more, the recursion formula (4.83) gives a direct and stable method for the evaluation of $N_{m,k}$, $k = 0, 1$, for $m \geq 3$. Though there is no such direct recursion formula to compute $\tilde{N}_{m,k}$ for $k = 0, 1$, Remark 5.5 imply that subdivision can be used to evaluate them in a stable and efficient manner. Moreover, it is proved in Theorem 4.14 that the mask p_m associated with $\mathbf{N_m}$ is unique. In contrast, though the unicity of the mask $\tilde{p}_m$ corresponding to $\mathbf{\tilde{N}_m}$ is not investigated, (4.61) and (4.62) give an explicit formula for a mask $p^{(m)}$ such that $\left(p^{(m)}, N_m\right)$, $m \geq 3$, form a refinement pair. Moreover, comparing (4.59) and (4.99), it should be observed that the recursion formula computing the mask symbol P_m is very simple, and generalizes the recursion for the computation of the scalar B-splines mask symbol. Finally, note from (4.40) that, for any integer $m \geq 2$, the components of $\tilde{\mathbf{N}}_{\mathbf{m}}$ present some symmetry properties with respect to each other, whereas the components of $\mathbf{N}_{\mathbf{m}}$ have such symmetry properties only for even integers $m \geq 2$, as discussed below Figure 4.4.

Numerical results

Given some particular examples of initial control points, we proceed in this section to apply the scalar and the matrix subdivision schemes defined by (3.4) and (5.1) with p replaced by $\tilde{p}_m$ or r_m. We then successively evaluate the subdivision error in terms of

$$\left\| \mathbf{\Phi}\left(\frac{\cdot}{2^r}\right) - \mathbf{c}^{(r)}_{(\cdot-1)} \right\|_{1,\infty}, \tag{5.57}$$

where the limit $\mathbf{\Phi}$ is either given by (5.13) or (3.8), depending on the type of subdivision scheme which is considered.

Remark 5.6. For a given sequence $\mathbf{v} : \mathbb{Z} \to \mathbb{R}^2$, let us use the notation

$$\mathbf{v}_j = (v_{j,0}, v_{j,1})^{\mathrm{T}}, \quad j \in \mathbb{Z}.$$

Observe from the definition (5.1) that, as opposed to scalar subdivision schemes, the sequences $C_0^{(r)} := \left\{c_{j,0}^{(r)} : \ j \in \mathbb{Z}\right\}$ and $C_1^{(r)} := \left\{c_{j,1}^{(r)} : \ j \in \mathbb{Z}\right\}$ are linked in the sense that the elements of $C_1^{(r-1)}$ contribute to the computation of $C_0^{(r)}$, and that the elements of $C_0^{(r-1)}$ also contribute to the computation of $C_1^{(r)}$. Consequently, the resulting limit curve, obtained by joining the elements of $c^{(r)} = \left(C_0^{(r)}, C_1^{(r)}\right)^{\mathrm{T}}$ as $r \to \infty$, forms a type of "average" curve between the initial sequences C_0 and C_1 representing the projections of the initial sequence $\mathbf{c}$ into respectively the $x-$ and the $y-$axis. Hence, instead of plotting the sequence c^r in $\mathbb{R}^2$, as done in Figures 3.1 and 3.3, we shall rather plot the projections of $\mathbf{c}$ and $\mathbf{c}^{(\mathbf{r})}$ onto the $x-$ and the $y-$axis.

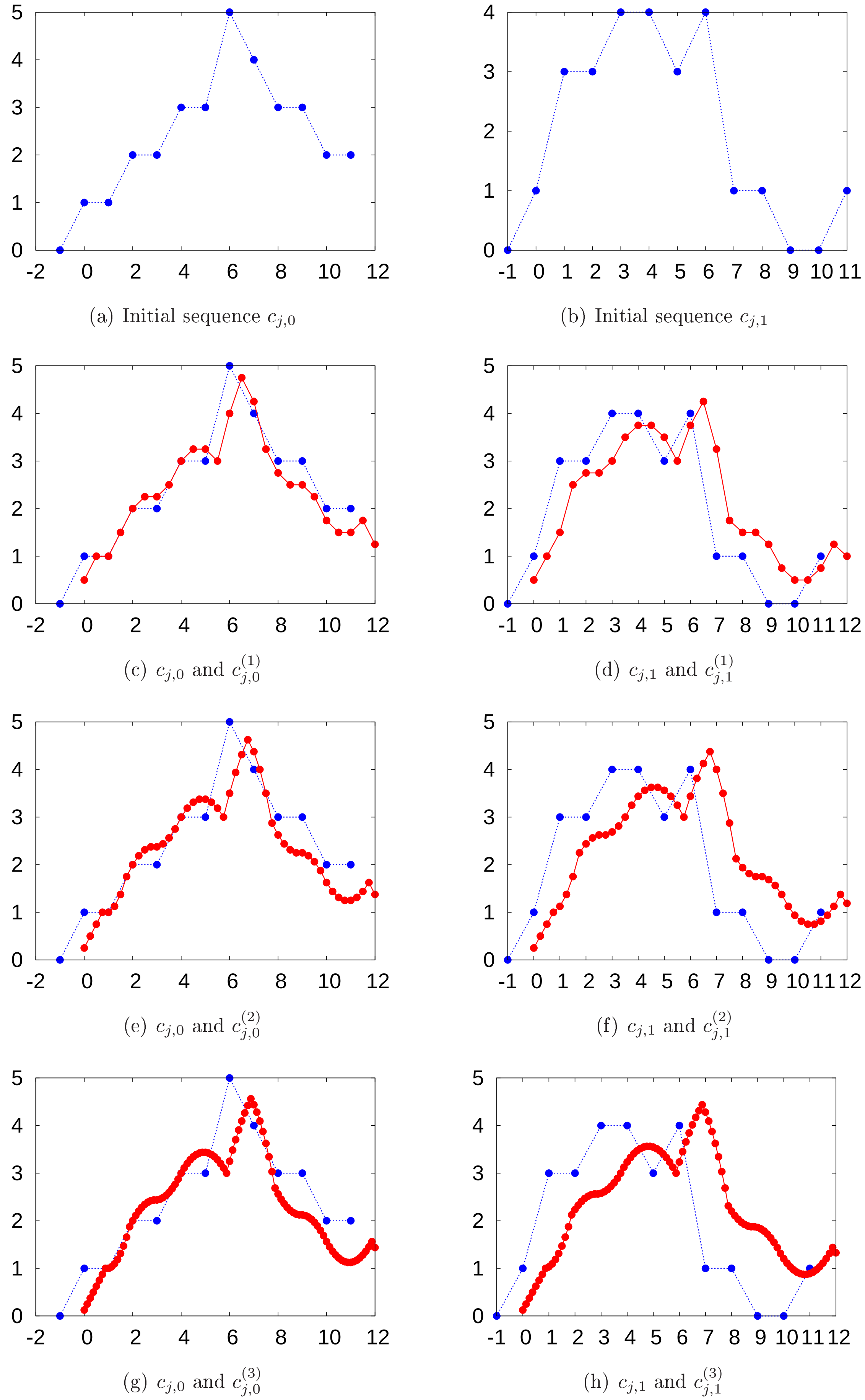
Example 1

With the control points sequence $\mathbf{c} = \{c_j : \ j \in \mathbb{Z}\}$ defined by

$$\begin{cases} \mathbf{c}_0 = (1,1)^{\mathrm{T}}, & \mathbf{c}_1 = (1,3)^{\mathrm{T}}, \qquad \mathbf{c}_2 = (2,3)^{\mathrm{T}}, \\ \mathbf{c}_3 = (2,4)^{\mathrm{T}}, & \mathbf{c}_4 = (3,4)^{\mathrm{T}}, \qquad \mathbf{c}_5 = (3,3)^{\mathrm{T}}, \\ \mathbf{c}_6 = (5,4)^{\mathrm{T}}, & \mathbf{c}_7 = (4,1)^{\mathrm{T}}, \qquad \mathbf{c}_8 = (3,1)^{\mathrm{T}}, \\ \mathbf{c}_9 = (3,0)^{\mathrm{T}}, & \mathbf{c}_{10} = (2,0)^{\mathrm{T}}, \qquad \mathbf{c}_{11} = (2,1)^{\mathrm{T}}, \\ \mathbf{c}_{12} = (1,1)^{\mathrm{T}}, & \mathbf{c}_j = \mathbf{0} \text{ otherwise,} \end{cases} \tag{5.58}$$

we apply the matrix subdivision schemes (SS) $(\tilde{p}_3, \mathbf{c})$, $(r_3, \mathbf{c})$ and the scalar SS $(p_3, \mathbf{c})$. We obtain respectively Figures 5.1, 5.2 and 5.3.

The corresponding subdivision error (5.57) is given in Table 5.1.



(a) Initial sequence $c_{j,0}$

(b) Initial sequence $c_{j,1}$

(c) $c_{j,0}$ and $c_{j,0}^{(1)}$

(d) $c_{j,1}$ and $c_{j,1}^{(1)}$

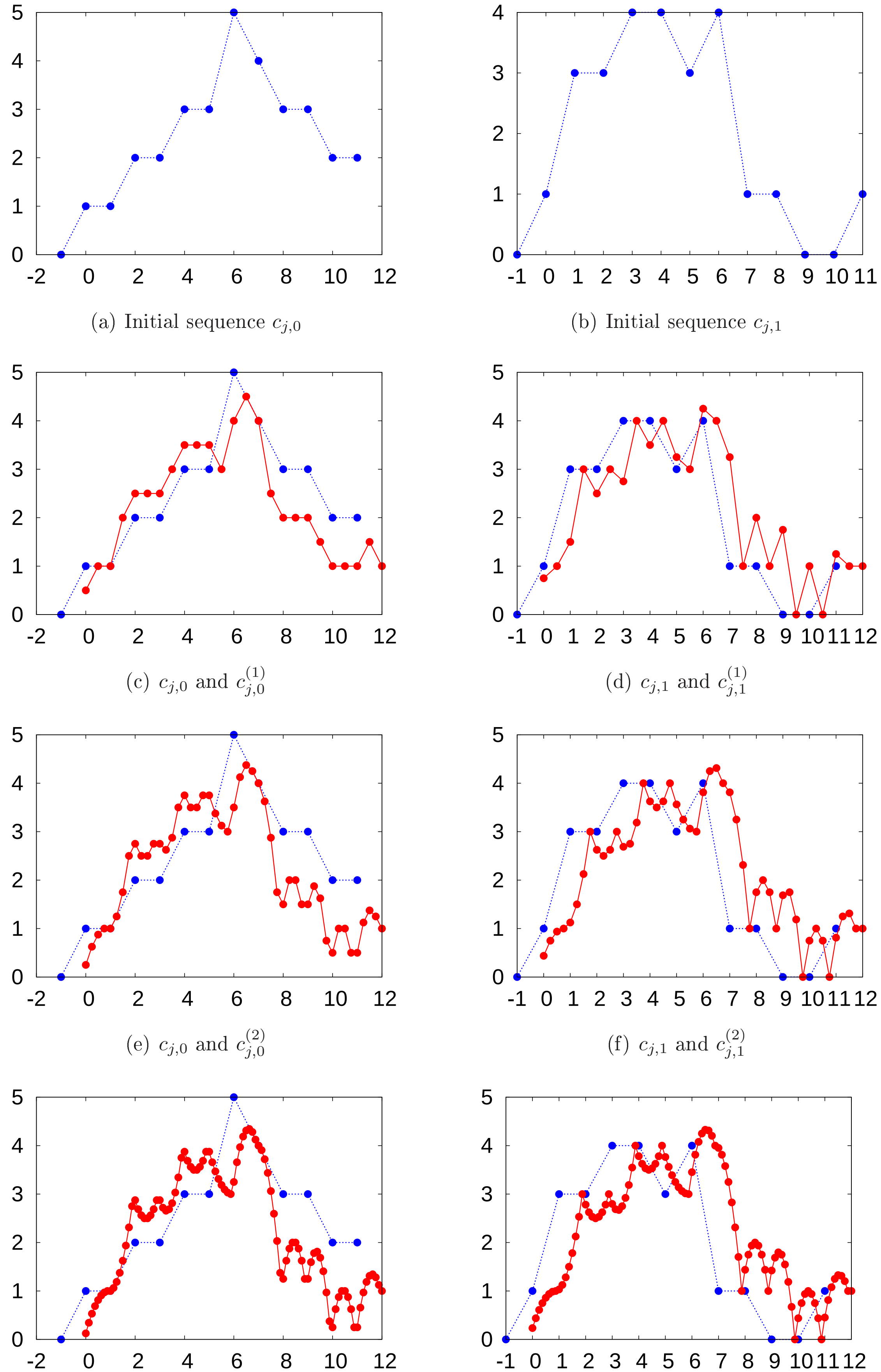
(e) $c_{j,0}$ and $c_{j,0}^{(2)}$

(f) $c_{j,1}$ and $c_{j,1}^{(2)}$

(g) $c_{j,0}$ and $c_{j,0}^{(3)}$

(h) $c_{j,1}$ and $c_{j,1}^{(3)}$

Figure 5.1: SS with subdivision mask given in (4.71) and control points given by (5.58)



(a) Initial sequence $c_{j,0}$

(b) Initial sequence $c_{j,1}$

(c) $c_{j,0}$ and $c^{(1)}_{j,0}$

(d) $c_{j,1}$ and $c^{(1)}_{j,1}$

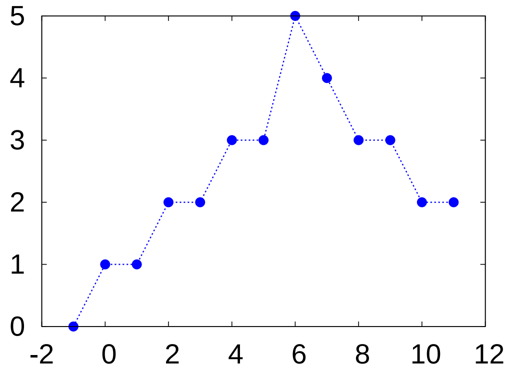
(e) $c_{j,0}$ and $c^{(2)}_{j,0}$

(f) $c_{j,1}$ and $c^{(2)}_{j,1}$

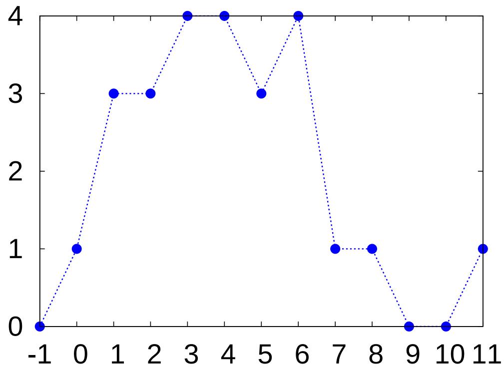
(g) $c_{j,0}$ and $c^{(3)}_{j,0}$

(h) $c_{j,1}$ and $c^{(3)}_{j,1}$

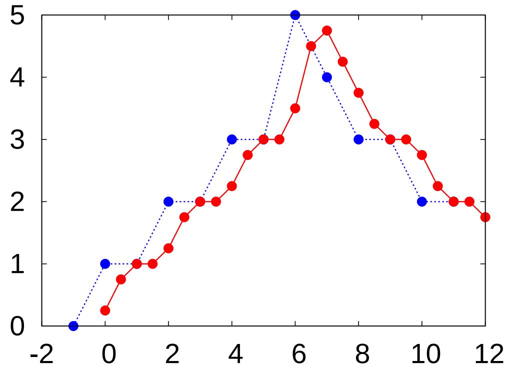
Figure 5.2: SS with subdivision mask given in (4.103) and control points given by (5.58)



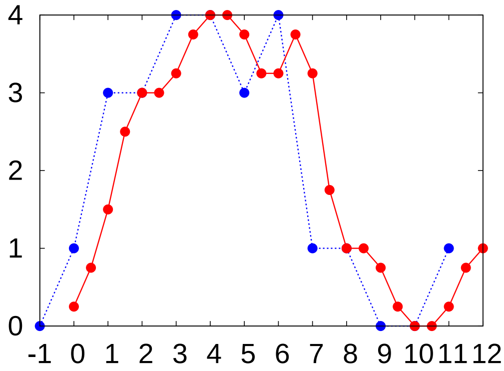
(a) Initial sequence $c_{j,0}$



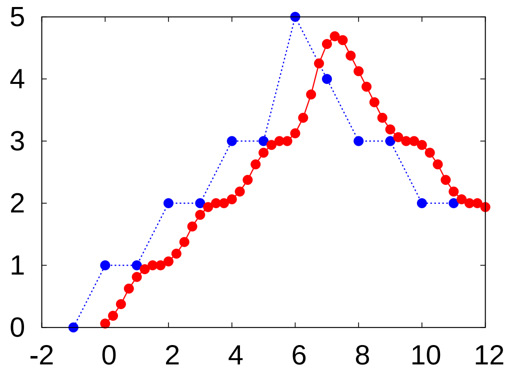
(b) Initial sequence $c_{j,1}$



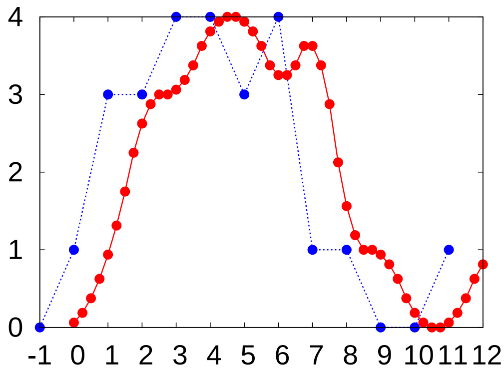
(c) $c_{j,0}$ and $c_{j,0}^{(1)}$



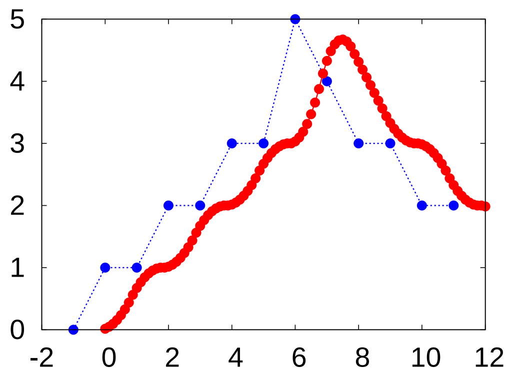
(d) $c_{j,1}$ and $c_{j,1}^{(1)}$



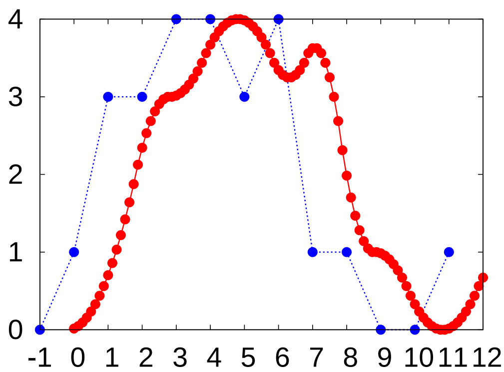
(e) $c_{j,0}$ and $c_{j,0}^{(2)}$



(f) $c_{j,1}$ and $c_{j,1}^{(2)}$



(g) $c_{j,0}$ and $c_{j,0}^{(3)}$



(h) $c_{j,1}$ and $c_{j,1}^{(3)}$

Figure 5.3: SS with subdivision mask given in Theorem 2.2 with $m = 3$ and control points given by (5.58)

Matrix subdivision mask as in Theorem 4.3

m	r=1	r=2	r=3	r=4	r=5
3	0.7500	0.3750	0.1875	0.0938	0.0469
4	1.1667	0.5651	0.2760	0.1368	0.0680
5	1.5313	0.7389	0.3615	0.1794	0.0893

Plonka's B-spline matrix subdivision mask

m	r=1	r=2	r=3	r=4	r=5
3	1.5000	0.7500	0.3750	0.1875	0.0938
4	0.7500	0.3750	0.1875	0.0938	0.0469
5	0.9375	0.4219	0.2237	0.1117	0.0558

Scalar B-spline subdivision mask

m	r=1	r=2	r=3	r=4	r=5
3	0.7500	0.3750	0.1875	0.0938	0.0469
4	1.1042	0.5417	0.2682	0.1340	0.0670
5	1.2708	0.6888	0.3475	0.1735	0.0868

Table 5.1: Subdivision error with control points given by (5.58)

Example 2

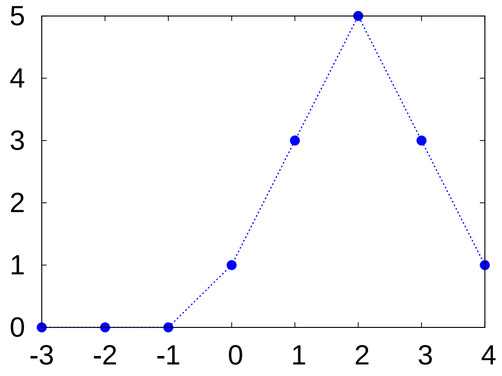
With the sequence $\mathbf{c} = \{\mathbf{c}_j : \ j \in \mathbb{Z}\}$ defined by

$$\begin{cases} \mathbf{c}_0 = (1,2)^{\mathrm{T}}, \quad \mathbf{c}_1 = (3,6)^{\mathrm{T}}, \quad \mathbf{c}_2 = (5,2)^{\mathrm{T}}, \\ \mathbf{c}_3 = (3,0)^{\mathrm{T}}, \quad \mathbf{c}_4 = (1,2)^{\mathrm{T}}, \quad \mathbf{c}_j = \mathbf{0} \text{ otherwise}, \end{cases} \tag{5.59}$$

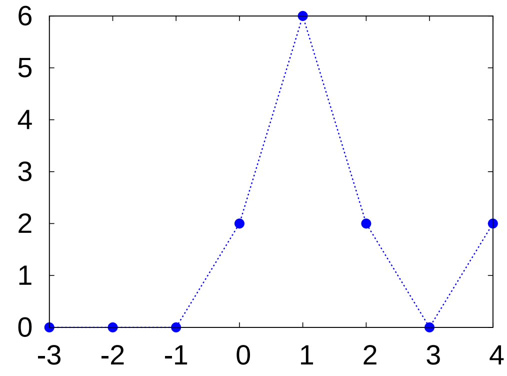
we obtain Figures 5.4, 5.5 and 5.6.

The subdivision error (5.57) is given in Table 5.2.

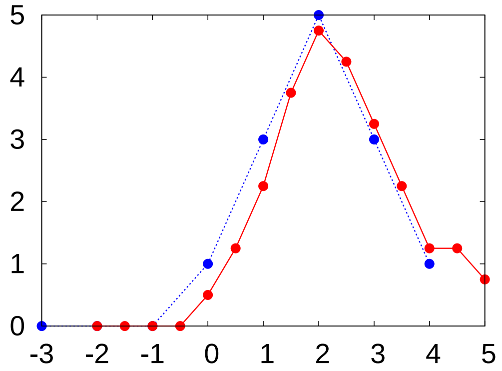
In the cases of scalar and matrix subdivision schemes which are studied here, observe that the subdivision error (5.57) decreases as r grows, as expected from Theorem 3.3 and Theorem 5.4 for the case of the Lane-Riesenfeld algorithm and the SS $(\tilde{p}_m, \mathbf{c})$. Moreover, as illustrated in the numerical results, it should be pointed out that the subdivision error increases with m, as suggested by the error bounds in the above-mentioned theorems for the SS $(p_m, \mathbf{c})$ and $(\tilde{p}_m, \mathbf{c})$. In Table 5.1 corresponding to Example 1, the B-spline matrix



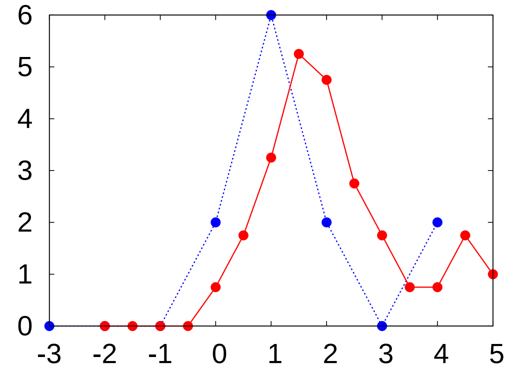
(a) Initial sequence $c_{j,0}$



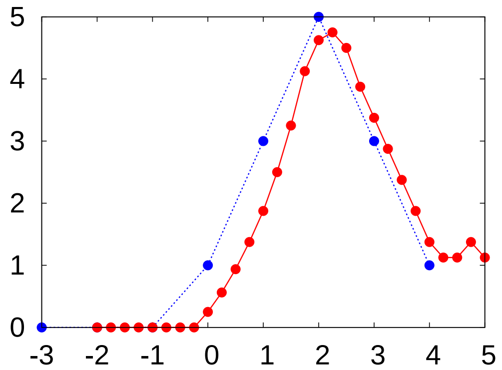
(b) Initial sequence $c_{j,1}$



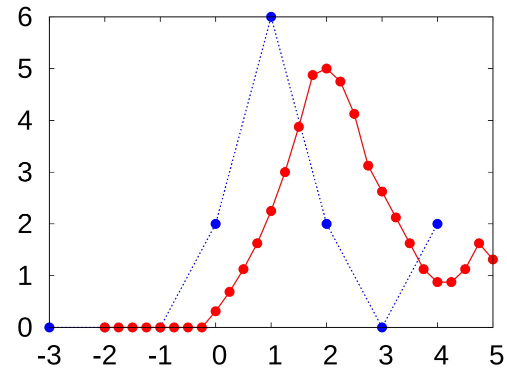
(c) $c_{j,0}$ and $c_{j,0}^{(1)}$



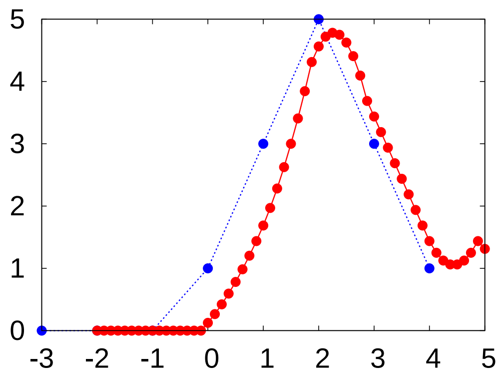
(d) $c_{j,1}$ and $c_{j,1}^{(1)}$



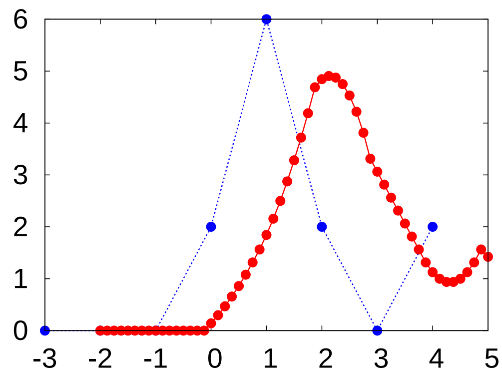
(e) $c_{j,0}$ and $c_{j,0}^{(2)}$



(f) $c_{j,1}$ and $c_{j,1}^{(2)}$

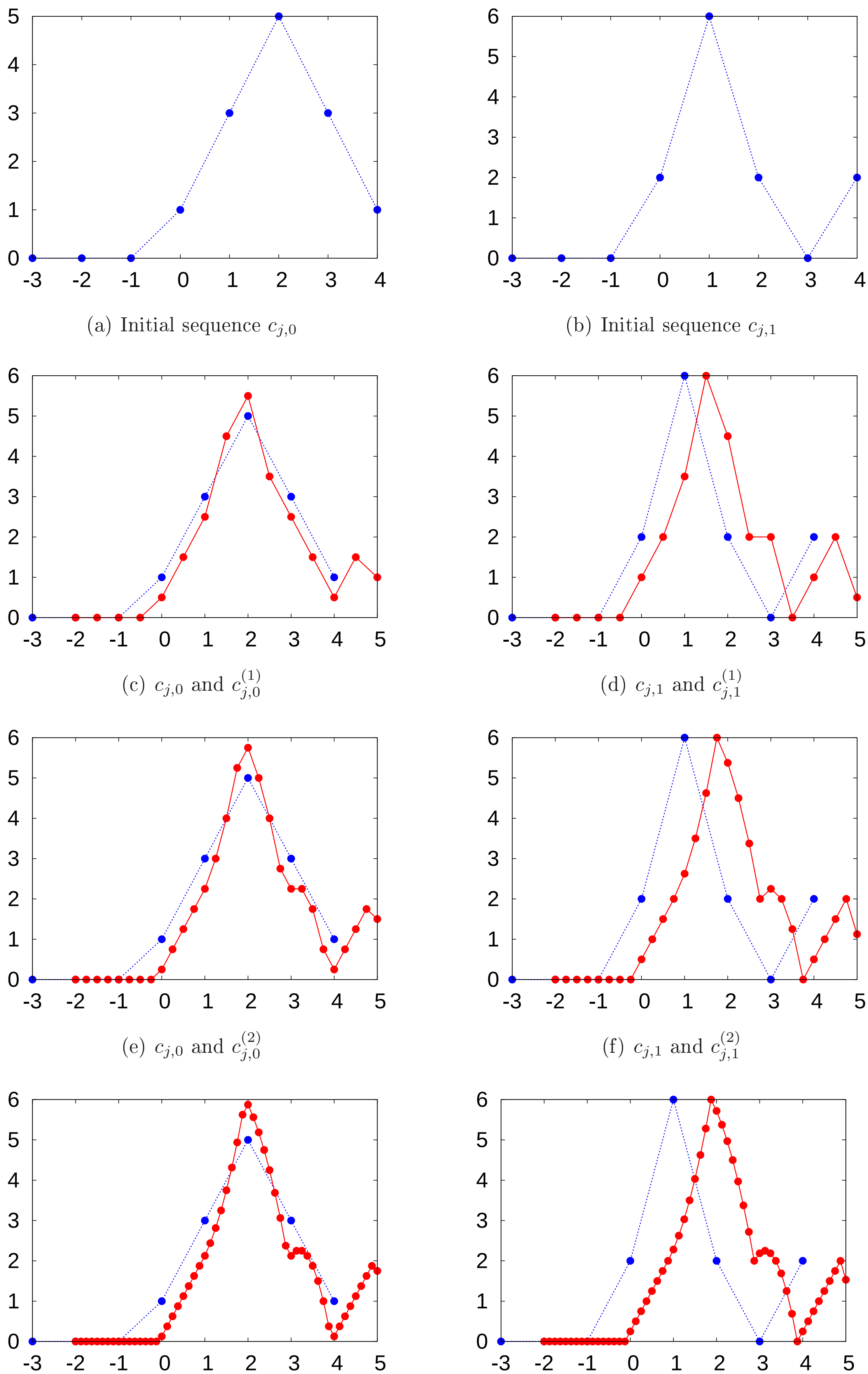


(g) $c_{j,0}$ and $c_{j,0}^{(3)}$



(h) $c_{j,1}$ and $c_{j,1}^{(3)}$

Figure 5.4: SS with subdivision mask given in (4.71) and control points given by (5.59)



(a) Initial sequence $c_{j,0}$

(b) Initial sequence $c_{j,1}$

(c) $c_{j,0}$ and $c_{j,0}^{(1)}$

(d) $c_{j,1}$ and $c_{j,1}^{(1)}$

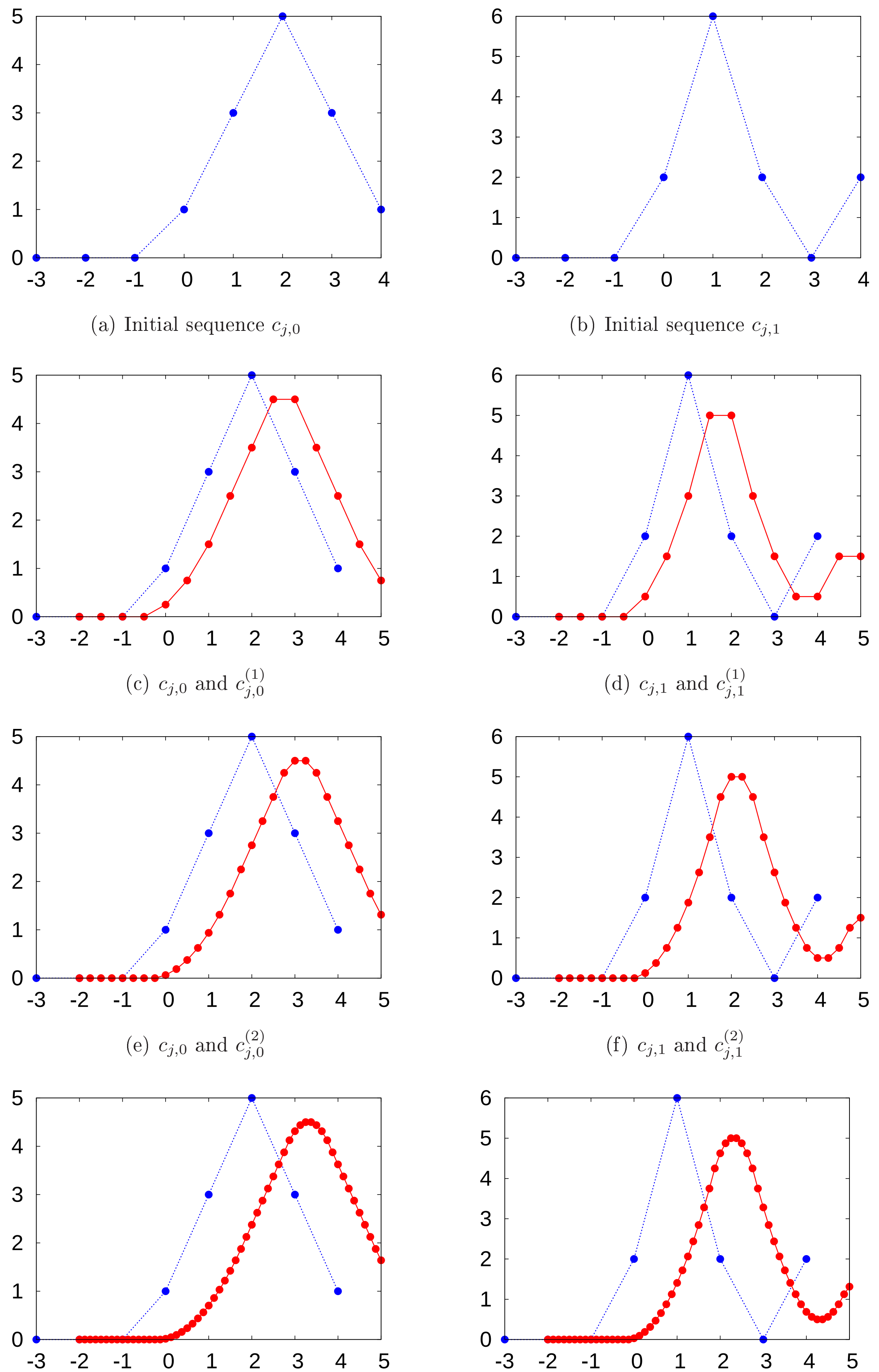
(e) $c_{j,0}$ and $c_{j,0}^{(2)}$

(f) $c_{j,1}$ and $c_{j,1}^{(2)}$

(g) $c_{j,0}$ and $c_{j,0}^{(3)}$

(h) $c_{j,1}$ and $c_{j,1}^{(3)}$

Figure 5.5: SS with subdivision mask given in (4.103) and control points given by (5.59)



(a) Initial sequence $c_{j,0}$

(b) Initial sequence $c_{j,1}$

(c) $c_{j,0}$ and $c_{j,0}^{(1)}$

(d) $c_{j,1}$ and $c_{j,1}^{(1)}$

(e) $c_{j,0}$ and $c_{j,0}^{(2)}$

(f) $c_{j,1}$ and $c_{j,1}^{(2)}$

(g) $c_{j,0}$ and $c_{j,0}^{(3)}$

(h) $c_{j,1}$ and $c_{j,1}^{(3)}$

Figure 5.6: SS with subdivision mask given in Theorem 2.2 with $m = 3$ and control points given by (5.59)

Matrix subdivision mask as in Theorem 4.3

m	r=1	r=2	r=3	r=4	r=5
3	0.7500	0.3750	0.1875	0.0938	0.0469
4	1.4167	0.6719	0.3281	0.1628	0.0811
5	1.8333	0.8984	0.4470	0.2223	0.1109

Plonka's B-spline matrix subdivision mask

m	r=1	r=2	r=3	r=4	r=5
3	1.5000	0.7500	0.3750	0.1875	0.0938
4	0.7500	0.3750	0.1875	0.0938	0.0469
5	1.1250	0.5313	0.2697	0.1350	0.0675

Scalar subdivision mask

m	r=1	r=2	r=3	r=4	r=5
3	1.000	0.5000	0.2500	0.1250	0.0625
4	1.6667	0.8073	0.4010	0.2001	0.1000
5	2.1250	1.0703	0.5317	0.2663	0.1333

Table 5.2: Subdivision error with control points given by (5.59)

subdivision gives the smallest error for $m = 4$ and $m = 5$. For $m = 3$, the error obtained with the alternative matrix masks $\tilde{p}_m$ and with the B-spline masks p_m are numerically closed, while the error corresponding to the B-spline matrix masks gives the less optimal results.

Moreover, one should note that, in Table 5.2 corresponding to Example 2 and with $m = 3$, the subdivision error (5.57) is the smallest whenever the matrix SS $(\tilde{p}_m, \mathbf{c})$ is used.

Chapter 6

Conclusions

The most important innovation of this thesis is to introduce a new extension of the scalar B-spline refinable functions to the vector setting.

Comparing the vector cardinal B-splines of [Plo95b] to the refinable vector splines discussed in Section 4.2, it should be observed that, apart from generalizing the efficient recursive formula (2.6) for the scalar B-spline mask, the latter also has the advantage of having an explicit and very simple formula for the corresponding refinement mask. What is more, the components of our vector splines are shown to present some symmetry with respect to each other, whereas the vector B-splines present a similar symmetry property only when the order of the spline is even.

It should also be pointed out that both the vector B-splines of [Plo95b] and our vector splines are one degree less smooth than the scalar B-splines. Besides, both of these two classes refinable vectors have a shorter support than the scalar B-splines.

Our graphical results, in particular the tables giving the subdivision error at the end of Chapter 5, tend to indicate that, for $m \geq 3$, the set

$$\left\{ \tilde{N}_{m,k}(\cdot - j);\ k = 0, 1;\ j \in \mathbb{Z} \right\}, \tag{6.1}$$

might define a stable spanning set for $S_{m,2}(\mathbb{Z})$. Hence, a pertinent question, which is not answered here, would be to study the properties of the set (6.1). Moreover, an interesting direction would be to investigate under which conditions the matrix subdivision scheme gives optimal results in terms of subdivision error, as in Table 5.2, for $m = 3$.

One should also bear in mind that our analytical results exposed in Theorem 5.4 assume that we are working with an infinitely many control sequence $\mathbf{c}$. In this thesis, the control points that we use are chosen to form a closed curve, allowing us to work with periodic data points, as discussed in Remark 3.1. However, many real-world applications involve working with finite data. In the case where the initial control points do not form a closed curve, some non-desirable features arise near the endpoints of the subdivision limit curve. Removing such deficiencies for the scalar subdivision schemes is investigated in [dVGH03]. In the case of the matrix subdivision schemes, further work could be done with the view to eliminate the algorithm weakness near the endpoints.

Finally, if we define the operator S by

$$S[A] := \sum_{i,j} a_{i,j}, \tag{6.2}$$

where $A = (a_{i,j})_{i,j}$ is a given real matrix, then equations (4.61) and (4.62) yield

$$\begin{cases} \sum_j S\left[p_j^{(m)}\right] &= 2^2, \\ \sum_j S\left[p_{2j}^{(m)}\right] &= \sum_j S\left[p_{2j+1}^{(m)}\right] = 2. \end{cases} \tag{6.3}$$

Further research can be done with the view to investigate whether (6.3) provides an extension of the sum rules to the vector setting.

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