Partitioning the hypercube $Q_n$ into $n$ isomorphic edge-disjoint trees

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1 Introduction

The problem of finding edge-disjoint trees in a hypercube e.g. arises in the context of parallel computing [3]. Independent of applications it is of high aesthetic appeal. A hypercube of dimension $n$, denoted by $Q_n$, comprises $2^n$ vertices each corresponding to a distinct binary string of length $n$. An edge exists between two vertices if and only if their corresponding binary strings differ in exactly one position. Since each vertex of $Q_n$ has degree $n$, the number of edges is $n2^{n-1}$. A variety of decomposability options derive from this fact. In the remainder of the introduction we focus on three of them. The first two have been dealt with before in the literature, the third is the topic of this article.

a) Roskind and Tarjan [7] found a polynomial-time algorithm for computing the spanning tree packing number $\sigma(G)$ of a connected graph $G$, i.e. the maximum number of edge-disjoint spanning trees of $G$. Since each spanning tree of $Q_n$ has $2^n - 1$ edges, one concludes

$$\sigma(Q_n) \leq \left\lfloor \frac{n2^{n-1}}{2^n - 1} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor.$$ 

It turns out that $\sigma(Q_n) = \left\lfloor \frac{n}{2} \right\rfloor$. This is an instance of the fact that a variety of graphs [4] which are regular of degree $n$ have $\sigma(G) = \left\lfloor \frac{n}{2} \right\rfloor$. For $n$ even, an explicit construction of $m = \frac{n}{2}$ edge-disjoint spanning trees $S_1, \ldots, S_m$ of $Q_n$ is given in [1]. These $S_i$'s are not isomorphic, and because of $m(2^n - 1) = n2^{n-1} - m$, there are $m$ edges left over (which constitute a path in $Q_n$). For $n$ odd, an analogous construction remains to be found.
b) So partitioning the edge set of $Q_n$ into spanning trees is impossible. What about a partitioning into isomorphic trees? Surprisingly, for every tree $T$ with $n$ edges, the edge set of $Q_n$ can by [2] be covered by $2^{n-1}$ trees all of which isomorphic to $T$. This has been extended in [5], [6], and will be followed up in Section 4.

c) As opposed to partitioning $Q_n$ into $2^{n-1}$ isomorphic trees of size $n$, in this article we partition $Q_n$ into $n$ trees of size $2^{n-1}$, all of which isomorphic to some tree $T_n$. Our tree $T_n$ needs to have a very specific shape.

As proposed by one referee, the authors at one time pondered to condense the whole article to Theorem 2 and its easy (once found) proof. Eventually we opted for a more leisurely approach which includes two equivalent definitions of $T_n$ and some more historical remarks. The direct definition of $T_n$ is useful in Theorem 2, whereas the recursive definition underlies our pictures of $T_1$ up to $T_5$.

\section{Construction of $T_n$}

Rather than dealing with 0, 1-strings, it will be more convenient to let the vertex set $V(Q_n)$ be the power set $P([n])$ of $[n] := \{1, 2, \ldots, n\}$, with $X, X' \in V(Q_n)$ being adjacent if and only if their symmetric difference has cardinality one. Before tackling the formal definition of $T_n$, let us do some small cases; of course, we must have $|V(T_n)| = 2^{n-1} + 1$ for every $n$.

\begin{center}
\begin{tikzpicture}
\node at (0,0) {$\emptyset$};
\node at (0,1) {$\{1\}$};
\node at (0,2) {$\{1, 2\}$};
\node at (0,3) {$\{1, 2, 3\}$};
\node at (0,4) {$\{1, 2, 3, 4\}$};
\node at (1,0) {$\emptyset$};
\node at (1,1) {$\{1\}$};
\node at (1,2) {$\{1, 2\}$};
\node at (1,3) {$\{1, 2, 3\}$};
\node at (1,4) {$\{1, 2, 3, 4\}$};
\node at (2,0) {$\emptyset$};
\node at (2,1) {$\{1\}$};
\node at (2,2) {$\{1, 2\}$};
\node at (2,3) {$\{1, 2, 3\}$};
\node at (2,4) {$\{1, 2, 3, 4\}$};
\node at (3,0) {$\emptyset$};
\node at (3,1) {$\{1\}$};
\node at (3,2) {$\{1, 2\}$};
\node at (3,3) {$\{1, 2, 3\}$};
\node at (3,4) {$\{1, 2, 3, 4\}$};
\node at (4,0) {$\emptyset$};
\node at (4,1) {$\{1\}$};
\node at (4,2) {$\{1, 2\}$};
\node at (4,3) {$\{1, 2, 3\}$};
\node at (4,4) {$\{1, 2, 3, 4\}$};
\node at (5,0) {$\emptyset$};
\node at (5,1) {$\{1\}$};
\node at (5,2) {$\{1, 2\}$};
\node at (5,3) {$\{1, 2, 3\}$};
\node at (5,4) {$\{1, 2, 3, 4\}$};
\node at (6,0) {$\emptyset$};
\node at (6,1) {$\{1\}$};
\node at (6,2) {$\{1, 2\}$};
\node at (6,3) {$\{1, 2, 3\}$};
\node at (6,4) {$\{1, 2, 3, 4\}$};
\node at (7,0) {$\emptyset$};
\node at (7,1) {$\{1\}$};
\node at (7,2) {$\{1, 2\}$};
\node at (7,3) {$\{1, 2, 3\}$};
\node at (7,4) {$\{1, 2, 3, 4\}$};
\node at (8,0) {$\emptyset$};
\node at (8,1) {$\{1\}$};
\node at (8,2) {$\{1, 2\}$};
\node at (8,3) {$\{1, 2, 3\}$};
\node at (8,4) {$\{1, 2, 3, 4\}$};
\node at (9,0) {$\emptyset$};
\node at (9,1) {$\{1\}$};
\node at (9,2) {$\{1, 2\}$};
\node at (9,3) {$\{1, 2, 3\}$};
\node at (9,4) {$\{1, 2, 3, 4\}$};
\node at (10,0) {$\emptyset$};
\node at (10,1) {$\{1\}$};
\node at (10,2) {$\{1, 2\}$};
\node at (10,3) {$\{1, 2, 3\}$};
\node at (10,4) {$\{1, 2, 3, 4\}$};
\node at (11,0) {$\emptyset$};
\node at (11,1) {$\{1\}$};
\node at (11,2) {$\{1, 2\}$};
\node at (11,3) {$\{1, 2, 3\}$};
\node at (11,4) {$\{1, 2, 3, 4\}$};
\node at (12,0) {$\emptyset$};
\node at (12,1) {$\{1\}$};
\node at (12,2) {$\{1, 2\}$};
\node at (12,3) {$\{1, 2, 3\}$};
\node at (12,4) {$\{1, 2, 3, 4\}$};
\node at (13,0) {$\emptyset$};
\node at (13,1) {$\{1\}$};
\node at (13,2) {$\{1, 2\}$};
\node at (13,3) {$\{1, 2, 3\}$};
\node at (13,4) {$\{1, 2, 3, 4\}$};
\node at (14,0) {$\emptyset$};
\node at (14,1) {$\{1\}$};
\node at (14,2) {$\{1, 2\}$};
\node at (14,3) {$\{1, 2, 3\}$};
\node at (14,4) {$\{1, 2, 3, 4\}$};
\end{tikzpicture}
\end{center}

The following notation shall be employed throughout the article. Permutations $\pi$ of $[n]$ will be written in cycle notation and to the right of the argument, separated by a
Thus if $\pi = (3, 4)(2, 5, 7)$ (for some $n \geq 7$), then $3 \cdot \pi = 4$ and $7 \cdot \pi = 2$. Each permutation of $[n]$ induces, in the obvious way, a permutation of the vertex set $V(Q_n)$. Carrying over the described notation, applying the permutation $\pi$ above to the vertex (say) $X = \{1, 2, 4\}$ of $Q_n$ yields $X \cdot \pi := \{1, 5, 3\}$. Furthermore, for $\{X_1, X_2, \cdots \} \subseteq V(Q_n)$ we set $\{X_1, X_2, \cdots \} \cdot \pi := \{X_1 \cdot \pi, X_2 \cdot \pi, \cdots \}$. If $T$ is a subgraph of $Q_n$ and $\pi$ a permutation, then $T \cdot \pi$ is defined as the subgraph of $Q_n$ which has vertex set $V(T) \cdot \pi$ and edge set $\{\{X \cdot \pi, Y \cdot \pi\} | \{X, Y\} \in E(T)\}$.

Let us resume the discussion of our trees $T_n$ and put $n = 3, \pi = (1, 2, 3)$. Then we have

$$T_3 \cdot \pi = \begin{array}{ccc} 123 \\ \downarrow \\ 2 \\
\downarrow \\ 3 \\
\downarrow \\ \emptyset \end{array} \quad T_3 \cdot \pi^2 = \begin{array}{ccc} 123 \\ \downarrow \\ 13 \\
\downarrow \\ 3 \\
\downarrow \\ \emptyset \end{array}$$

where for simplicity we wrote e.g. 23 instead of $\{2, 3\}$. The edge-disjoint decomposition of $Q_3$ into the isomorphic rooted trees $T_3, T_3 \cdot \pi,$ and $T_3 \cdot \pi^2$ is now apparent:

$$Q_3 = \begin{array}{ccc} 123 \\ \downarrow \\ 12 \\\n\downarrow \\ 13 \\
\downarrow \\ 23 \\
\downarrow \\ \emptyset \end{array}$$

The corresponding decomposition of $Q_4$ into the trees $T_4 \cdot \pi^i \ (0 \leq i < 4, \ \pi := (1, 2, 3, 4))$ is already more surprising:
Let us now define $T_n$ in general. For any positive integer $n$, we set $V(T_n) = \mathcal{P}([n-1]) \cup \{n\}$, i.e. the vertices are all subsets of $[n-1]$ and the root $[n]$. It is obvious from this definition that $|V(T_n)| = 2^{n-1} + 1$ for all $n$. Furthermore, we assign a parent vertex $p(v)$ to every vertex $v \neq [n]$, namely the set $p(v) = v \cup \{x(v)\}$, where

\[(1) \quad x(v) = \min(\mathbb{N} \setminus v).\]

In other words, to obtain the parent vertex, we add the smallest positive integer that is not yet contained in the set. If $v \neq [n-1]$, this is an element of $[n-1]$; if $v = [n-1]$, then $x(v) = n$ and $p(v) = [n]$. Now the edge set $E(T_n)$ consists of all pairs $(v, p(v))$. The fact that every set (= vertex) $v \neq [n]$ is adjacent to a unique superset, implies that the graph $(V(T_n), E(T_n))$ has no cycles, i.e. is a forest. Since every vertex $v \neq [n]$ is connected to $[n-1]$ via $p(v), p^2(v), \cdots, p^k(v) = [n-1]$, the forest is in fact a tree. The levels (which consist of all those vertices whose distance from the root is some given integer) correspond exactly to the subsets of $[n-1]$ of fixed cardinality.

It should be noted that $T_n$ can be constructed in a recursive manner as well. Consider the subforest $S_{n,i}$ of $T_n$ ($1 \leq i \leq n-1$) that is induced by the vertex set

\[(2) \quad V(S_{n,i}) = \{v \in \mathcal{P}([n-1]) : \{i+1, i+2, \ldots, n-1\} \subseteq v, \ i \notin v\} \cup \{n-1\}\]

which consists of all those sets that contain $i + 1, i + 2, \ldots, n - 1$ but do not contain $i$, together with the set $[n-1]$. The picture for $n = 5$ is as follows:
Notice that

(3) \( S_{5,4} = T_4, \) \( S_{5,3} \simeq T_3, \) \( S_{5,2} \simeq T_2, \) \( S_{5,1} \simeq T_1 \)

**Proposition 1:** The subforest \( S_{n,i} \) of \( T_n \) \( (1 \leq i \leq n - 1) \) is a tree isomorphic to \( T_i. \) It follows that \( T_n \) is isomorphic to the tree that is obtained by gluing \( T_1, \ldots, T_{n-1} \) at their roots and then attaching a new root.

*Proof.* The second statement of Proposition 1 follows from the first because \( \mathcal{P}([n-1]) \setminus \{[n-1]\} \) is by (2) partitioned by the sets \( V(S_{n,1}) \setminus \{[n-1]\} \) up to \( V(S_{n,n-1}) \setminus \{[n-1]\}. \)

As to the first statement, fix \( n \) and \( i \in \{1, 2, \cdots, n-1\}. \)

**Case 1:** \( i = n - 1. \) As a special case of (2),

(4) \( V(S_{n,n-1}) = \{v \in \mathcal{P}([n-1]) : n-1 \notin v\} \cup [n-1], \)

and so \( V(S_{n,n-1}) \) coincides with \( V(T_{n-1}). \) Thus \( S_{n,n-1} \) is not just isomorphic to but actually equals \( T_{n-1} \) (cf. (3)).

**Case 2:** \( i \in \{1, 2, \cdots, n-2\}. \) Then

(5) \( V(S_{i+1,i}) = \{v \in \mathcal{P}([i]) : i \notin v\} \cup [i], \)

and so the map \( f \) from \( V(S_{n,i}) \) to \( V(S_{i+1,i}) \) defined by

\[
f(v) := v \setminus \{i+1, i+2, \cdots, n-1\}
\]

is again bijective. If \( v \) and \( v \cup \{x(v)\} \) are adjacent in \( S_{n,i}, \) then their images are \( f(v) \) and \( f(v) \cup \{x(v)\} \) (since \( x(v) < i \)), which are adjacent in \( S_{i+1,i}. \) Analogously, adjacent
vertices \( w \) and \( w \cup \{x(w)\} \) in \( S_{i+1,i} \) yield adjacent vertices \( f^{-1}(w) \) and \( f^{-1}(w) \cup \{x(w)\} \) in \( S_{n,i} \). Hence \( S_{n,i} \cong S_{i+1,i} \cong T_i \) where the second isomorphism follows by Case 1. \( \square \)

We mention that the recursive construction was the way the second author discovered \( T_n \). The equivalent view in terms of parent vertices, which enabled to properly prove the speculated theorem below, is due to the first author.

## 3 Main result

Our main result states that the edges of the \( n \)-dimensional hypercube \( Q_n \) can be partitioned into isomorphic copies of \( T_n \):

**Theorem 2:** Let \( \pi = (1, 2, \ldots, n) \). If \( \{X, Y\} \in E(T_n) \), then it follows that \( \{X, Y\} \notin E(T_n \cdot \pi^i) \) for all \( 0 < i < n \). Therefore, the edge sets of the trees \( T_n, T_n \cdot \pi, \cdots, T_n \cdot \pi^{n-1} \) form a partition of the edge set of \( Q_n \).

**Proof:** Let \( i \) be an integer with \( 0 < i < n \). We show the following: if a set \( X \) belongs to the vertex sets of both \( T_n \) and \( T_n \cdot \pi^i \), then the parent vertices of \( X \) in \( T_n \) and \( T_n \cdot \pi^i \) are distinct. Since parent vertices are unique, this implies immediately that \( T_n \) and \( T_n \cdot \pi^i \) have no common edges. The vertex set of \( T_n \cdot \pi^i \) is precisely

\[
(6) \quad (\mathcal{P}(\{n-1\}) \cup \{n\}) \cdot \pi^i = \mathcal{P}(\{n\} \setminus \{i\}) \cup \{n\}.
\]

Hence, a set \( X \) can only belong to the vertex sets of both \( T_n \) and \( T_n \cdot \pi^i \) if \( i, n \notin X \) or if \( X = [n] \). Since \( X = [n] \) has no parent vertex, it suffices to consider the case that \( i, n \notin X \). Let \( X \cup \{x\} \) and \( X \cup \{y\} \) be the parent vertices of \( X \) in \( T_n \) and \( T_n \cdot \pi^i \) respectively.

Since \( i \notin X \), we have \( 1 \leq x \leq i \) by the definition of \( T_n \). Let \( Z \in V(T_n) \) be the preimage of \( X \in V(T_n \cdot \pi^i) \). From \( n \notin X \) follows \( (n-i) \notin Z \), and so the parent vertex \( Z \cup \{z\} \) of \( Z \) satisfies \( z \in \{1, 2, \ldots, n-i\} \). But this forces \( y = z \cdot \pi^i \in \{i+1, \ldots, n\} \). Altogether, this shows that \( x \neq y \), and so the parents \( X \cup \{x\} \) and \( X \cup \{y\} \) are distinct, as claimed.

Therefore, the edge sets of \( T_n \) and \( T_n \cdot \pi^i \) are disjoint for all \( 0 < i < n \). This also implies that the edge sets of \( T_n \cdot \pi^i \) and \( T_n \cdot \pi^j \) are disjoint for all \( 0 \leq i < j < n \): if they were not disjoint, the edge sets of \( T_n \) and \( T_n \cdot \pi^{j-i} \) would not be disjoint either, a contradiction.

Since \( T_n \) has \( 2^{n-1} \) edges and the isomorphic copies \( T_n, T_n \cdot \pi, \cdots, T_n \cdot \pi^{n-1} \) are pairwise edge-disjoint by the above argument, it follows immediately that they partition the edge set of \( Q_n \) that comprises of \( n2^{n-1} \) edges. \( \square \)
4 Related matters

Let $E$ be a subset of $E(Q_n)$. Following Ramras call $E$ a fundamental set for $Q_n$ with group $G$, if $G$ is a subgroup of $\text{Aut}(Q_n)$ such that $\{g(E) \mid g \in G\}$ is an edge decomposition of $Q_n$. It is shown in [5] that if $|E| = n$ and the graph induced by $E$ is connected with at most one cycle (e.g. a tree), then $E$ is a fundamental set for $Q_n$. Results about fundamental $2n$-element sets are contained in [6]. Our tree $T_n$ fits into this framework in that $E(T_n)$ is a fundamental $2^{n-1}$-set for the group $G \subseteq \text{Aut}(Q_n)$ that is induced by the cyclic group $\langle \pi \rangle \subseteq S_n$.

In another vein, every rooted tree $T$ (e.g. $T = T_n$) becomes a unique partially ordered set $(T, \leq)$ when the root is postulated as largest element. It is an open problem to find necessary or sufficient conditions for a rooted tree $(T, \leq)$ to be cover preserving order embeddable into $(Q_n, \subseteq)$. That is, we want a map $\phi : T \to Q_n$ that satisfies

\[(7) \quad (\forall x, y \in T) \quad x \leq y \iff \phi(x) \subseteq \phi(y)\]
\[(8) \quad (\forall x, y \in T) \quad x < y \quad \Rightarrow \quad \phi(x) < \phi(y)\]

Here $x < y$ means that $x < y$ and $(x < z \leq y \Rightarrow z = y)$. We mention that for 0,1-posets $(P, \leq)$ the problem is settled in [8] in terms of the chromatic number of some auxiliary graph. Notice that e.g. $(T_4, \leq)$ is not order embedded in $Q_4$: while $\{2\} \subseteq \{2, 3\}$ in $Q_4$, the corresponding vertices are not comparable in $(T_4, \leq)$.

References


