Lorentz Symmetry in Non-Commutative Field Theories: Commutative/Non-Commutative Dualities and Manifest Covariance

by

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Declaration

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Abstract

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We cover two approaches to building Lorentz invariant non-commutative field theories.

First we construct dualities between commutative and non-commutative field theories. This construction exploits a generalization of the exact renormalization group equation (ERG). We review ERG dualities for the two dimensional quantum mechanical Landau problem. We also review the idea of non-canonical field theories. From this we build an ERG duality for the free non-canonical complex scalar field theory. This approach allows us to track the Lorentz symmetry, we show this explicitly for the free theory. Finally we construct dualities for the ϕ^4 interacting theory.

Second we build a manifestly Lorentz invariant $2 + 1$ dimensional field theory living in $\mathrm{SU}(1,1)$ fuzzy space-time. Here the commutation relations themselves respect the Lorentz symmetry. We start by reviewing the non-relativistic construction of $\mathrm{SU}(2)$ fuzzy space and quantum mechanics. We briefly discuss a potential field theory extension of this. We then begin our $\mathrm{SU}(1,1)$ construction by reviewing the $\mathrm{SU}(1,1)$ group and make the connection to space-time. We then build a quantum theory living on this space-time and introduce dynamics with the Klein-Gordon equation. From this we can move on to a scalar field theory in terms of functions on the group manifold. Finally we make the connection to commutative theories by introducing the symbols of operators. From this we are able to compute different correlators and compare with commutative theories.

Uittreksel

Lorentz Simmetrie in Nie-kommutatiewe Veldeteorieë: Kommutatiewe/Nie-kommutatiewe Dualiteite en Eksplisiete Kovariansie

("Lorentz Symmetry in Non-Commutative Field Theories: Commutative/Non-Commutative Dualities and Manifest Covariance")

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Ons beskryf twee benaderings tot die konstruksie van Lorentz invariante nie-kommutatiewe veldeteorieë.

Eerstens beskryf ons dualiteite tussen kommutatiewe en nie-kommutatiewe veldeteorieë. Hierdie konstruksie berus op 'n veralgemening van die Eksakte Renormerings Groep (ERG). Ons hersien dualiteite vir die twee-dimensionele kwantum meganiese Landau probleem. Dit word opgevolg deur 'n hersiening van nie-kanoniese veldeteorieë. Met dit as basis, konstrueer ons 'n ERG dualiteit vir die vrye nie-kanoniese komplekse skalare veld. Hierdie benadering stel ons in staat om die Lorentz simmetrie na te volg, en dit word eksplisiet vir die vrye teorie gedemonstreer. Ten slotte, konstrueer ons dualiteite vir die ϕ^4 wisselwerkende teorie.

Tweedens, konstrueer ons 'n eksplisiete Lorentz invariante $2+1$ dimensionele veldeteorie in SU(1,1) wasige ruimte-tyd. In hierdie geval respekteer die kommutasieverbande self die Lorentz simmetrie. Ons begin met 'n hersiening van die nie-relativistiese konstruksie van wasige ruimte en die formulering van kwantum meganika op die ruimtes. Ons bespreek dan kortliks 'n veldteoretiese veralgemening hiervan. Ons begin dan met die SU(1,1) konstruksie deur die hersiening van die SU(1,1) groep en die konneksie met ruimte-tyd. Ons konstrueer dan 'n kwantum veldeteorie op die ruimte-tyd en voer dinamika in deur die Klein-Gordon vergelyking. Met dit as basis, kan ons dan 'n vrye skalare teorie opbou in terme van funksies op die groep manifold. Ten slotte

maak ons die verbintenis met kommutatiewe toerië deur die invoering van die simbole van operatore. Die gebruik hiervan stel ons in staat om verskillende korrelators te bereken en met kommutatiewe teorieë te vergelyk.

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Contents

Chapter 1

Introduction

The structure of space-time at short length scales and the emergence of space-time as we perceive it at long length scales are probably the most challenging problems facing modern physics [1]. In order to combine gravity and quantum mechanics into a unified theory, we will need a more sophisticated understanding of physics at very short length scales. The compelling arguments of Doplicher et al [2] highlighted the need for a revised notion of space-time at short length scales and gave strong arguments in favour of a non-commutative geometry. It is theorized that any attempt to localize events at a scale shorter than the Planck length will lead to gravitational instability. Naturally we should seek to build this maximal localization, or rather minimum length scale, into our theories. One realization of this is non-commutative space-time. Non-commutative space-time has received considerable attention in the past few decades, however, it is not a new idea. It was originally proposed by Snyder [3] in an attempt to avoid the ultra-violet divergences of field theories. The discovery of renormalization pushed these ideas to the background until more recently when they resurfaced in the search for a consistent theory of quantum gravity. Non-commutative coordinates also occur quite naturally in certain string theories [4], generally perceived to be the best candidate for a theory of quantum gravity.

This sparked renewed interest in non-commutative space-time and the formulation of quantum mechanics [5], and quantum field theories [6], on such spaces. Even though there have been many investigations into the possible physical implications of non-commutativity in quantum mechanics, quantum many-body systems [7; 8; 9; 10; 11], quantum electrodynamics [12; 13; 14], the standard model [15] and cosmology [16; 17], a systematic formulation of non-commutative quantum mechanics was lacking, until [5] where an operator valued formulation for two dimensional quantum mechanics was developed. Here the standard quantum mechanical formalism and interpretation was shown to carry over. The only modification required is that more general Hilbert spaces are needed to represent the states of the system, and to find the unitary representations of the non-commutative Heisenberg algebra. Here the simplest form of non-commutativity

$$[\hat{x}, \hat{y}] = i\theta, \tag{1.0.1}$$

for the space-time commutation relations was adopted, where θ is a parameter related to the minimum length scale. This is what we, from here on, refer to as the constant commutation relations, where the coordinates commutate to a multiple of the identity, when discussing types of non-commutative theories. These commutation relations break rotational and Lorentz symmetry. For field theories living in this space-time a twisted implementation of the Lorentz symmetry was suggested in [12].

The twisted implementation of the Lorentz symmetry [18] is still controversial and gave rise to several outstanding issues. The first difficulty is to carry out the standard Noether analysis and identify conserved charges for the twisted Lorentz symmetry. The correct quantization of these theories is also debated in the literature. Should one use the standard quantization procedure or also deform the canonical commutation relations? On the level of the path integral this means we need to alter the measure. In fact UV/IR mixing may be related to a quantum anomaly of the Lorentz group [18].

In this thesis we investigate two attempts at building a systematic and consistent framework for non-commutative field theories. There have been many studies of non-commutative field theories [6; 19; 20], however, we seek a framework where properties like symmetries, in particular Lorentz symmetry, unitary and renormalizabilty are manifest and interactions can be included naturally. Often these elements are incorporated in an ad-hoc way. The two methods we investigate use two very different approaches to ensure that the Lorentz symmetry is intact. Both of them, however, provide interesting results.

The first approach we follow is based on the theories defined in [21; 22], often referred to as non-canonical quantum field theories. In these papers non-commutativity is implemented on the level of modified canonical commutation relations of the fields and momenta, rather than the space-time coordinates as is conventionally done [6]. This is close in spirit to non-commutative quantum mechanics as formulated in [5], where the degrees of freedom are the non-commutative coordinate and momenta. In non-canonical quantum field theory the Lorentz symmetry also gets violated, but a partial mapping to a commutative theory can restore the Lorentz invariance. However, this introduces non-local interactions for interacting theories and it is an open question whether these theories are unitary and renormalizable [23]. In [24] the exact renormalization group (ERG) was used to construct dualities between commutative and non-commutative quantum systems in two dimensions for the constant commutations relations. This construction was carried out for different quantum systems, in particular the Landau problem. We seek to generalize this to non-canonical quantum field theories. We note that although the resulting non-commutative quantum field theories may be non-local and not obviously renormalizable, unitary or Lorentz invariant, the duality must ensure that these properties are still present, albeit in a non-manifest way. A possible benefit of these dualities may be the weakening of interactions.

This is the motivation for part 1, which aims to demonstrate and construct commutative/non-commutative dualities for field theories using the ERG program. The focus here is on dualities between commutative and non-canonical quantum field theories as described in [21; 22; 23]. These dualities are straightforward generalizations of the ones constructed in [24].

Part 1 is therefore organized as follows: in chapter 2 we review non-commutative quantum mechanics with constant commutation relations and non-canonical field theories. Chapter 3 reviews and expands on the duality construction of [24] for the Landau problem. Finally in chapter 4 we discuss the duality construction for non-canonical theories, we investigate the fate of the symmetries and construct the duality for an interacting theory.

The second approach we follow is based on the fuzzy space constructions as explored in [25]. In three dimensional fuzzy space the coordinates obey the su(2) algebra. Here the commutation relations clearly preserve rotational symmetry. This is in a similar spirit to non-commutative quantum mechanics as formulated in [5], but this time in the sense that we can build a similar operator valued formulation. In these papers a unitary representation of SU(2) is found on the non-commutative Hilbert space, instead of the Heisenberg algebra as was done in 2 dimensions. These commutation relations were studied for various systems including the free particle [26], spherical well [27] and hydrogen atom [28]. However, these theories currently do not have a relativistic extension. In these theories time is commutative, which is incompatible with a relativistic theory where time and space should be on an equal footing. There have been attempts to introduce non-commutative time with constant commutation relations [29], but we seek a theory where the commutation relations are manifestly Lorentz invariant i.e. a fuzzy space-time. This is what we proceed to construct in part 2.

Part 2 attempts to build a non-commutative field theory living on a Lorentz invariant non-commutative fuzzy space-time. In $2+1$ dimensions the most natural set of Lorentz invariant commutation relations would be $\mathrm{su}(1,1)$, so we will build our space-time using these commutation relations. We also look at a SU(2) field theory as a prototype for our construction. We note that a slightly different construction of Snyder-space has already been explored in [30]. We begin by building a relativistic non-commutative quantum mechanical theory, as was done for the non-relativistic case in [25], and then use this to build a field theory.

Part 2 is organized as follows: in chapter 5 we review three dimensional fuzzy space quantum mechanics and discuss possible field theory constructions. In chapter 6 we review the group $\mathrm{SU}(1,1)$ and discuss the connection to fuzzy space-time. In chapter 7 we discuss how we can build our relativistic non-commutative quantum mechanics and move on to a field theory. Finally, in chapter 8 we describe ways in which we can connect our non-commutative theory to measurable quantities and commutative theories.

Before we begin the thesis proper, we should spend some time reviewing the standard "textbook" descriptions of quantum physics, as these are the theories we wish to adapt in an attempt to create a modified description of physics at short length scales. We must make sure that our modifications disappear in the limits where the standard theories match measurements well i.e. long length scales. For the entire thesis we work in natural units with

$$\hbar = c = 1. \tag{1.0.2}$$

1.1 Commutative Quantum Mechanics

We begin with non-relativistic quantum mechanics. In quantum mechanics, unlike classical physics, we cannot find the description of a particle's location as a function of time, rather if we know the quantum state of a system, we can extract the information to be compared with measurements. Textbooks which cover the topics in this section in detail are [31; 32].

Mathematically we represent the state ϕ of our system as a vector in a Hilbert space $\mathcal{H}$. The Hilbert space is a complete inner product vector space. We have the usual mathematical structure on the Hilbert space, namely an inner product

$$(\cdot, \cdot) : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}. \tag{1.1.1}$$

A linear functional is defined as a linear map from the Hilbert space to the complex numbers

$$F : \mathcal{H} \to \mathbb{C}. \tag{1.1.2}$$

The space of linear functionals denoted by $\mathcal{H}^*$ is also a Hilbert space, called the dual space of $\mathcal{H}$. We know from Riesz's theorem that we can write any linear functional as an inner product, so for every vector in the dual space there exists a linear functional

$$F_\phi(\psi) = (\phi, \psi), \tag{1.1.3}$$

and vice versa. We can then use the Dirac bra-ket notion where we denote vectors in $\mathcal{H}$ by $|\psi\rangle$ kets, and vectors in the dual space by bras $F_\phi \equiv \langle\phi|$. In this notation the inner product is written as $\langle\phi|\psi\rangle \equiv (\phi, \psi)$.

Now that we have the states of our system we need to introduce the observables i.e. the quantities we are able to measure. From the correspondence principle we know that associated with every observable is a linear hermitian operator. These operators act on the states in our Hilbert space

$$\hat{A} : \mathcal{H} \to \mathcal{H}. \tag{1.1.4}$$

We can define the adjoint of an operator,

$$(\phi, \hat{A}\psi) = (\hat{A}^\dagger\phi, \psi), \tag{1.1.5}$$

then an operator is hermitian when $\hat{A} = \hat{A}^\dagger$ and the domain of A is the same as the domain of $A^\dagger$. We can also define outer products $|\psi\rangle\langle\phi|$ which are operators.The eigenstates of hermitian operators, $\hat{A}|a\rangle = a|a\rangle$, form an orthonormal basis for the Hilbert space and all their eigenvalues are real $a \in \mathbb{R}$. Via the spectral theorem we can write, if the spectrum is discrete

$$\hat{A} = \sum_a a|a\rangle\langle a| \tag{1.1.6}$$

or if the spectrum is continuous

$$\hat{A} = \int da\, a|a\rangle\langle a|\,. \tag{1.1.7}$$

To extract the physical information from these states and observables, we begin with a more general construction: mixed states. If we say the system is in particular mixed state we mean it has probability p_i of being in a state $|\psi_i\rangle$. We can then define the density matrix as

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|\,. \tag{1.1.8}$$

Now we have introduced not only quantum but also statistical uncertainty into the system. If the system is known to be in a state ψ_n then the density matrix is simply

$$\hat{\rho} = |\psi_n\rangle\langle\psi_n|\,. \tag{1.1.9}$$

This is known as a pure state. When we make a measurement of an observable A, the possible values we can obtain are the eigenvalues a. The probability $P(a)$ of obtaining the value a when measuring the observable A for a system with a density matrix $\hat{\rho}$ is

$$P(a) = \langle a|\hat{\rho}|a\rangle\,. \tag{1.1.10}$$

In this way we introduce the probabilistic interpretation of quantum mechanics, we extract the probabilities of measuring values of observables from the state the system is in. We see even for pure states we have the quantum uncertainty. We note this uncertainty is not a statement about our ignorance like statistical uncertainty, but rather is a fundamental property of nature. This is known as a strong measurement, and after these measurements the wavefunction collapses into the state $|a\rangle$ associated with the value measured. To make a sensible probabilistic interpretation requires our states to be normalized. Therefore we require the states in the Hilbert space to have finite norm i.e. $\langle\phi|\phi\rangle < \infty$.

Two typical observables are position $\hat{x}$ and momentum $\hat{p}$. These operators satisfy the Heisenberg algebra:

$$[\hat{x}^i, \hat{p}^j] = i\delta^{ij}. \tag{1.1.11}$$

All other commutators are zero, namely $[\hat{x}^i, \hat{x}^j] = [\hat{p}^i, \hat{p}^j] = 0$. This means that we cannot find simultaneous eigenstates for these operators, since they do not commute, and we cannot simultaneously measure these observables. From this commutator we know that we have an uncertainty principle for $\hat{x}$ and $\hat{p}$

$$\Delta x \Delta p \geq \frac{1}{2}. \tag{1.1.12}$$

Where the $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$ is the standard deviation of an observable. When quantizing a classical system a key step is to find a unitary representation of the Heisenberg algebra. Since $\hat{\vec{x}}$ is a hermitian operator we can use its eigenstates $|\vec{x}\rangle$ as a basis for our Hilbert space. Note x is a continuous label. We can find a unitary representation of the Heisenberg algebra, commonly referred to as the Schrödinger representation

$$\begin{aligned} \hat{\vec{x}}\,|\vec{x}\rangle &= \vec{x}\,|\vec{x}\rangle\,, \\ \hat{\vec{p}}\,|\vec{x}\rangle &= -i\frac{\partial}{\partial \vec{x}}\,|\vec{x}\rangle\,, \end{aligned} \tag{1.1.13}$$

then $[x^i, -i\partial_{x^j}] = i\delta^{ij}$. In the position basis the inner product is $\int d^3x\ \phi^*(\vec{x})\psi(\vec{x})$ and the Hilbert space is now L^2, which are the square integrable functions, as required for a sensible probabilistic interpretation. The Stone-von Neumann theorem ensures that this is a unique representation up to unitary transformations.

Another important set of operators are the orbital angular momentum operators, which are the generators of rotations. In three dimensions the rotation transformations are elements of SO(3). Then the angular momentum operators satisfy the su(2) algebra

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k. \tag{1.1.14}$$

This implies the uncertainty

$$\Delta L_i \Delta L_j \geq \frac{1}{2}|\,\langle \epsilon_{ijk}\hat{L}_k\rangle\,|. \tag{1.1.15}$$

There is a well-known homomorphism between SO(3) and SU(2). In fact SU(2) is a double cover of SO(3). This is how spin is introduced. Spin being an angular momentum observable with no classical analogue. Then we have operators for spin S_i and total angular momentum J_i, which also satisfy the su(2) algebra. The position representation of the orbital angular momentum is given by the cross product

$$\hat{\vec{L}}\,|\vec{x}\rangle = \frac{1}{i}\vec{x} \times \frac{\partial}{\partial \vec{x}}\,|\vec{x}\rangle\,. \tag{1.1.16}$$

1.1.1 The Schrödinger Equation

We can introduce dynamics into our quantum theory. Non-relativistically the time-dependent Schrödinger equation governs how states change in time:

$$i\frac{\partial}{\partial t}\left|\psi,t\right\rangle = \hat{H}\left|\psi,t\right\rangle. \tag{1.1.17}$$

$\hat{H}$ is the Hamiltonian, it contains the information about the factors which influence our dynamics. For a particle interacting with a static electromagnetic field the Hamiltonian, from minimal substitution, is $\hat{H} = \frac{(\hat{\vec{p}}-q\vec{A}(\hat{\vec{x}}))^2}{2m} + qV(\hat{\vec{x}})$. The time-dependent Schrödinger equation is then

$$i\frac{\partial}{\partial t}\left|\psi,t\right\rangle = \left(\frac{(\hat{\vec{p}}-q\vec{A}(\hat{\vec{x}}))^2}{2m} + qV(\hat{\vec{x}})\right)\left|\psi,t\right\rangle. \tag{1.1.18}$$

If we multiply from the left by $\langle\vec{x}|$ in the position basis

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = \left(\frac{(-i\vec{\partial}_x - q\vec{A}(\vec{x}))^2}{2m} + qV(\vec{x})\right)\psi(\vec{x},t). \tag{1.1.19}$$

If the Hamiltonian is not time dependent we can write $\psi(x,t) = e^{-i\hat{H}t}\psi(x)$, then we choose $\psi(x)$ to be an eigenstate of $\hat{H}$ with eigenvalue E. This then produces the time-independent Schrödinger equation

$$E\psi(\vec{x}) = \left(\frac{(-i\vec{\partial}_x - q\vec{A}(\vec{x}))^2}{2m} + qV(\vec{x})\right)\psi(\vec{x}). \tag{1.1.20}$$

This is an eigenvalue equation for E, which determines the allowed energies of our system.

$\psi(\vec{x})$ is a function $\psi(\vec{x}) : \mathbb{R}^3 \to \mathbb{C}$. We call $\mathbb{R}^3$ the configuration space, which is also a Hilbert space, the space that the physical system occupies. We note that free particle solutions $e^{i\vec{k}\cdot\vec{x}}$ are not actually in the Hilbert space. They are limit points of this space and thus extra care must be taken when making a probabilistic interpretation.

1.1.2 The Path Integral

The propagator is defined as

$$K(x_0,t_0,x_f,t_f) = \langle x_f,t_f|x_0,t_0\rangle. \tag{1.1.21}$$

We can calculate this by the usual time slicing method that yields in the continuous limit the path integral in phase space

$$K(x_0,t_0,x_f,t_f) = \int_{(x_0,t_0)}^{(x_f,t_f)}[dx][dp]\; e^{i\int_{t_0}^{t_f} dt\, p\dot{x} - H(p,x)}. \tag{1.1.22}$$

If we perform the $[dp]$ integral we obtain

$$K(x_0, t_0, x_f, t_f) = \int_{(x_0,t_0)}^{(x_f,t_f)} [dx]\ e^{iS}, \tag{1.1.23}$$

where

$$S = \int_{t_0}^{t_f} dt\ \left(\frac{\dot{x}^2}{2m} - V(x)\right). \tag{1.1.24}$$

This is an integral over all the configurations the function $x(t)$ can take i.e. over all the possible paths a particle can take from $x_0 = x(t_0)$ to $x_f = x(t_f)$. We note that this is the classical action. In fact, the path integral is one way to quantize classical systems by starting with a classical action.

We can calculate normalized time-ordered expectation values (correlators) using the path integral

$$\frac{\langle x_0, t_0|\, T[\hat{x}(t_n)\ldots\hat{x}(t_1)]\, |x_f, t_f\rangle}{\langle x_0, t_0|x_f, t_f\rangle} = \frac{\int_{(x_0,t_0)}^{(x_f,t_f)}[dx][dp]\ x(t_n)\ldots x(t_1)e^{i\int_{t_0}^{t_f} dt\ p\dot{x}-H(p,x)}}{K(x_0, t_0, x, t)}. \tag{1.1.25}$$

If we perform the appropriate analytic continuation of time and then take the limit where $t_f \to \infty e^{-i\phi}$ and $t_0 \to -\infty e^{-i\phi}$ this projects out the ground state, then we can use this path integral to calculate vacuum to vacuum expectation values. One way to do this is the Wick rotation here $\phi = \frac{\pi}{2}$.

We can define the normalized generating functional by adding a source to the action

$$Z[J] = \frac{\int_{(x_0,t_0)}^{(x_f,t_f)}[dx]\ e^{iS+\int_{t_0}^{t_f} dt\ J(t)x(t)}}{\int_{(x_0,t_0)}^{(x_f,t_f)}[dx]\ e^{iS}}, \tag{1.1.26}$$

often written in terms of the unnormalized generating functional $Z[J] = \frac{W[J]}{W[0]}$. Then we can compute any correlator by taking functional derivatives with respect to the normalized generating functional

$$\frac{\langle x_0, t_0|\, T[\hat{x}(t_n)\ldots\hat{x}(t_1)]\, |x_f, t_f\rangle}{\langle x_0, t_0|x_f, t_f\rangle} = \frac{\delta}{J(t_n)}\cdots\frac{\delta}{J(t_1)}Z[J]\bigg|_{J=0}. \tag{1.1.27}$$

From now on we are only interested in vacuum to vacuum correlators.

As an example let us consider the Landau problem. In 2 dimensions we introduce complex coordinates $z(t) = x(t) + iy(t)$. For a particle of charge $-e < 0$ moving in a magnetic field $B > 0$ pointing in the positive z-direction, the action in the symmetric gauge is given by

$$S = \int_{-\infty}^{\infty} dt\ \left[m\left\{\dot{z^*} + \frac{ieB}{2m}z^*\right\}\left\{\dot{z} - \frac{ieB}{2m}z\right\} - \frac{e^2B^2}{4m}z^*z\right]. \tag{1.1.28}$$

Electromagnetic interactions were added by minimal substitution. The simplest way to quantize this action is to follow [32]. For this it is convenient to return to the phase-space functional integral for which the action in the commutative case is given by, in the symmetric gauge,

$$S = \int_{-\infty}^{\infty} dt \; \left[\pi^* \left(\dot{z} - \frac{ieB}{2m} z \right) + \left(\dot{z}^* + \frac{ieB}{2m} z^* \right) \pi \right. \\ \left. - \frac{1}{m} \pi^* \pi - \frac{e^2 B^2}{4m} z^* z \right]. \tag{1.1.29}$$

This gives the commutation relations

$$\left[\hat{z}, \hat{\pi}^\dagger \right] = \left[\hat{z}^\dagger, \hat{\pi} \right] = i, \tag{1.1.30}$$

with all other commutators vanishing. Using these commutation relations the Hamiltonian is found to be

$$\hat{H} = \frac{1}{m} \hat{\pi}^\dagger \hat{\pi} + \frac{e^2 B^2}{4m} \hat{z}^\dagger \hat{z} + \frac{ieB}{2m} \left(\hat{\pi}^\dagger \hat{z} - \hat{z}^\dagger \hat{\pi} \right) - \frac{eB}{2m}, \tag{1.1.31}$$

where care with ordering is taken so that we get the correct form for the Zeeman term. We can factorize $\hat{H}$ as

$$\hat{H} = \frac{1}{m} \left(\hat{\pi}^\dagger - \frac{ieB}{2} \hat{z}^\dagger \right) \left(\hat{\pi} + \frac{ieB}{2} \hat{z} \right) - \frac{eB}{2m}. \tag{1.1.32}$$

By introducing the boson creation and annihilation operators

$$\hat{b} = \frac{1}{\sqrt{eB}} \left(\hat{\pi}^\dagger - \frac{ieB}{2} \hat{z}^\dagger \right), \quad \hat{b}^\dagger = \frac{1}{\sqrt{eB}} \left(\hat{\pi} + \frac{ieB}{2} \hat{z} \right) \tag{1.1.33}$$

we can write

$$\hat{H} = \omega_c \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right) \tag{1.1.34}$$

where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency. As usual there is an infinite degeneracy in each Landau level. This is due to another set of creation and annihilation operators, constructed in terms of the guiding centre coordinates, which commute with $\hat{H}$.

1.2 Commutative Field Theory

Now we can move on to a relativistic version of quantum mechanics. Textbooks which cover the topics in this section in detail are [33; 34].

We can write down a relativistic version of the Schrödinger equation, motivated by the relativistic dispersion relation

$$k^\mu k_\mu - m^2 = 0. \tag{1.2.1}$$

When electromagnetic interactions are included the corresponding equation, obtained from minimal substitution, is given by

$$((\partial - qA(x^\nu))^\mu(\partial - qA(x^\nu))_\mu - m^2)\psi(t,\vec{x}) = 0. \tag{1.2.2}$$

This equation, which describes spin-0 particles, is called the Klein-Gordon equation, the Dirac equation for spin-$\frac{1}{2}$ particles can be obtained by linearizing the dispersion relation. In this thesis we only deal with spin-0 theories.

Due to problematic features, such as negative norms for the solutions of the Klein-Gordon equation and negative energies in both the Klein-Gordon and Dirac theories, the single particle probabilistic interpretation of ψ fails. A proper interpretation, as fields, can be obtained when we consider many-particle systems i.e. quantum field theories.

We start with a classical field theory in $3+1$ dimensions. We have a classical Lagrangian, which now depends on the classical field $\phi(\vec{x},t)$ depending on coordinate labels. We can introduce the momentum conjugate to this field, usually $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$. From this we can find the classical Hamiltonian. For the complex scalar field this is

$$H = \int d^3x\, \pi^*(\vec{x},t)\pi(\vec{x},t) + (\vec{\nabla}\phi^*(\vec{x},t))\cdot(\vec{\nabla}\phi(\vec{x},t)) + m^2\phi^*(\vec{x},t)\phi(\vec{x},t). \tag{1.2.3}$$

If we write the path integral action

$$S = \int d^4x\, \phi^*(\vec{x},t)\left(-\partial_t^2 + \vec{\nabla}^2 - m^2\right)\phi(\vec{x},t), \tag{1.2.4}$$

we can quantize using the standard method of Dirac brackets or the method presented in [35]. After quantizing the field operators satisfy the Heisenberg algebra (at equal times)

$$[\hat{\phi}(\vec{x},t),\hat{\pi}(\vec{y},t)] = i\delta(\vec{x}-\vec{y}), \tag{1.2.5}$$

$$[\hat{\phi}(\vec{x},t),\hat{\phi}(\vec{y},t)] = [\hat{\pi}(\vec{x},t),\hat{\pi}(\vec{y},t)] = 0. \tag{1.2.6}$$

Moving to the quantum field theory, we consider our observables $\hat{\phi}$ and $\hat{\pi}$ to be operators which act on a Hilbert space. Now these fields are not our physical states but rather our observables and coordinates are now relegated to labels. The Hilbert space the $\hat{\phi}$ and $\hat{\pi}$ will act on is a Fock space with multi-particle states. We can write our field in terms of creation and annihilation operators

$$\hat{\phi}(\vec{x},t) = \int d^3k\, \left[e^{i\vec{k}\cdot\vec{x}}\hat{a}_{\vec{k}}(t) + e^{-i\vec{k}\cdot\vec{x}}\hat{b}^\dagger_{\vec{k}}(t)\right]. \tag{1.2.7}$$

Here the $\hat{a}_{\vec{k}}$ and $\hat{b}_{\vec{k}}$ create and destroy particle and anti-particle states with momentum $\vec{k}$. We can diagonalize the Hamiltonian as

$$\hat{H} = \int d^3k\, \left[\omega_k\hat{a}^\dagger_{\vec{k}}(t)\hat{a}_{\vec{k}}(t) + \omega_k\hat{b}^\dagger_{\vec{k}}(t)\hat{b}_{\vec{k}}(t) + \omega_k\right], \tag{1.2.8}$$

where $\omega_k = \sqrt{k^2 + m^2}$. We see that the Hamiltonian counts all the momentum contributions to the energy for the different particles and anti-particles. We also see the divergent vacuum contribution. Every point in space acts like the ground state of a harmonic oscillator, even though the space contains no particles. Since space is continuous even in a finite volume the vacuum contribution is divergent. This divergence is something we hope to remove by introducing a minimum length scale, like in our non-commutative theories.

We can find a "position" basis similar to the quantum mechanical case in terms of eigenstates of the fields $|\phi(x,t)\rangle$, the eigenvalues are possible field configurations $\phi(x,t)$,

$$\langle \phi | F \rangle = F[\phi]. \tag{1.2.9}$$

Now $F[\phi]$ is a functional of the field configuration ϕ. The representation of the Heisenberg algebra on these functionals is given uniquely by

$$\begin{aligned} \langle \phi | \, \hat{\phi} \, | F \rangle &= \phi F[\phi], \\ \langle \phi | \, \hat{\pi} \, | F \rangle &= -i \frac{\delta}{\delta \phi} F[\phi]. \end{aligned} \tag{1.2.10}$$

We can think of the quantum mechanical case as field theory in 0+1 dimensions with coordinates $x_i(t)$ being fields discretely labeled by i.

Part I

Non-commutative/Commutative Dualities

Chapter 2

Non-Commutative and Non-Canonical Theories

In part 1 our goal is to develop dualities between different non-canonical field theories, as defined in [21], and standard field theories. By dualities we mean that the generating functionals of the two theories are the same and hence also all the correlators. It could also allow us to track symmetries, and other properties such as unitarity and renormalizability. It could give insight into how to correctly implement the symmetries in non-canonical theories, about which there is still debate [12]. Finally, it could provide a systematic way to introduce interactions in non-canonical theories, the correct way to do this is not obvious [21; 22]. Another potential advantage as mentioned in the introduction, is that these dualities could make it easier to calculate quantities, by calculating them for the dual systems first. One way this simplifies calculations is by weakening interactions in the dual system.

In this chapter we review non-commutative quantum mechanics, specifically focusing on the two-dimensional case. This is what we generalize when we study non-canonical theories. After this we review non-canonical field theories, as our aim is to build dualities for these theories.

2.1 Non-Commutative Quantum Mechanics

We believe there must exist a minimum length scale and that we should build this into our theories. In non-commutative theories our coordinates are operators, then their commutation relations introduce, from the uncertainty principle, a minimum length scale. In two dimensions, the simplest choice for the commutation relations are the constant commutation relations:

$$[\hat{x}^i, \hat{x}^j] = i\epsilon^{ij}\theta. \tag{2.1.1}$$

This leads to the uncertainty

$$\Delta x^i \Delta x^j \geq \frac{|\epsilon^{ij}\theta|}{2}, \tag{2.1.2}$$

where θ, which has the dimension of length squared, is related to the minimum length scale. These commutation relations where studied in [5], where an operator-based formulation for quantum mechanics was developed. It was found that mathematically it is structured exactly in the same way as standard quantum mechanics. It is then possible to find a unitary representation of the non-commutative Heisenberg algebra. The only modification required is that more general Hilbert spaces are needed to represent the states of the system. Since the coordinates do not commute we cannot simultaneously measure both coordinates. The way to deal with this is to introduce weak, rather then strong, measurements when position is measured.

The non-commutative coordinate algebra (2.1.1) acts on a Hilbert space $\mathcal{H}_c$, which we call the configuration space or classical Hilbert space. We use the angular ket for states in this space: $|n\rangle \in \mathcal{H}_c$. We can realize the constant commutation relations by introducing a pair of creation and annihilation operators,

$$\hat{b} = \frac{1}{\sqrt{2\theta}}(\hat{x} + i\hat{y}), \qquad \hat{b}^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x} - i\hat{y}), \tag{2.1.3}$$

such that $[\hat{b}, \hat{b}^\dagger] = \hat{\mathbb{I}}_c$. The configuration space is then simply the boson Fock space

$$\mathcal{H}_c = \text{span}\{|n\rangle = \frac{1}{\sqrt{n}}(\hat{b}^\dagger)^n\,|0\rangle\}_{n=0}^{n=\infty}. \tag{2.1.4}$$

Now our quantum states (wave-functions) are operators which are elements of the algebra generated by the coordinate operators i.e. $\psi(\hat{x}^1, \hat{x}^2)$. These states live in a Hilbert space called the quantum Hilbert space. Note that this is different from the configuration space. The most natural choice for the quantum Hilbert space is the space of Hilbert-Schmidt operators.

$$\mathcal{H}_q = \{\psi(\hat{x}^1, \hat{x}^2) : \text{Tr}_c(\psi^\dagger(\hat{x}^1, \hat{x}^2)\psi(\hat{x}^1, \hat{x}^2)) < \infty\}. \tag{2.1.5}$$

We use a subscript q for the quantum Hilbert space to distinguish it from the configuration space $\mathcal{H}_c$. The finite trace condition is a generalization of square integrability. However, we can still consider scattering theory [36], in the same way as commutatively, by considering operators outside this Hilbert space (free particle solutions), which are limit points of $\mathcal{H}_q$. We use the rounded ket to represent states $|\hat{\psi}) \in \mathcal{H}_q$ and reserve the symbol $\dagger$ specifically for the operator adjoint on $\mathcal{H}_c$. The natural inner product on this space is

$$(\hat{\phi}|\hat{\psi}) = \text{Tr}_c(\phi^\dagger(\hat{x}^1, \hat{x}^2)\psi(\hat{x}^1, \hat{x}^2)). \tag{2.1.6}$$

As in standard quantum mechanics we have linear functionals and a dual space which are defined using the trace inner product.

Now we need to introduce observables i.e. operators which act on states in the quantum Hilbert space. The non-commutative Heisenberg algebra acting on the quantum Hilbert space, obey the commutation relations

$$
\begin{aligned}
[\hat{X}^i, \hat{X}^j] &= i\epsilon^{ij}\theta, \\
[\hat{X}^i, \hat{P}^i] &= i\delta^{ij}, \\
[\hat{P}^i, \hat{P}^i] &= 0.
\end{aligned} \tag{2.1.7}
$$

We use capital letters for these operators $\hat{X}^i$,$\hat{P}^i$ to distinguish them from the coordinate operators acting on configuration space. These are often called super operators since they operate on operators. We use the symbol $\ddagger$ for the adjoint on $\mathcal{H}_q$ defined with respect to the trace inner product

$$(\hat{\phi}|\hat{O}\hat{\psi}) = (\hat{O}^{\ddagger}\hat{\phi}|\hat{\psi}). \tag{2.1.8}$$

We can find a unitary representation of the non-commutative Heisenberg algebra in the following way [5]

$$
\begin{aligned}
\hat{X}^i|\hat{\psi}) &= |\hat{x}^i\hat{\psi}), \\
\hat{P}^i|\hat{\psi}) &= \frac{1}{\theta}\epsilon^{ij}|[\hat{x}^j, \hat{\psi}]).
\end{aligned} \tag{2.1.9}
$$

This is similar to the position representation of commutative quantum mechanics. The non-commutative time independent Schrödinger equation is

$$\left(\frac{(\hat{\vec{P}} - q\vec{A}(\hat{\vec{X}}))^2}{2m} + qV(\hat{\vec{X}})\right)|\hat{\psi}) = E|\hat{\psi}), \tag{2.1.10}$$

where electromagnetic interactions have been added by minimal substitution as in the commutative case.

Instead of the operator-based formulation above, the constant commutation relations can also be realized in terms of scalar functions. This is similar to the Wigner phase space construction of quantum mechanics using a star product [37] and quasiprobabilities. However, this is much less physically intuitive. A star product which implements the non-commutative coordinate algebra is the Moyal product given by

$$*_M = e^{\frac{i}{2}\theta\epsilon^{ij}\overleftarrow{\partial^i}\overrightarrow{\partial^j}}, \tag{2.1.11}$$

$$x^i *_M x^j - x^j *_M x^i = i\epsilon^{ij}\theta. \tag{2.1.12}$$

Then the scalar functions depend on the numbers x^i, but all regular products of scalar functions now become composed with star products. However, these commuting coordinates cannot be interpreted as true physical coordinates; they have no clear physical meaning. This star product is also not unique, but there are infinitely many that would lead to the same commutation relations. Another star product commonly used is the Voros product

$$*_V = e^{\frac{i}{2}\theta\epsilon^{ij}\overleftarrow{\partial^i}\overrightarrow{\partial^j} + \frac{i}{2}\theta\overleftarrow{\partial^i}\overrightarrow{\partial^i}}. \tag{2.1.13}$$

In this formulation the time independent Schrödinger equation can be written in terms of functions as

$$\left(\frac{(\hat{\vec{p}}-q\vec{A}(\hat{\vec{x}}))^2}{2m}+qV(\hat{\vec{x}})\right)*_{V/M}\psi(x)=E\psi(x), \tag{2.1.14}$$

Later it will become apparent why this is potentially the more natural choice. Mathematically these formulations are equivalent [38] , however, care must be taken with the physical interpretation of these star products and in particular the space of functions on which they are defined, which is not simply L^2. These domains are even different for the Voros and Moyal products and great care must be taken when any comparison of theories built on these two products is made [38].

The Moyal product can also be thought of as a transformation that mixes position and momentum, since the star product shifts a coordinate by a derivative. The coordinates in the scalar functions are then combinations of position and momentum and can therefore not be interpreted as coordinates in position space. The Bopp shift does exactly this on the operator level

$$\hat{x}_i=\hat{\tilde{x}}_i+i\frac{\theta}{2}\epsilon^{ij}\hat{\tilde{p}}_j, \tag{2.1.15}$$

this is one way to build our non-commutative algebra. Here the tilde operators are operators which obey the standard position and momentum commutation relations. It is then possible to build eigenstates for these tilde coordinates but they are not physical and care must be taken when using these states for calculations.

2.2 Position and the Path Integral

We would like to introduce a "position state", however, since we cannot find simultaneous eigenstates for $\hat{x}$ and $\hat{y}$ the best we can do is a minimum uncertainty state. It is possible to define a one mode Glauber coherent state in configuration space [39]

$$|z\rangle=e^{-\frac{|z|^2}{2}}e^{z\hat{b}^\dagger}|0\rangle\,. \tag{2.2.1}$$

In the quantum Hilbert space we define a state, which is known to have the interpretation of a minimum uncertainty state [5]

$$|z)=|z\rangle\langle z|\,, \tag{2.2.2}$$

for $\theta>0$

$$z=\frac{1}{\sqrt{2\theta}}(x+iy) \tag{2.2.3}$$

is a dimensionless complex number. These states form an over-complete basis for the configuration space and it is possible to write the identity as

$$\hat{\mathbb{I}}_c = \frac{1}{\pi}\int dzdz^* \ |z\rangle\langle z| \,. \tag{2.2.4}$$

As these are minimum uncertainty states, the real and imaginary parts of z are the closest we can get to points in space obtained from a measurement. We can write the identity on the quantum Hilbert space as

$$\hat{\mathbb{I}}_q = \frac{1}{\pi}\int dzdz^* \,|z) *_V (z|. \tag{2.2.5}$$

The Voros product, which in terms of the complex coordinates is given by $*_V = e^{\overleftarrow{\partial_z}\overrightarrow{\partial_{z^*}}}$, is necessary for this to be the identity. Motivating why this is a more natural choice than the Moyal product.

We can think of quantum mechanics as field theory in $0+1$ dimensions, the coordinates are the fields which depend on time $x^i(t)$. A path integral approach to non-commutative quantum mechanics been developed in [40], we now mention the relevant results. Here we time slice using the $|z)$ states, with our propagator being $(z_f, t_f|z_0, t_0)$, then the calculation proceeds using the completeness of these states in terms of the star product. This propagator is given by

$$(z_f, t_f|z_0, t_0) = Ne^{-\partial_{z_f}\partial_{z_0^*}}\int_{(z_0,t_0)}^{(z_f,t_f)} [dz][dz^*]e^{iS}. \tag{2.2.6}$$

As calculated in [40], for the free theory the action in the above propagator is given by

$$S = \int_{t_0}^{t_f} dt \,\frac{1}{2}\dot{z}^*\left(\frac{1}{2m}+\frac{i\theta}{2}\partial_t\right)^{-1}\dot{z}. \tag{2.2.7}$$

2.2.1 The Landau Problem

One of the simplest non-trivial dualities we can build is for the Landau problem [41]. This construction will also be useful for non-canonical field theories due to the similar form of the non-canonical and the non-commutative Landau problem actions. We now briefly review the non-commutative Landau problem. The action was derived in [40; 42], and reads

$$\begin{aligned} S = \int_{-\infty}^{\infty} dt \ \Bigg[&\frac{1}{2}\left\{\dot{z}^* + \frac{ieB}{2m}z^*\right\}\left(\frac{1}{2m}+\frac{i\theta}{2}\partial_t\right)^{-1}\left\{\dot{z}-\frac{ieB}{2m}z\right\} \\ &-\frac{e^2B^2}{4m}z^*z\Bigg]. \end{aligned} \tag{2.2.8}$$

In phase space the non-commutative action is given by

$$S = \int_{-\infty}^{\infty} dt \, \left[\pi^* \left(\dot{z} - \frac{ieB}{2m} z \right) + \left(\dot{z}^* + \frac{ieB}{2m} z^* \right) \pi \right.$$
$$\left. - \pi^* \left(\frac{1}{m} + i\theta\partial_t \right) \pi - \frac{e^2 B^2}{4m} z^* z \right]. \tag{2.2.9}$$

The functional integral is now over π, π^* and z, z^*.

The above action can be quantized, the simplest way to do so is to follow [35]. The commutation relations are found to be

$$\left[\hat{z}, \hat{\pi}^\dagger\right] = \left[\hat{z}^\dagger, \hat{\pi}\right] = i, \quad \left[\hat{z}, \hat{z}^\dagger\right] = \theta, \tag{2.2.10}$$

with all other commutators vanishing. This calculation is shown in more detail later in this chapter for the case of the non-canonical field theory. Now the momenta are still commuting, but the coordinates are non-commuting. The Hamiltonian is still given by (1.1.31) as in the commutative case. We can factorize in the same way as in (1.1.32). Now the creation and annihilation operators are

$$\hat{b} = \frac{1}{\sqrt{e|\tilde{B}|}} \left(\hat{\pi}^\dagger - \frac{ie\tilde{B}}{2} \hat{z}^\dagger \right), \quad \hat{b}^\dagger = \frac{1}{\sqrt{e|\tilde{B}|}} \left(\hat{\pi} + \frac{ie\tilde{B}}{2} \hat{z} \right), \tag{2.2.11}$$

where the effective magnetic field,

$$\tilde{B} = B \left(1 - \frac{eB\theta}{4} \right), \tag{2.2.12}$$

is positive. For negative $\tilde{B}$ the definitions of $\hat{b}$ and $\hat{b}^\dagger$ are exchanged. The Hamiltonian becomes

$$H_{NC} = \tilde{\omega}_c \hat{b}^\dagger \hat{b} + \frac{\tilde{\omega}_c}{2}, \tag{2.2.13}$$

where $\tilde{\omega}_c$ is the effective cyclotron frequency $\tilde{\omega}_c = \frac{e|\tilde{B}|}{m}$. There is again an infinite degeneracy of each Landau level.

Now we have reviewed all we need to construct dualities for these non-commutative quantum mechanical theories. This is the subject of the next chapter. The rest of this chapter is devoted to reviewing non-canonical field theories, keeping in mind the ultimate goal of this part of the thesis is to construct dualities for non-canonical field theories.

2.3 Non-Canonical Field Theory

One naive way found in the literature to build non-commutative field theories, is to take standard Lagrangians and insert star products between the fields.

This, however, produces several problems. Which star product to use is not obvious, and the space of fields is now no longer L^2 so one would have to modify the path integral measure [18], which is quite difficult. However, there are more systematic ways to build non-commutative field theories [42], but even then implementing the symmetries is not obvious [12]. We discuss in part 2 a way to build a theory in non-commutative space-time that, by construction, respects Lorentz symmetry, but for now we discuss a different kind of non-commutative theory.

In non-canonical field theories, space-time is not non-commutative, but rather we have non-commutative fields. This is more in the spirit of non-commutative quantum mechanics where the degrees of freedom are the coordinates. In field theories the fields are the degrees of freedom where as position is simply what we use to label the fields. We label the coordinates x^i with i, while the fields $\phi(x)$ are labeled by x.

We change the canonical commutation relation of our observables, hence for a non-canonical field theory, we postulate, at equal times,

$$[\hat{\Phi}_a(x), \hat{\Phi}_b(y)] = i\theta_{ab}(x-y), \tag{2.3.1}$$

$$[\hat{\Pi}_a(x), \hat{\Pi}_b(y)] = i\delta_{a,b}\delta(x-y). \tag{2.3.2}$$

The a, b denote different species of fields and θ_{ab} is some function of $x - y$. These operators associated with the observable fields will be the super-operators, denoted by capitals. Following the quantum mechanical case they act on states in H_q. These states are operators which are elements of the algebra generated by the non-commutative field configurations i.e. $F[\hat{\phi}] \equiv |\hat{F})$. The field configurations act on the configuration space $|f\rangle \in H_c$. The non-commutative field configurations satisfy modified commutation relations, at equal times,

$$[\hat{\phi}_a(x), \hat{\phi}_b(y)] = i\theta_{ab}(x-y), \tag{2.3.3}$$

and play the role of the non-commutative coordinates.

2.3.1 Complex Field Theory

Now we consider the complex scalar field theory i.e. a field theory with two different real fields. We consider non-commutativity of the form [21], at equal times,

$$[\hat{\Phi}(x), \hat{\Phi}^{\ddagger}(y)] = \theta\delta(x-y), \tag{2.3.4}$$

the fields also obey the standard equal time commutation relations

$$[\hat{\Phi}(x), \hat{\Pi}(y)] = i\delta(x-y). \tag{2.3.5}$$

Recalling that ‡ is the adjoint on the quantum Hilbert space. Again we can find a "position" representation in the following way, similar to the non-commutative quantum mechanical case,

$$\hat{\Phi}(x,t)|\hat{F}) = |\hat{\phi}(x,t)\hat{F}), \tag{2.3.6}$$

$$\hat{\Pi}(x,t)|\hat{F}) = \frac{1}{\theta}|[\hat{\phi}^{\dagger}(y,t),\hat{F}]), \tag{2.3.7}$$

the same for ‡ fields.

All other commutation relations are standard. The free Hamiltonian is given by

$$\hat{H} = \int d^3x\ \hat{\Pi}^{\ddagger}(\vec{x},t)\hat{\Pi}(\vec{x},t) + (\vec{\nabla}\hat{\Phi}^{\ddagger}(\vec{x},t))\cdot(\vec{\nabla}\hat{\Phi}(\vec{x},t)) + m^2\hat{\Phi}^{\ddagger}(\vec{x},t)\hat{\Phi}(\vec{x},t). \tag{2.3.8}$$

We can write it in Fourier modes, by introducing the Fourier transform

$$\begin{aligned}\hat{\Phi}(x) &= \int d^3k\ e^{-ikx}\Phi_k \\ \hat{\Pi}(x) &= \int d^3k\ e^{-ikx}\Pi_k\end{aligned} \tag{2.3.9}$$

then we have

$$\hat{H} = \int d^3k\ \hat{\Pi}_k^{\ddagger}\hat{\Pi}_k + \omega_k^2\hat{\Phi}_k^{\ddagger}\hat{\Phi}_k, \tag{2.3.10}$$

these operators satisfy the following commutation relations

$$[\hat{\Phi}_k,\hat{\Pi}_{k'}] = i\delta(k+k'), \tag{2.3.11}$$

$$[\hat{\Phi}_k^{\ddagger},\hat{\Pi}_{k'}^{\ddagger}] = i\delta(k+k'), \tag{2.3.12}$$

$$[\hat{\Phi}_k,\hat{\Phi}_{k'}^{\ddagger}] = 2\frac{g_k}{\omega_k}\delta(k-k'), \tag{2.3.13}$$

where

$$\omega_k = \sqrt{k^2+m^2}, \qquad g_k = \frac{\omega_k\theta}{2}. \tag{2.3.14}$$

We can diagonalize the Hamiltonian by introducing creation and annihilation operators acting on $\mathcal{H}_q$, the details of this calculation are in [21],

$$\begin{aligned}\hat{H} = \int d^3k \quad & \Big[\omega_k\left(\sqrt{1+g_k^2}+g_k\right)\hat{A}_k^{\ddagger}\hat{A}_k + \omega_k\left(\sqrt{1+g_k^2}-g_k\right)\hat{B}_k^{\ddagger}\hat{B}_k \\ & +\omega_k\sqrt{1+g_k^2}\Big].\end{aligned} \tag{2.3.15}$$

This has the very interesting property of matter anti-matter asymmetry, the physical consequences of this is discussed in [23].

The path integral action in position space is given by

$$S = \int d^4x\ \phi^*(t,\vec{x})\left(\frac{-\partial_t^2}{1+i\theta\partial_t - i\epsilon} + \vec{\nabla}^2 - m^2\right)\phi(t,\vec{x}). \tag{2.3.16}$$

This is derived in [21] using a method similar to the one discussed above, the difference is that the field theory equivalent of Bopp shift states (2.1.15) are used instead of coherent states. Note the similarity with (2.2.7).

We will need this action when we build our dual theory. As a consistency check, we quantize this theory to verify that the commutation relations (2.3.4) are indeed reproduced. Introducing auxiliary fields χ and χ^* we can rewrite the action as

$$S = \int d^4x \quad \Big[\chi^* \left(1 + i\theta\partial_t - i\epsilon\right)\chi + \chi^*\partial_t\phi + \chi\partial_t\phi^* + \phi^*(\vec{\nabla}^2 - m^2)\phi\Big] . \tag{2.3.17}$$

The epsilon term ensures the convergence of the path integral, but hereafter we drop it. We follow the Faddeev-Jackiw method [35] to quantize the system. We denote

$$\xi^i = \{\phi, \phi^*, \chi, \chi^*\}, \tag{2.3.18}$$

and write the Lagrangian as

$$L = \int d^3x \left[\frac{1}{2}\xi^i\omega_{ij}\partial_t\xi^j - V(\xi)\right], \tag{2.3.19}$$

with

$$\omega_{ij} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & i\theta \\ 1 & 0 & -i\theta & 0 \end{pmatrix} . \tag{2.3.20}$$

Following [35] the commutation relations (2.3.4) are indeed recovered from the entries of $i\omega^{ij}$ with ω^{ij} the inverse of ω_{ij}. Following quantization the Hamiltonian has the form

$$\hat{H} = V(\xi) = \int d^3x \left[\hat{\chi}^\ddagger\hat{\chi} - \hat{\Phi}^\ddagger(\vec{\nabla}^2 - m^2)\hat{\Phi}\right]. \tag{2.3.21}$$

The auxiliary fields are constrained to the conjugate momenta, yielding the final Hamiltonian

$$\hat{H} = \int d^3x \left[\hat{\Pi}^\ddagger\hat{\Pi} + \vec{\nabla}\hat{\Phi}^\ddagger \cdot \vec{\nabla}\hat{\Phi} + m^2\hat{\Phi}^\ddagger\hat{\Phi}\right]. \tag{2.3.22}$$

This does indeed match the non-canonical Hamiltonian we started with.

2.4 Chapter Summary

We have reviewed the non-commutative theories we seek to develop dualities for. We do this because we believe that these non-commutative theories are a better description of nature at short length scales. However, we need the dualities to track symmetries and the renormalizabilty of the non-commutative theories. We covered the two dimensional quantum mechanical case including the path integral formulation. We also studied the very similar in spirit non-canonical field theories. We now can move on to constructing the dualities for our non-canonical field theories. First, however, we review the dualities for quantum mechanical theories, this will provide some additional insights in the next chapter.

Chapter 3

The Exact Renormalization Group (ERG) and Non-Commutative Dualities

In this chapter we introduce and review dualities for the non-commutative quantum mechanical case of the Landau problem. The goal of part one is to generalize the ERG dualities to the non-canonical field theory case. With a duality we mean that the generating functionals of the two theories are the same and hence also all the correlators. Note, however, in the context of the ERG a weaker form of duality is normally implied in that the generating functionals and correlators are the same below a certain momentum scale.

In the case of a complex scalar field, the generating functional is given by

$$Z[J] = \frac{W[J]}{W[0]} \tag{3.0.1}$$

where

$$W[J] = \int [d\phi^*][d\phi] e^{i[S[\phi^*,\phi]+J^*\phi+J\phi^*]}. \tag{3.0.2}$$

We discuss two ways to build these dualities. First we use a change of variables that mixes momentum and position similar to the Bopp shift. This is physically more intuitive and mathematically transparent, but it is difficult to apply this to systems with more general interactions. This is why we use the second method based on the exact renormalization group (ERG). In particular, the ERG will be useful when studying interacting systems, which we discuss in the next chapter.

3.1 Bopp Shift

We start with a generalized Bopp shift, and derive a slightly more general duality then just a commutative/non-commutative one. We introduce the change of variables

$$z = Z - ia\pi, \quad z^* = Z^* + ia\pi^*, \quad a \in \mathbb{R} \tag{3.1.1}$$

into the *non-commutative* phase-space generating functional integral with action (2.2.9). The action then becomes

$$S = \int_{-\infty}^{\infty} dt \left[\pi^* \left(\dot{Z} - \frac{ie\tilde{B}}{2\tilde{m}} Z \right) + \left(\dot{Z}^* + \frac{ie\tilde{B}}{2\tilde{m}} Z^* \right) \pi - \pi^* \left(\frac{1}{\tilde{m}} + i\tilde{\theta}\partial_t \right) \pi \right.$$
$$\left. - \frac{e^2\tilde{B}^2}{4\tilde{m}} Z^* Z + J^* \left(Z - ia\pi \right) + \left(Z^* + ia\pi^* \right) J \right], \tag{3.1.2}$$

where

$$\tilde{B} = \frac{2B}{2 + aeB}, \quad \tilde{m} = \frac{4m}{(2 + aeB)^2}, \quad \tilde{\theta} = \theta + 2a. \tag{3.1.3}$$

If we compare this with (2.2.9), we see that this is just another non-commutative theory with $\tilde{\theta} = \theta + 2a$. The only difference from the standard non-commutative generating functional is the coupling between π and the source J.

If we introduce a further change of variables

$$\pi = \tilde{\pi} + ia\Delta^{-1} J, \tag{3.1.4}$$

where

$$\Delta = \left(\frac{1}{\tilde{m}} + i\tilde{\theta}\partial_t \right), \tag{3.1.5}$$

then we have standard renormalized source terms with an additional quadratic source term, provided we set

$$a = \frac{1}{e\tilde{B}} - \frac{\theta}{2}. \tag{3.1.6}$$

The action is then

$$S = \int_{-\infty}^{\infty} dt \left[\tilde{\pi}^* \left(\dot{Z} - \frac{ie\tilde{B}}{2\tilde{m}} Z \right) + \left(\dot{Z}^* + \frac{ie\tilde{B}}{2\tilde{m}} Z^* \right) \tilde{\pi} \right.$$
$$- \tilde{\pi}^* \left(\frac{1}{\tilde{m}} + i\tilde{\theta}\partial_t \right) \tilde{\pi} - \frac{e^2\tilde{B}^2}{4\tilde{m}} Z^* Z$$
$$\left. + \left(1 - \frac{a}{\tilde{\theta}} \right) J^* Z + \left(1 - \frac{a}{\tilde{\theta}} \right) Z^* J + a^2 J^* \Delta^{-1} J \right]. \tag{3.1.7}$$

If we solve equations (3.1.3) and (3.1.6) we find

$$\tilde{\theta} = \frac{4 - \theta eB}{eB} \qquad \tilde{B} = \frac{2B}{4 - \theta eB} \tag{3.1.8}$$

$$\tilde{m} = \frac{4m}{(4 - \theta eB)^2} \qquad a = \frac{2 - \theta eB}{eB}. \tag{3.1.9}$$

We now have the exact relation between the generating functionals of two non-commutative theories

$$Z_\theta [J] = e^{ia^2 \int_{-\infty}^{\infty} dt \left[J^* \Delta^{-1} J\right]} \tilde{Z}_{\tilde{\theta}} [\kappa J] \tag{3.1.10}$$

with

$$\kappa = \frac{2}{4 - \theta e B}. \tag{3.1.11}$$

In the special case of $\theta = 0$, this gives the commutative/non-commutative duality

$$Z_c [J] = e^{ia^2 \int_{-\infty}^{\infty} dt \left[J^* \Delta^{-1} J\right]} Z_{nc} \left[\frac{1}{2} J\right], \tag{3.1.12}$$

where the non-commutative theory has the non-commutative parameter $\tilde{\theta} = \frac{4}{eB}$. Now we have a complete duality between the commutative and non-commutative theories. No zero mass limit or infinite magnetic field limit had to be taken as is normally done. This duality is exact.

From (3.1.12) the following simple rule tells how to compute the correlators of the non-commutative theory in terms of the correlators of the commutative theory, and vice versa,

$$\begin{aligned} \frac{\delta}{\delta J} &\rightarrow \frac{\delta}{\delta J} + ia^2 \left(\Delta^*\right)^{-1} J^*, \\ \frac{\delta}{\delta J^*} &\rightarrow \frac{\delta}{\delta J^*} + ia^2 \Delta^{-1} J. \end{aligned} \tag{3.1.13}$$

since these equations can be easily inverted. We now have a complete dictionary for the computation of correlation functions.

3.2 The Exact Renormalization Group (ERG)

The duality above can be constructed in a more general way [41], and is discussed in detail for the Landau problem in [41], where the ERG is used to construct dual families of non-commutative theories. We now review this construction which we will use in the next chapter for the complex scalar field.

The basic idea of the ERG procedure is to modify the kinetic energy in a particular way [43] to avoid ultraviolet divergences by introducing a cut-off. This modification is defined by a parameter called the flow parameter. In the ERG construction the interacting part of the action is also modified in such a way that the generating functional doesn't depend on the flow parameter, at least below the cut-off scale. The change of the interaction part of the action is defined by a set of flow equations. In this way we have not changed the physics below the cut-off. We also have to restrict the sources so they are zero above the cut-off. When the flow parameter is zero we obtain the standard action which defines the initial conditions of the flow equations. A detailed discussion can be found in the textbook [43]. We can use the ERG construction to build our non-commutative/commutative duality.

Essentially the commutative and non-commutative theories differ in the kinetic energy terms, this is clear when comparing (1.1.28) with (2.2.8). We can use the ERG to derive a duality between these theories, if we identify the flow parameter as the non-commutative parameter θ. The only difference from the standard ERG approach is that the restricting assumptions on the modified kinetic energy term and source terms do not necessarily apply in this case [43]. We can compensate for this by introducing flow equations for the source terms, derived in [41] and summarized in appendix D.

We consider a complex scalar field theory in $0+1$-dimensions with Fourier transformed action

$$S[\phi,\phi^*] = \int d\omega\, \phi^*(\omega)K(\omega,\ell)\phi(\omega) + S_I[\phi,\phi^*] + J_\ell[\phi,\phi^*]. \tag{3.2.1}$$

Here $K(\omega,\ell)$ is our modified kinetic term and ℓ the flow parameter. The kinetic energy takes the standard form

$$K(\omega,0) = m\omega^2, \tag{3.2.2}$$

when the flow is switched off. The second term $S_I[\phi,\phi^*]$ is the interacting part of the action, it is a function of the flow parameter ℓ. $J_\ell[\phi,\phi^*]$ is a generalized source term, which is a functional of the fields, determined by the requirement of invariance of the generating functional. We will see, when we flow the source, that it is sufficient to take this source term to be linear if the interaction is at most quadratic in the fields

$$J_\ell[\phi,\phi^*] = \int d\omega\, \left[J_0(l) + J_0^*(l) + J_1(l)\phi^*(\omega) + J_1^*(l)\phi(\omega)\right]. \tag{3.2.3}$$

The initial conditions imply $J_0(0) = J_0^*(0) = 0$ and $J_1(0)$ is an arbitrary function of ω that acts as the standard source in the bare ($\ell = 0$) action.

We now apply the logic of the ERG as set out in [44] and [43], and require $Z[J] = \frac{W[J]}{W[0]}$ to be invariant under the flow, i.e. independent of ℓ. However, we relax the usual conditions imposed on $K(\omega,\ell)$ and the sources, that lead to the weaker form of duality mentioned earlier, which necessitates the flow of the source terms as well. A derivation of the generalized flow equations can be found in appendix D. The flow equations for the interacting part and source terms are then given by

$$\partial_\ell S_I = \int d\omega\, \partial_\ell K^{-1} \left\{ \frac{\delta S_I}{\delta\phi^*(\omega)}\frac{\delta S_I}{\delta\phi(\omega)} - \frac{\delta^2 S_I}{\delta\phi^*(\omega)\delta\phi(\omega)} \right\} \tag{3.2.4}$$

$$\begin{aligned} \partial_\ell J_\ell &= \int d\omega\ \partial_\ell K^{-1} \left\{ \frac{\delta S_I}{\delta\phi(\omega)} \frac{\delta J_\ell}{\delta\phi^*(\omega)} + \frac{\delta S_I}{\delta\phi^*(\omega)} \frac{\delta J_\ell}{\delta\phi(\omega)} \right. \\ &\left. + \frac{\delta J_\ell}{\delta\phi^*(\omega)} \frac{\delta J_\ell}{\delta\phi(\omega)} - \frac{\delta^2 J_\ell}{\delta\phi^*(\omega)\delta\phi(\omega)} \right\}. \end{aligned} \tag{3.2.5}$$

These equations can easily be solved when the interaction term is quadratic in the fields, i.e.

$$S_I[\phi, \phi^*] = \int d\omega\ \phi^*(\omega) g(\omega, \ell) \phi(\omega), \tag{3.2.6}$$

with $g(\omega, \ell)$ real. Focusing on the source terms for the moment and using (3.2.5) and (3.2.3)

$$\begin{aligned} \partial_\ell J_1(\ell) &= \partial_\ell K^{-1}(\omega, \ell) g(\omega, \ell) J_1(\ell), \\ \partial_\ell [J_0(\ell) + J_0^*(\ell)] &= \partial_\ell K^{-1}(\omega, \ell) |J_1(\ell)|^2. \end{aligned} \tag{3.2.7}$$

Integrating these equations and using the initial conditions on the sources, we obtain

$$\begin{aligned} J_1(\ell) &= J_1(0) \exp\left(\int_0^\ell d\ell'\ g(\omega, \ell') \partial_{\ell'} K^{-1}(\omega, \ell') \right), \\ J_0(\ell) + J_0^*(\ell) &= |J_1(0)|^2 \int_0^\ell d\ell'\ \partial_{\ell'} K^{-1}(\omega, \ell') \\ &\times \exp\left(2 \int_0^{\ell'} d\ell''\ g(\omega, \ell'') \partial_{\ell''} K^{-1}(\omega, \ell'') \right). \end{aligned} \tag{3.2.8}$$

We now apply this result to the Landau problem (1.1.28), for which the Fourier transformed action reads

$$\begin{aligned} S &= \int d\omega\, [z^*(\omega) m\omega^2 z(\omega) - eB\omega z^*(\omega) z(\omega) \\ &+ J(\omega) z^*(\omega) + J^*(\omega) z(\omega)]. \end{aligned} \tag{3.2.9}$$

Motivated by the form of the action for the non-commutative Landau problem (2.2.7) we take

$$K(\omega, \ell) = \frac{m\omega^2}{(1 - m\omega\ell - i\epsilon)}, \tag{3.2.10}$$

the epsilon term ensures the convergence of the path integral, but hereafter we drop it. The interacting part of the action (3.2.6) has the initial condition

$$g(\omega, 0) = -eB\omega. \tag{3.2.11}$$

Inserting (3.2.9) into (3.2.4) we obtain the following equation for the interacting term

$$\frac{\partial g(\omega, \ell)}{\partial \ell} = -\frac{1}{\omega} g^2(\omega, \ell). \tag{3.2.12}$$

Note we have ignored a vacuum term generated from the right hand side of (3.2.4), since this just adds a term independent of z to the interaction. Solving the equation with the initial conditions imposed produces

$$g(\omega,\ell) = e\tilde{B}(\ell)\omega, \qquad \tilde{B}(\ell) = \frac{B}{1+eB\ell}. \tag{3.2.13}$$

From (3.2.5) we obtain for the source,

$$\begin{aligned} J_1(\ell) &= \frac{J_1(0)}{1-eB\ell}, \\ J_0(\ell) + J_0^*(\ell) &= -\frac{|J_1(0)|^2\ell}{\omega(1-eB\ell)}. \end{aligned} \tag{3.2.14}$$

However, this action still does not quite have the same form as the non-commutative action so we rescale the fields in an ω-independent way

$$\tilde{z}(\omega) = \sqrt{\frac{1}{1-eB\ell}}z(\omega). \tag{3.2.15}$$

This yields the dual action [41]

$$\begin{aligned} S &= \int d\omega \left[\tilde{z}^*(\omega)K(\omega,\ell)\tilde{z}(\omega) - \frac{eB\omega}{(1-m\omega\ell)}\tilde{z}^*(\omega)\tilde{z}(\omega) \right. \\ &\left. -\frac{|J_1(0)|^2\ell}{\omega(1-eB\ell)} + \left(\frac{J_1(0)}{\sqrt{1-eB\ell}}\tilde{z}^*(\omega) + c.c.\right)\right]. \end{aligned} \tag{3.2.16}$$

Now the generating functional is seemly not independent of ℓ, however this is just a change of variables in the path integral, if done correctly this does not change the propagator. Care must be taken when computing correlators, as one needs to keep in mind which fields we are calculating the correlators of. Since we have a dictionary between the commutative and non-commutative fields this poses no problem. Now this is the action of a non-commutative Landau problem with

$$\theta = \ell \tag{3.2.17}$$

and a magnetic field $\tilde{B}$ determined by

$$B = \tilde{B}\left(1 - \frac{e\tilde{B}\ell}{4}\right). \tag{3.2.18}$$

We have a duality similar in form to (3.1.12), and thus (3.2.16) represents a family of dual non-commutative theories. By duality we mean that for every commutative system with magnetic field B there is a corresponding family of non-commutative systems with magnetic field $\tilde{B}$.

Using these values (3.2.18) of the non-commutative parameter and magnetic field in the excitation energy (2.2.12) indeed yields the commutative cyclotron frequency $\omega_c = \frac{eB}{m}$ as is required by the duality. We have explicitly computed the generating functionals for the non-commutative and commutative theories, which is simple to do since we have a Gaussian path integral. We do indeed find the generating functions are identical. It should also be noted that the flow equations are not unique, as explained in [45]. In this paper the relationship between coarse graining and the ERG is discussed. In fact, there is a map between different kinds of coarse graining and different flow equations. We discuss this in the next chapter when we investigate the fate of symmetries in the ERG procedure.

3.3 Chapter Summary

In this chapter we reviewed the ERG construction for the quantum mechanical Landau problem, as a field theory in $0+1$ dimensions. Also in this chapter we have constructed a more general duality between non-commutative theories using a change of variables. This was interesting in its own right and also allowed us to benchmark our ERG construction. Now we are well prepared to tackle the ERG construction for field theories and also interacting theories. This is the subject of the next chapter.

Chapter 4

Non-Canonical ERG Dualities

We can now proceed to the ultimate goal of this part of the thesis which is to develop dualities for non-canonical field theories. This work appears in the publication [46]. We start with the free complex scalar theory that proceeds in a very similar way to the 0+1 dimensional field theory of quantum mechanics in the previous chapter. We also study the fate of the symmetries in this duality using a more general flow [45]. This is important as tracking symmetries, at least in principle, is one of the motivations for these dualities. Finally we study the most complicated duality covered in this thesis for the interacting complex scalar field.

4.1 Free Complex Scalar Field Theory

We now follow the ERG procedure as described in section 2.2.1. We again modify the kinetic energy term, based on the action of non-canonical field theories. We then let the interaction flow, thus leaving the normalized generating functional unchanged. In four dimensions we can write down the action that results from the ERG procedure as

$$S[\phi,\phi^*] = \int d^4k\, \phi^*(k^0,\vec{k})K(k^0,\theta)\phi(k^0,\vec{k}) + S_I[\phi,\phi^*] + J_\theta[\phi,\phi^*], \qquad (4.1.1)$$

we have also preemptively identified the flow parameter with the non-commutative length scale θ as in (3.2.17). For the non-canonical complex scalar field theories [21; 22] in four dimensional Minkowski space, the action in Fourier space is given by

$$S = \int d^4k\; \phi^*(k^0,\vec{k}) \left(\frac{(k^0)^2}{1-\theta k^0} - \left(\vec{k}^2 + m^2\right) \right) \phi(k^0,\vec{k}), \qquad (4.1.2)$$

which amounts to the modification

$$(k^0)^2 \to K(k^0,\theta) = \frac{(k^0)^2}{1-\theta k^0}. \qquad (4.1.3)$$

Some discussion of the critical point $k^0 = \frac{1}{\theta}$ can found in [21]. Solving the ERG as in [41], we have the flow equations, (3.2.4) and (3.2.5), adapted to four dimension, these are derived in appendix D

$$\partial_\theta S_I = \int d^4k\ \partial_\theta K^{-1} \left\{ \frac{\delta S_I}{\delta\phi^*(k)} \frac{\delta S_I}{\delta\phi(k)} - \frac{\delta^2 S_I}{\delta\phi^*(k)\delta\phi(k)} \right\}, \tag{4.1.4}$$

$$\begin{aligned} \partial_\theta J_\theta \quad &= \quad \int d^4k\ \partial_\ell K^{-1} \left\{ \frac{\delta S_I}{\delta\phi(k)} \frac{\delta J}{\delta\phi^*(k)} + \frac{\delta S_I}{\delta\phi^*(k)} \frac{\delta J}{\delta\phi(k)} \right. \\ &\left. + \frac{\delta J}{\delta\phi^*(k)} \frac{\delta J}{\delta\phi(k)} - \frac{\delta^2 J}{\delta\phi^*(k)\delta\phi(k)} \right\}. \end{aligned} \tag{4.1.5}$$

Using the flow equations the remaining quadratic term flows as

$$g(\theta) = \frac{k^0\left(\vec{k}^2 + m^2\right)}{\theta(\vec{k}^2 + m^2) - k^0}. \tag{4.1.6}$$

For the free theory we have

$$S_I[\phi, \phi^*] = \phi^*(k^0, \vec{k}) g(\theta) \phi(k^0, \vec{k}) \tag{4.1.7}$$

which has the initial condition

$$g(0) = -\left(\vec{k}^2 + m^2\right). \tag{4.1.8}$$

Note we have ignored a vacuum term generated from the right hand side of (4.1.4), since this just adds a term independent of ϕ and J to the interaction. The source term becomes

$$J_1(0) \to J_1(\theta) = \frac{k^0 J_1(0)}{k^0 - \theta(\vec{k}^2 + m^2)}, \tag{4.1.9}$$

with an additional quadratic term

$$J_0(\theta) = -\frac{\theta |J_1(0)|^2}{k^0 - \theta(\vec{k}^2 + m^2)}. \tag{4.1.10}$$

The full action with source terms is therefore

$$S_\theta = \int d^4k\ \left[\phi^* K \phi + \phi^* g \phi + J[\phi, \phi^*]\right]. \tag{4.1.11}$$

This action is not quite that of the non-canonical system, this same effect occurred in the quantum mechanical case. Again it can be brought into the right form if we rescale the fields. However, this time it is a momentum dependent rescaling of the fields

$$\phi(k^0, \vec{k}) \to \sqrt{\frac{K(\theta) + g(0)}{K(\theta) + g(\theta)}}\phi(k^0, \vec{k}). \tag{4.1.12}$$

This of course also modifies the sources. Now we have again a complete duality between the commutative and non-canonical field theories with generating functionals independent of the non-commutative parameter θ, just like section 2.2.1.

As we mentioned earlier, and as explained in [24] and [45] we can think of the ERG as a coarse graining or blocking procedure i.e a transformation of the fields. To see this, recall that it is possible to interpret the renormalization flow as a change of variables in the path integral. If we ignore the source terms, then in our case the blocking procedure is a simple momentum dependent rescaling of the commutative fields

$$\phi_0(k^0, \vec{k}) = \sqrt{\frac{K(\theta) + g(\theta)}{K(0) + g(0)}}\phi_\theta(k^0, \vec{k}). \tag{4.1.13}$$

This immediately produces the non-commutative action with modified sources without having to solve the flow equations. However, in position space the momentum dependent rescaling above is highly non-local. In the interacting case, this blocking procedure becomes a much more complicated transformation, not even necessarily a linear one, and so in this case the flow equations become a necessity.

4.2 Symmetries

Before moving on to the interacting case, we investigate the fate of the Lorentz symmetry under the dualities constructed via the ERG and the corresponding coarse graining. We investigate the simplest case of a momentum dependent rescaling of the fields, such as in (4.1.13).

The two theories are dual, as is ensured by the ERG flow equations, and therefore the symmetry must be present in both. It may, however, not be manifest in the dual theory but instead be implemented in a highly non-trivial way due to the coarse graining not being manifestly Lorentz covariant. In fact, it should already be clear from (4.1.12) that the Lorentz symmetry cannot be manifest in the dual theory and must be implemented in a highly non-local way.

A more general formulation of the flow equations which establishes the link to coarse graining [45] is

$$-\partial_\theta e^{-S_\theta[\phi]} = \int d^4k \, \frac{\delta}{\delta\phi(k)} \left(\Psi_\theta[\phi(k)]e^{-S_\theta[\phi]}\right), \tag{4.2.1}$$

where Ψ_θ is a general functional of the fields. If the action satisfies this flow equation, the normalized generating functional is again invariant, since the right-hand side of (4.2.1) leads to a total derivative of the integrand of the path integral. If we relax the conditions on the kinetic term and the sources, we must also make sure the sources are adjusted appropriately.

Every allowed Ψ_θ is related to some blocking transformation (coarse graining) [45]. If we write the blocking transformation as $f_\theta[\phi_0](k) = \phi(k)$, where ϕ_0 and ϕ are the fields before and after the blocking respectively. Then we can write an equation to calculate the corresponding Ψ_θ:

$$\Psi_\theta[\phi(k)]e^{-S_\theta[\phi]} = \int [d\phi_0]\ \delta\left[\phi - f_\theta[\phi_0]\right] \left(\partial_\theta f_\theta[\phi_0]\right) e^{-S[\phi_0]}. \tag{4.2.2}$$

For the simplest coarse graining which is just multiplicative and invertible, for example (4.1.13), then

$$f_\theta[\phi_0](k) = f_\theta(k)\phi_0(k), \tag{4.2.3}$$

implies

$$\Psi_\theta[\phi(k)] = |f_\theta^{-1}(k)|\partial_\theta f_\theta(k)\phi(k). \tag{4.2.4}$$

Then equation (4.2.1) gives a simple equation for the flow of the sources, if we include them in the action,

$$\partial_\theta J_\theta = |f_\theta^{-1}(k)|\partial_\theta f_\theta(k) J_\theta. \tag{4.2.5}$$

The symmetries are implemented differently in the two theories. If the fields transform as $\phi_0 \to \tilde{\phi}_0$ under a symmetry transformation in the original theory, the transformation induced in the dual theory is

$$\phi \to \tilde{\phi} = f_\theta(k)\tilde{\phi}_0(k). \tag{4.2.6}$$

Applying this to the Lorentz generators, we find them to be given in the dual theory by

$$M^{ij} = k^i\tilde{\partial}_{k^j} - k^j\tilde{\partial}_{k^i} \tag{4.2.7}$$

where

$$\tilde{\partial}_{k^i} = \partial_{k^i} - \frac{\partial_{k^i} f(k)}{f(k)}. \tag{4.2.8}$$

Note that in real space this corresponds to a highly non-local implementation of the Lorentz symmetry. As expected, the symmetry exists in the dual theory, however, implemented in a different way. When we have a more complicated coarse graining, such as the interacting case, it may prove impractical to track the symmetry in this way.

4.3 Interactions

Since we have covered all the relevant details of the free theory, and also investigated the fate of the symmetries, we can move onto our duality for non-canonical field theory with interactions. We add a ϕ^4 theory interaction term to the action

$$\begin{aligned} \lambda \quad \int \quad & d^4k^1\, d^4k^2\, d^4k^3\, d^4k^4\ \phi^*(k^1)\phi^*(k^2)\phi(k^3)\phi(k^4) \\ \times \quad & \delta(k^1+k^2-k^3-k^4). \end{aligned} \tag{4.3.1}$$

We will denote the flowed quartic coupling by $g_4(k^1, k^2, k^3, k^4, \theta, \lambda)$, for which the initial conditions are

$$\begin{aligned} g_4(k^1, k^2, k^3, k^4, 0, \lambda) &= \lambda\delta(k^1+k^2-k^3-k^4) \\ g_4(k^1, k^2, k^3, k^4, \theta, 0) &= 0. \end{aligned} \tag{4.3.2}$$

The quadratic coupling is denoted by $g_2(k, \theta, \lambda)$, which reduces to (4.1.6) in the case of a free theory

$$\begin{aligned} g_2(k, \theta, 0) = \frac{k^0\left(\vec{k}^2+m^2\right)}{\theta(\vec{k}^2+m^2)-k^0} \\ g_2(k, 0, \lambda) = -(\vec{k}^2+m^2). \end{aligned} \tag{4.3.3}$$

To simplify the notation we will also set

$$\partial_\theta K^{-1}(\theta) = F(k). \tag{4.3.4}$$

The individual coupling strengths and sources flow according to (4.1.4) and (4.1.5). Interactions to all orders in ϕ can be generated during the flow, since the interaction is not just quadratic anymore. The general form of the interaction term S_I in the action is given in [43] as

$$\begin{aligned} S_I = \sum_{m=0}^{\infty} \int d^4k^1 \ldots d^4k^{2m}\ g_{2m}(k^1, \ldots, k^{2m}, \theta, \lambda) \\ \times\phi^*(k^1)\ldots\phi^*(k^m)\phi(k^{m+1})\ldots\phi(k^{2m}). \end{aligned} \tag{4.3.5}$$

Note this includes everything in the action that isn't the kinetic term, thus it also includes the quadratic couplings. The flow equation produces terms containing an even number of ϕ and ϕ^* factors only. We recall for $n > 4$ the initial conditions are

$$g_{n>4}(\theta = 0) = 0. \tag{4.3.6}$$

Inserting the expression for S_I above into the flow equation (4.1.4) gives a set of coupled equations for the g_n functions

$$\begin{gathered}
\partial_\theta g_2(k,\theta,\lambda) = F(k)\, g_2^2(k,\theta,\lambda) - 4\int d^4p\, F(p)\, g_4(k,p,k,p,\theta,\lambda) \\
\partial_\theta g_4(k^1,k^2,k^3,k^4) = \textstyle\sum_{i=1}^4 F(k^i) g_2(k^i) g_4(k^1,k^2,k^3,k^4) \\
-9\int d^4p\, F(p)\, g_6(k^1,k^2,p,k^3,k^4,p) \\
\partial_\theta g_6(k^1,k^2,k^3,k^4,k^5,k^6) = \textstyle\sum_{i=1}^6 F(k^i) g_2(k^i) g_6(k^1,k^2,k^3,k^4,k^5,k^6) \\
+4\int d^4p\, F(p)\, g_4(k^1,k^2,k^3,(k^1+k^2-k^3)) g_4(k^4,k^5,k^6,(k^4+k^5-k^6)) \\
-16\int d^4p\, F(p)\, g_8(k^1,k^2,k^3,p,k^4,k^5,k^6,p) \\
\vdots
\end{gathered} \tag{4.3.7}$$

The θ and λ dependence of the functions have been suppressed in the second line. While solving the full set of equations is not possible, we can proceed perturbatively in orders of the interaction strength λ, note this is not a commutative expansion as we are not expanding in θ, instead we are considering the interactions in our non-commutative theory to be small. To do so, we expand each g_n as $g_n = g_n^{(0)} + \lambda g_n^{(1)} + \ldots$. For $n > 4$ we find that g_n is always at least of order λ^2, and due to the initial conditions these functions must vanish when $\theta = 0$. g_4 remains only linear in λ i.e. $g_4 = \lambda g_4^{(1)}$, this notation may differ from the usual perturbation expansion. Solving the coupled equations up to linear order produces

$$g_2^{(0)}(k) = \frac{k^0\left(\vec{k}^2 + m^2\right)}{\theta(\vec{k}^2+m^2) - k^0}, \tag{4.3.8}$$

$$g_2^{(1)}(k) = \frac{-4c\theta(k^0)^2}{(\theta(\vec{k}^2+m^2)-k^0)^2}, \tag{4.3.9}$$

$$g_4^{(1)}(k^1,k^2,k^3,k^4) = \lambda\prod_{i=1}^{4}\frac{(k^0)^i}{\theta((\vec{k}^i)^2+m^2)-(k^0)^i} \times\delta(k^1+k^2-k^3-k^4), \tag{4.3.10}$$

where

$$c = \int d^4p\, \frac{p^0}{(\vec{p}^2+m^2-p^0)^2} \tag{4.3.11}$$

is a divergent integral that has to be regularized appropriately.

We also need to compute the flow of the sources to the same order in λ, i.e.

$$J[\phi,\phi^*] = J^{(0)}[\phi,\phi^*] + \lambda J^{(1)}[\phi,\phi^*]. \tag{4.3.12}$$

After solving the flow equations (4.3.7) for the couplings to order λ, the interacting action has the form

$$\begin{aligned} S_I = & \int d^4k \ \left(g_2^{(0)}(k) + \lambda g_2^{(1)}(k)\right) \phi^*(k)\phi(k) \\ & + \int d^4k^1\, d^4k^2\, d^4k^3\, d^4k^4\, \lambda\, g_4^{(1)}(k^1,k^2,k^3,k^4) \\ & \qquad \times \phi^*(k^1)\phi^*(k^2)\phi(k^3)\phi(k^4). \end{aligned} \tag{4.3.13}$$

Inserting (4.3.12) and (4.3.13) into the flow equation for the sources (4.1.5) produces the equations considered in appendix E. We note that to zeroth order in λ the sources have the form

$$J^{(0)}[\phi,\phi^*] = \int d^4k \left[J_0^{(0)}(k) + J_1^{(0)}(k)\phi^*(k) + c.c.\right], \tag{4.3.14}$$

which, as expected, agrees with the free case (4.1.9) and (4.1.10). To first order in λ we find

$$\begin{aligned} J^{(1)}[\phi,\phi^*] = & \int d^4k \ \left[J_0^{(1)}(k) + J_1^{(1)}(k)\phi^*(k) + c.c.\right] + \\ & \int d^4k^1 d^4k^2 \ \left[J_2(k^1,k^2)\phi^*(k^1)\phi^*(k^2) + c.c.\right] + \\ & \int d^4k^1 d^4k^2 \ \left[\tilde{J}_2(k^1,k^2)\phi^*(k^1)\phi(k^2)\right] + \\ & \int d^4k^1 d^4k^2 d^4k^3 \ \left[J_3(k^1,k^2,k^3)\phi^*(k^1)\phi^*(k^2)\phi(k^3) + c.c.\right]. \end{aligned} \tag{4.3.15}$$

Here $\tilde{J}_2$ is real and we also see terms with an unequal number of ϕ and ϕ^* factors appearing. In principle, all the equations appearing in appendix E can be solved to obtain expressions for the various functions in (4.3.15) above. For example:

$$\begin{aligned} J_3(k^1,k^2,k^3) = \quad & \prod_{i=1}^{3} \frac{(k^i)^0}{\theta((\vec{k}^i)^2 + m^2) - (k^i)^0} \times \\ & \ln\left(\frac{\kappa^0}{\theta(\vec{\kappa}^2 + m^2) - \kappa^0}\right) J_1^{(0)}(\kappa) \end{aligned}$$

where

$$\kappa = k^1 + k^2 - k^3. \tag{4.3.16}$$

Again we need to rescale the fields as in (4.1.12) so that the quadratic part of the non-commutative action takes the standard form (2.3.16), finally we have the complete duality between the interacting commutative and non-canonical theories, albeit in a perturbative way. Note that although the interactions of the non-canonical theory are highly non-local the duality should ensure, at least in principle, that both the Lorentz symmetry and unitarity are maintained. Now we have our duality, what to do with it requires further investigation. The advantages of the usual form of the sources and interaction are not overtly apparent.

4.4 Chapter Summary

The aim of this part was to use the ERG philosophy to construct dualities for non-canonical field theories. This was done after realizing the actions of non-canonical field theories have different kinetic terms to canonical theories. In an ERG construction, this type of change to the action can be countered by allowing the interacting part of the theory to flow. This flow is defined by the ERG flow equations. In this way we ensure that the normalized generating functional remains invariant. The difference for the non-canonical and non-commutative dualities from the standard ERG, is that no limiting assumptions are made on the sources as is usually done in the ERG. We are required to let the source terms flow as well according to [41].

We have achieved this goal for the free theory and perturbatively for the interacting theory. We see that the free theory proceeds in the same way as the quantum mechanical case. Then we tracked the symmetries after the ERG procedure for the free theory, we found they of course still exist but are implemented in a different way. Lastly we constructed the duality for the interacting complex scalar field, we found the change in interaction and also the change in the sources. The mere existence of such dualities does not necessarily bring an obvious benefit. Further exploration is required to determine whether this constructive approach may offer real advantages. Such investigations may also have to move beyond the idea of purely commutative/non-commutative dualities. The work in this part has appeared for publication in [46].

Part II

SU(1,1) Field Theory

Chapter 5

3-D Non-Commutative Quantum Mechanics

In part 2 our goal is to build a non-commutative field theory that is Lorentz invariant by construction. We want to introduce a minimum length scale into field theories as this makes physical sense. This could also alleviate some of the divergences we encounter in field theories. A successful non-relativistic non-commutative theory that preserves rotational symmetry in three dimensions is fuzzy space [25]. In this chapter we review the construction of fuzzy space quantum mechanics, as this will guide us in our attempt to build a relativistic version of this theory. As the title of this part suggests, this will involve using the $\mathrm{su}(1,1)$ algebra. This will be discussed in detail in chapter 6.

In this chapter we introduce fuzzy space, where the coordinates obey the $\mathrm{su}(2)$ commutation relations. For this reason we start by briefly reviewing some of the properties of the $\mathrm{SU}(2)$ group and its representations which are relevant to us. Using this we build our non-commutative quantum theory. We continue by discussing how we introduce dynamics into our quantum theory and formulate our non-commutative version of the Schrödinger equation. We introduce the notion of symbols and a particular family of symbols which are not unique to fuzzy space. At the end we briefly mention the idea of field theories built with $\mathrm{su}(2)$ commutation relations and commutative time in 3+1 dimensions.

5.1 SU(2) Group

The classical matrix group $\mathrm{SU}(2)$, the two by two special unitary matrices, preserves the metric

$$m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.1.1}$$

and so we can write the matrices as

$$g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1, \tag{5.1.2}$$

then $g^\dagger m g = m$. The Pauli spin $\frac{1}{2}$ matrices are the two dimensional irreducible representation of the group. The generators of SU(2) satisfy

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k, \tag{5.1.3}$$

where ϵ_{ijk} is the Levi-Civita symbol. We can define a set of ladder operators

$$\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2. \tag{5.1.4}$$

As usual the basis for the Hilbert spaces is labeled by the eigenvalues of $\hat{J}_3, \hat{J}^2$. Here $J^2 = \hat{\vec{J}} \cdot \hat{\vec{J}}$ is the Casimir operator. The unitary irreps have the following structure

$$\begin{aligned} \hat{J}_3 \left| j, m \right\rangle &= m \left| j, m \right\rangle, \\ \hat{J}_\pm \left| j, m \right\rangle &= \sqrt{j(j+1) - m(m \pm 1)} \left| j, m \pm 1 \right\rangle, \end{aligned} \tag{5.1.5}$$

and the Casimir acts as

$$\hat{J}^2 \left| j, m \right\rangle = j(j+1) \left| j, m \right\rangle. \tag{5.1.6}$$

The ranges of the labels are,

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \quad \text{and} \quad m = -j, -j+1, -j+2, \ldots j-2, j-1, j. \tag{5.1.7}$$

5.1.1 Two Mode Representations

We can introduce the Schwinger representation in terms of creation and annihilation operators $\hat{a}_1^{(\dagger)}$ and $\hat{a}_2^{(\dagger)}$ as

$$\begin{aligned} \hat{J}_i &= \frac{1}{2}\left(\hat{a}_\alpha \sigma^i_{\alpha\beta} \hat{a}^\dagger_\beta\right), \\ \hat{J}^2 &= \frac{1}{4}\hat{a}^\dagger_\alpha \hat{a}_\alpha (\hat{a}^\dagger_\beta \hat{a}_\beta + 2). \end{aligned} \tag{5.1.8}$$

Where as usual the sum of over repeated indices is implied. Then we can write the basis states in terms of number states

$$\left| j = \frac{n_1 + n_2}{2}, m = \frac{n_1 - n_2}{2} \right\rangle = \left| n_1, n_2 \right\rangle. \tag{5.1.9}$$

5.2 SU(2) Fuzzy Space

To introduce our non-commutative theory, we choose our coordinates, not to be scalar numbers which label points in space, but rather to be operators. Due to the Heisenberg uncertainty principle we hereby introduce a minimum length scale proportional to the non-commutative parameter θ. For fuzzy space our coordinates are the generators of the SU(2) group,

$$\begin{aligned} \hat{x}_i &= \theta \hat{J}_i, \\ [\hat{x}_i, \hat{x}_j] &= i\theta\epsilon_{ijk}\hat{x}_k, \\ \hat{x}^2 + \hat{y}^2 + \hat{z}^2 &= \theta^2 \hat{J}^2. \end{aligned} \tag{5.2.1}$$

Unlike the constant commutation relations, the su(2) commutation relations preserve rotational symmetry. We can define the radial distance operator $\hat{r}$, which is approximately the square root of $\hat{r}^2$, in the two mode representation as

$$\begin{aligned} \hat{r}^2 &= \frac{\theta^2}{4}\hat{a}^\dagger_\alpha \hat{a}_\alpha(\hat{a}^\dagger_\beta \hat{a}_\beta + 2), \\ \hat{r} &= \frac{\theta}{2}(\hat{a}^\dagger_\alpha \hat{a}_\alpha + 1). \end{aligned} \tag{5.2.2}$$

We can then cover space with the different irreps of SU(2). We have an onion-like structure where the radii of the spheres that cover our space are discrete and labeled by j or $n_1 + n_2$.

The Hilbert space these different representations act on is called the configuration space $\mathcal{H}_c$. For the two mode representation, $\mathcal{H}_c$ is just a two mode boson-fock space.

5.3 Quantum States

The states of our quantum system are operators which are elements of the algebra generated by the coordinates. Our wavefunctions $\hat{\phi} \equiv \phi(\hat{x}_i)$ are then operators acting on $\mathcal{H}_c$. The Hilbert space of the quantum states we shall call the quantum Hilbert space $\mathcal{H}_q$. We can denote these states with a rounded ket

$$\phi(\hat{x}_i) = |\hat{\phi}). \tag{5.3.1}$$

We can define an inner product on this Hilbert space as

$$(\hat{\psi}|\hat{\phi}) = \theta^2 \,\mathrm{Tr}_c(\hat{\psi}^\dagger \hat{r} \hat{\phi}), \tag{5.3.2}$$

where $\hat{r}$ is a volume factor, which is necessary for this trace to act as the non-commutative analog of an integral over space [25]. An important property of the quantum states is that they only depend on the coordinates and so must satisfy

$$[\hat{J}^2, \hat{\phi}] = 0. \tag{5.3.3}$$

In this case we can define $\mathcal{H}_q$ as

$$\mathcal{H}_q = \left\{ \hat{\psi} : [\hat{\psi}, \hat{J}^2] = 0 \quad \text{and} \quad \mathrm{Tr}_c(\hat{\psi}^\dagger \hat{r} \hat{\psi}) < \infty \right\}. \tag{5.3.4}$$

All the outer products of the states in $\mathcal{H}_c$ do not necessarily commute with $\hat{J}^2$. In fact they form a larger Hilbert space which we denote as $\tilde{\mathcal{H}}_q$, then $\mathcal{H}_q \subset \tilde{\mathcal{H}}_q$.

We can define operators that act on the quantum Hilbert space. These will be the non-commutative observables. For example, we can define a position operator that acts by left multiplication by the coordinate operator

$$\hat{X}_i|\hat{\phi}) = |\hat{x}_i \hat{\phi}). \tag{5.3.5}$$

We see that the $\hat{X}_i$ satisfy the su(2) commutation relations, and so have the desired minimum uncertainty on the order of θ.

5.3.1 Rotations

We can also define angular momentum operators. These are the generators of rotations. They act with the adjoint action of the coordinates

$$\hat{R}_i|\hat{\phi}) = \left| \frac{1}{i\theta} \left[\hat{x}_i, \hat{\phi} \right] \right) = \left| \left[\hat{J}_i, \hat{\phi} \right] \right) = |\, \mathrm{ad}_{\hat{J}_i} \hat{\phi}), \tag{5.3.6}$$

we can see that the coordinates will transform correctly as vectors since

$$\hat{R}_i|\hat{x}_j) = \left| \frac{1}{i\theta} [\hat{x}_i, \hat{x}_j] \right) = |\epsilon_{ijk} \hat{x}_k). \tag{5.3.7}$$

Also the adjoint action is a representation of SU(2) on $\mathcal{H}_q$. Using the Jacobi identity we have

$$[\hat{R}_i, \hat{R}_j]|\hat{\phi}) = \left| \left[\hat{J}_i, \left[\hat{J}_j, \hat{\phi} \right] \right] - \left[\hat{J}_j, \left[\hat{J}_i, \hat{\phi} \right] \right] \right) = |[i\epsilon_{ijk} \hat{J}_k, \hat{\phi}]) = i\epsilon_{ijk} \hat{R}_k |\hat{\phi}). \tag{5.3.8}$$

5.3.2 Plane Waves

In fuzzy space we parametrize group elements [47] as $e^{i\vec{p}\cdot\hat{\vec{J}}}$, the group elements are the natural choice for our plane waves. Introducing "momenta" (k_1, k_2, k_3) we can write

$$\hat{U}(k) = e^{i\vec{k}\cdot\hat{\vec{x}}}, \tag{5.3.9}$$

then we can express any operator as

$$\hat{\phi} = \int d\mu(k)\ \phi(k)\hat{U}(k). \tag{5.3.10}$$

Here $d\mu(k)$ is the invariant (Haar) measure, which is invariant under group transformations [48]. We use this when we want to integrate over the group manifold. The invariant measure above is given by

$$d\mu(k) = \frac{4\sin^2\left(\frac{\theta|k|}{2}\right)}{\theta^2|k|^2} dk_1 dk_2 dk_3, \tag{5.3.11}$$

where $|k| = \sqrt{\vec{k}\cdot\vec{k}}$.

5.4 Schrödinger Equation

We can write a non-commutative Schrödinger equation using a double commutator, which acts as a non-commutative Laplacian,

$$\hat{\Delta}\hat{\psi} = -\frac{2}{\theta\hat{r}}[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \hat{\psi}]]. \tag{5.4.1}$$

The $\frac{1}{\hat{r}}$ must be included for the operator to be hermitian w.r.t the inner product in (5.3.2). Then the time-independent Schrödinger equation becomes

$$\left(-\frac{\hbar^2}{2m}\hat{\Delta} + \hat{V}\right)\hat{\psi} = E\hat{\psi} \tag{5.4.2}$$

If we let the Laplacian act on a plane wave, we find that

$$-\frac{2}{\theta\hat{r}}[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, e^{ik\cdot\hat{x}}]] = -\frac{16}{\theta^2}\frac{1}{\hat{r}}\hat{r}\left(\sin^2\left(\frac{|k|\theta}{4}\right)\right)e^{ik\cdot\hat{x}} \tag{5.4.3}$$

This can be shown easily using BCH formulas. A more general calculation is found in appendix A. We also see that due to the factor $\frac{1}{\hat{r}}$ the plane wave is an eigenstate of the Laplacian. For the free particle we find the dispersion relation

$$E = \frac{8\hbar^2}{m\theta^2}\sin^2\left(\frac{|k|\theta}{4}\right). \tag{5.4.4}$$

An interesting property of this equation is the upper bound on the energy for the free particle. Thorough discussions for the free particle [26], spherical well [27] and hydrogen atom [28] exist in the literature.

5.5 Symbols

To introduce the notion of space we need to introduce commutative coordinates in some way. This is how we can make the connection to measurable quantities. The symbol is a map $\hat{\phi} \to \phi(x, y, z)$, we discuss these maps in more detail in chapter 8. For now we only discuss one class of symbols.

5.5.1 Two Mode Coherent States

Since we have a two mode representation we can construct two mode Glauber coherent states

$$|z_1, z_2\rangle = e^{-\left(\frac{|z_1|^2+|z_2|^2}{2}\right)} e^{z_1 \hat{a}_1^\dagger} e^{z_2 \hat{a}_2^\dagger} |0,0\rangle . \tag{5.5.1}$$

The two z's introduce four real parameters, so there is a redundant unphysical coordinate introduced. What these z's represent physically is unclear, however we can calculate the expectation values of the coordinates in terms of these as

$$\begin{aligned} x = \langle z_1, z_2| \hat{x} |z_1, z_2\rangle &= \frac{\theta}{2}(z_1 z_2^* + z_1^* z_2), \\ y = \langle z_1, z_2| \hat{y} |z_1, z_2\rangle &= i\frac{\theta}{2}(z_1 z_2^* - z_1^* z_2), \\ z = \langle z_1, z_2| \hat{z} |z_1, z_2\rangle &= \frac{\theta}{2}(|z_1|^2 - |z_2|^2), \\ r = \sqrt{x^2+y^2+z^2} &= \frac{\theta}{2}(|z_1|^2 + |z_2|^2). \end{aligned} \tag{5.5.2}$$

We can calculate the symbol of the plane wave, written in terms of the expectation values of the coordinates. This produces

$$\langle z_1, z_2| e^{i\vec{k}\cdot\hat{\vec{x}}} |z_1, z_2\rangle = e^{-\frac{4r}{\theta}\sin^2\left(\frac{\theta|k|}{4}\right)} e^{i\frac{2\sin\left(\frac{\theta|k|}{2}\right)}{k\theta}\vec{k}\cdot\vec{x}}. \tag{5.5.3}$$

We calculate the symbol of the free non-commutative Schrödinger equation

$$\langle z_1, z_2| -\frac{\hbar^2}{2m}\hat{\Delta}\hat{\psi} |z_1, z_2\rangle = \langle z_1, z_2| E\hat{\psi} |z_1, z_2\rangle , \tag{5.5.4}$$

to produce a scalar differential equation for the symbols $\langle z_1, z_2| \hat{\psi} |z_1, z_2\rangle = \psi(z_1, z_2)$,

$$-\frac{\hbar^2}{2m}\Delta\psi(z_1, z_2) = E\left(1 + \frac{\theta}{2}\left(\frac{1}{r} + \hat{e}_r \cdot \vec{\nabla}\right)\right)\psi(z_1, z_2). \tag{5.5.5}$$

If we insert solutions of the form

$$\psi(z_1, z_2) = e^{-\frac{Em\theta}{2\hbar^2}r}\tilde{\psi}, \tag{5.5.6}$$

we obtain

$$-\frac{\hbar^2}{2m}\Delta\tilde{\psi} = E\left(1 - \frac{E}{E_{\max}}\right)\tilde{\psi}. \tag{5.5.7}$$

Where $E_{\max} = \frac{8\hbar^2}{\theta^2 m}$. This matches the maximum energy found using (5.4.4).

5.6 Generalized Two Mode Symbols

We can generalize the two mode coherent states to a family of symbols labeled by a parameter δ. The symbols considered above correspond to the lower symbol [49; 50; 51] $\langle z_1, z_2| \hat{\phi} |z_1, z_2\rangle = \phi_{-1}(z_1, z_2)$. We define

$$\phi_\delta = V_\delta \phi_{-1} = e^{-\frac{1}{2}(\delta+1)\partial_{z^*_\alpha}\partial_{z_\alpha}} \phi_{-1}. \tag{5.6.1}$$

The symbol of the product of two operators is given by

$$(\hat{\phi}\hat{\psi})_\delta = \phi_\delta *_\delta \psi_\delta, \tag{5.6.2}$$

where

$$*_\delta = e^{-\frac{1}{2}[(\delta+1)\overleftarrow{\partial}_{z^*_\alpha}\overrightarrow{\partial}_{z_\alpha}+(\delta-1)\overleftarrow{\partial}_{z_\beta}\overrightarrow{\partial}_{z^*_\beta}]}, \tag{5.6.3}$$

is a star product. The case $\delta = -1$ is called the Voros product. We can calculate the trace of two operators in terms of these two mode coherent states as

$$\begin{aligned} \mathrm{Tr}(\hat{\phi}\hat{\psi}) &= \frac{1}{\pi}\int dz^*_\alpha dz_\alpha \ \langle z_1, z_2| \hat{\phi}\hat{\psi} |z_1, z_2\rangle \\ &= \frac{1}{\pi}\int dz^*_\alpha dz_\alpha \ \phi_{-1} *_{-1} \psi_{-1} \\ &= \frac{1}{\pi}\int dz^*_\alpha dz_\alpha \ \phi_0 \psi_0, \end{aligned} \tag{5.6.4}$$

this is done by partial integration assuming these fields vanish at the boundaries in z-space. What this physically means is not clear, since z's are not the physical coordinates. This seems like a reasonable assumption though, otherwise the trace is not finite. The special case of $\delta = 0$ is called the Weyl symbol. We can map between different symbols in the following way

$$\phi_\delta(z_1, z_2) = e^{-\frac{1}{2}(\delta-\delta')\partial_{z^*_\alpha}\partial_{z_\alpha}} \phi_{\delta'}(z_1, z_2), \tag{5.6.5}$$

if $\delta < \delta'$ then this is a smoothing operation. Another special case $\delta = 1$ is known as the upper symbol. It has the property

$$\hat{\phi} = \frac{1}{\pi}\int dz^*_\alpha dz_\alpha \ \phi_1(z_1, z_2) |z_1, z_2\rangle \langle z_1, z_2| \,. \tag{5.6.6}$$

The above discussion of generalized two mode symbols will follow in exactly the same way for SU(1, 1), or any theory where the two mode coherent states are used.

5.7 SU(2) Field Theories

Before we move on to our SU(1, 1) field theory, it is worthwhile to see how one would build a field theory out of SU(2), as this doesn't appear to exist in the literature. We want to construct a field theory using our SU(2) Laplacian. We assume time behaves commutatively, which explicitly breaks Lorentz symmetry. We can write an equation of motion of the form

$$(\partial_\mu \partial^\mu + m^2)\hat{\phi} = (\partial_t^2 - \hat{\Delta} + m^2)\phi(t, \hat{x}^i). \tag{5.7.1}$$

The action that gives this equation of motion for the scalar field is

$$S = \int dt \ \mathrm{Tr}_c \left[\hat{\phi}^\dagger \hat{r}(\partial_t^2 - \hat{\Delta} + m^2)\hat{\phi}\right]. \tag{5.7.2}$$

If we write the Fourier transform of our operators

$$\hat{\phi} = \int d\omega d\mu(k) \ \phi(\omega, k) e^{-i\omega t} e^{i\vec{k}\cdot\hat{\vec{x}}} \tag{5.7.3}$$

then the trace is given by a calculation that can be found in appendix B

$$\mathrm{Tr}_c[\hat{\psi}^\dagger \hat{r}\hat{\phi}] = \int d\mu(k) \ \psi^*(t, k)\phi(t, k). \tag{5.7.4}$$

This allows us to write the action as

$$S = \int d\omega d\mu(k) \ \psi^*(\omega, k) \left(-\frac{16}{\theta^2}\sin^2\left(\frac{\theta|k|}{4}\right) + \omega^2 - m^2\right)\psi(\omega, k). \tag{5.7.5}$$

If we write the standard path integral

$$Z[\psi(\omega, k)] = \int [d\psi^*(k)][d\psi(k)] \ e^{iS[\psi]}, \tag{5.7.6}$$

we can calculate the correlator of fields in momentum space

$$\langle \phi(\omega', k')^* \phi(\omega, k)\rangle = i\delta(\omega' - \omega)\delta_\mu(k' - k)\left[\omega^2 - \frac{16}{\theta^2}\sin^2\left(\frac{\theta|k|}{4}\right) - m^2\right]^{-1}. \tag{5.7.7}$$

Where $\delta_\mu(k' - k)$ is a delta function w.r.t the group measure.

5.7.1 Weyl Symbols

We calculate the correlator of Weyl symbols using the symbols of the plane waves,

$$V_0 \langle z_1, z_2| e^{i\vec{k}\cdot\hat{\vec{x}}} |z_1, z_2\rangle = U_0(k, z_1, z_2), \tag{5.7.8}$$

since our scalar field can then be written as

$$\phi_0(t, z_1, z_2) = \int d\mu(k)\ \phi(t, k) U_0(k, z_1, z_2). \tag{5.7.9}$$

Then the correlator becomes

$$\begin{aligned}\langle \phi_0^*(t', z_1', z_2')\phi_0(t, z_1, z_2)\rangle = & \int d\mu(k) d\mu(k')\ U_0^*(k, z_1, z_2) U_0(k, z_1', z_2') \\ & \times \langle \phi^*(t, k)\phi(t', k')\rangle\,. \end{aligned} \tag{5.7.10}$$

We can calculate the Weyl symbol of the plane wave by acting on the lower symbol 5.5.3 with V_0

$$\begin{aligned} U_0(k, z_1, z_2) &= e^{-\frac{1}{2}\partial_{z_\alpha^*}\partial_{z_\alpha}} \langle z_1, z_2 |\, e^{i\vec{k}\cdot\hat{\vec{x}}}\, | z_1, z_2\rangle \\ &= \sec^2\left(\frac{\theta|k|}{4}\right) e^{i\frac{4\tan\left(\frac{\theta|k|}{4}\right)}{\theta|k|}\vec{k}\cdot\vec{x}}. \end{aligned} \tag{5.7.11}$$

If we define

$$\vec{s} = \frac{4\tan\left(\frac{\theta|k|}{4}\right)}{\theta}\hat{k}, \tag{5.7.12}$$

we can compute the correlator and write it in terms of s

$$\langle \phi_0^*(t', z_1', z_2')\phi_0(t, z_1, z_2)\rangle = \int d\omega d^3s\ \frac{e^{-i\omega(t'-t)} e^{i\vec{s}\cdot(\vec{x}'-\vec{x})}}{\left[(\omega^2 - m^2)\left(1 + \frac{\theta|s|^2}{16}\right) - |s|^2\right]}. \tag{5.7.13}$$

We stop at this point, as this is a sufficient warm up for the SU(1, 1) case we cover in the rest of the thesis.

5.8 Chapter Summary

In this chapter we have reviewed the essentials of SU(2) fuzzy space. We covered the group SU(2) and its representations. Using this we introduced dynamics using a non-commutative Laplacian, and then a non-commutative Schrödinger equation. We introduced the notion of symbols which will be expanded upon in chapter 8. We also included a general discussion of a family of symbols in section 5.6 that will be useful later. All of this will guide us with the construction of a relativistic theory. We also very briefly mention SU(2) field theories. This has the potential to be expanded upon in a future work, however, this is not in the scope of thesis which seeks to investigate theories where Lorentz symmetries are intact.

Chapter 6

SU(1, 1) Fuzzy Space-Time

Now that have we reviewed SU(2) fuzzy space we can proceed to develop our relativistic version of this theory. The end goal is to have a manifestly Lorentz covariant non-commutative relativistic theory with a minimum length scale. Then we also need to connect this non-commutative theory to measurable quantities. This is the goal of the rest of the thesis.

In SU(2) fuzzy space we introduced commutation relations which in three dimensions preserves rotational symmetry. This fixes the rotational symmetry breaking of constant commutation relations and also bypasses the problems of constant commutation relations in three dimensions [52].

Our relativistic version begins by building a non-commutative space-time based on the su(1, 1) algebra. In this way we will have a relativistic theory which preserves Lorentz symmetry in $2 + 1$ dimensions. Note that SO(2, 1), SU(1, 1) and SL$(2, \mathbb{R})$ are homomorphic [53]. A major difference from the non-relativistic fuzzy space is that SU(1, 1) is a non-compact group.

The goal of this chapter is to review the properties of SU(1, 1) and its representations, which differ significantly from SU(2). We then use this to build our non-commutative space-time. The classical matrix group SU(1, 1) which are two by two special matrices, preserve the metric

$$m = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.0.1}$$

and so we can write the matrices as [54; 55]

$$g = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \quad \text{with} \quad |\alpha|^2 - |\beta|^2 = 1. \tag{6.0.2}$$

6.1 Algebra

The generators of SU(1, 1) satisfy

$$[\hat{K}_1, \hat{K}_2] = -i\hat{K}_0, \quad [\hat{K}_0, \hat{K}_1] = i\hat{K}_2, \quad [\hat{K}_2, \hat{K}_0] = i\hat{K}_1. \tag{6.1.1}$$

The two dimensional irreducible representation is

$$\begin{aligned}
\hat{K}_0 &= \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\sigma_3}{2},\\
\hat{K}_1 &= \frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = i\frac{\sigma_2}{2},\\
\hat{K}_2 &= \frac{1}{2i}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = -i\frac{\sigma_1}{2}.
\end{aligned} \tag{6.1.2}$$

We note that, unlike SU(2), the above representation is non-unitary. Since SU(1,1) is a non-compact group, all its unitary irreps are infinite dimensional. As with SU(2) we can define a set of ladder operators

$$\hat{K}_\pm = \hat{K}_1 \pm i\hat{K}_2. \tag{6.1.3}$$

The Casimir is given by

$$\hat{C}^2 = \hat{K}^\mu \hat{K}_\mu = (K_0)^2 - (K_1)^2 - (K_2)^2. \tag{6.1.4}$$

6.2 Unitary Representations

There are two types of unitary irreps [54; 55; 56] that are of particular interest to us. Later we will discuss how we use these to cover our space-time, in similar fashion to SU(2). We can label the irreps with χ. As usual the basis for the Hilbert spaces are labeled by the eigenvalues of $\hat{K}_0$ and $\hat{C}^2$.

6.2.1 Discrete Series $\chi = (k, \pm)$

The operators act on the basis states as

$$\begin{aligned}
\hat{K}_0 |k,m\rangle &= m\,|k,m\rangle\,,\\
\hat{K}_\pm |k,m\rangle &= \sqrt{(m\pm k)(m\mp k\pm 1)}\,|k,m\pm 1\rangle\,,
\end{aligned} \tag{6.2.1}$$

and the Casimir as

$$\hat{C}^2 |k,m\rangle = k(k-1)\,|k,m\rangle\,. \tag{6.2.2}$$

The possible ranges of the labels [56] give two different representations:

6.2.1.1 Positive discrete $(k,+)$ series

$$k = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \quad \text{and} \quad m = k, k+1, k+2, \ldots \tag{6.2.3}$$

6.2.1.2 Negative discrete $(k,-)$ series

$$k = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \quad \text{and} \quad m = -k, -k-1, -k-2, \ldots \tag{6.2.4}$$

6.2.2 Continuous Series $\chi = (\frac{1}{2} + i\rho, \epsilon)$

The operators act on the basis states as

$$\begin{aligned} \hat{K}_0 \left|\rho, m\right\rangle &= m \left|\rho, m\right\rangle, \\ \hat{K}_\pm \left|\rho, m\right\rangle &= \sqrt{\left(m \mp \frac{1}{2} \mp i\rho\right)} \left|\rho, m \pm 1\right\rangle, \end{aligned} \tag{6.2.5}$$

and the Casimir as

$$\hat{C}^2 \left|\rho, m\right\rangle = -\left(\frac{1}{2} + i\rho\right)\left(\frac{1}{2} - i\rho\right)\left|\rho, m\right\rangle. \tag{6.2.6}$$

In a similar labeling scheme to the positive and negative discrete series, we could have written $k = \frac{1}{2} + i\rho$, but this would then be complex. The ranges of the labels [56] again give two representations:

6.2.2.1 Even parity $(\frac{1}{2} + i\rho, 0)$

Here

$$m \in \mathbb{Z} \quad \text{and} \quad \rho \geq 0. \tag{6.2.7}$$

6.2.2.2 Odd parity $(\frac{1}{2} + i\rho, \frac{1}{2})$

Here

$$m \in \frac{1}{2} + \mathbb{Z} \quad \text{and} \quad \rho > 0. \tag{6.2.8}$$

The $(\rho = 0, \frac{1}{2})$ representation is often excluded from the continuous series since this decomposes into the direct sum of $(\frac{1}{2}, +)$ and $(\frac{1}{2}, -)$. Negative ρ provides no new representations since $(\frac{1}{2} + i\rho, \epsilon)$ and $(\frac{1}{2} - i\rho, \epsilon)$ can be shown to be equivalent [56].

6.2.3 Two Mode Representations

In a similar way to SU(2), we can introduce a Schwinger representation in terms of bosonic operators $\hat{a}_1^{(\dagger)}$ and $\hat{a}_2^{(\dagger)}$. However, in this case we need different constructions for the continuous and discrete series.

6.2.3.1 Discrete Series

$$\begin{aligned} \hat{K}_0 &= \pm\frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1\right), \\ \hat{K}_1 &= \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_1 \hat{a}_2\right), \\ \hat{K}_2 &= \pm\frac{1}{2i}\left(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2\right), \\ \hat{C}^2 &= \frac{1}{4}\left(\left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2\right)^2 - 1\right). \end{aligned} \tag{6.2.9}$$

The $\pm$ refer to the positive and negative discrete series respectively. We can see that $\hat{C}^2$ has discrete eigenvalues.

We can then write the basis states in terms of number states for the discrete series,

$$|n_1, n_2\rangle = \left|k = \frac{|n_1 - n_2| + 1}{2}, m = \pm\frac{n_1 + n_2 + 1}{2}\right\rangle. \tag{6.2.10}$$

We see that this representation actually covers all the irreps twice, since the basis states are symmetric in n_1 and n_2, expect for the case $k = \frac{1}{2}$ which only occurs when $n_1 = n_2 = n$ and $m = n + \frac{1}{2}$. For this, and other reasons that will become apparent, we usually exclude the $k = \frac{1}{2}$ representation from the discrete series.

6.2.3.2 Continuous Series

Here we define

$$\begin{aligned}
\hat{K}_0 &= \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\right), \\
\hat{K}_1 &= \frac{1}{4}\left(\hat{a}_1^\dagger\hat{a}_1^\dagger + \hat{a}_1\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2^\dagger + \hat{a}_2\hat{a}_2\right), \\
\hat{K}_2 &= \frac{1}{i4}\left(\hat{a}_1^\dagger\hat{a}_1^\dagger - \hat{a}_1\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2^\dagger + \hat{a}_2\hat{a}_2\right), \\
C^2 &= -\frac{1}{4}\left(\left(\hat{a}_1\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2^\dagger\right)^2 + 1\right).
\end{aligned} \tag{6.2.11}$$

In this case all the continuous series representations are covered twice. We can see that $\hat{C}^2$ has a continuous spectrum. If we include $(\rho = 0, \frac{1}{2})$ here then it is also covered twice.

6.2.4 Other Representations

We briefly mention some other unitary representations, but the physical significance of these is unclear.

6.2.4.1 Supplementary Series $\chi = (k, m\pm)$

We use exactly the same labeling scheme as the discrete series, then operators act on the basis states as

$$\begin{aligned}
\hat{K}_0 |k, m\rangle &= m |k, m\rangle, \\
\hat{K}_\pm |k, m\rangle &= \sqrt{(m \pm k)(m \mp k \pm 1)}\, |k, m \pm 1\rangle,
\end{aligned} \tag{6.2.12}$$

and the Casimir as

$$\hat{C}^2 |k, m\rangle = k(k-1) |k, m\rangle. \tag{6.2.13}$$

However now the possible ranges of the labels are

$$0 < k < \frac{1}{2} \quad \text{and} \quad m = 0, \pm 1, \pm 2, \ldots. \tag{6.2.14}$$

6.2.4.2 Singleton Representations $\chi = (k, \pm)$

The singleton representation follows from a single mode representation

$$\begin{aligned}
\hat{K}_0 &= \pm\frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \\
\hat{K}_1 &= \frac{1}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right), \\
\hat{K}_2 &= \frac{1}{2i}\left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}\right), \\
\hat{C}^2 &= -\frac{3}{16}\hat{\mathcal{I}}. \qquad (6.2.15)
\end{aligned}$$

The basis states are then just eigenstates of the number operator $|n\rangle$. Since the Casimir acts as

$$\hat{C}^2\,|n\rangle = -\frac{3}{16}\,|n\rangle = k(k-1)\,|n\rangle\,, \qquad (6.2.16)$$

k can take the values $\frac{1}{4}$ or $\frac{3}{4}$. Then n is even or odd, respectively.

In fact we can build the two mode representations out of the singleton representation [57]. If we write the singleton representation as

$$S_\pm = \left(\frac{1}{4}, \pm\right) \oplus \left(\frac{3}{4}, \pm\right), \qquad (6.2.17)$$

then we can write the positive and negative discrete series as

$$S_\pm \otimes S_\pm = \left(\frac{1}{2}, \pm\right) \oplus 2\,(1,\pm) \oplus 2\left(\frac{3}{2}, \pm\right) \oplus \ldots \qquad (6.2.18)$$

We can see the $k = \frac{1}{2}$ occurring once and all others twice. We can also write the continuous series as

$$S_+ \otimes S_- = 2\int^{\oplus} d\rho\,\left[\left(\frac{1}{2} + i\rho, 0\right) \oplus \left(\frac{1}{2} + i\rho, \frac{1}{2}\right)\right], \qquad (6.2.19)$$

where the above is a direct integral, with this we cover the entire continuous series twice.

6.3 Parametrization of the Group

On the fuzzy sphere we parametrized group elements as $e^{i\vec{p}\cdot\hat{\vec{J}}}$. However, the exponential $e^{ip_\mu \hat{K}^\mu}$ does not produce all the elements of SU(1,1) [53].

Consider the 2×2 representation for which we can write an operator

$$\hat{A} = a^\mu \hat{K}_\mu = \frac{1}{2}\begin{pmatrix} a_0 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 \end{pmatrix}. \qquad (6.3.1)$$

The eigenvalues of $\hat{A}$ are $\lambda_\pm = \pm\frac{i}{2}\sqrt{a_0^2 - a_1^2 - a_2^2}$. Then

$$\mathrm{Tr}(e^{i\hat{A}}) = \begin{cases} 2 & a^\mu \text{ lightlike} \\ 2\cos\left(\frac{\sqrt{a_0^2-a_1^2-a_2^2}}{2}\right) & a^\mu \text{ timelike} \\ 2\cosh\left(\frac{\sqrt{-a_0^2+a_1^2+a_2^2}}{2}\right) & a^\mu \text{ spacelike} \end{cases} . \tag{6.3.2}$$

For the single exponential $\mathrm{Tr}(e^{iA}) \geq -2$ always. But there are 3 classes [58; 59] of group elements for SU(1,1)

$$\begin{aligned} \mathrm{Tr}(g) &\in (-2,2) \quad \text{Elliptic}, \\ \mathrm{Tr}(g) &= \pm 2 \quad \text{Parabolic}, \\ |\,\mathrm{Tr}(g)| &> 2 \quad \text{Hyperbolic}. \end{aligned} \tag{6.3.3}$$

To cover the entire group we cannot therefore use just one exponential [53]. We can use the following parametrization instead

$$g(p_0,p_1,p_2) = e^{-i\frac{1}{2}p_0\hat{K}_0} e^{ip_1\hat{K}_1 + ip_2\hat{K}_2} e^{-i\frac{1}{2}p_0\hat{K}_0}. \tag{6.3.4}$$

If we introduce polar coordinates

$$\begin{gathered} p = \sqrt{p_1^2 + p_2^2}, \\ p_1 = p\sin\left(\tfrac{\phi}{2}\right) \quad p_2 = p\cos\left(\tfrac{\phi}{2}\right), \end{gathered} \tag{6.3.5}$$

then, in the 2×2 representation, the group element reads

$$g(p_0,p,\phi) = \begin{pmatrix} \cosh\left(\frac{p}{2}\right)e^{-i\frac{p_0}{2}} & \sinh\left(\frac{p}{2}\right)e^{i\frac{\phi}{2}} \\ \sinh\left(\frac{p}{2}\right)e^{-i\frac{\phi}{2}} & \cosh\left(\frac{p}{2}\right)e^{i\frac{p_0}{2}} \end{pmatrix}. \tag{6.3.6}$$

Here

$$p \in [0,\infty), \qquad p_0 \in [0,4\pi), \qquad \phi \in [0,4\pi), \tag{6.3.7}$$

and we note that p_0 is restricted due to the discreteness of the spectrum of $\hat{K}_0$. Another possible parameterization often used is

$$g(\alpha,\beta,p) = e^{-i\alpha\hat{K}_0} e^{ip\hat{K}_2} e^{-i\beta\hat{K}_0}. \tag{6.3.8}$$

In the 2×2 representation this reads

$$g(\alpha,\beta,p) = \begin{pmatrix} \cosh\left(\frac{p}{2}\right)e^{-i\frac{\alpha+\beta}{2}} & \sinh\left(\frac{p}{2}\right)e^{-i\frac{\alpha-\beta}{2}} \\ \sinh\left(\frac{p}{2}\right)e^{i\frac{\alpha-\beta}{2}} & \cosh\left(\frac{p}{2}\right)e^{i\frac{\alpha+\beta}{2}} \end{pmatrix}, \tag{6.3.9}$$

where

$$p \in [0,\infty), \qquad \alpha \in [0,2\pi), \qquad \beta \in [0,4\pi). \tag{6.3.10}$$

The different parameterizations are related by

$$p_0 = \alpha + \beta, \qquad \phi = -\alpha + \beta. \tag{6.3.11}$$

The invariant (Haar) measure of a group is invariant under group transformations [48]. We use this when we want to integrate over the group manifold. The invariant measures for the above parametrizations are given by

$$d\mu(g) = \frac{\sinh(p)}{16\pi^2} dp\, d\alpha\, d\beta = \frac{\sinh(p)}{32\pi^2} dp\, dp_0\, d\phi. \tag{6.3.12}$$

6.3.1 Matrix Elements

We can compute the matrix elements of the group elements [60], these are the Wigner D-functions for $SU(1,1)$

$$\begin{aligned}\mathcal{P}^{\chi}_{m,n}(g) &= \langle m| e^{i\alpha \hat{K}_0} e^{ip\hat{K}_2} e^{i\beta\hat{K}_0} |n\rangle \\ &= e^{-i(\alpha m+\beta n)} \langle k,m| e^{ip\hat{K}_2} |k,n\rangle \\ &= e^{-i(\alpha m+\beta n)} P^{\chi}_{m,n}(p).\end{aligned} \tag{6.3.13}$$

Where $|m\rangle$ is a state in the representation χ. To write down $P^{\chi}_{m,n}(p)$, we first define,

$$B^{\tau}_{m,n}(t) = \operatorname{sech}^{m+n}\left(\frac{t}{2}\right) \sinh^{m-n}\left(\frac{t}{2}\right) \times \frac{{}_2F_1\left(1-\tau-n, \tau-n, 1+m-n, -\sinh^2\left(\frac{t}{2}\right)\right)}{(m-n)!}, \tag{6.3.14}$$

and

$$N^{\tau}_{m,n} = \frac{\Gamma(1-\tau+m)}{\Gamma(1-\tau+n)}. \tag{6.3.15}$$

${}_2F_1$ is the ordinary hypergeometric function. Then we can write [56; 60],

$$\begin{aligned}P^{(k,+)}_{m,n}(t) &= \sqrt{N^{1-k}_{m,n} N^{k}_{m,n}} B^{k}_{m,n}(t) \qquad (m \geq n), \\ P^{(k,-)}_{m,n}(t) &= P^{(k,+)}_{-m,-n}(t) \qquad (m \geq n), \\ P^{(k,\pm)}_{m,n}(t) &= (-1)^{n-m} P^{(k,\pm)}_{m,n}(t) \qquad (m < n).\end{aligned} \tag{6.3.16}$$

and

$$\begin{aligned}P^{(\frac{1}{2}+i\rho,\epsilon)}_{m,n}(t) &= N^{(\frac{1}{2}+i\rho,\epsilon)}_{m,n} B^{(\frac{1}{2}+i\rho,\epsilon)}_{(m,n)}(t) \qquad (m \geq n), \\ P^{(\frac{1}{2}+i\rho,\epsilon)}_{m,n}(t) &= (-1)^{m-n} N^{(\frac{1}{2}-i\rho,\epsilon)}_{m,n} B^{(\frac{1}{2}+i\rho,\epsilon)}_{(m,n)}(t) \qquad (m < n).\end{aligned} \tag{6.3.17}$$

We note that the matrix element for $(\rho = 0, \frac{1}{2})$ can be written in terms of the matrix elements of $(\frac{1}{2}, +)$ and $(\frac{1}{2}, -)$,

$$P^{(\frac{1}{2},\frac{1}{2})}_{m,n}(t) = \begin{cases} P^{(\frac{1}{2},+)}_{m,n}(t) & m,n \geq \frac{1}{2} \\ (-1)^{m-n} P^{(\frac{1}{2},-)}_{m,n}(t) & m,n \leq \frac{1}{2} \end{cases}, \tag{6.3.18}$$

which illustrates how $(\rho = 0, \frac{1}{2})$ decomposes into the direct sum of $(\frac{1}{2}, +)$ and $(\frac{1}{2}, -)$.

The matrix elements have the following orthogonality relations for the discrete series

$$\int d\mu(g) \ \left[\mathcal{P}^{(k,\sigma)}_{m,n}(g)\right]^* \mathcal{P}^{(k',\sigma')}_{m',n'}(g) = \frac{\delta_{m,m'}\delta_{n,n'}\delta_{k,k'}\delta_{\sigma,\sigma'}}{2k-1}, \tag{6.3.19}$$

which is ill defined for the $k = \frac{1}{2}$ representation. For the continuous series we have

$$\int d\mu(g) \ \left[\mathcal{P}^{(\frac{1}{2}+i\rho,\epsilon)}_{m,n}(g)\right]^* \mathcal{P}^{(\frac{1}{2}+i\rho',\epsilon')}_{m',n'}(g) = \frac{\delta_{m,m'}\delta_{n,n'}\delta(\rho-\rho')\delta_{\epsilon,\epsilon'}}{2\rho\tanh(\pi(\rho+i\epsilon))}. \tag{6.3.20}$$

6.4 SU$(1,1)$ Fuzzy Space-Time

As in two dimensional non-commutative quantum mechanics, and fuzzy space quantum mechanics, we choose our coordinates to be operators and not scalars that label points in space-time. Then due to the uncertainty principle we induce a minimum length scale proportional to the non-commutative parameter θ.

In this relativistic theory our coordinates are the generators of the SU(1,1) group

$$\hat{t} = \theta\hat{K}^0, \quad \hat{x} = \theta\hat{K}^1, \quad \hat{y} = \theta\hat{K}^2. \tag{6.4.1}$$

The Casimir is proportional to the invariant interval $\hat{s}^2$

$$\hat{s}^2 = \hat{t}^2 - \hat{x}^2 - \hat{y}^2 = \theta^2\hat{C}^2. \tag{6.4.2}$$

These commutation relations preserve Lorentz symmetry in $2+1$ dimensions.

We can cover the different regions of Minkowski space with the unitary irreps of SU$(1,1)$ as we did with SU(2). Unlike SU(2) we can not use a single set of irreps $|j,m\rangle$ to cover all of Minkowski space. We need both the discrete and continuous series. It possible that there exists a larger reducible representation that includes both. However this is not easy to construct.

The Hilbert space that coordinates act on we call the configuration space $\mathcal{H}_c$. For the two mode representation the configuration space is two mode boson-fock space.

6.4.1 Interior Forward and Backward Light Cone

We can cover the forward and backward interiors of the light cone with the positive discrete series and the negative discrete series respectively. We note that $k = \frac{1}{2}$ is excluded for both, as this corresponds to the forward and backward cones. This representation cause difficulties when we try to introduce dynamics and we discuss this later. It also appears only once in the two mode representations as mentioned earlier. If we compute the expectation value of the invariant interval $\hat{x}^\mu \hat{x}_\mu$ in terms of the usual basis states for the irreps we obtain

$$\langle \hat{t}^2 - \hat{x}^2 - \hat{y}^2 \rangle = -\frac{\theta^2}{4} + \theta^2 \left(k - \frac{1}{2}\right)^2. \tag{6.4.3}$$

Calculating this expectation value gives us an idea of how the irreps cover the fuzzy space-time. The natural analogue of the space-time interval is then

$$s^2 = \theta^2 \left(k - \frac{1}{2}\right)^2. \tag{6.4.4}$$

In this case we have discrete hyperbolic sheets, labeled by the irrep label k, that cover the forward and backward cones. This is reminiscent of the onion structure of SU(2).

6.4.2 Space-like Region

We cover the space-like region with the continuous series, both odd and even. For these irreps the expectation value of the invariant interval is given by

$$\langle \hat{t}^2 - \hat{x}^2 - \hat{y}^2 \rangle = -\frac{\theta^2}{4} + \theta^2 (i\rho)^2. \tag{6.4.5}$$

The natural analogue of the space-time interval in this case is

$$s^2 = \theta^2 (i\rho)^2. \tag{6.4.6}$$

In this case the interval is not discrete, but rather we have a continuous label for the sheets in the space-like region. This structure differs significantly from the forward and backward cones. The physical meaning of the odd and even irreps is unclear and it is not obvious how they manifest when we cover the space-like region.

6.4.3 Light Cones

We can cover the light cones by including $(0, \frac{1}{2})$ as this decomposes into $(\frac{1}{2}, +)$ and $(\frac{1}{2}, -)$ as mentioned previously.

6.5 Chapter Summary

In this chapter we reviewed the properties of SU(1,1) and its representations. We described the unitary irreps, and displayed some quantities we will use later. We explained how we can parametrize the group. We then discussed how we can use the different irreps to cover our non-commutative space-time. Now we can move to building a quantum theory living on this non-commutative space-time.

Chapter 7

SU(1, 1) Fuzzy Quantum Mechanics and Field Theory

Now that we have defined a fuzzy space-time, we would like to build a quantum field theory living on this space-time. This is the goal of this chapter. We first introduce our quantum Hilbert space and quantum states and from this build a relativistic quantum mechanical theory. After this we introduce dynamics, and write the analogue of the Klein-Gordon equation. Finally in the chapter we discuss how we can use this to build a field theory.

7.1 Quantum States

The states of our quantum system are operators which are elements of the algebra generated by the coordinates. These "wave-functions" then act on the configuration space $\mathcal{H}_c$. The Hilbert space of the quantum states we call the quantum Hilbert space $\mathcal{H}_q$. We denote these states with a rounded ket

$$\phi(\hat{x}_\mu) = |\hat{\phi}), \tag{7.1.1}$$

and define an inner product on this Hilbert space by

$$(\hat{\psi}|\hat{\phi}) = \mathrm{Tr}(\hat{\psi}\hat{\Gamma}\hat{\phi}), \tag{7.1.2}$$

where $\hat{\Gamma}$ plays the role of a volume factor, which is necessary if we use this trace as the non-commutative analog of an integral over space-time. This is similar to the role $\hat{r}$ played in chapter 5 [25]. An important property of the quantum states is that they only depend on the coordinates and so must satisfy

$$[\hat{C}^2, \hat{\phi}] = 0, \tag{7.1.3}$$

all the outer products of the states in $\mathcal{H}_c$ do not necessarily satisfy this, and so are not necessarily in $\mathcal{H}_q$. For example the two mode boson-fock space does not produce this, and the outer products are in fact a larger Hilbert space which we denote as $\tilde{\mathcal{H}}_q$, then $\mathcal{H}_q \subset \tilde{\mathcal{H}}_q$. We can define operators that act on the quantum Hilbert space. For example, we can define a position operator that acts by left multiplication of the coordinate operator

$$\hat{X}_\mu|\hat{\phi}) = |\hat{x}_\mu\hat{\phi}). \tag{7.1.4}$$

We see that $\hat{X}_\mu$ satisfy the su$(1,1)$ commutation relations, and measurement of these will have a minimum uncertainty on the order of θ. Note the $\hat{X}$ are super operators since they act on operators, and are thus denoted by capitals

7.1.1 Lorentz Transformations

The Lorentz generators act via the adjoint action of the coordinates

$$\hat{L}_\nu|\hat{\phi}) = \left|\frac{1}{i\theta}\left[\hat{x}_\nu,\hat{\phi}\right]\right\rangle = \left|\left[\hat{K}_\nu,\hat{\phi}\right]\right\rangle = |\,\mathrm{ad}_{\hat{K}_\nu}\hat{\phi}), \tag{7.1.5}$$

and we can see that the coordinates will transform correctly as vectors since, for example,

$$\hat{L}_0|\hat{x}_1) = \left|\frac{1}{i\theta}\left[\hat{x}_0,\hat{x}_1\right]\right\rangle = |\hat{x}_2). \tag{7.1.6}$$

Also the adjoint action is a representation of SU$(1,1)$ on $\mathcal{H}_q$. Using the Jacobi identity we have

$$[\hat{L}_0,\hat{L}_1]|\hat{\phi}) = \left|\left[\hat{K}_0,\left[\hat{K}_1,\hat{\phi}\right]\right] - \left[\hat{K}_1,\left[\hat{K}_0,\hat{\phi}\right]\right]\right\rangle = |[i\hat{K}_2,\hat{\phi}]) = i\hat{L}_2|\hat{\phi}). \tag{7.1.7}$$

7.1.2 Plane Waves

The group elements are the natural choice for our plane waves. We introduce dimensionful "momenta" (k_0,k_1,k_2) and write for a group element

$$\hat{U}(k) = e^{i\frac{1}{2}k_0\hat{t}}e^{-ik_1\hat{x}-ik_2\hat{y}}e^{i\frac{1}{2}k_0\hat{t}}. \tag{7.1.8}$$

Then we can write any operator as [61],

$$\hat{\phi} = \int d\mu(k)\;\phi(k)\hat{U}(k). \tag{7.1.9}$$

We must be careful, however, interpreting the labels k_μ as some sort of physical momenta. These labels transform in a very complicated way under Lorentz transformations and the parameterization is not unique.

Now we have our quantum theory with states $|\hat{\phi})$ living in our quantum Hilbert space $\mathcal{H}_q$, they are operators that act on the configuration space $\mathcal{H}_c$. We have the notion of observables as the super operators that act on $\mathcal{H}_q$, and the Lorentz symmetry is explicitly realized. Now we want to introduce dynamics.

7.2 Tensor Operators

To define the dynamics of our relativistic quantum mechanical theory we need to find an operator which acts on $\mathcal{H}_q$, which is the non-commutative analogue of the d'Alembertian. From this we can define the non-commutative analogue of the Klein-Gordon equation. Eventually we will use this Klein-Gordon equation to define a non-commutative field theory for scalar fields.

To do this we first introduce four tensor operators $\hat{T}_+, \hat{T}_-, \hat{\hat{T}}_+, \hat{\hat{T}}_-$ such that

$$[\hat{T}_+, \hat{\hat{T}}_-] = [\hat{\hat{T}}_+, \hat{T}_-] = 1, \tag{7.2.1}$$

while all other commutators are zero. We do this because, unlike SU(2), we cannot use a single two mode representation to cover all of space-time. The tensor operators will prove more general and can easily be related to the two mode representations.

A representation of su(1, 1) is then

$$\hat{K}_\pm = \hat{T}_\pm \hat{\hat{T}}_\pm, \qquad \hat{K}_0 = -\frac{1}{2}\left(\hat{T}_+\hat{\hat{T}}_- + \hat{T}_-\hat{\hat{T}}_+\right), \qquad \hat{C}^2 = -\frac{1}{4} + \hat{\Gamma}^2, \tag{7.2.2}$$

where

$$\hat{\Gamma} = \frac{1}{2}\left(\hat{T}_+\hat{\hat{T}}_- - \hat{\hat{T}}_+\hat{T}_-\right). \tag{7.2.3}$$

These operators are spinors with respect to SU(1, 1)

$$[\hat{K}_+, \hat{T}_-] = \hat{T}_+, \qquad [\hat{K}_-, \hat{T}_+] = -\hat{T}_-, \qquad [\hat{K}_0, \hat{T}_\pm] = \pm\frac{1}{2}\hat{T}_\pm, \tag{7.2.4}$$

and similarly with $\hat{\hat{T}}_\pm$. If we introduce a map

$$\hat{U}_\pm = \pm\hat{\hat{T}}_\mp, \qquad \hat{\hat{U}}_\pm = \pm\hat{T}_\mp, \tag{7.2.5}$$

then the representations are related by

$$\hat{K}^{(U)}_\pm = \hat{K}^{(T)}_\mp, \quad \hat{K}^{(U)}_0 = -\hat{K}^{(T)}_0. \tag{7.2.6}$$

For unitarity there are four possibilities

$$\hat{T}^\dagger_\pm = \hat{\hat{T}}_\mp, \tag{7.2.7}$$

$$\hat{T}^\dagger_\pm = -\hat{\hat{T}}_\mp, \tag{7.2.8}$$

$$\hat{T}^\dagger_\pm = \hat{T}_\mp \quad \text{and} \quad \hat{\hat{T}}^\dagger_\pm = \hat{\hat{T}}_\mp, \tag{7.2.9}$$

$$\hat{T}^\dagger_\pm = -\hat{T}_\mp \quad \text{and} \quad \hat{\hat{T}}^\dagger_\pm = -\hat{\hat{T}}_\mp. \tag{7.2.10}$$

Here (7.2.7) and (7.2.8) are related by (7.2.6) and so are (7.2.9) and (7.2.10).

Using these four tensor operators we can define a scalar operator

$$\hat{\Box} = \frac{2}{\theta^2 \hat{\Gamma}}(\mathrm{ad}_{\hat{T}_+}\, \mathrm{ad}_{\hat{\bar{T}}_-} - \mathrm{ad}_{\hat{T}_-}\, \mathrm{ad}_{\hat{\bar{T}}_+}). \tag{7.2.11}$$

This operator is a good candidate for the non-commutative version of the d'Alembertian, since it has the right transformation properties under Lorentz transformations . The factor $\frac{2}{\theta^2\hat{\Gamma}}$ is included for similar reasons to SU(2), it ensures that this operator is hermitian w.r.t the trace inner product above. We have to be careful that $\hat{\Box}$ does not throw us out of $\mathcal{H}_q$ i.e. by acting on it we do not produce a state that violates (7.1.3). Since

$$[\hat{\Box}, \mathrm{ad}_{\hat{\Gamma}}] = 0, \tag{7.2.12}$$

acting on a "wavefunction" with $\hat{\Box}$ cannot produce non-physical states i.e. in the larger algebra of $\tilde{\mathcal{H}}_q$ and the tensor operators, $\mathcal{H}_q$ is closed under the action of $\hat{\Box}$.

7.3 Klein-Gordon Equation

Let us consider the action of $\hat{\Box}$ on a group element, a calculation that can be found in appendix A,

$$\begin{aligned} &\hat{\Box} e^{i\frac{1}{2}k_0\hat{t}} e^{-ik_1\hat{x}-ik_2\hat{y}} e^{i\frac{1}{2}k_0\hat{t}} \\ = \;& \tfrac{2}{\theta^2\hat{\Gamma}}\left(2\left(1-\cos\left(\tfrac{\theta k_0}{2}\right)\cosh\left(\tfrac{\theta k}{2}\right)\right)(2\hat{\Gamma})\right) e^{i\frac{1}{2}k_0\hat{t}} e^{-ik_1\hat{x}-ik_2\hat{y}} e^{i\frac{1}{2}k_0\hat{t}}, \end{aligned} \tag{7.3.1}$$

where the $k = \sqrt{k_1^2 + k_2^2}$ in the cosh. We see the $\hat{\Gamma}$ operators cancel, and the plane wave is therefore an eigenstate as desired. We therefore take the free Klein-Gordon equation as

$$\left(\hat{\Box} - m^2\right)\hat{\phi} = 0. \tag{7.3.2}$$

Inspired by the commutative case, m is then the mass of the particle that is described by $\hat{\phi}$. We have a dispersion relation that holds for both the discrete and continuous series,

$$d(k) = \frac{8}{\theta^2}\left(1 - \cos\left(\frac{\theta k_0}{2}\right)\cosh\left(\frac{\theta k}{2}\right) - \tilde{m}^2\right) = 0 \tag{7.3.3}$$

where

$$\tilde{m}^2 = \frac{\theta^2 m^2}{8}. \tag{7.3.4}$$

In the $\theta \to 0$ limit this does indeed produce the commutative $k^\mu k_\mu - m^2 = 0$ dispersion relation. An interesting property of this equation is that if we have a mass such that

$$|1 - \tilde{m}^2| > \cosh\left(\frac{\theta k}{2}\right), \tag{7.3.5}$$

we cannot find a k_0 which solves the equation. So if we have a too high mass and a too low spacial momentum, e.g. in the rest frame of the particle, we can never be on shell. If $\tilde{m}^2 < 2$ (low mass) this is never true and we can always be on shell. In SI units this condition is

$$m < \frac{4\hbar}{\theta c}. \tag{7.3.6}$$

Note that we can also rewrite this condition as

$$\theta < \frac{4\hbar}{mc} = 4\lambda_c \tag{7.3.7}$$

with λ_c is the Compton wave-length. This condition therefore amounts to the physically sensible condition that the Compton wave-length of a particle must always be larger than the non-commutative length scale. For example if $\theta = l_p$ the Planck length, then low mass is $m < 87\mu g$ which is physically very large, so the dispersion relation seems to be fine for describing microscopic particles.

However, this poses a problem in the macroscopic (classical) limit where m is large. In this limit, we expect the functional integral to be dominated by the saddle point corresponding to the on-shell solution. However, if the solution cannot be obtained, this signals some instability. One way to bypass this problem [62] is to introduce a mass dependent non-commutative parameter $\theta = \frac{\alpha}{m}$ then the condition (7.3.6) becomes

$$1 < \frac{4\hbar}{\alpha c} \tag{7.3.8}$$

so now we just have a restriction on our new non-commutative parameter α. Then θ is a not universal parameter for all space-time, but now the effect of the non-commutativity depends on the particles that occupy space-time. This argument makes sense for macroscopic objects since we would use effective degrees of freedom to describe them for example, center of mass coordinates in SU(2) fuzzy space. We would then explicitly see a factor of the total mass appearing in the effective SU(2) commutation relations. The effective degrees of freedom for our relativistic theory is not as obvious, however, it would make sense for them to also scale as the number of particles increases. Now our limit is only on the mass of fundamental particles.

7.3.1 Two Mode Representations

We can write the tensor operators in terms of the creation and annihilation operators. There is more than one way to build the tensors operators out of $\hat{a}_i$, the different possibilities correspond to (7.2.7),(7.2.8),(7.2.9) and (7.2.10). In each two mode representation the Klein-Gordon equation still takes the form of a double commutator, which matches SU(2) fuzzy space and other non-commutative theories [47].

7.3.1.1 Discrete Series

The possibility (7.2.8) can be realized by

$$\begin{aligned} \hat{T}_+ &= \hat{a}_1^\dagger, \quad & \hat{T}_- &= -\hat{a}_2, \\ \hat{\hat{T}}_+ &= \hat{a}_2^\dagger, \quad & \hat{\hat{T}}_- &= -\hat{a}_1. \end{aligned} \tag{7.3.9}$$

Then the two representations (6.2.9) and (7.2.2) are the same. The negative discrete series is then found using the transformation (7.2.6). The Klein-Gordon equation in this representation reads

$$\frac{2}{\theta^2\hat{\Gamma}}\left([\hat{a}_1^\dagger,[\hat{a}_1,\hat{\psi}]] - [\hat{a}_2^\dagger,[\hat{a}_2,\hat{\psi}]]\right) + m^2\hat{\psi} = 0 \tag{7.3.10}$$

where,

$$\hat{\Gamma}^{-1} = -\frac{1}{2\left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\right)}. \tag{7.3.11}$$

Note that we have used the Jacobi identity and the fact $[[\hat{a}_i^\dagger,\hat{a}_i],\hat{\psi}] = 0$. This $\hat{\Gamma}^{-1}$ is well-defined if we exclude the $k=\frac{1}{2}$ irrep from the discrete series, which is another reason we excluded this representation. For the $k=\frac{1}{2}$ discrete series the Klein-Gordon equation is ill-defined, since $\hat{\Gamma}=0$.

7.3.1.2 Continuous Series

The possibility (7.2.9) can also be realized by

$$\begin{aligned} \hat{T}_+ &= \hat{a}_1^\dagger + i\hat{a}_2, \quad & \hat{T}_- &= i\hat{a}_2^\dagger - \hat{a}_1, \\ \hat{\hat{T}}_+ &= \hat{a}_1^\dagger - i\hat{a}_2, \quad & \hat{\hat{T}}_- &= -i\hat{a}_2^\dagger - \hat{a}_1. \end{aligned} \tag{7.3.12}$$

In this case (6.2.11) and (7.2.2) are the same representation. As mentioned this is equivalent to (7.2.10) which just reflects that fact that the continuous series is invariant under the transformation between the positive and negative discrete series (7.2.6). Again we can write the Klein-Gordon equation

$$\frac{2}{\theta^2\hat{\Gamma}}\left([\hat{a}_1^\dagger,[\hat{a}_1,\hat{\psi}]] - [\hat{a}_2^\dagger,[\hat{a}_2,\hat{\psi}]]\right) + m^2\hat{\psi} = 0 \tag{7.3.13}$$

where

$$\hat{\Gamma}^{-1} = -\frac{1}{2\left(\hat{a}_1\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2^\dagger\right)}. \tag{7.3.14}$$

7.4 Field Theory on the Group Manifold

We have our non-commutative Klein-Gordon equation, we would now like to promote our quantum mechanical theory to a quantum field theory. We do not know of a way to canonically quantize this on the operator level. The approach we follow to developing a field theory is to write an action and then quantize via a path integral. An action which depends on operators can be written in terms of the inner product, in this case the trace, as

$$S = \mathrm{Tr}(\hat{\phi}^\dagger \hat{\Gamma} \left(\hat{\Box} - m^2 \right) \hat{\phi}). \tag{7.4.1}$$

This does produce the correct equations of motion under variation due to the properties of the inner product. This can be done in the SU(2) case as mentioned earlier, where the trace is then over all the standard basis states i.e. a sum over both j and m. For SU(1,1) there is the complication of having both the continuous and discrete series, each with different $\hat{\Gamma}$. In particular trying to use the two mode representation becomes much more involved, since we have two different two mode representations. Even for just the discrete series using $\hat{\Gamma}$ (7.3.11) gives a zero trace since we cover all the irreps twice, and (7.3.11) weights with opposite sign. The correct volume factor should instead be $|\hat{\Gamma}|$, but calculating this trace is very difficult. In principle there should be some reducible representation, that includes both the discrete and continuous series, that we can trace over. However, constructing this representation if it exists would be a difficult task, so we instead use a different approach.

Our approach is to instead use the fields that live on the group manifold $\phi(k)$, it depends on coordinates k on the group manifold i.e. for every set of coordinates k there is associated a group element. Recalling (7.1.9)

$$\hat{\phi} = \int d\mu(k)\ \phi(k) \hat{U}(k), \tag{7.4.2}$$

where

$$d\mu(k) = \frac{\sinh\left(\theta|k|\right)}{|k|} dk_0 dk_1 dk_2. \tag{7.4.3}$$

If we know the correlators of $\phi(k)$, we can calculate other correlators which are of interest to us. This is discussed in detail in the next chapter.

From this approach, the dispersion relation is a function on the group manifold that must be Lorentz invariant. The group characters have this property. The group characters are traces of the group elements in a particular representation. The simplest is the trivial one dimensional case. Then for the 2×2 non-unitary representation (6.3.6) the character is

$$\mathrm{Tr}\left[\begin{pmatrix} \cosh\left(\frac{p}{2}\right) e^{-i\frac{p_0}{2}} & \sinh\left(\frac{p}{2}\right) e^{i\frac{\phi}{2}} \\ \sinh\left(\frac{p}{2}\right) e^{-i\frac{\phi}{2}} & \cosh\left(\frac{p}{2}\right) e^{i\frac{p_0}{2}} \end{pmatrix} \right] = \cosh\left(\frac{p}{2}\right) \cos\left(\frac{p_0}{2}\right). \tag{7.4.4}$$

If we combined the one and two dimensional characters with the appropriate weights we obtain (7.3.3). This matches the dispersion calculated using the tensor operators, this shows a connection between the operators and fields on the manifold. This gives a motivation for a momentum space action for the functions on the group manifold, using the dispersion relation (7.3.3),

$$S[\phi] = \int d\mu(k)\ \phi^*(k)d(k)\phi(k). \tag{7.4.5}$$

If we use this action we avoid the problems of calculating the trace. The fact that the dispersion relation is the same for both discrete and continuous series suggests that there exists a representation where the above is equal to the trace. We write the trace in terms of (7.4.2), to show the difficulty with calculating it,

$$S[\phi] = \int d\mu(k')d\mu(k)\ \phi^*(k')d(k)\phi(k)\,\mathrm{Tr}(\hat{U}^\dagger(k')\hat{\Gamma}\hat{U}(k)). \tag{7.4.6}$$

The trace and our momentum space integral are equal exactly when

$$\mathrm{Tr}(\hat{U}^\dagger(k')\hat{\Gamma}\hat{U}(k)) = \delta_\mu(k'-k), \tag{7.4.7}$$

where this is a Dirac delta with respect to the group measure i.e.

$$\int d\mu(k')\ f(k')\delta_\mu(k'-k) = f(k). \tag{7.4.8}$$

Unfortunately we are not able to construct this representation. In appendix B we discuss this in more detail.

However, for now we assume there exists such representation. We follow the standard path integral approach, and write

$$Z[J] = \int [d\phi][d\phi^*]\ e^{iS[\phi]+i\int d\mu(k)(J^*(k)\phi(k)+J(k)\phi^*(k))}, \tag{7.4.9}$$

with the above action. The correlator

$$\langle\phi^*(k)\phi(k')\rangle = \frac{\int[d\phi][d\phi^*]\ \phi^*(k)\phi(k')e^{iS[\phi]}}{Z[0]}, \tag{7.4.10}$$

is then simply given by

$$\langle\phi^*(k)\phi(k')\rangle = i\frac{\theta^2}{8}\delta_\mu(k-k')\left[1-\cos\left(\frac{\theta k_0}{2}\right)\cosh\left(\frac{\theta k}{2}\right)-\tilde{m}^2\right]^{-1}. \tag{7.4.11}$$

Recalling we have defined $\tilde{m} = \frac{\theta^2 m^2}{8}$. That this is the correct momentum space correlator is not immediately obvious, as the path integral measure might need to be modified [18]. However, it is still interesting to calculate space-time correlators using the above, which is the goal of the next chapter. To do this we find maps between operators and scalar functions that have space-time labels. Then we can calculate correlators of these scalar functions.

7.5 Chapter Summary

In this chapter we have formulated our quantum mechanical theory by introducing the quantum Hilbert space which contains the states of our system. On this space we can find a representation of the coordinate algebra and the Lorentz transformations in terms of super operators. We found a Klein-Gordon equation by introducing tensor operators. These tensor operators can be realized in terms of creation and annihilation operators. Using the Klein-Gordon equation we can introduce a scalar field theory. We can write this field theory in terms of functions that live on the group manifold, from this we calculated the correlator of these fields.

Chapter 8

Connection to Commutative Theories

Now that we have our SU$(1,1)$ field theory, we need to connect this to physical quantities. Our original goal was to develop a Lorentz invariant non-commutative theory. This could solve the problems we have with commutative field theories such as divergent correlators. Our hope being to introduce a minimum length scale to our field theories, without breaking Lorentz symmetry. In the previous chapter we have achieved this goal. However, we cannot just write down such a theory, we must also calculate some observable quantities to potentially verify our non-commutative theory. We can then compare with commutative theories, and in the limit where the minimum length scale becomes irrelevant make sure the theories are equivalent since we would not like to change any physics at large length scales where commutative theories match experimental results very well. The goal of this chapter is to make this connection to commutative theories. We calculate quantities that could potentially be measured, and check if certain quantities are still divergent such as corrrelators at equal space-time points.

8.1 Symbols

Consider a map from the operators $\hat{\phi}$ to scalar fields:

$$\hat{\phi} \to \phi(o). \tag{8.1.1}$$

The o is a completely general set of labels which are the argument of the scalar field, e.g. later o will be a set of space-time coordinates. We can calculate the correlator of the scalar fields using the transformation of the group elements

$$e^{i\frac{1}{2}k_0\hat{t}}e^{-ik_1\hat{x}-ik_2\hat{y}}e^{i\frac{1}{2}k_0\hat{t}} = \hat{U}(k) \to U_k(o), \tag{8.1.2}$$

since our scalar field can be written as

$$\phi(o) = \int d\mu(k)\ \phi(k)U_k(o), \tag{8.1.3}$$

this map can now be thought of as a generalized integral transform of the fields on the group manifold. Then the correlator becomes

$$\langle \phi^*(o)\phi(o')\rangle = \int d\mu(k)d\mu(k')\, U_k^*(o)U_{k'}(o')\, \langle \phi^*(k)\phi(k')\rangle \,. \tag{8.1.4}$$

If $o = \{t, x, y\}$ is the set of coordinates we choose to parameterize our space-time, then this correlator is our space-time correlator. The scalar fields $\phi(t, x, y)$ are the degrees of freedom in the measurable theories we are trying to describe, and in the θ goes to zero limit these should be our commutative fields. We call these scalar fields $\phi(t, x, y)$ the symbols of the operator $\hat{\phi}$.

An obvious way to produce a scalar quantity from an operator is to calculate an expectation value. This will be our first attempt at defining a symbol.

$$\begin{aligned} \phi(o) &= \langle t, x, y|\, \hat{\phi}\, |t, x, y\rangle = \phi(t, x, y), \\ U_k(o) &= \langle t, x, y|\, \hat{U}(k)\, |t, x, y\rangle = U_k(t, x, y). \end{aligned} \tag{8.1.5}$$

where the arguments of the functions are given by the expectations values $t = \langle \hat{t}\rangle$, $x = \langle \hat{x}\rangle$ and $y = \langle \hat{y}\rangle$ w.r.t the state $|t, x, y\rangle$.

8.2 SU(1,1) Coherent States

Ideally we would like to use some kind of minimum uncertainty state to define our symbol. This is physical sensible since maximally localized states are the closest we can get to commutative states, i.e. fields which have well-defined coordinates. Then correlators of this symbol should match commutative correlators at large length scales, since the uncertainty is on the order of θ. This is done in both SU(2) fuzzy space and the two dimensional constant commutation relations [5]. $SU(1, 1)$ coherent states [63] have fixed s^2, and minimize the uncertainty in the hyperbolic angles ϕ and τ. For the positive and negative discrete series we can define

$$|z, k\rangle = (1 - |z|^2)^k e^{z\hat{K}_\pm}\, |k, \pm k\rangle \quad \text{with} \quad z = e^{\mp i\phi} \tanh\left(\frac{\tau}{2}\right). \tag{8.2.1}$$

where z is a complex number inside the unit disk, $D = \{z, |z| < 1\}$. The symbols of the coordinates are then

$$\begin{aligned} \langle z, k|\, \hat{t}\, |z, k\rangle &= \pm\theta k \cosh(\tau), \\ \langle z, k|\, \hat{x}\, |z, k\rangle &= \theta k \sinh(\tau) \cos(\phi), \\ \langle z, k|\, \hat{y}\, |z, k\rangle &= \theta k \sinh(\tau) \sin(\phi). \end{aligned} \tag{8.2.2}$$

Instead of the onion structure of SU(2), we cover the forward and backward light cones with discrete layers of hyperbolic sheets. We note

$$\langle \hat{K}_0\rangle^2 - \langle \hat{K}_1\rangle^2 - \langle \hat{K}_2\rangle^2 = k^2, \qquad \langle \hat{C}^2\rangle = k(k-1), \tag{8.2.3}$$

we can then see that $s^2 = \theta^2 \left(k - \frac{1}{2}\right)^2$ as defined in (6.4.4).

We can write the identity in terms of these coherent states

$$\hat{\mathbb{I}}_c = \int_D dzdz^* \; \frac{2k-1}{\pi} \frac{1}{(1-|z|^2)^2} \left|z,k\right\rangle \left\langle z,k\right| . \tag{8.2.4}$$

We see for the $k = \frac{1}{2}$ representation of the discrete series this is not a valid representation of the identity. So we only use these states for the interior light cones.

We can calculate the group elements expectation values, for example

$$\begin{aligned} \left\langle z,k\right| e^{ik_0\hat{t}} \left|z,k\right\rangle &= (1-|z|^2)^{2k} \left\langle k,k\right| e^{z^*\hat{K}_-} e^{ik_0\hat{t}} e^{z\hat{K}_+} \left|k,k\right\rangle \\ &= (1-|z|^2)^{2k} \left\langle k,k\right| e^{A_+\hat{K}_+} e^{A_0\hat{K}_0} e^{A_-\hat{K}_-} \left|k,k\right\rangle \\ &= (1-|z|^2)^{2k} \left(\frac{e^{ik_0\theta}}{(1-e^{ik_0\theta}|z|^2)^2} \right)^k , \end{aligned} \tag{8.2.5}$$

where

$$A_+ = \frac{e^{i\theta k_0} z}{1-e^{i\theta k_0}|z|^2}, \quad A_- = \frac{e^{i\theta k_0} z^*}{1-e^{i\theta k_0}|z|^2}, \quad A_0 = \log\left(\frac{e^{i\theta k_0}}{(1-e^{i\theta k_0}|z|^2)^2} \right) \tag{8.2.6}$$

Calculating correlators using the above approach is difficult. The symbol of a general group element is even more cumbersome. Therefore we will instead perform some numerical calculations.

If we know the correlators (w.r.t the path integral (7.4.9)) of matrix elements

$$M^{k,k'}_{m,n,m',n'}(k) = \left\langle (\left\langle k,m\right| \hat{U}(k) \left|k,n\right\rangle)^* \left\langle k',m'\right| \hat{U}(k) \left|k',n'\right\rangle \right\rangle , \tag{8.2.7}$$

we can in principle sum up and calculate correlators of other symbols. Since we can always write

$$\left\langle o\right| \hat{U}(k) \left|o\right\rangle = U_k(o) = \sum_{k,m,n} c^*_{k,m,o} c_{k,n,o} \left\langle k,m\right| \hat{U}(k) \left|k,n\right\rangle , \tag{8.2.8}$$

where

$$c_{k,n,o} = \left\langle o|k,n\right\rangle . \tag{8.2.9}$$

We begin by expanding the coherent states (8.2.1) in terms of the standard basis states (6.2.1), for the positive discrete series, as

$$\left|z,k\right\rangle = \sum_{m=k}^{\infty} \sqrt{\frac{\Gamma(k+m)}{(m-k)!\Gamma(2k)}} z^{(m-k)} \left|k,m\right\rangle . \tag{8.2.10}$$

As shown before, in equation (6.3.13) ,the matrix elements are given by

$$\left\langle m\right| \hat{U}(k) \left|n\right\rangle = e^{-\frac{i}{2}(n+m)p_0} e^{-\frac{i}{2}(n-m)\phi} P^k_{n,m}(p). \tag{8.2.11}$$

The correlator of matrix elements (8.2.7) is

$$M^{k,k'}_{m,n,m',n'}(k) = \int \sinh(p)dp_0dpd\phi \;\; P^k_{n,m}(p)P^{k'}_{n',m'}(p) \times \frac{e^{-\frac{i}{2}(n+m-n'-m')p_0}e^{-\frac{i}{2}(n-m-n'+m')\phi}}{1-\cos\left(\frac{p_0}{2}\right)\cosh\left(\frac{p}{2}\right)-\tilde{m}^2}, \tag{8.2.12}$$

we can perform the ϕ integral which simply produces $4\pi\delta_{n-m,n'-m'}$. We notice that correlators of matrix elements that are not equally off-diagonal are always zero, i.e. only correlators between matrix elements where $n-m=n'-m'$ are non-zero.

To perform the p_0 integral we make a change of variables, $z=e^{-i\frac{p_0}{2}}$, then $\int_0^{4\pi} dp_0 = \oint dz\frac{2i}{z}$ and write $q=n+m-n'-m'$. So we are integrating along a unit circle in the complex plane, then

$$\int dp_0 \;\; \frac{e^{-\frac{i}{2}(n+m-n'-m')p_0}}{1-\cos\left(\frac{p_0}{2}\right)\cosh\left(\frac{p}{2}\right)-\tilde{m}^2} = \left(4i\,\mathrm{sech}\left(\frac{p}{2}\right)\right)\oint dz\frac{z^q}{(z-z_+)(z-z_-)}, \tag{8.2.13}$$

where

$$z_\pm = \mathrm{sech}\left(\frac{p}{2}\right)\left(1-\tilde{m}^2 \pm \sqrt{(1-\tilde{m}^2)^2-\cosh^2\left(\frac{p}{2}\right)}\right). \tag{8.2.14}$$

If $|1-\tilde{m}^2| > \cosh\left(\frac{p}{2}\right)$ then the integral is well-defined as it stands since one pole lies inside the unit circle and other outside. Using the residue theorem we obtain for the $\oint dz$ integral

$$\begin{aligned}\oint dz\frac{z^q}{(z-z_+)(z-z_-)} &= (2\pi i)\frac{z_-^q}{z_--z_+} \\ &= \pi(-1)^{|q|}\frac{e^{-|q|\epsilon_p}}{\sinh\left(\epsilon_p\right)},\end{aligned} \tag{8.2.15}$$

where

$$\cosh\left(\epsilon_p\right) = \frac{|1-\tilde{m}^2|}{\cosh\left(\frac{p}{2}\right)}. \tag{8.2.16}$$

When $|1-\tilde{m}^2| < \cosh\left(\frac{p}{2}\right)$ the poles are on the integration contour i.e. the values of z that make the denominator zero are on the unit circle. To continue we need an ϵ-prescription to move the poles off the unit circle to give the integral meaning. This is what is done in the commutative case, where we move the poles off the real line.

For the Feynman propagators we simply shift $\tilde{m}^2 \to \tilde{m}^2 - i\epsilon$, which also shifts the poles. If $\tilde{m} \leq 1$ then z_- lies inside the unit circle while z_+ lies outside. We perform the $\oint dz$ integral to obtain

$$\left(4i\,\mathrm{sech}\left(\frac{p}{2}\right)\right)\oint dz\frac{z^q}{(z-z_+)(z-z_-)} = \begin{cases} 4i\,\mathrm{sech}\left(\frac{p}{2}\right)(2\pi i)\frac{z_-^q}{z_--z_+} & \text{if } \tilde{m}\leq 1 \\ -4i\,\mathrm{sech}\left(\frac{p}{2}\right)(2\pi i)\frac{z_+^q}{z_--z_+} & \text{if } \tilde{m}>1 \end{cases}. \tag{8.2.17}$$

Now we can take the $\epsilon \to 0$ limit and combine all the results above we obtain the correlator of matrix elements (8.2.7):

$$M^{k,k'}_{m,n,m',n'}(k) = 4\pi \frac{i\theta^2}{8}(-4\pi)\delta_{n-m,n'-m'} \int_0^\infty dp\ \frac{\sinh(p)}{\cosh\left(\frac{p}{2}\right)} P^k_{n,m}(p) P^{k'}_{n',m'}(p)\Delta(p,q), \tag{8.2.18}$$

where

$$\Delta(q,p) = \begin{cases} \frac{e^{-i|q|E_p}}{\sin(E_p)} & \text{if } \cosh\left(\frac{p}{2}\right) > |\tilde{m}^2 - 1| \\ -i\frac{(-1)^{|q|}e^{-|q|\epsilon_p}}{\sinh(\epsilon_p)} & \text{if } \cosh\left(\frac{p}{2}\right) > |\tilde{m}^2 - 1|. \end{cases} \tag{8.2.19}$$

and we define

$$\cos\left(E_p\right) = \frac{|1-\tilde{m}^2|}{\cosh\left(\frac{p}{2}\right)}. \tag{8.2.20}$$

The last integral cannot be done analytically and so we do this numerically. We start by calculating the simplest case $|z = 0, k\rangle = |k, k\rangle$ i.e. correlators between fields at different points along the time axis. In this case the matrix element is

$$P^k_{k,k}(p) = \operatorname{sech}^{2k}\left(\frac{p}{2}\right). \tag{8.2.21}$$

We note that q plays the role of the time difference, which appears in the commutative propagator, in fact for this simplified case, it is exactly proportional since $\langle k,k|\,\hat{t}\,|k,k\rangle = \theta k$ and so $q = 2(k-k') = -\frac{2}{\theta}\Delta t$.

For $\tilde{m} = 0.03$ we can produce a plot using *Mathematica*:

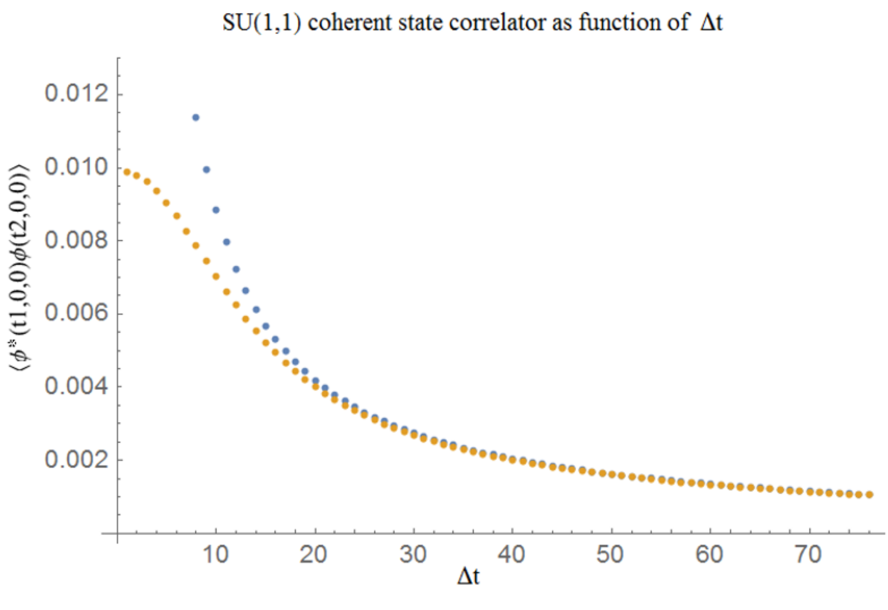

Figure 8.1: Correlator

The correlator here is of symbols calculated using SU(1, 1) coherent states along the time axis i.e $\phi(t, 0, 0) = \langle z = 0, k| \hat{\phi} |z = 0, k\rangle$. The correlator does not diverge for $t_1 = t_2$, and for large Δt we obtain the commutative result, represented by the blue line.

However, the problem with these coherent states is that they only work in the interior forward and backward light cones. For the space-like region in the literature [64] the idea of using states $K_1 |\rho, \lambda\rangle = \lambda |\rho, \lambda\rangle$ is proposed. However, building sensible coherent states out of these proved to be quite difficult and calculating symbols is intractable. Even calculating matrix elements in this basis proved tedious.

8.2.1 Other Propagators

There are other choices of ϵ-prescriptions. If we return to the p_0 integral we can introduce another change of variables $w = e^{i\frac{p_0}{2}}$, and write

$$\oint dz \frac{z^q}{(z - z_+)(z - z_-)} = \oint dw \frac{w^{-q}}{(w - w_-)(w - w_+)}. \tag{8.2.22}$$

The choice between z and w is the analogue of closing the integration in the top or bottom half of the complex plane in the commutative case. The pole lies at

$$z_\pm = w_\mp = \operatorname{sech}\left(\frac{p}{2}\right)\left(1 - \tilde{m}^2 \pm \sqrt{(1 - \tilde{m}^2)^2 - \cosh^2\left(\frac{p}{2}\right)}\right). \tag{8.2.23}$$

If we let p_0 run from $-2\pi \leq p_0 < 2\pi$ then we see $w_- = z_+$ is the positive energy pole, while $w_+ = z_-$ is the negative energy pole.

In the $m^2 \to m^2 - i\epsilon$ choice we moved one pole inside and one outside the unit circle. To illustrate the other ways to move the poles we introduce η_+ and η_-, which are either infinitesimally smaller or larger than one. Then we can write

$$\oint dz \frac{z^q}{(z - \eta_+^{-1} z_+)(z - \eta_-^{-1} z_-)} = \oint dw \frac{w^{-q}}{(w - \eta_- w_-)(w - \eta_+ w_+)}. \tag{8.2.24}$$

8.2.1.1 The Retarded Propagator

Here we choose $\eta_+, \eta_- > 1$. If $q \leq 0$ we use the w integral which will clearly be zero since both poles lie outside the integration contour. If $q > 0$, we can think of this as negative Δt therefore retarded propagator , then the z integral produces

$$4\pi \frac{z_+^q - z_-^q}{z_+ - z_-} = 4\pi \frac{\sin(qE_p)}{\sin(E_p)}. \tag{8.2.25}$$

8.2.1.2 The Advanced Propagator

Here we choose $\eta_+, \eta_- < 1$. If $q \geq 0$ we can use the z integral which will clearly be zero since both poles lie outside the integration contour. If $q < 0$, we can think of this as positive Δt therefore advanced propagator, then the w integral produces

$$-4\pi \frac{z_+^q - z_-^q}{z_+ - z_-} = -4\pi \frac{\sin(qE_p)}{\sin(E_p)}. \tag{8.2.26}$$

8.2.1.3 The Feynman Propagator

If $\eta_+ > 1, \eta_- < 1$, depending on the sign of q we use either z or w and obtain

$$4\pi \frac{z_-^{|q|}}{z_+ - z_-} = 2\pi i \frac{e^{-i|q|E_p}}{\sin(E_p)}. \tag{8.2.27}$$

This is equivalent to $m^2 \to m^2 - i\epsilon$, since if we look at the poles

$$|z_\pm|^2 = 1 \pm \frac{2\epsilon}{\sqrt{1 - \frac{1-\tilde{m}^2}{\cosh\left(\frac{p}{2}\right)}}}, \tag{8.2.28}$$

we see one pole moved outside and one inside.

8.2.1.4 The Dyson Propagator

If $\eta_+ < 1, \eta_- > 1$ depending on the sign of q we use either z or w and obtain

$$-4\pi \frac{z_+^{|q|}}{z_+ - z_-} = -2\pi i \frac{e^{i|q|E_p}}{\sin(E_p)}. \tag{8.2.29}$$

This is equivalent to $m^2 \to m^2 + i\epsilon$, since if we look at the poles

$$|z_\pm|^2 = 1 \mp \frac{2\epsilon}{\sqrt{1 - \frac{1-\tilde{m}^2}{\cosh\left(\frac{p}{2}\right)}}}. \tag{8.2.30}$$

8.2.2 Naive Symbol

On a side note, we could have also considered the most naive choice of

$$U_k(t, x, y) = e^{ik_\mu x^\mu}, \tag{8.2.31}$$

and

$$\phi(t, x, y) = \int d\mu(k)\, e^{ik_\mu x^\mu} \phi(k), \tag{8.2.32}$$

i.e. the standard Fourier transform of the non-commutative fields which live on the group manifold, so we are only actually integrating over the manifold. Then we can continue in a similar way to the above. Proceeding to do the k_0 integral, this produces the same Δ function. We go to polar coordinates for the spacial k, the angular integral gives $J_0(k\Delta r)$, where J_0 is a Bessel function. To obtain

$$N \int dp \, \frac{\sinh(\theta k)}{\cosh\left(\frac{\theta k}{2}\right)} \Delta\left(\theta k, 2\frac{\Delta t}{\theta}\right) J_0(k\Delta r). \tag{8.2.33}$$

When we computed this numerically we found divergent answers.

8.3 Two Mode Coherent States

Using the two mode representation for the su$(1,1)$ algebra in terms of creation and annihilation operators, we can introduce two mode Glauber coherent states

$$|z_1, z_2\rangle = e^{\frac{-|z_1|^2-|z_2|^2}{2}} e^{z_1 \hat{a}_1^\dagger} e^{z_2 \hat{a}_2^\dagger} |0,0\rangle \,. \tag{8.3.1}$$

These states exist for both the discrete and continuous series, since they both can be constructed using a two mode representation. However, it is difficult to think of this as a minimum uncertainty state for position. In fact, the two z's introduce four real parameters, so there is a redundant unphysical coordinate introduced. Nevertheless it will prove interesting to study symbols defined by these coherent states. We denote the unnormalized states as

$$|z_1, z_2\rangle_U = e^{z_1 \hat{a}_1^\dagger} e^{z_2 \hat{a}_2^\dagger} |0,0\rangle \,. \tag{8.3.2}$$

8.3.1 Discrete series

Using the two mode representation of the discrete series we calculate the two mode symbols of the coordinates as:

$$\begin{aligned}
\langle z_1, z_2| \, \hat{t} \, |z_1, z_2\rangle &= \frac{\theta}{2}\left(|z_1|^2 + |z_2|^2 + 1\right) = t, \\
\langle z_1, z_2| \, \hat{x} \, |z_1, z_2\rangle &= \frac{\theta}{2}\left(z_1 z_2 + z_1^* z_2^*\right) = x, \\
\langle z_1, z_2| \, \hat{y} \, |z_1, z_2\rangle &= \frac{i\theta}{2}\left(z_1 z_2 - z_1^* z_2^*\right) = y, \\
\langle z_1, z_2| \, \hat{\Gamma} \, |z_1, z_2\rangle &= -\frac{\theta^2}{4}\left(|z_1|^2 - |z_2|^2\right).
\end{aligned} \tag{8.3.3}$$

To calculate the expectation value of group elements we note the $\hat{K}_0$ acts like a dilatation operator on the coherent states:

$$\begin{aligned}
&\langle z_1, z_2|\, e^{i\gamma \hat{K}_0}\, |z_1, z_2\rangle \\
&= e^{-|z_1|^2-|z_2|^2} e^{i\gamma}{}_U\langle z_1, z_2|\, e^{i\gamma\left(z_1\partial_{z_1}+z_2\partial_{z_2}\right)}\, |z_1, z_2\rangle_U \\
&= e^{-|z_1|^2-|z_2|^2} e^{i\gamma}{}_U\langle z_1, z_2|(e^{i\gamma}z_1), (e^{i\gamma}z_2)\rangle_U \\
&= e^{i\gamma} e^{(e^{i\gamma}-1)(|z_1|^2+|z_2|^2)}.
\end{aligned} \tag{8.3.4}$$

For the group element:

$$\begin{aligned}
U_k(z_1, z_2) &= \langle z_1, z_2|\, e^{i\frac{1}{2}k_0\hat{t}} e^{-ik_1\hat{x}-ik_2\hat{y}} e^{i\frac{1}{2}k_0\hat{t}}\, |z_1, z_2\rangle \\
&= \langle z_1, z_2|\, e^{Y_-\hat{K_-}} e^{-\log(Y_0)\hat{K_0}} e^{Y_+\hat{K_+}}\, |z_1, z_2\rangle \\
&= \tfrac{1}{Y_0} e^{Y_+z_1^*z_2^*+Y_-z_1z_2+\left(\frac{1}{Y_0}-1\right)\left(|z_1|^2+|z_2|^2\right)} \\
&= N e^{i\kappa^\mu x_\mu},
\end{aligned} \tag{8.3.5}$$

where

$$\begin{aligned}
\kappa^0 &= \theta^{-1}(-1 + e^{\frac{i\theta k_0}{2}}\operatorname{sech}\left(\frac{\theta}{2}\sqrt{k_1^2+k_2^2}\right)), \\
\kappa^1 &= \frac{k_1 e^{\frac{i\theta k_0}{2}} \tanh\left(\frac{\theta}{2}\sqrt{k_1^2+k_2^2}\right)}{\theta\sqrt{k_1^2+k_2^2}}, \\
\kappa^2 &= \frac{k_2 e^{\frac{i\theta k_0}{2}} \tanh\left(\frac{\theta}{2}\sqrt{k_1^2+k_2^2}\right)}{\theta\sqrt{k_1^2+k_2^2}}, \\
N &= e^{-\frac{i\theta\kappa_0}{2}} e^{\frac{i\theta k_0}{2}}\operatorname{sech}\left(\frac{\theta}{2}\sqrt{k_1^2+k_2^2}\right).
\end{aligned} \tag{8.3.6}$$

We used 8.3.3 to deduce

$$\begin{aligned}
\kappa^0 &= \frac{1}{Y_0} - 1, \\
\kappa^1 &= \frac{Y_+ + Y_-}{\theta}, \\
\kappa^2 &= \frac{Y_+ - Y_-}{i\theta}, \\
N &= \frac{e^{-\frac{i\theta\kappa_0}{2}}}{Y_0}.
\end{aligned} \tag{8.3.7}$$

8.3.2 Continuous Series

Using the two mode representation of the continuous series we calculate the two mode symbols of the coordinates as

$$\begin{aligned} \langle z_1, z_2|\, \hat{t}\, |z_1, z_2\rangle &= \frac{\theta}{2}\left(|z_1|^2 - |z_2|^2\right) = t, \\ \langle z_1, z_2|\, \hat{x}\, |z_1, z_2\rangle &= \frac{\theta}{4}\left(z_1^2 + (z_1^*)^2 + z_2^2 + (z_2^*)^2\right) = x, \\ \langle z_1, z_2|\, \hat{y}\, |z_1, z_2\rangle &= \frac{i\theta}{4}\left(-z_1^2 + (z_1^*)^2 + z_2^2 - (z_2^*)^2\right) = y, \\ \langle z_1, z_2|\, \hat{\Gamma}\, |z_1, z_2\rangle &= -\frac{\theta^2}{4}\left(z_1 z_2 + z_1^* z_2^*\right). \end{aligned} \tag{8.3.8}$$

When we compute the symbol of the group element, we see that none of the coordinates mix the different ladder operators, there are no terms that include both $a_1^{(\dagger)}$ and $a_2^{(\dagger)}$. So we can split the symbol by noting $|z_1, z_2\rangle = |z_1\rangle \otimes |z_2\rangle$,

$$\begin{aligned} &\langle z_1, z_2|\, e^{i\frac{1}{2}k_0\hat{t}} e^{-ik_1\hat{x} - ik_2\hat{y}} e^{i\frac{1}{2}k_0\hat{t}}\, |z_1, z_2\rangle = \langle z_1|\, e^{i\frac{p_0}{2}\hat{K}_0} e^{ip\hat{K}_- + ip^*\hat{K}_+} e^{i\frac{p_0}{2}\hat{K}_0}\, |z_1\rangle \times \\ &\langle z_2|\, e^{-i\frac{p_0}{2}\hat{K}_0} e^{ip^*\hat{K}_- + ip\hat{K}_+} e^{-i\frac{p_0}{2}\hat{K}_0}\, |z_2\rangle . \end{aligned} \tag{8.3.9}$$

Where $p = p_1 + ip_2$. The two expectations values above give two exponentials each only having z_1 or z_2. The contain κ's (8.3.6) that have the same form as the discrete series above. However when we try to recombine the exponentials we find, since $\kappa(k_0) \neq -\kappa(-k_0)$, we cannot combine the z's to write this as a function of only the symbols of the coordinates $\langle x^\mu\rangle$. So the way we should map $\hat{\phi} \to \phi(\langle t\rangle, \langle x\rangle, \langle y\rangle)$ is unclear. Thus, using this as our symbol is not viable, since now the correlator would depend on the fourth unphysical coordinate.

8.4 Weyl Symbols

Using the two mode coherent states to cover all space-time seems impossible. In the regions we can cover calculations proved difficult. There are no other obvious states with which we can compute expectations values. It becomes necessary to introduce an even more general map to symbols,

$$\phi(t, x, y) = \mathrm{Tr}\left(\hat{O}(t, x, y)\hat{\phi}\right). \tag{8.4.1}$$

Now the different symbols are defined by choices of $\hat{O}$. The expectation values were the special case where $\hat{O}$ is just an outer product.

One example would be the generalized symbols in section 5.6 then $\hat{O} = V_\delta\, |z_1, z_2\rangle \langle z_1, z_2|$. The Weyl symbol we defined earlier (5.6.1)

$$\psi_0(z_1, z_2) = V_0\psi(z_1, z_2) = e^{-\frac{1}{2}\left(\partial_{z_1}\partial_{z_1^*} + \partial_{z_2}\partial_{z_2^*}\right)}\psi(z_1, z_2), \tag{8.4.2}$$

has the property,

$$\mathrm{Tr}_c(\hat{\phi}\hat{\psi}) = \frac{1}{\pi}\int dz^* dz\ \phi_0\psi_0. \tag{8.4.3}$$

We can then define infinitesimal Lorentz transformations of the symbols as

$$L_\nu\phi_\delta(t,x,y) = V_\delta \left\langle z_1, z_2\right| \hat{L}_\nu\hat{\phi}\left|z_1, z_2\right\rangle \tag{8.4.4}$$

if we calculate this, see appendix C, and write it in terms of t, x, y then the generators look completely standard, for the Weyl symbol. For this reason it would be interesting to calculate the correlator of two Weyl symbols.

To calculate the Weyl symbol, we calculate the effect of V_0 on the two mode symbols. We do this by using a Fourier transform of $U_k(z_1, z_2)$ (8.3.5) in terms of z_1 and z_2. First we write U as

$$U_k(z_1, z_2) = \frac{1}{Y_0}e^{\mathbf{z}^\dagger M_0 \mathbf{z}}, \tag{8.4.5}$$

with (8.3.7)

$$M_0 = \begin{pmatrix} Y_0 & Y_- \\ Y_+ & Y_0 \end{pmatrix}, \qquad \mathbf{z} = \{z_1, z_2^*\}. \tag{8.4.6}$$

The Fourier transform is

$$U_k(z_1, z_2) = \int dk_1 dk_1^* dk_2 dk_2^*\ \frac{\det(M_0)}{Y_0}e^{\mathbf{k}^\dagger M_0^{-1}\mathbf{k}}e^{i\mathbf{k}^\dagger\mathbf{z}+i\mathbf{k}\mathbf{z}^\dagger}. \tag{8.4.7}$$

Then we can apply V_0

$$\begin{aligned} V_0 e^{\mathbf{k}^\dagger M_0^{-1}\mathbf{k}}e^{i\mathbf{k}^\dagger\mathbf{z}+i\mathbf{k}\mathbf{z}^\dagger} &= e^{\mathbf{k}^\dagger\mathbf{k}}e^{\mathbf{k}^\dagger M_0^{-1}\mathbf{k}}e^{i\mathbf{k}^\dagger\mathbf{z}+i\mathbf{k}\mathbf{z}^\dagger} \\ &= e^{\mathbf{k}^\dagger M_v\mathbf{k}}e^{i\mathbf{k}^\dagger\mathbf{z}+i\mathbf{k}\mathbf{z}^\dagger}, \end{aligned} \tag{8.4.8}$$

where

$$M_v = M_0^{-1} + \mathbb{I}_2. \tag{8.4.9}$$

Taking the inverse Fourier transform we obtain the Weyl symbol

$$U_0(k, z_1, z_2) = \frac{\det(M_0)}{\det(M_v)Y_0}e^{\mathbf{z}^\dagger M_v^{-1}\mathbf{z}}. \tag{8.4.10}$$

In the forward light cone $V_0(Ne^{i\kappa^\mu x_\mu}) = Te^{s^\mu x_\mu}$ we obtain:

$$\begin{aligned}
s_0 &= \frac{4\sin\left(\frac{\theta k_0}{2}\right)\cosh\left(\frac{\theta|k|}{2}\right)}{\theta+\theta\cos\left(\frac{\theta k_0}{2}\right)\cosh\left(\frac{\theta|k|}{2}\right)}, \\
s_1 &= \frac{4(k_1)\sinh\left(\frac{\theta|k|}{2}\right)}{|k|\left(\theta+\theta\cos\left(\frac{\theta k_0}{2}\right)\cosh\left(\frac{\theta|k|}{2}\right)\right)}, \\
s_2 &= \frac{4(k_2)\sinh\left(\frac{\theta|k|}{2}\right)}{|k|\left(\theta+\theta\cos\left(\frac{\theta k_0}{2}\right)\cosh\left(\frac{\theta|k|}{2}\right)\right)}, \\
T &= e^{-\frac{is_0}{2}}\frac{2}{1+\cos\left(\frac{\theta k_0}{2}\right)\cosh\left(\frac{\theta|k|}{2}\right)}.
\end{aligned} \tag{8.4.11}$$

We can also compute the Weyl symbol in the space-like region by letting $e^{-\frac{1}{2}\left(\partial_{z_i}\partial_{z_i^*}\right)}$ act on the separate expectation values which depend on only either z_1 or z_2. We find when recombining the exponentials, we obtain the appropriate combinations (8.3.8) of z_1 and z_2 to introduce our space-time coordinates as we did in the time-like region.

We can now calculate the correlator of two Weyl symbols, we define the notation $z(\Delta x^\mu) \equiv \langle\phi_0^*(z)\phi_0(z')\rangle$ then

$$z(\Delta x^\mu) \equiv \int d\mu(k)\; TT^* e^{is_\mu(x-x')^\mu}\left[1-\cos\left(\frac{\theta k_0}{2}\right)\cosh\left(\frac{\theta k}{2}\right)-\frac{\theta^2 m^2}{8}\right]^{-1}, \tag{8.4.12}$$

and if we do a change of variables from the $k \to s$ we find

$$z(\Delta x^\mu) = A(\theta)\int d^3s\ \mathrm{sgn}\left(s^2+\left(\frac{4}{\theta}\right)^2\right)\frac{e^{is^\mu\Delta x_\mu}}{s^2-M(\theta)^2}, \tag{8.4.13}$$

where

$$\begin{aligned}
A(\theta) &= 1-\frac{\theta^2 m^2}{16}, \\
M(\theta) &= \frac{m}{\sqrt{1-\frac{\theta^2 m^2}{16}}}.
\end{aligned} \tag{8.4.14}$$

This was done in *Mathematica* since this is an intricate calculation, by writing the dispersion relation and the T in terms of s. We note the sgn function is left over from the Jacobian

We have found a correlator that almost matches the commutative result except for the sign function. There is also a different normalization A. We also see a modified mass which has a critical point $m = \frac{4}{\theta}$.

To calculate the integral (8.4.13) we go to polar coordinates for the spacial s. Then we can use hyperbolic coordinates for the magnitude of spacial s and s^0, however, we will need different hyperbolic coordinates to cover the time-like and space-like s regions so we have to split the integral domain. We will also consider time-like and space-like separated events individually, this simplifies the calculation. We also drop the arguments of A and M to simplify notation.

8.4.1 Time-Like Separated Events

We can simplify the exponential $e^{is^\mu \Delta x_\mu}$ by performing a Lorentz transformation on the coordinates, this will allow us to transform some of the momenta away. For time-like separated events we can perform a Lorentz transformation $e^{is^\mu \Delta x_\mu} \to e^{is^0 \Delta t}$. Now we can start calculating the different parts of the integral.

8.4.1.1 Time-like s

We have to compute the forward and backward cones separately. For the forward cone we introduce the hyperbolic coordinates:

$$\begin{aligned} s^0 &= q\cosh(\alpha), \\ s &= q\sinh(\alpha), \\ ds^0\, sds &= q^2\sinh(\alpha)\, dqd\alpha, \\ \operatorname{sgn}\left(s^2 + \left(\frac{4}{\theta}\right)^2\right) &= \operatorname{sgn}\left(q^2 + \left(\frac{4}{\theta}\right)^2\right). \end{aligned} \tag{8.4.15}$$

The sign is always positive. The propagator is then

$$\begin{aligned} z(\Delta x^\mu)_{\text{forward cone}} &= A\int_0^\infty dq \int_0^\infty d\alpha\, \frac{q^2}{q^2 - M^2}\sinh(\alpha) e^{iq\cosh(\alpha)\Delta t} \\ &= A\int_0^\infty dq\, \frac{q^2}{q^2 - M^2}\int_1^\infty du\, e^{iqu\Delta t} \\ &= A\int_0^\infty dq\, \frac{q^2}{q^2 - M^2} e^{iq\Delta t}\frac{1}{q\Delta t}. \end{aligned} \tag{8.4.16}$$

For the backward cone $s^0 = -q\cosh(\alpha)$ then the propagator is

$$z(\Delta x^\mu)_{\text{backward cone}} = A\int_0^\infty dq\, \frac{q^2}{q^2 - M^2} e^{-iq\Delta t}\frac{-1}{q\Delta t}. \tag{8.4.17}$$

We can combine the forward and backward cones. If we want to perform the q integral we will need to introduce an ϵ prescription. Then the time-like s contribution to the correlator is

$$\begin{aligned} z(\Delta x^\mu)_{\text{time-like } s} &= \frac{A}{\Delta t}\int_{-\infty}^\infty dq\, \frac{q}{q^2 - M^2 + i\epsilon} e^{iq\Delta t} \\ &= A\pi \frac{e^{iM\Delta t}}{\Delta t}. \end{aligned} \tag{8.4.18}$$

8.4.1.2 Space-like s

The set of hyperbolic coordinates we need for space-like s is:

$$\begin{aligned} s^0 &= q\sinh(\alpha), \\ s &= q\cosh(\alpha), \\ ds^0\, sds &= q^2\cosh(\alpha)\, dqd\alpha, \\ \mathrm{sgn}\left(s^2+\left(\frac{4}{\theta}\right)^2\right) &= \mathrm{sgn}\left(-q^2+\left(\frac{4}{\theta}\right)^2\right). \end{aligned} \tag{8.4.19}$$

The integral becomes

$$\begin{aligned} & z(\Delta x^\mu)_{\text{space-like}\,s} = A\int dq\ \mathrm{sgn}\left(-q^2+\left(\frac{4}{\theta}\right)^2\right) \\ & \int_{-\infty}^{\infty} d\alpha\ \frac{-q^2}{q^2+M^2}\cosh(\alpha)e^{iq\sinh(\alpha)\Delta t} \\ = & -A\int dq\ \mathrm{sgn}\left(-q^2+\left(\frac{4}{\theta}\right)^2\right)\int_{-\infty}^{\infty} du\ \frac{q^2}{q^2+M^2}e^{iqu\Delta t} \\ = & -A\int dq\ \mathrm{sgn}\left(-q^2+\left(\frac{4}{\theta}\right)^2\right)\frac{q^2}{q^2+M^2}\delta(q\Delta t). \end{aligned} \tag{8.4.20}$$

To do the q integral, we note that when $q > \frac{4}{\theta}$ the sign function is negative. Therefore we can separate the q integral into $\int_0^{\frac{4}{\theta}} - \int_{\frac{4}{\theta}}^{\infty}$. We can also instead write this integral as $\int_0^{\infty} - 2\int_{\frac{4}{\theta}}^{\infty}$, this way we can separate the integral into implicitly θ dependent terms, where the θ dependence only sits in the A and M and explicitly θ dependent terms. However both the explicitly and implicitly θ dependent integrals are always zero for space-like s, because of the delta function, and thus make no contribution to the correlator.

So for time-like separated events the only contribution comes from the time-like s,

$$z(\Delta x^\mu) = z(\Delta x^\mu)_{\text{time-like}\,s} = A\pi\frac{e^{iM\Delta t}}{\Delta t}. \tag{8.4.21}$$

It has the same form as commutatively, except with a modified $M(\theta)$ and the prefactor $A(\theta)$.

8.4.2 Space-like Separated Events

For space like events we can perform a Lorentz transformation $e^{is^\mu\Delta x_\mu} \to e^{is^1\Delta x + is^2\Delta y}$. We go to polar coordinates for the spacial s, the angular integral gives $J_0(s\Delta r)$, where J_0 is a Bessel function.

8.4.2.1 Time-like s

The forward and backward cones contribute the same, since the exponential does not depend on s^0, then the hyperbolic coordinates we use are given by:

$$\begin{aligned} s^0 &= \pm q\cosh(\alpha), \\ s &= q\sinh(\alpha), \\ ds^0\, sds &= q^2\sinh(\alpha)\, dqd\alpha, \\ \mathrm{sgn}\left(s^2+\left(\frac{4}{\theta}\right)^2\right) &= \mathrm{sgn}\left(q^2+\left(\frac{4}{\theta}\right)^2\right). \end{aligned} \tag{8.4.22}$$

The propagator is then

$$\begin{aligned} z(\Delta x^\mu)_{\text{time-like } s} &= 2A\int_0^\infty dq \int_0^\infty d\alpha\, \frac{q^2}{q^2-M^2}\sinh(\alpha)J_0(q\sinh(\alpha)\Delta r) \\ &= 2A\int_0^\infty dq \int_0^\infty du\, \frac{q^2}{q^2-M^2}\frac{u}{\sqrt{u^2+1}}J_0(qu\Delta r) \\ &= 2A\int_0^\infty dq\, \frac{q^2}{q^2-M^2+i\epsilon}\frac{e^{-q\Delta r}}{q\Delta r} \\ &= 2A\int_0^\infty dq\, \frac{q}{q^2+M^2}\frac{e^{-iq\Delta r}}{\Delta r}. \end{aligned} \tag{8.4.23}$$

In the last line we rotated q to $-iq$ this will be more convenient later.

8.4.2.2 Space-like s

The set of hyperbolic coordinates we need for space-like s is:

$$\begin{aligned} s^0 &= q\sinh(\alpha), \\ s &= q\cosh(\alpha), \\ ds^0\, sds &= q^2\cosh(\alpha)\, dqd\alpha, \\ \mathrm{sgn}\left(s^2+\left(\frac{4}{\theta}\right)^2\right) &= \mathrm{sgn}\left(-q^2+\left(\frac{4}{\theta}\right)^2\right). \end{aligned} \tag{8.4.24}$$

The propagator is then

$$\begin{aligned} z(\Delta x^\mu)_{\text{space-like } s} &= A\int dq \int_{-\infty}^\infty d\alpha\, \frac{-q^2}{q^2+M^2}\cosh(\alpha)J_0(q\cosh(\alpha)\Delta r) \\ &= -2A\int dq \int_1^\infty du\, \frac{q^2}{q^2+M^2}\frac{u}{\sqrt{u^2-1}}J_0(qu\Delta r) \\ &= -2A\int dq\, \frac{q^2}{q^2+M^2}\frac{\cos(q\Delta r)}{q\Delta r}. \end{aligned} \tag{8.4.25}$$

If we combine the time-like and space-like parts that are explicitly θ independent,

$$\begin{aligned} &2A \int_0^\infty dq \, \tfrac{q}{q^2+M^2} \left(\tfrac{e^{-iq\Delta r}}{\Delta r} - \tfrac{\cos(q\Delta r)}{\Delta r} \right) \\ &= -2A \int_0^\infty dq \, \tfrac{q}{q^2+M^2} \tfrac{\sin(q\Delta r)}{\Delta r} \\ &= -A\pi \tfrac{e^{-M\Delta r}}{\Delta r}, \end{aligned} \tag{8.4.26}$$

this has the same form as the commutative result.

If we include the remaining θ explicit term, the correlator for space-like separated events is given by

$$z(\Delta x^\mu) = -A\pi \frac{e^{-M\Delta r}}{\Delta r} + 4A \int_{\frac{4}{\theta}}^\infty dq \, \frac{q}{q^2+M^2} \frac{\cos(q\Delta r)}{\Delta r}, \tag{8.4.27}$$

the cos integral is a special function which for large Δr tends to zero, and the first term matches the commutative result except with a modified $M(\theta)$ and the prefactor $A(\theta)$. So, what we have found for the double z symbols is that the correlators have the same form as commutatively, except for space-like separate events where a special functions is added. Unfortunately this special function does not remove any divergences. It does however vanish for large length scales so we have the same long length scale physics as commutative field theories.

8.5 Chapter Summary

The purpose of this chapter was to investigate the different ways to relate operators to scalar fields. We needed a map from the operators to scalar fields, because these are the quantities we can hopefully measure. We called these scalar fields the symbols of the operators. The two successful symbols were the SU$(1,1)$ coherent state symbols and the Weyl symbols. The former works well in the forward and backward interior light cones. It has the right commutative limits. Numerically we can see that it removes the divergences of correlators at the same point in space-time. However defining symbols for the rest of space-time using these states was not successful, however, future investigations into this problem could be warranted. The Weyl symbols have the property that Lorentz transformations have the standard form. We successfully calculated the symbol in all the regions of space-time, they had the right commutative limits, however, the divergences remain and it is not obvious how the minimum length scales manifests itself.

Further investigation into these and other symbols could be a fruitful area of study, as the underlying SU$(1,1)$ field theory is interesting on a purely mathematical level, but could also be potentially physically relevant. This is the important message from this chapter even though we can have a well-defined operator valued theory, it is still necessary to have some way to relate these theories to scalar quantities, as these are the things we can measure. As the ideas of non-commutativity become more popular, $\mathrm{SU}(1,1)$ field theories should not be overlooked. For this thesis however we stop at this point and leave the rest for future work.

Chapter 9

Conclusion and Outlook

In this thesis we investigated two possible approaches to developing non-commutative field theories, with emphasis on preserving Lorentz invariance in some way. We now briefly summarize what we achieved for each approach and give the potential outlook for both approaches.

The first approach we followed was the idea of ERG dualities. With the ERG dualities, the Lorentz symmetry might not be manifest but must exist by virtue of the duality. We began with a review of two dimensional non-commutative quantum mechanics [5]. First we built a duality using a coordinate transform namely, the Bopp shift. In fact we used a more general Bopp shift than is usually encountered in the literature. We did this as a way to benchmark our ERG dualities. We then successfully built an ERG duality for the complex non-canonical scalar field. Using this duality we were able to track the fate of the Lorentz symmetry in the non-canonical theory, for the free theory. Following this an interacting duality was built for a ϕ^4 theory. This was a systematic way of constructing interactions in the dual theory, the apparent lack of Lorentz symmetry makes this difficult to do consistently for non-canonical theories [23]. The work in this part was accepted for publication in [46].

The benefit of these dualities is not immediately obvious, particularly for the interacting theory which has some unconventional source terms. Further exploration of these dualities may shed some light on the meaning of these theories and the advantages they may offer. Higher spin theories could be studied, a Dirac equivalent of the non-canonical theories is currently lacking in literature. Finally, investigations of ERG dualities could move beyond the idea of purely commutative/non-commutative dualities.

The second approach built a $\mathrm{SU}(1,1)$ fuzzy space-time, we then constructed field theories, in $2+1$ dimensions, living on this space-time. The Lorentz symmetry is then manifest by construction since the commutation relations are Lorentz invariant. We began with a review of $\mathrm{SU}(2)$ fuzzy space, here quantum mechanics is built in three dimensions that preserves rotational symmetry [25]. Also briefly we mentioned the idea of scalar fields living on this space, they however do not respect Lorentz symmetry due to time being commutative. Using this as a base, we reviewed the properties of the $\mathrm{SU}(1,1)$ group and used this to build our fuzzy space-time successfully. On this space-time we introduced the notion of states, observables and built a relativistic quantum mechanical theory. We then introduced dynamics, by using tensor operators we wrote down a non-commutative Klein-Gordon equation. Using this we built a scalar field theory by using the functions on the group manifold as our fields, and calculated correlators for the free theory. Finally, we considered different ways to attach scalar space-time labels to our operators, since we needed to make contact with measurable quantities. To do this we can introduced maps from our operators to scalar fields called symbols, we then calculated the correlators of these symbols. We studied two different symbols which both have their own advantages and disadvantages. The first is the $\mathrm{SU}(1,1)$ coherent state symbols. These can be thought of as minimum uncertainty states, and they remove divergences present in commutative theories when computing correlators. However, their symbols are only easily defined in the forward and backward light cones. The second was the two mode coherent states symbols. Actually we built a family of symbols based on the two mode coherent states. The most promising was the Weyl symbol, it is easily defined for all of space-time and Lorentz transformations take a completely standard form. For the correlators calculated, however, it is not obvious how the non-commutative length scale manifests itself physically. It is also not clear if it is possible to think of these as minimum uncertainty states, as the physical interpretation of the coordinates is currently not obvious.

Even though $\mathrm{SU}(1,1)$ theory is consistent and offers some interesting results even on just the operator level, further work is necessary to determine which symbol is the best candidate to represent the physical fields we measure in experiments. This is not necessarily one of the symbols discussed in this thesis and future work could investigate other kinds of symbols. Other potentially interesting topics that could be investigate would be adding interactions into our $\mathrm{SU}(1,1)$ scalar field theory. One could also try to extend this to a spinor field theory, by trying to write the analogue of Dirac equation. Finally it would be intriguing to extend these theories to $3+1$ dimensions, this would involve the $SO(3,1)$ group, here there are more generators than coordinates operators. This would provide an interesting challenge, although some attempts at a Snyder space construction do exist in literature [30]. We would seek a consistent theory built from the operator level up.

In this thesis we have successfully illustrated two approaches to constructing non-commutative field theories. If we want to combine gravity and quantum mechanics into a unified theory we need to revise the notion of space-time at short length scales. Since we have strong arguments in favour of a non-commutative geometry, the development non-commutative field theories is essential.

Appendices

Appendix A

Dispersion Relation

If we consider generally a unitary group transformation of a tensor operator

$$\hat{D}(R)\hat{T}_{\pm}\hat{D}^{\dagger}(R) = D^{\frac{1}{2}}_{q,\pm}(R)\hat{T}_q, \tag{A.0.1}$$

then we can calculate the action of a scalar operator on a group element $\left(\mathrm{ad}_{\hat{T}_+}\,\mathrm{ad}_{\hat{\tilde{T}}_-} - \mathrm{ad}_{\hat{T}_-}\,\mathrm{ad}_{\hat{\tilde{T}}_+}\right)\hat{D}(R)$. We start with

$$\begin{aligned}\mathrm{ad}_{\hat{T}_{\pm}}\,\hat{D}(R) &= \hat{T}_{\pm}\hat{D}(R) - \hat{D}(R)\hat{T}_q(\hat{D}^{\dagger}(R)\hat{D}(R)) \\ &= [\hat{T}_{\pm} - D^{\frac{1}{2}}_{q,\pm}(R)\hat{T}_q]\hat{D}(R),\end{aligned} \tag{A.0.2}$$

similarly for the $\hat{\tilde{T}}$. These $\hat{D}$ are the group elements of some group, the notation we used for SU(2) and SU(1,1) was $\hat{U}(k)$, and $D^{\frac{1}{2}}_{q,\pm}(R)$ are the group elements in the $j = \frac{1}{2}$ representation since the $\hat{T}$ are spinors or rank $\frac{1}{2}$ tensor operators. We can now calculate the action of the scalar operator, starting with the first term

$$\begin{aligned}&\mathrm{ad}_{\hat{T}_+}\,\mathrm{ad}_{\hat{\tilde{T}}_-}\,\hat{D} = \mathrm{ad}_{\hat{T}_+}[\hat{\tilde{T}}_- - D^{\frac{1}{2}}_{q,-}\hat{\tilde{T}}_q]\hat{D} \\ &= [-1 + D^{\frac{1}{2}}_{-,-}]\hat{D} + [\hat{\tilde{T}}_- - D^{\frac{1}{2}}_{q,-}\hat{\tilde{T}}_q][\hat{T}_+ - D^{\frac{1}{2}}_{q',+}\hat{T}_{q'}]\hat{D},\end{aligned} \tag{A.0.3}$$

and then the second

$$\begin{aligned}&\mathrm{ad}_{\hat{T}_-}\,\mathrm{ad}_{\hat{\tilde{T}}_+}\,\hat{D} = [\hat{\tilde{T}}_+ - D^{\frac{1}{2}}_{q,+}\hat{\tilde{T}}_q]\hat{D} \\ &= [1 - D^{\frac{1}{2}}_{+,+}]\hat{D} + [\hat{\tilde{T}}_+ - D^{\frac{1}{2}}_{q,+}\hat{\tilde{T}}_q][\hat{T}_- - D^{\frac{1}{2}}_{q',-}\hat{T}_{q'}]\hat{D}.\end{aligned} \tag{A.0.4}$$

To simplify notation we introduce

$$M_{q,q'} = \delta_{q,q'} - D^{\frac{1}{2}}_{q,q'}. \tag{A.0.5}$$

Then we can write (A.0.3) as

$$(-M_{-,-} + M_{q,-}M_{q',+}\hat{\tilde{T}}_q\hat{T}_{q'})\hat{D}, \tag{A.0.6}$$

and (A.0.4) as

$$(M_{+,+} + M_{q,+}M_{q',-}\hat{\tilde{T}}_q\hat{T}_{q'})\hat{D}. \tag{A.0.7}$$

Then we can write (A.0.3) − (A.0.4)

$$\begin{aligned}
&(-M_{-,-} - M_{+,+} + (M_{q,-}M_{q',+} - M_{q,+}M_{q',-})\hat{\tilde{T}}_q\hat{T}_{q'})\hat{D} \\
&= (-M_{-,-} - M_{+,+} + (M_{+,-}M_{-,+} - M_{+,+}M_{-,-})\hat{\tilde{T}}_+\hat{T}_-)\hat{D} \\
&+((M_{-,-}M_{+,+} - M_{-,+}M_{+,-})\hat{\tilde{T}}_-\hat{T}_+)\hat{D}.
\end{aligned} \tag{A.0.8}$$

After simplification and noting the determinant is $(D^{\frac{1}{2}}_{+,+}D^{\frac{1}{2}}_{-,-} - D^{\frac{1}{2}}_{+,-}D^{\frac{1}{2}}_{-,+}) = 1$ we obtain

$$\begin{aligned}
&(-2 + D^{\frac{1}{2}}_{-,-} + D^{\frac{1}{2}}_{+,+} + (-2 + D^{\frac{1}{2}}_{+,+} + D^{\frac{1}{2}}_{-,-})\hat{\tilde{T}}_+\hat{T}_-)\hat{D} \\
&+((2 - D^{\frac{1}{2}}_{+,+} - D^{\frac{1}{2}}_{-,-})\hat{\tilde{T}}_-\hat{T}_+)\hat{D} \\
&= (D^{\frac{1}{2}}_{+,+} + D^{\frac{1}{2}}_{-,-} - 2)(\hat{\tilde{T}}_+\hat{T}_- - \hat{\tilde{T}}_-\hat{T}_+ + 1)\hat{D}.
\end{aligned} \tag{A.0.9}$$

this is a completely general answer which only depends on the fact that the the operators transform as tensors, and not which unitary group our scalar operator acts on.

For SU(2) the creation and annihilation operators transform as spinors operators under SU(2). We can write

$$\begin{pmatrix} \hat{T}_+ \\ \hat{T}_- \end{pmatrix} = \begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}, \quad \begin{pmatrix} \hat{\tilde{T}}_+ \\ \hat{\tilde{T}}_- \end{pmatrix} = \begin{pmatrix} -\hat{a}_2 \\ \hat{a}_1 \end{pmatrix}. \tag{A.0.10}$$

We wrote the Laplacian as

$$-\frac{2}{\theta\hat{r}}[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \hat{\psi}]] = -\frac{2}{\theta\hat{r}}\left(\mathrm{ad}_{\hat{T}_+}\,\mathrm{ad}_{\hat{\tilde{T}}_-} - \mathrm{ad}_{\hat{T}_-}\,\mathrm{ad}_{\hat{\tilde{T}}_+}\right)\hat{\psi}. \tag{A.0.11}$$

We note that we can write $\hat{r}$ as

$$\begin{aligned}
\hat{r} &= \frac{\theta}{2}(\hat{a}_\alpha^\dagger\hat{a}_\alpha + 1) \\
&= -\frac{\theta}{2}(\hat{\tilde{T}}_+\hat{T}_- - \hat{\tilde{T}}_-\hat{T}_+ + 1).
\end{aligned} \tag{A.0.12}$$

Then we have the Laplacian on a SU(2) group element

$$\hat{\nabla}\hat{D} = \frac{4}{\theta^2}[\mathrm{Tr}(D^{\frac{1}{2}}) - 2]\hat{D}. \tag{A.0.13}$$

We note for the different SU(2) parameterizations we have for $\mathrm{Tr}(D^{\frac{1}{2}})$

$$\mathrm{Tr}(e^{i\phi\hat{J}_0}e^{i\alpha\hat{J}_2}e^{i\psi\hat{J}_0}) = 2\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\phi+\psi}{2}\right) \tag{A.0.14}$$

and

$$\mathrm{Tr}(e^{p\cdot\hat{J}}) = 2\cos\left(\frac{|p|}{2}\right). \tag{A.0.15}$$

If we insert (A.0.15) into the Laplacian we obtain (5.4.3)

$$\hat{\nabla}e^{ik\cdot\hat{x}} = -\frac{16}{\theta^2}\left(\sin^2\left(\frac{|k|\theta}{4}\right)\right)e^{ik\cdot\hat{x}}. \tag{A.0.16}$$

For SU$(1,1)$ we wrote the d'Alembertian as

$$\hat{\Box}\hat{\psi} = \frac{2}{\theta^2\hat{\Gamma}}(\mathrm{ad}_{\hat{T}_+}\,\mathrm{ad}_{\hat{\hat{T}}_-} - \mathrm{ad}_{\hat{T}_-}\,\mathrm{ad}_{\hat{\hat{T}}_+})\hat{\psi}, \tag{A.0.17}$$

with

$$\hat{\Gamma} \;=\; -\frac{1}{2}(\hat{\hat{T}}_+\hat{T}_- - \hat{\hat{T}}_-\hat{T}_+ + 1), \tag{A.0.18}$$

and then we have again

$$\hat{\Box}\hat{D} = \frac{4}{\theta^2}[\mathrm{Tr}(D^{\frac{1}{2}}) - 2]\hat{D}. \tag{A.0.19}$$

Then $\mathrm{Tr}(D^{\frac{1}{2}})$ for our SU$(1,1)$ group elements is

$$\mathrm{Tr}(e^{-i\frac{1}{2}p_0\hat{K}_0}e^{ip_1\hat{K}_1+ip_2\hat{K}_2}e^{-i\frac{1}{2}p_0\hat{K}_0}) = 2\cos\left(\frac{p_0}{2}\right)\cosh\left(\frac{|p|}{2}\right), \tag{A.0.20}$$

and we obtain the dispersion relation

$$\frac{8}{\theta^2}\left(1-\cos\left(\frac{\theta k_0}{2}\right)\cosh\left(\frac{\theta k}{2}\right)\right)e^{-i\frac{1}{2}p_0\hat{K}_0}e^{ip_1\hat{K}_1+ip_2\hat{K}_2}e^{-i\frac{1}{2}p_0\hat{K}_0} = \\ \hat{\Box}e^{-i\frac{1}{2}p_0\hat{K}_0}e^{ip_1\hat{K}_1+ip_2\hat{K}_2}e^{-i\frac{1}{2}p_0\hat{K}_0}. \tag{A.0.21}$$

We see the dispersion relation doesn't depend on which particular representation we choose, as long as the representation can be built out of tensor operators.

Appendix B

Fourier Analysis on the Group Manifold

We want to illustrate the difficulties in finding a representation and corresponding $\hat{\Gamma}$ factor, such that

$$\mathrm{Tr}(\hat{U}^{\dagger}(p')\hat{\Gamma}\hat{U}(p)) = \delta_{\mu}(p'-p), \tag{B.0.1}$$

where this is a Dirac delta with respect to the group measure.

Let us first consider this for SU(2) since in this case we actually know the correct representation. Recalling that we can write any group element as

$$\hat{\phi} = \int d\mu(p)\ \phi(p)\hat{U}(p). \tag{B.0.2}$$

Now $\phi(p)$ is a function on the group manifold. It depends on coordinates p which parametrize group manifold i.e. for every set of coordinates p is associated a group element. We can then expand the $\phi(p)$ in terms of matrix elements

$$\phi(p) = \sum_{j}(2j+1)\sum_{m,n} c^{j}_{m,n}[\mathcal{P}^{j}_{m,n}(p)]^{*}, \tag{B.0.3}$$

where the expansions coefficients are given by

$$\begin{aligned} c^{j}_{m,n} &= \int d\mu(p)\ \phi(p)\mathcal{P}^{j}_{m,n}(p) \\ &= \langle j,m|\,\hat{\phi}\,|j,n\rangle\,, \end{aligned} \tag{B.0.4}$$

noting the $\mathcal{P}$ are SU(2) matrix elements

$$\mathcal{P}^{j}_{m,n}(p) = \langle j,m|\,\hat{U}(p)\,|j,n\rangle\,. \tag{B.0.5}$$

The factor $2j+1$ arises since

$$\int d\mu(p)\ [\mathcal{P}^{j}_{m,n}(p)]^{*}\mathcal{P}^{j'}_{m',n'}(p) = \frac{\delta_{m,m'}\delta_{n,n'}\delta_{j,j'}}{2j+1}. \tag{B.0.6}$$

This sum (B.0.3) can be written instead as a trace over the SU(2) configuration space described in the thesis

$$\begin{aligned}\phi(p) &= \mathrm{Tr}_c[\hat{U}^\dagger(p)\hat{r}\hat{\phi}] \\ &= \int d\mu(p')\ \phi(p')\,\mathrm{Tr}_c[\hat{U}^\dagger(p)\hat{r}\hat{U}(p')],\end{aligned} \tag{B.0.7}$$

recalling

$$\hat{r} = \frac{\theta}{2}(\hat{a}^\dagger_\alpha \hat{a}_\alpha + 1). \tag{B.0.8}$$

Now we can see this clearly implies

$$\mathrm{Tr}_c[U^\dagger(p)\hat{r}\hat{U}(p')] = \delta_\mu(p - p'). \tag{B.0.9}$$

Here the $\hat{r}$ compensates exactly for the factor $2j+1$ in the expansion for $\phi(p)$. For SU(2) we see that with the trace over the usual configuration space we get the desired result.

However for SU(1,1) the decomposition of functions on the group manifold is given in [56], using (6.3.19) and (6.3.20),

$$\begin{aligned}\phi(p) = \sum_k (2k-1) \sum_{\sigma=\pm 1} \sum_{m,n}{}^{(k,\sigma)} c^{(k,\sigma)}_{m,n} [\mathcal{P}^{(k,\sigma)}_{m,n}(p)]^* \\ + \sum_{\epsilon=0,\frac{1}{2}} \sum_{m,n}{}^{(\epsilon)} \int_0^\infty d\rho\ 2\rho \tanh(\pi(\rho + i\epsilon)) c^{(\frac{1}{2}+i\rho,\epsilon)}_{m,n} [\mathcal{P}^{(\frac{1}{2}+i\rho,\epsilon)}_{m,n}(p)]^*.\end{aligned} \tag{B.0.10}$$

Now it is unclear how to choose the configuration space and $\hat{\Gamma}$ so that

$$\mathrm{Tr}_c[U^\dagger(p)\hat{\Gamma}\hat{U}(p')] = \delta_\mu(p - p'). \tag{B.0.11}$$

Appendix C

Lorentz Transfomation for Weyl Symbols

We write the Lorentz generators for symbols as

$$L_\nu \phi_\delta(z_1, z_2) = V_\delta \langle z_1, z_2 | \, \hat{L}_\nu \hat{\phi} \, | z_1, z_2 \rangle \, . \tag{C.0.1}$$

For the lower symbol $\delta = -1$ in terms of the z we can write the Lorentz generators as

$$\begin{aligned} L_0 &= \frac{1}{2}\left((z_1^* \partial_{z_1^*} - z_1 \partial_{z_1}) + (z_2^* \partial_{z_2^*} - z_2 \partial_{z_2}) \right), \\ L_1 &= \frac{1}{2}\left((z_2^* \partial_{z_1^*} - z_2 \partial_{z_1}) + (z_1^* \partial_{z_2^*} - z_1 \partial_{z_2}) + (\partial_{z_1^*} \partial_{z_2^*} - \partial_{z_1} \partial_{z_2}) \right), \\ L_2 &= \frac{1}{2}\left((z_2^* \partial_{z_1^*} + z_2 \partial_{z_1}) + (z_1^* \partial_{z_2^*} + z_1 \partial_{z_2}) + (\partial_{z_1^*} \partial_{z_2^*} + \partial_{z_1} \partial_{z_2}) \right). \end{aligned} \tag{C.0.2}$$

If we introduce the t, x, y we see that, L_0 already has the standard form, so for lower symbols we have standard rotations.

For the Weyl symbol $\delta = 0$ in terms of z

$$\begin{aligned} L_0 &= \frac{1}{2}\left((z_1^* \partial_{z_1^*} - z_1 \partial_{z_1}) + (z_2^* \partial_{z_2^*} - z_2 \partial_{z_2}) \right), \\ L_1 &= \frac{1}{2}\left((z_2^* \partial_{z_1^*} - z_2 \partial_{z_1}) + (z_1^* \partial_{z_2^*} - z_1 \partial_{z_2}) \right), \\ L_2 &= \frac{1}{2}\left((z_2^* \partial_{z_1^*} + z_2 \partial_{z_1}) + (z_1^* \partial_{z_2^*} + z_1 \partial_{z_2}) \right). \end{aligned} \tag{C.0.3}$$

we see that L_0 doesn't change and in terms of t, x, y

$$\begin{aligned} L_0 &= -i(x\partial_y - y\partial_x), \\ L_1 &= -i(t\partial_y - y\partial_t), \\ L_2 &= -i(x\partial_t - t\partial_x). \end{aligned} \tag{C.0.4}$$

now all the generators have the standard form. So for the Weyl symbols Lorentz transformations behave in the standard commutative way. Also importantly this only depends on t, x, y and not the fourth unphysical coordinate, this is not necessarily true for the other symbols.

Appendix D

Generalised Flow

We want $Z[J] = \frac{W_\ell[J]}{W_\ell[0]}$ to be invariant under the flow, i.e. independent of the flow parameter ℓ, where

$$W_\ell[J] = \int [d\phi][d\phi^*] \; e^{-(S_0+S_I+J)} \tag{D.0.1}$$

and in four dimensions

$$S_0 = \int d^4p \; \phi^*(p)\phi(p)K(p,\ell). \tag{D.0.2}$$

However, we relax the usual conditions imposed on $K(p,\ell)$ and the sources [43] and this necessitates the flow of the source terms as well. We will make use of the ansatz

$$\partial_\ell e^{-S_I} = -\int d^4p \; \partial_\ell K^{-1} \frac{\delta^2 e^{-S_I}}{\delta\phi^*(p)\delta\phi(p)}, \tag{D.0.3}$$

$$\partial_\ell e^{-(S_I+J)} = -\int d^4p \; \partial_\ell K^{-1} \frac{\delta^2 e^{-(S_I+J)}}{\delta\phi^*(p)\delta\phi(p)}, \tag{D.0.4}$$

which we will verify by calculating $\partial_\ell Z[J]$ directly. To do so we use

$$\begin{aligned} \partial_\ell W_\ell[J] = &-\int [d\phi][d\phi^*] \; e^{-S_0} \int d^4p \\ &\left[\phi^*(p)\phi(p)e^{-(S_I+J)}\partial_\ell K + \partial_\ell K^{-1} \frac{\delta^2 e^{-(S_I+J)}}{\delta\phi^*(p)\delta\phi(p)}\right] \end{aligned} \tag{D.0.5}$$

which, by partial integration, can be rewritten as

$$\begin{aligned} \partial_\ell W_\ell[J] &= -\int [d\phi][d\phi^*] \; e^{-(S_0+S_I+J)} \int d^4p \; [\phi^*(p)\phi(p)\partial_\ell K \\ &+\partial_\ell K^{-1}\left(-K\delta^4(0) + K^2\phi^*(p)\phi(p)\right)] \\ &= \int [d\phi][d\phi^*] \; e^{-(S_0+S_I+J)} \left[\int d^4p \; \left(\partial_\ell K^{-1}\right) K\delta^4(0)\right]. \end{aligned} \tag{D.0.6}$$

If we set $\Gamma = \int d^4p\ (\partial_\ell K^{-1})\, K\delta^4(0)$, this factor of the space-time volume corresponds to the renormalization of the vacuum energy density from modes that are dropped if we lower ℓ [43], then the result above implies that

$$\partial_\ell Z[J] = \frac{\Gamma W_\ell[J]}{W_\ell[0]} - \frac{W_\ell[J]}{W_\ell[0]^2}\Gamma W_\ell[0] = 0, \tag{D.0.7}$$

as desired.

Returning to (D.0.3), we see that taking the derivatives and dividing by $-e^{-S_I}$ produces

$$\partial_\ell S_I = -\int d^4p\ \partial_\ell K^{-1} \left\{ \frac{\delta S_I}{\delta\phi^*(p)} \frac{\delta S_I}{\delta\phi(p)} - \frac{\delta^2 S_I}{\delta\phi^*(p)\delta\phi(p)} \right\}, \tag{D.0.8}$$

which is the desired flow for the action. The same procedure applied to (D.0.4) produces

$$\partial_\ell(S_I + J) = \int d^4p\ \partial_\ell K^{-1} \left\{ \frac{\delta(S_I + J)}{\delta\phi^*(p)} \frac{\delta(S_I + J)}{\delta\phi(p)} - \frac{\delta^2(S_I + J)}{\delta\phi^*(p)\delta\phi(p)} \right\}, \tag{D.0.9}$$

which, combined with (D.0.8), produces the flow equation for the sources

$$\begin{aligned}\partial_\ell J = \int d^4p\ \partial_\ell K^{-1} \Bigg\{ & \frac{\delta S_I}{\delta\phi(p)} \frac{\delta J}{\delta\phi^*(p)} + \frac{\delta S_I}{\delta\phi^*(p)} \frac{\delta J}{\delta\phi(p)} \\ & + \frac{\delta J}{\delta\phi^*(p)} \frac{\delta J}{\delta\phi(p)} - \frac{\delta^2 J}{\delta\phi^*(p)\delta\phi(p)} \Bigg\}.\end{aligned} \tag{D.0.10}$$

We have shown that our generalized flow equations (D.0.9) and (D.0.10) do leave the generating function invariant i.e. independent of the flow parameter ℓ.

Appendix E

Flow of the Sources for the Interacting Complex Scalar Theory

We will treat the sources to linear order in the coupling, and write

$$J[\phi,\phi^*] = J^{(0)}[\phi,\phi^*] + \lambda J^{(1)}[\phi,\phi^*]. \tag{E.0.1}$$

To the same order the action is given by eq. (4.3.13) as

$$\begin{aligned} S_I = &\int d^4k \; \left(g_2^{(0)}(k) + \lambda g_2^{(1)}(k)\right) \phi^*(k)\phi(k) \\ &+ \int d^4k^1\, d^4k^2\, d^4k^3\, d^4k^4\, \lambda\, g_4^{(1)}(k^1,k^2,k^3,k^4,\theta) \\ &\qquad \times \phi^*(k^1)\phi^*(k^2)\phi(k^3)\phi(k^4). \end{aligned} \tag{E.0.2}$$

For the zeroth order term in λ we introduce the ansatz

$$J^{(0)}[\phi,\phi^*] = \int d^4k \left[J_0^{(0)}(k) + J_1^{(0)}(k)\phi^*(k) + c.c.\right], \tag{E.0.3}$$

while for the first order term we will use

$$\begin{aligned} J^{(1)}[\phi,\phi^*] = &\int d^4k \; \left[J_0^{(1)}(k) + J_1^{(1)}(k)\phi^*(k) + c.c.\right] + \\ &\int d^4k^1\, d^4k^2 \; \left[J_2(k^1,k^2)\phi^*(k^1)\phi^*(k^2) + c.c.\right] + \\ &\int d^4k^1\, d^4k^2 \; \left[\tilde{J}_2(k^1,k^2)\phi^*(k^1)\phi(k^2)\right] + \\ &\int d^4k^1\, d^4k^2\, d^4k^3 \; \left[J_3(k^1,k^2,k^3)\phi^*(k^1)\phi^*(k^2)\phi(k^3) + c.c.\right]. \end{aligned} \tag{E.0.4}$$

Here $\tilde{J}_2$ is real. We insert (E.0.2) and (E.0.3) into the flow equation (4.1.5) and note that the ϕ and ϕ^* terms on the left and right hand side of the equation match to first order in λ, justifying our ansatz above. Comparing term by term we obtain a set of coupled equations for the various functions. At zeroth order we find

$$\begin{aligned}\partial_\theta J_0^{(0)}(k) &= F(k)|J_1^{(0)}(k)|^2,\\ \partial_\theta J_1^{(0)}(k) &= F(k)g_2^{(0)}(k)J_1^{(0)}(k),\end{aligned} \tag{E.0.5}$$

while the first order equations are

$$\begin{aligned}
&\partial_\theta J_0^{(1)}(k) = F(k)\left(\left((J^*)_1^{(1)}(k)J_1^{(0)}(k) + c.c.\right) + \tilde{J}_2(k,k)\right),\\
&\partial_\theta J_1^{(1)}(k) = F(k)\left(g_2^{(0)}(k)J_1^{(1)}(k) + g_2^{(1)}(k)J_1^{(0)}(k)\right) +\\
&\int d^4p\ F(p)\left[(J^*)_1^{(0)}(p)\left(J_2(p,k)+J_2(k,p)\right) + J_1^{(0)}(p)\tilde{J}_2(k,p) + J_3(p,k,p) + J_3(k,p,p)\right],\\
&\partial_\theta \tilde{J}_2(k^1,k^2) = 2\sum_{i=1}^{2} F(k^i)g_2^{(0)}(k^i)\tilde{J}_2(k^1,k^2) +\\
&\int d^4p\ \left(F(p)J_1^{(0)}(p)\left((J^*)_3(k^1,p,k^2) + (J^*)_3(p,k^1,k^2)\right) + c.c.\right),\\
&\partial_\theta J_2(k^1,k^2) = \sum_{i=1}^{2} F(k^i)g_2^{(0)}(k^i)J_2(k^1,k^2) + \int d^4p\ F(p)J_1^{(0)}(p)J_3(k^1,k^2,p),\\
&\partial_\theta J_3(k^1,k^2,k^3) = \sum_{i=1}^{3} F(k^i)g_2^{(0)}(k^i)J_3(k^1,k^2,k^3) +\\
&2\int d^4p\ J_1^{(0)}(p)g_4(k^1,k^2,p,k^3)\delta(k^1+k^2-p-k^3).
\end{aligned} \tag{E.0.6}$$

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