Concrete Foundations of the Theory of Noetherian Forms

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Dissertation presented for the degree of Doctor of Philosophy in the Faculty of Science at Stellenbosch University.

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December 2019

Declaration

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Date: December 2019

Abstract

This thesis concerns certain investigations in abstract algebra that bring together the ideas of the category of algebraic structures and the lattice of substructures. A central notion in such investigation is that of a noetherian form. Originally, noetherian forms were introduced to provide a self-dual axiomatic context for establishing homomorphism theorems for (non-abelian) group-like structures. It is known that the form of "subobjects" over any variety is a noetherian form exactly when the variety is semi-abelian. An unexpected result in this thesis is that there is a noetherian form over any variety. In particular, this shows that the context of a noetherian form is much wider than originally thought. One of the aims of the thesis is to explore methods of constructing new noetherian forms out of existing forms; the mentioned result is obtained as an application of one of these constructions. Another aim is to show how the self-dual analogue of products in noetherian forms, called "biproducts" (first introduced in the author's MSc thesis), are related to products. Finally, in this thesis we study the notion of an n-complemented lattice. This notion arose from studying subgroup lattices of finite abelian groups.

Opsomming

Hierdie tesis handel oor sekere ondersoeke in abstrakte algebra wat die idees van die kategorie van algebraïse strukture en die tralie van substrukture by mekaar bring. ‘n Sentrale idee van so ‘n ondersoek is dié van ‘n noetherse vorm. Noetherse vorms was oorspronklik bekendgestel om ‘n selfduale konteks te bied vir die skepping van homomorfisme stellings vir (nie-abelse) groepagtige strukture. Dit is bekend dat die vorm van “sub-objekte” oor ‘n variëteit ‘n noetherse vorm is presies wanneer die variëteit semi-abels is. ‘n Onverwagte resultaat in hierdie tesis is dat daar ‘n noetherse vorm oor enige variëteit bestaan. In besonders wys dit dat die konteks van noetherse vorms baie wyer strek as oorspronklik gedink. Een van die doelwitte van die tesis is om metodes van konstruksies van nuwe noetherse vorms uit bestaande vorms te verken; die genoemde resultaat is verkry deur ’n toepassing van een van hierdie konstruksies. ‘n Ander doelwit is om die verwantskap tussen die selfduale analoog van produkte in noetherse vorms, genoem “biprodukte” (soos bekendgestel in die skrywer se MSc tesis) en kategoriese produkte aan te toon. Laastens, in hierdie tesis bestudeer ons die idee van ‘n n-komplemente tralie. Hierdie idee het ontstaan deur om subgroep tralies van eindige abelse groepe te bestudeer.

Acknowledgements

I wish to thank:

- Professor Zurab Janelidze for your role as my supervisor.
- Dr. James Gray for your role as co-supervisor.
- Professor Marino Gran who hosted me during my stay at Université Catholique de Louvain during the first half of 2018.
- Professor Leon van Wyk for your support and encouragement throughout the years.
- The Harry Crossley Scholarship fund for financial support over the past two years.
- The Erasmus Plus Programme for financial support during my stay in Belgium.

Finally, I express my gratitude to my parents, Pieter and Christel van Niekerk, for their support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis.

Contents

Chapter 1

Introduction

1.1 Background and outline

As explained below, the origins of the theory of noetherian forms can be traced back to the beginning of category theory; but, in fact, it can be traced even further back to the ideas of E. Noether who placed an emphasis on homomorphism theorems in abstract algebra and means of establishing them in a language that attempts to avoid use of underlying operations of algebraic structures. In 20th century, various general axiomatic contexts have emerged where these homomorphism theorems can be established. Two major ones are that of a semi-abelian category in the sense of [12] and that of a Grandis exact category (see [9, 10]). The context of a noetherian form brings these two together, and is applicable far beyond these contexts, as we show in this thesis.

In his Review of S. Mac Lane's work from 1948 on "Groups, categories and duality" (see [18, 19]), published by the American Mathematical Society, P. Hall writes:

> The direct product and the free product of two groups are defined abstractly in terms of homomorphisms, the two definitions being formally deducible one from the other by applying the following "duality rules": invert the direction of each homomorphism, invert the order of all products of homomorphisms, interchange homomorphisms onto with isomorphisms into. The same duality is observed to hold between free Abelian groups and infinitely divisible Abelian groups. The author aims to formulate these and other similar duality relations of group theory axiomatically. This is done by a refinement of the notion of category, originally introduced by Eilenberg and MacLane [Trans. Amer. Math. Soc. 58, 231-294 (1945); MR0013131]. A category is a class of entities called "mappings" (e.g., homomorphisms) in which the products of certain pairs of mappings are defined and satisfy certain axioms (conditional existence and associativity of products, existence of "identities"). A bicategory is now defined to be a category with two (dual) distinguished classes of mappings, called injections and projections, which satisfy certain simple

additional postulates. ...

Today the term "bicategory" means something entirely different. While duality has become a central tool in category theory, research on duality relations in (non-abelian) group theory has been halted since 1970's. This direction of research has been recently revived in the work of Z. Janelidze and his coauthors. In particular, in [8] the authors propose a self-dual context, which can be seen as a further "refinement" of the context of a "bicategory" and in which, the usual homomorphism theorems of group theory, such as the isomorphism theorems and the homological diagram lemmas, can be established (see also [14–17]). This context is called a *noetherian form*. The term was introduced in my MSc thesis [28], where among other things, I developed a self-dual approach to direct product of groups. As a start, in this thesis we revisit this work in a slightly less restrictive context (which we still refer to as a noetherian form). As it turns out, the minor relaxation of the axioms for a noetherian form leads to new significant examples. In particular, as we show in this thesis, not just the category of groups, but in fact any category of algebraic structures (in a variety of algebras) provides an example of the context of a noetherian form. This significantly increases the scope of application of the theory, which was originally designed for the self-dual study of specifically group-like structures.

Apart from what was mentioned above, which is carried out in Chapters 2 and 5, in this thesis we clarify the relationship between noetherian forms and semi-abelian categories (Chapter 3) and describe various general methods for constructing noetherian forms (Chapter 4).

The last chapter of the thesis (Chapter 6) is slightly detached from the rest of the thesis. In it, we study generalizations of the notion of a complement in a lattice and apply this notion to lattices of subgroups of finite abelian groups. Yet, this is not entirely off topic, as it can be seen as an investigation of specific phenomena in the noetherian form of finite abelian groups.

1.2 Preliminaries

In this thesis we assume that the reader is familiar with basic notions from (with references to books in brackets)

- set theory (see [21]): set and the element relation, the empty set, subset, intersection and union of two sets, (binary) relation, function, direct and inverse image of a subset under a function, injection, surjection, bijection, equivalence relation, quotient set;
- lattice theory (see [21]): poset, meet and join, top and bottom element, complement, modular lattice, distributive lattice, Galois connection, homomorphism of lattices;
- group theory (see [21]): group, subgroup, normal subgroup, product of groups, quotient group, morphism of groups, cyclic group, (finite) abelian group;

- universal algebra (see for example [2]): algebra, variety, subalgebra, congruence, morphism of algebras, term;
- category theory (see [20]): category, functor, opposite category and opposite functor, natural transformation, cone and cocone, limit and colimit (this includes terminal and initial object, equalizer and coequalizer, product and coproduct, and, pullback and pushout), finitely complete/cocomplete category (i.e., category having finite limits/colimits), commutative diagram, monomorphism and epimorphism, jointly monomorphic and epimorphic pair, split epimorphism and split monomorphism, isomorphism, zero object, pointed category, kernel and cokernel.

In the thesis, when we speak of a variety, we often mean the category of its algebras and algebra homomorphisms.

Below we have included definitions of some notions that may be new to non-experts of category theory.

For a large part of this thesis, we are studying functors which have at least the following two properties:

Definition 1.2.1. A functor $F\colon \mathbb{C} \to \mathbb{D}$ is called

- *faithful* when any two parallel morphisms mapping to the same morphism are equal. In symbols, for any $f, g\colon A \to B$ in $\mathbb{C}$,
$$Ff = Fg \implies f = g;$$
- *amnestic* when the only isomorphisms mapping to identity morphisms are the identity morphisms. In symbols, for any isomorphism $f\colon A \to B$ in $\mathbb{C}$,
$$Ff = 1_{FA} \implies A = B \text{ and } f = 1_A.$$

Forgetful functors from most categories of mathematical structures to the category of sets, which assign to a mathematical structure the underlying set, are both faithful and amnestic.

Below we recall the definitions of "regular", "Barr-exact", "protomodular", and "semi-abelian" categories. These concepts are only needed in Subsections 3.3 and 3.4 of Chapter 3, and no further knowledge other than their definition is assumed in the thesis.

A morphism is called a *regular epimorphism* when it is the coequalizer of some parallel pair of morphisms. The *kernel pair* of morphism $f\colon X \to Y$ is a pair (k_1, k_2) of parallel morphisms $k_1, k_2\colon K \to X$ such that

$$\begin{tikzcd} K \arrow[r, "k_2"] \arrow[d, "k_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a pullback. In a variety seen as a category, regular epimorphisms are the surjective morphisms of algebras, and the kernel pair of a morphism is given by the "kernel congruence".

Definition 1.2.2. A finitely complete category $\mathbb{C}$ is *regular* [1] if

- coequalizers of kernel pairs exist in $\mathbb{C}$;
- regular epimorphisms are pullback stable. That is, if the following diagram is a pullback

$$\begin{tikzcd} \cdot \arrow[r, "q"] \arrow[d, "g"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "p"] & \cdot \end{tikzcd}$$

  where p is a regular epimorphism, then q is a regular epimorphism.

One can define so-called "internal equivalence relations" in a category. The intuition behind the definition is that in the category of sets, the internal equivalence relations will correspond to the usual equivalence relations. Let $\mathbb{C}$ be a finitely complete category. A *relation from X to Y* is a graph

$$\begin{tikzcd} & R \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\ X & & Y \end{tikzcd}$$

such that $(d_1, d_2)\colon R \to X \times Y$ is a monomorphism (equivalently, d_1 and d_2 are jointly monomorphic). Further, for $X = Y$, the relation (R, d_1, d_2) is called

- *reflexive* if there is an arrow $s\colon X \to R$ such that $d_1 s = 1_X = d_2 s$;
- *symmetric* if there is an arrow $\sigma\colon R \to R$ such that $d_1\sigma = d_2$ and $d_2\sigma = d_1$;
- *transitive* if for the pullback

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "d_2"] \\ R \arrow[r, "d_1"] & X \end{tikzcd}$$

  there is an arrow $\tau\colon P \to R$ such that $d_1\tau = d_1 p_1$ and $d_2\tau = d_2 p_2$.

Notice that kernel pairs can be seen as (internal) equivalence relations.

Definition 1.2.3. A category $\mathbb{C}$ is *Barr-exact* [1] if

- $\mathbb{C}$ is regular;
- any equivalence relation (R, d_1, d_2) in $\mathbb{C}$ is *effective*: that is, (R, d_1, d_2) is the kernel pair of some morphism.

Any variety is a Barr-exact category where equivalence relations are (up to isomorphism) congruences.

Definition 1.2.4. A pointed category $\mathbb{C}$ is *protomodular* [4] when the Split Short Five Lemma holds in $\mathbb{C}$: for any commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"] & A \arrow[r, "f"] \arrow[d, "v"] & B \arrow[d, "w"] \\ K' \arrow[r, "k'"] & A' \arrow[r, "f'"] & B' \end{tikzcd}$$

where k and k' are the respective categorical kernels of f and f', and both f and f' are split epimorphisms, we have

$$u \text{ and } w \text{ are isomorphisms} \implies v \text{ is an isomorphism.}$$

Definition 1.2.5. A finitely complete category $\mathbb{C}$ is *semi-abelian* [12] if it is pointed, Barr-exact and regular, and the coproduct of any two objects exists.

As shown in [5], a variety is protomodular if and only if it has constants $e_1, \ldots, e_n$, binary terms $d_1, \ldots, d_n$, and an $(n+1)$-ary term p such that the following identities hold:

$$d_1(x, x) = e_1, \ldots, d_n(x, x) = e_n,$$
$$p(d_1(x, y), \ldots, d_n(x, y), y) = x.$$

A variety is semi-abelian if and only if it is protomodular with $e_1 = \cdots = e_n$ being the unique constant of the variety (uniqueness of the constant corresponds to pointedness of the category). This shows that essentially all group-like algebraic structures form protomodular/semi-abelian varieties, as, when an algebraic structure has (say, additive) group operations we can define the above terms as follows, with $n = 1$:

$$e_1 = 0, \quad d_1(x, y) = x - y, \quad p(x, y) = x + y.$$

1.3 The concept of a noetherian form

The notion of a noetherian form evolved in the following papers: [8, 14–17]. Intuitively, a form is a category equipped with a data of abstract "substructure" posets, with a relation of how the morphisms of the category interact with these posets. In a noetherian form the posets will become lattices. The principal example (thus far) of a noetherian form is given by the category of groups, where the substructure lattices are the subgroup lattice. Each homomorphism $f\colon G \to H$ from a group G to a group H determines a relation between the subgroup lattice of G and the subgroup lattice of H: a subgroup A of G is related to a subgroup B of H when f maps A into B. This relation in the general context will be written as "$\leq_f$".

We start off with a formal definition of a form, and slowly add conditions until we get the formal definition of a noetherian form.

Definition 1.3.1. A *form* over $\mathbb{C}$ is a faithful amnestic functor $F\colon \mathbb{B} \to \mathbb{C}$.

Two forms $F\colon \mathbb{B} \to \mathbb{C}$ and $G\colon \mathbb{A} \to \mathbb{C}$ are *isomorphic* when there exists an isomorphism $H\colon \mathbb{A} \to \mathbb{B}$ such that $FH = G$.

For each object $X \in \mathbb{C}$, we define $\mathsf{sub}^F X$ to be all those objects $A \in \mathbb{B}$ for which $FA = X$. We will drop the superscript F when the form F is clear from context. Elements of $\mathsf{sub}^F X$ are called *subobjects* of X. For each morphism $f\colon X \to Y$ in $\mathbb{C}$, we define a relation $\leq_f^F$ (and drop the superscript when the form F is clear from context) from $\mathsf{sub}X$ to $\mathsf{sub}Y$ as $A \leq_f B$ if and only if there is a morphism $A \to B$ such that $F(A \to B) = f$. For $f = 1_X$, we will denote $\leq_{1_X}$ by $\leq_X$ or just $\leq$. This relation satisfies the following:

(F1) For any object X, $\leq_X$ is reflexive.

(F2) For any two composable arrows $f\colon X \to Y$ and $g\colon Y \to Z$, and subobjects $A \in \mathsf{sub}X$, $B \in \mathsf{sub}Y$ and $C \in \mathsf{sub}Z$, we have

$$A \leq_f B \leq_g C \implies A \leq_{gf} C.$$

(F3) For any object X, $\leq_X$ is anti-symmetric.

Any functor will satisfy (F1) and (F2). Only (F3) needs F to be faithful and amnestic. Conversely, if a category $\mathbb{C}$ has such data of $\mathsf{sub}X$ and $\leq_f$ satisfying the above, then one can show that there is a unique form (up to isomorphism) over $\mathbb{C}$ such that the $\mathsf{sub}X$ and $\leq_f$ as defined by the form coincide with the given $\mathsf{sub}X$ and $\leq_f$ (up to possibly renaming of subobjects). For this reason one could think of a form as an "indexed" poset.

Duality will be a central theme when working with forms.

Definition 1.3.2. The dual of a form is the dual functor $F^{\mathrm{op}}\colon \mathbb{B}^{\mathrm{op}} \to \mathbb{C}^{\mathrm{op}}$, which is again a form.

Consequently, in the dual form all arrows are reversed and the relation $\leq_f$ is reversed.

Definition 1.3.3. A form $F\colon \mathbb{B} \to \mathbb{C}$ is called *orean* when $A \leq_{gf} C$ implies there is some B such that $A \leq_f B \leq_g C$ for any composable morphisms f and g and subobjects A and C, and for any morphism $f\colon X \to Y$ and finite subsets $S \subseteq \mathsf{sub}X$ and $T \subseteq \mathsf{sub}Y$, we have:

- the set $\{B' \in \mathsf{sub}Y \mid \forall_{A \in S}(A \leq_f B')\}$ has a minimum element, and
- the set $\{A' \in \mathsf{sub}X \mid \forall_{B \in T}(A' \leq_f B)\}$ has a maximum element.

In particular, $\mathsf{sub}X$ is a bounded lattice for any object X in an orean form. Using the last two points, select $f = 1_X$, and by selecting $S = \emptyset = T$, we get boundedness, and by selecting $S = \{A, B\} = T$, we will get the join $A \vee B$ and meet $A \wedge B$ of A and B.

Further, for any morphism $f\colon X \to Y$ and subobjects A of X and B of Y, we have the following concepts and notation:

- $fA = f_*A = f \cdot^F A = \mathsf{min}\{B' \in \mathsf{sub}Y \mid A \leq_f B'\}$ is the *direct image* of A under f, and
- $f^{-1}B = f^*B = B \cdot^F f = \mathsf{max}\{A' \in \mathsf{sub}X \mid A' \leq_f B\}$ is the *inverse image* of B under f.

The first of these notation will be the most commonly used on. The last of these notation, the ones with the $\cdot$, will be used to distinguish between the direct and inverse images of different forms.

With these concepts, we have an alternative formulation of the first condition on orean forms.

Proposition 1.3.4. *In any form where direct and inverse images exist, the following are equivalent, for any two composable morphisms $f\colon X \to Y$ and $g\colon Y \to Z$:*

(1) $A \leq_{gf} C \implies \exists_B(A \leq_f B \leq_g C)$, for any subobjects A of X and C of Z;

(2) $(gf)_* = g_*f_*$;

(3) $(gf)^* = f^*g^*$.

Proof. (1) $\implies$ (2): Take any subobject A of X. We wish to show that $(gf)_*A = g_*f_*A$. We have

$$A \leq_f f_*A \leq_g g_*f_*A.$$

And so $A \leq_{gf} g_*f_*A$. So $(gf)_*A \leq g_*f_*A$. Further, $A \leq_{gf} (gf)_*A$, and so there is a B such that $A \leq_f B \leq_g (gf)_*A$. We have $f_*A \leq B$, and further

$$f_*A \leq B \leq_g (gf)_*A.$$

Thus $g_*f_*A \leq (gf)_*A$, and so $g_*f_*A = (gf)_*A$.

(2) $\implies$ (1): Suppose $A \leq_{gf} C$. We have

$$A \leq_f f_*A \leq_g g_*f_*A = (gf)_*A \leq C,$$

and consequently f_*A is our desired B.

With dual arguments, we will get (1) $\iff$ (3). $\square$

Notice that in any orean form direct and inverse images forms a monotone Galois connection: for any $f\colon X \to Y$ and subobjects A of X and B of Y, we have

$$f_*A \leq B \iff A \leq_f B \iff A \leq f^*B.$$

From this observation and the above proposition, we can further adjust the definition of an orean form. Any form F in which subobject posets are bounded lattices, and direct and inverse images exist and are "functorial" is an orean form. To show this, take any morphism $f\colon X \to Y$ and finite subsets $S = \{s_1 \ldots, s_n\}$ of $\mathsf{sub}X$ and $T = \{t_1 \ldots, t_m\}$ of $\mathsf{sub}Y$. Then $f_*s_1 \vee \ldots \vee f_*s_n$ will be the minimum element of the set $\{B' \in \mathsf{sub}Y \mid \forall_{A\in S}(A \leq_f B')\}$, and $f^*t_1 \wedge \ldots \wedge f^*t_m$ is the maximum element of the set $\{A' \in \mathsf{sub}X \mid \forall_{B\in T}(A' \leq_f B)\}$.

An orean form also allows us to further define the following concepts. For a morphism f in $\mathbb{C}$, where there is an orean form F over $\mathbb{C}$,

- $\mathsf{Ker}^F f = f^*0$ is the *kernel* of f, and
- $\mathsf{Im}^F f = f_*1$ is the *image* of f.

A subobject is called *normal* when it appears as the kernel of some morphism, and is called *conormal* when it appears as the image of some morphism. So in the form of groups and subgroups, normal subobjects are the usual normal subgroups, while every subobject is conormal.

Lastly, an *embedding* of a conormal subobject S of X is a morphism $\iota_S\colon S' \to X$ whose image is S, and which has the property that for morphism $f\colon Y \to X$ whose image is contained in S, there is a unique morphism $h\colon Z \to S'$ such that $f = \iota_S h$. And dually, a *projection* of a normal subobject N of X is a morphism $\pi_N\colon X \to X/N'$ whose kernel is N, and which has the property that for any morphism $f\colon X \to Y$ whose kernel contains N, there is a unique morphism $h\colon X/N' \to Y$ such that $h\pi_N = f$.

$$\begin{tikzcd}
S' \arrow[r, "\iota_S"] & X \\
Y \arrow[u, dashed, "\exists!"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\pi_N"] \arrow[dr, "f"'] & X/N' \arrow[d, dashed, "\exists!"] \\
& Y
\end{tikzcd}$$

Definition 1.3.5. A *noetherian* form is an orean form such that

(N1) For any $f\colon X \to Y$, and $A \in \mathsf{sub}X$ and $B \in \mathsf{sub}Y$, we have

$$f^*f_*A = A \vee \mathsf{Ker}f \quad \text{and} \quad f_*f^*B = B \wedge \mathsf{Im}f;$$

(N2) Any morphism f factorizes as $f = em$ where e is a projection of the kernel of f and m is an embedding of the image of f;

(N3) The join of any two normal subobjects is normal and the meet of any two conormal subobjects is conormal.

Alternatively, instead of (N1), we could have used: For any $f\colon X \to Y$, and $A \geq \mathsf{Ker}f$ and $B \leq \mathsf{Im}f$, we have

$$f^*f_*A = A \quad \text{and} \quad f_*f^*B = B.$$

This is just a seemingly special case of (N1). Conversely, assuming this, we have for any morphism f and subobject A of its domain,

$$f^*f_*A = f^*f_*(A \vee \mathsf{Ker}f) = A \vee \mathsf{Ker}f,$$

and dually we can show the other equation.

What was studied in [8], are noetherian forms. There, an equivalent formulation of a noetherian form was used without referring to functors (and without using the term "noetherian form"). There the structure of a noetherian form is given by a category $\mathbb{C}$, posets $\mathsf{sub}X$ and maps $f_*\colon \mathsf{sub}X \to \mathsf{sub}Y$ and $f^*\colon \mathsf{sub}Y \to \mathsf{sub}X$ such that the following axioms hold:

(1) For any object X, $\mathsf{sub}X$ is a bounded lattice. Further, for any two composable morphisms f and g, $(fg)_* = f_*g_*$ and $(fg)^* = g^*f^*$, and for any identity morphism $(1_A)_* = 1_{\mathsf{sub}A} = 1_A^*$. Further for any morphism $f\colon X \to Y$, f_* and f^* forms a monotone Galois connection: for any $A \in \mathsf{sub}X$ and $B \in \mathsf{sub}Y$,

$$f_*A \leq B \iff A \leq f^*B.$$

(2) For any morphism $f\colon X \to Y$ and $A \in \mathsf{sub}X$ and $B \in \mathsf{sub}Y$,

$$f_*f^*B = B \wedge f_*1 \quad \text{and} \quad f^*f_*A = A \vee f^*0.$$

(3) For any "conormal" $S \in \mathsf{sub}X$, there is a morphism $\iota_S\colon S \to X$ such that $(\iota_S)_*1 = S$ and is universal morphism into X with $(\iota_S)_*1 \leq S$. Further, for any "normal" $N \in \mathsf{sub}X$, there is a morphism $\pi_N\colon X \to X/N$ such that $\pi_N^*0 = N$, and is universal morphism from X such that $\pi_N^*0 \geq N$.

(4) Any morphism f factorizes as $f = \iota_{f_*1}h\pi_{f^*0}$ for some isomorphism h.

(5) the join of normal subobjects is normal, and dually the meet of conormal subobjects is conormal.

We can construct a noetherian form F over $\mathbb{C}$ such that the F-subobjects are those given subobjects (possibly renamed), and the arising direct and inverse images from the form F are those given direct and inverse images. First, construct a category $\mathbb{B}$, where objects are pairs (A, X) with $X \in \mathbb{C}$ and $A \in \mathsf{sub}X$. Morphisms $f\colon (A, X) \to (B, Y)$ are morphisms $f\colon X \to Y$. Composition of morphisms f and g is defined as in $\mathbb{C}$. Then, define $F\colon \mathbb{B} \to \mathbb{C}$ as the functor sending $f\colon (A, X) \to (B, Y)$ to $f\colon X \to Y$. Further, one could even show that this is the unique such noetherian form up to isomorphism.

1.4 Preliminary results on noetherian forms

This section only contains some very elementary results, but which are very often used throughout this thesis. All these results were established in [8].

Here is a list of properties:

- $f_*0 = 0$ and $f^*1 = 1$;
- f_* preserves joins and f^* preserves meets;
- $f_*f^*f_* = f_*$ and $f^*f_*f^* = f^*$;
- any morphism f is a projection (of some subobject) if and only if $\mathsf{Im}f = 1$, and dually it is an embedding if and only if $\mathsf{Ker}f = 0$;
- any morphism f is an isomorphism if and only if $\mathsf{Ker}f = 0$ and $\mathsf{Im}f = 1$;
- any projection is an epimorphism, and dually any embedding is a monomorphism;
- for any normal subobject N and projection p, pN is normal, and dually for any conormal subobject C and embedding m, $m^{-1}C$ is conormal.

Note that the first three points are true for any monotone Galois connection f_* and f^*. For interest sake, last point can be shown to be equivalent to axiom (5).

Although the following result was not explicitly stated in [8], the proof of this result appeared in the proof in [8] of the Restricted Modular Law (the proposition hereafter). Since this result has some usefulness on its own, we make it explicit.

Proposition 1.4.1. *For a morphism f and a subobject X below the image of f and a normal subobject N of the codomain of f, we have*

$$f^{-1}(X \vee N) = f^{-1}X \vee f^{-1}N.$$

Proof. Suppose $N = g^{-1}0$, for some morphism g. We then have

$$\begin{aligned} f^{-1}X \vee f^{-1}N &= f^{-1}X \vee f^{-1}g^{-1}0 \\ &= f^{-1}X \vee (gf)^{-1}0 \\ &= (gf)^{-1}(gf)f^{-1}X \\ &= f^{-1}g^{-1}gff^{-1}X \\ &= f^{-1}g^{-1}gX \\ &= f^{-1}(X \vee g^{-1}0) \\ &= f^{-1}(X \vee N). \end{aligned}$$

□

Proposition 1.4.2 (Restricted Modular Law)**.** *For any three subobjects X, Y, and Z of an object G, if Y is normal and Z is conormal (or dually, if Y is conormal and X is normal), then*

$$X \leq Z \implies X \vee (Y \wedge Z) = (X \vee Y) \wedge Z.$$

Proof. Suppose $Y = g^{-1}0$ and $Z = f1$ for some morphisms g and f, and suppose $X \leq Z$. We have

$$\begin{aligned} X \vee (Y \wedge Z) &= X \vee (g^{-1}0 \wedge f1) \\ &= ff^{-1}X \vee ff^{-1}g^{-1}0 \\ &= f(f^{-1}X \vee f^{-1}g^{-1}0) \\ &= ff^{-1}(X \vee g^{-1}0) \\ &= (X \vee g^{-1}0) \wedge f1 \\ &= (X \vee Y) \wedge Z. \end{aligned}$$

□

Chapter 2

Biproducts and commutators

2.1 Introduction

Biproducts and commutators were first introduced and studied in [28]. In this instance a stronger notion of a noetherian form was used with a requirement that all subobjects have embeddings and projections, and not just the conormal and normal ones. It turns out that the theory of biproducts and commutators is not significantly affected by dropping this requirement.

In this chapter we recover all of the results obtained on biproducts and commutators from [28], up to a minor reformulation for some of them. The main obstacle in generalizing the results from [28] lay in finding a proof of Proposition 2.2.12 below, which did not make use of the additional requirement on a noetherian form assumed in [28]. Our notion of a biproduct is a generalization of the notion of a biproduct from abelian categories to noetherian forms. Commutator theory in a noetherian form developed in this section generalizes the theory of "Huq commutators" from the context of a semi-abelian category.

2.2 Biproducts

Throughout this section, we assume that we are working in a noetherian form.

2.2.1 Introduction to biproducts

Definition 2.2.1. A *split product* of A and B is an object G equipped with four maps

$$A \underset{p_1}{\overset{e_1}{\rightleftarrows}} G \underset{p_2}{\overset{e_2}{\leftrightarrows}} B$$

such that

$$\begin{aligned} \mathsf{Ker} p_1 &= \mathsf{Im} e_2, \quad p_1 e_1 = 1 \\ \mathsf{Ker} p_2 &= \mathsf{Im} e_1, \quad p_2 e_2 = 1 \end{aligned}$$

Sometimes we will just refer to G as a split product of A and B, and assume that their respective embeddings are given by e_1 and e_2, and their respective projections are given by p_1 and p_2.

The two additional conditions on a split product in the following definition were suggested to me by my supervisor Professor Zurab Janelidze.

Definition 2.2.2. A *biproduct* of A and B is a split product G of A and B such that for the following diagrams

$$\begin{tikzcd} A & G \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\ & W \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} & W' & \\ A \arrow[ur, "f'"] \arrow[r, "e_1"'] & G & B \arrow[ul, "g'"'] \arrow[l, "e_2"] \end{tikzcd}$$

the left one has a limit for any f and g, and the right one has a colimit for any f' and g'.

See Section 3.4 for comparison of biproducts with the usual categorical products and see Section 5.3 for examples of biproducts that are not usual categorical products.

To make it easier to refer to the diagrams in the definition above, the left one will be denoted by $\mathsf{L}_G(f, g)$ and the right one by $\mathsf{C}_G(f', g')$. The subscript G may be dropped when it is clear to which biproduct we are referring to.

Note that both the notions of a split- and biproduct are self-dual.

Some trivial properties of split products:

Proposition 2.2.3. *If G is a split product of A and B, then we have*

(1) $e_1 1 \vee e_2 1 = 1 = p_1^{-1} 0 \vee p_2^{-1} 0$;

(2) $e_1 1 \wedge e_2 1 = 0 = p_1^{-1} 0 \wedge p_2^{-1} 0$;

Proof. (1) Since $p_2 e_2 1 = 1$, we have

$$e_1 1 \vee e_2 1 = p_2^{-1} 0 \vee e_2 1 = p_2^{-1} p_2 e_2 1 = p_2^{-1} 1 = 1.$$

(2) is the dual of (1). □

Corollary 2.2.4. *If the split product of any two object exists, then the top subobject 1 is normal and the bottom subobject 0 is conormal for any object.*

Proof. By (1) of the previous proposition, 1 in $A \oplus A$ is normal, since it is the join of normal subobjects. Since $p_1 \colon A \oplus A \to A$ is a projection, $1 = p_1 1$ is normal. Dually 0 is a conormal subobject of A. □

Having biproducts forces pointedness:

Theorem 2.2.5. *If the biproduct of any two objects exists in a non-empty noetherian form, then the category is pointed. The zero objects* 0 *are exactly those for which* $\mathsf{sub}0$ *has one element, and zero morphisms are exactly those with image* 0 *or with kernel* 1*.*

Proof. Take any object G. By Corollary 2.2.4, $1 = G$ is a normal subobject of G. Let $T = G/G$. We have

$$1^T = \pi_G \pi_G^{-1} 1 = \pi_G 1 = 0^T.$$

Thus $\mathsf{sub}T$ has exactly one element. Let B be a biproduct of T and T. Then

$$1 = e_1 1 \vee e_2 1 = e_1 0 \vee e_2 0 = 0.$$

Thus $\mathsf{sub}B$ also has one element. From this in particular follows that both e_1 and e_2 are isomorphisms. For any object A, there is at least a morphism from T to A; for example, compose the embedding from T to a biproduct of T and A with the projection from the same biproduct to A. We would like to show that there is at most one morphism $T \to A$. Consider any $f, g\colon T \to A$. Let $(C, e\colon A \to C, m\colon B \to C)$ be a colimit (or even a cocone) of $C_B(f,g)$. Since $\mathsf{sub}(T)$ only has one element, both f and g are embeddings of 0 in A. Thus there exists an isomorphism $h\colon T \to T$ such that $fh = g$. We have

$$me_2 = eg = efh = me_1 h,$$

and since m is (trivially) an embedding, $e_2 = e_1 h$. Consequently $fe_1^{-1} = ge_2^{-1}$ for any $f, g\colon T \to A$. In the case when $f = g = 1_T$, $e_1 = e_2$. So in the general case, $f, g\colon T \to A$ implies $f = g$. So T is an initial object. With dual arguments, one can show that T is also a terminal object. Thus T is a zero object.

It can be readily observed that any other object is a zero object as well if and only if its subobject lattice has one element.

For any zero morphism $me\colon A \to B$, where $e\colon A \to T$ and $m\colon T \to B$, we have

$$\mathsf{Im}\, me = me1 = m0 = 0,$$

and therefore also $\mathsf{Ker}(me) = 1$. For any morphism $f\colon A \to B$, if $\mathsf{Ker}f = 1$ (equivalently $\mathsf{Im}f = 0$), then f factors through $\pi_A\colon A \to A/A$. As observed in the beginning, A/A is a zero object, therefore f is a zero morphism. □

The remaining results of this section are some general properties of biproducts.

Proposition 2.2.6. *Suppose* G *is a biproduct of* A *and* B*. For any* $f\colon A \to W$ *and* $g\colon B \to W$*, if* $(C, e\colon W \to C, m\colon G \to C)$ *is a colimit of* $\mathsf{C}(f,g)$*, then* e *is a projection.*

Proof. Suppose (C, e, m) is a colimit of $\mathsf{C}(f, g)$. Let $I = \mathsf{Im}e$, and let $\iota_I \colon I \to C$ be an embedding corresponding to I. We have the following diagram

$$\begin{tikzcd}
 & & W \arrow[dl, "e"'] \arrow[dr, dashed, "a"] & & \\
 & C \arrow[rr, dashed, "k"'] & & I \arrow[ll, "\iota_I"'] & \\
A \arrow[rr, "e_1"'] \arrow[uurr, bend left, "f"] & & G \arrow[ul, "m"] \arrow[ur, dashed, "b"'] & & B \arrow[ll, "e_2"] \arrow[uull, bend right, "g"']
\end{tikzcd}$$

Morphism a exists such that $\iota_I a = e$, since ι_I is an embedding and $\mathsf{Im}e \leq \mathsf{Im}\iota_I$. We (always) have $\mathsf{Im}m \leq I$, since

$$m1 = m(e_1 1 \vee e_2 1) = me_1 1 \vee me_2 1 = ef1 \vee eg1 \leq \mathsf{Im}e = I.$$

Thus morphism b exists such that $\iota_I b = m$.
We have

$$\iota_I be_1 = me_1 = ef = \iota_I af,$$

which implies $be_1 = af$, since ι_I is a monomorphism. Similarly $be_2 = ag$. Thus (I, a, b) is a cocone of $\mathsf{C}(f, g)$. Since C is a colimit, there exists a morphism $k \colon C \to I$ such that $ke = a$ and $km = b$. Composing k and ι_I, we get a morphism $\iota_I k \colon C \to C$ such that $(\iota_I k)e = e$ and $(\iota_I k)m = m$. But $1_C \colon C \to C$ is the unique such morphism. Thus $\iota_I k = 1_C$, and so ι_I is a projection. Consequently e is a projection. □

Proposition 2.2.7. *If G is a biproduct of A and B, then e_1 and e_2 are jointly epi, and dually p_1 and p_2 are jointly mono.*

Proof. Suppose $(C, e \colon G \to C, m \colon G \to C)$ is a colimit of $\mathsf{C}_G(e_1, e_2)$. Consider the cocone $(G, 1_G, 1_G)$. There is an $h \colon C \to G$ such that $he = 1_G$ and $hm = 1_G$; consequently, e is an embedding. By Proposition 2.2.6, e is also a projection, thus e is an isomorphism. Consequently, h is also an isomorphism, and $(G, 1_G, 1_G)$ is also a colimit of $\mathsf{C}(e_1, e_2)$. Suppose for some $u, v \colon G \to W$, $ue_1 = ve_1$ and $ue_2 = ve_2$. Then (W, u, v) forms a cocone of $\mathsf{C}(e_1, e_2)$. Thus there is an $h \colon G \to W$ such that $u = h1_G$ and $v = h1_G$, from which we get $u = v$. □

Proposition 2.2.8. *Suppose G is a biproduct of A and B. For any $f \colon A \to W$ and $g \colon B \to W$, any cocone $(C, e \colon W \to C, m \colon G \to C)$ of $\mathsf{C}(f, g)$ is a colimit if and only if*

- *e is a projection;*
- *for any cocone $(D, d \colon W \to D, n \colon G \to D)$ we have $\mathsf{Ker}e \leq \mathsf{Ker}d$.*

Proof. Suppose (C, e, m) is a colimit. Then by Proposition 2.2.6, e is a projection. To show the second point, consider any cocone (D, d, n) of $\mathsf{C}(f, g)$. Since (C, e, m) is a colimit of $\mathsf{C}(f, g)$, there is an h such that $he = d$. Consequently $\mathsf{Ker}e \leq \mathsf{Ker}d$.

Conversely suppose there is such a cocone (C, e, m) with those properties. Let

$$(D, d\colon W \to D, n\colon G \to D)$$

be another cocone. Since e is a projection and $\mathsf{Ker}e \leq \mathsf{Ker}d$, there is a unique $h\colon C \to D$ such that $he = d$. We further have

$$hme_1 = hef = df = ne_1,$$

and similarly $hme_2 = ne_2$. By Proposition 2.2.7, e_1 and e_2 are jointly epi, and thus $hm = n$. Thus (C, e, m) is indeed a colimit of $\mathsf{C}(f, g)$. ☐

2.2.2 Biproducts of morphisms

The following theorem is the main theorem of this subsection.

Theorem 2.2.9. *Suppose G is a biproduct of A and B, and H is a split product of C and D. For any pair of morphisms $f\colon A \to C$ and $g\colon B \to D$, there is a unique morphism $h\colon G \to H$ such that*

$$\begin{aligned} he_1 &= e_1 f, \quad p_1 h = f p_1 \\ he_2 &= e_2 g, \quad p_2 h = g p_2. \end{aligned}$$

Furthermore

$$\begin{aligned} &\mathsf{Im}h = e_1\mathsf{Im}f \vee e_2\mathsf{Im}g, \\ &\mathsf{Ker}h = p_1^{-1}\mathsf{Ker}f \wedge p_2^{-1}\mathsf{Ker}g. \end{aligned}$$

Proof. Let $(L, e\colon H \to L, m\colon G \to L)$ be a colimit of $\mathsf{C}_G(e_1 f, e_2 g)$. To make the proof easier to follow, here is a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[dd, "f"] & G \arrow[d, "m"] & B \arrow[l, "e_2"'] \arrow[dd, "g"] \\ & L & \\ C \arrow[r, "e_1"] & H \arrow[u, "e"] & D \arrow[l, "e_2"'] \end{tikzcd}$$

The aim will be to deduce that e is an isomorphism; then $h = e^{-1}m$ is our desired morphism. Both $(C, p_1\colon H \to C, fp_1\colon G \to C)$ and $(D, p_2\colon H \to D fp_2\colon G \to D)$ are cocones of $\mathsf{C}(e_1 f, e_2 g)$. So $\mathsf{Ker}e \leq \mathsf{Ker}p_1, \mathsf{Ker}p_2$, thus $\mathsf{Ker}e = 0$. By Proposition 2.2.6, e is a projection and thus an isomorphism. Let $h = e^{-1}m$. By choice of h, $he_1 = e_1 f$ and $he_2 = e_2 g$. We further have

$$p_1 he_1 = p_1 e_1 f = f = fp_1 e_1 \quad \text{and} \quad p_1 he_2 = p_1 e_2 g = 0 = fp_1 e_2.$$

Since e_1 and e_2 are jointly epic by Proposition 2.2.7, $p_1 h = f p_1$. Similarly $p_2 h = g p_2$. To compute the image of h, we have

$$\mathsf{Im}h = h1 = h(e_1 1 \vee e_2 1) = he_1 1 \vee he_2 1 = e_1 f 1 \vee e_2 g 1 = e_1 \mathsf{Im} f \vee e_2 \mathsf{Im} g.$$

By a dual argument, we get the formula for the kernel of h. □

Corollary 2.2.10. *Biproducts are unique up to canonical isomorphisms.*

Proof. Let both G and H be biproducts of A and B. Take $f = 1_A$ and $g = 1_B$. Then the induced morphism h in the proposition above, is an isomorphism which commutes with the projections and with the embeddings. □

Notation. The biproduct of A and B will be denoted by $A \oplus B$. For $f \colon A \to C$ and $g \colon B \to D$, the unique $h \colon A \oplus B \to C \oplus D$ in the statement of Theorem 2.2.9 will be denoted by $f \oplus g$.

Biproducts are "functorial" in the following sense:

Corollary 2.2.11. *For any objects A and B, and for any morphisms f, g, u, and v, we have:*

- $1_A \oplus 1_B = 1_{A \oplus B}$;
- $(f \oplus g)(u \oplus v) = fu \oplus gv$, *whenever the compositions are defined.*

Further basic results that follow from Theorem 2.2.9:

Proposition 2.2.12. *In any biproduct $A \oplus B$, we have:*

(1) For normal subobjects N of A and M of B, $p_1^{-1}N \wedge p_2^{-1}M$ is a normal subobject of $A \oplus B$.

(2) For conormal subobjects C of A and D of B, $e_1 C \vee e_2 D$ is a conormal subobject of $A \oplus B$.

(3) if N is a normal subobject of A, then $e_1 N$ is a normal subobject of $A \oplus B$;

(4) if C is a conormal subobject of $A \oplus B$, then $p_1^{-1}C$ is a conormal subobject of A;

(5) For any $X \in \mathsf{sub}A$ and $Y \in \mathsf{sub}B$, if X is normal or conormal, or Y is normal or conormal, then

$$e_1 X \vee e_2 Y = p_1^{-1} X \wedge p_2^{-1} Y.$$

Proof. (1) The subobject $p_1^{-1}N \wedge p_2^{-1}M$ is normal, since it is the kernel of $\pi_N \oplus \pi_M$.

(2) is the dual of (1).

(3) Since $p_1 e_1 = 1$, we get $N = 1^{-1}N = e_1^{-1} p_1^{-1} N$. Then we have

$$e_1 N = e_1 e_1^{-1} p_1^{-1} N = e_1 1 \wedge p_1^{-1} N = p_1^{-1} N \wedge p_2^{-1} 0,$$

thus e_1N is normal by (1).

(4) is the dual of (3).

(5) As already noticed in the proof of (3), $e_1X = e_1 1 \wedge p_1^{-1}X$ and similarly $e_2Y = e_2 1 \wedge p_2^{-1}Y$. These are in particular just special cases of (5) for $Y = 0$ or $X = 0$. Suppose X is normal. Then we have, making use of the restricted modular law twice,

$$\begin{aligned} & e_1X \vee e_2Y \\ =& (e_1 1 \wedge p_1^{-1}X) \vee (e_2 1 \wedge p_2^{-1}Y) \\ =& ((e_1 1 \wedge p_1^{-1}X) \vee e_2 1) \wedge p_2^{-1}Y \\ =& (p_1^{-1}X \wedge (e_1 1 \vee e_2 1)) \wedge p_2^{-1}Y \\ =& p_1^{-1}X \wedge p_2^{-1}Y. \end{aligned}$$

The second equality follows, since $p_2^{-1}Y \geq p_2^{-1}0 = e_1 1 \geq e_1 1 \wedge p_1^{-1}X$, and $e_1 1 \wedge p_1^{-1}X = e_1X$ is normal by (3) and $e_2 1$ is conormal. The third follows, since $p_1^{-1}X \geq p_1^{-1}0 = e_2 1$, and $e_2 1$ is normal and $e_1 1$ is conormal.

So for X normal the result is true. By duality it is also true if X is conormal. The case for when Y is normal or conormal is similar. □

The remaining results of this subsection are about when an object is a biproduct of two other objects. All this still relies on Theorem 2.2.9.

Theorem 2.2.13. *For any A and B, any split product of A and B is a biproduct, assuming the biproduct of A and B exists.*

Proof. Let G be a split product of A and B. Theorem 2.2.9 guarantees an isomorphism h (taking $f = 1_A$ and $g = 1_B$) between G and $A \oplus B$ which commutes with the projections and with the embeddings. From this, it can be readily checked that the split product G will satisfy the remaining biproduct conditions. □

Corollary 2.2.14. *If G has two subobjects A and B such that*

- *A and B are both normal and conormal;*
- *$A \vee B = 1$ and $A \wedge B = 0$,*

then $G \cong A \oplus B$, assuming $A \oplus B$ exists.

Proof. Let $e_1 = \iota_A \colon A \to G$ and $e_2 = \iota_B \colon B \to G$. Notice that

$$\pi_B \iota_A 1 = \pi_B A = \pi_B(A \vee B) = \pi_B 1 = 1,$$

and

$$\iota_A^{-1}\pi_B^{-1}0 = \iota_A^{-1}B = \iota_A^{-1}(A \wedge B) = \iota_A^{-1}0 = 0.$$

Thus $\pi_B\iota_A$ is an isomorphism. Denote the inverse by h_1. Similarly $\pi_A\iota_B$ is an isomorphism. Denote the inverse by h_2. Define $p_1 = h_1\pi_B$ and $p_2 = h_2\pi_A$. A straightforward verification shows that G together with e_1, e_2, p_1, and p_2 forms a split product of A and B, thus a biproduct of A and B by the above theorem. □

Corollary 2.2.15. *If $f\colon B \to A$ has a right inverse $s\colon A \to B$, and $\mathsf{Im}s$ is normal and $\mathsf{Ker}f$ is conormal, then $B \cong \mathsf{Im}s \oplus \mathsf{Ker}f$, assuming that biproduct exists.*

Proof. Both $\mathsf{Im}S$ and $\mathsf{Ker}f$ are both normal and conormal. We have

$$f^{-1}0 \vee s1 = f^{-1}fs1 = f^{-1}1 = 1 \quad \text{and} \quad f^{-1}0 \wedge s1 = ss^{-1}f^{-1}0 = s0 = 0.$$

Thus by Corollary 2.2.14 the result follows. □

2.2.3 Monoidality of biproducts

Throughout this subsection, we are working in a noetherian form.

Notation. For $a\colon A \to C$ and $b\colon A \to D$, if there is a morphism $h\colon A \to C \oplus D$ such that $p_1 h = a$ and $p_2 h = b$, then it is unique by Proposition 2.2.7, and h will be denoted by (a, b).

Notice that any morphism $h\colon A \to C \oplus D$ can be written as $h = (p_1 h, p_2 h)$. Some basic properties:

Proposition 2.2.16. *For any morphisms a, b, f, and g, we have, whenever the composites are defined:*

(1) $(a, b)f = (af, bf)$*;*

(2) $f \oplus g = (fp_1, gp_2)$*;*

(3) $(f \oplus g)(a, b) = (fa, gb)$*.*

(4) $(p_1, p_2) = 1$ *for the projections of any biproduct.*

Proof. Since p_1 and p_2 are jointly monic, we only need to verify that both sides are equal when composing with p_1 and p_2 on both sides.

(1) $p_1(a, b)f = af = p_1(af, bf)$ and $p_2(a, b)f = p_2(af, bf)$.

(2) $p_1(f \oplus g) = fp_1 = p_1(fp_1, gp_2)$ and $p_2(f \oplus g) = p_2(fp_1, gp_2)$.

(3) $(f \oplus g)(a, b) = (fp_1, gp_2)(a, b) = (fp_1(a, b), gp_2(a, b)) = (fa, gb)$.

(4) $p_1 = p_1(p_1, p_2)$ and $p_2 = p_2(p_1, p_2)$.

□

The dual of the above will be:

Notation. For $a\colon A \to C$ and $b\colon B \to C$, if there is a morphism $h\colon A \oplus B \to C$ such that $he_1 = a$ and $he_2 = b$, it is unique, and h will be denoted by $[a, b]$.

We also have dual properties, which are true by duality:

Proposition 2.2.17. *For any morphisms a, b, f, and g, we have, whenever the composites are defined:*

(1) $f[a,b] = [fa, fb]$;

(2) $f \oplus g = [e_1 f, e_2 g]$;

(3) $[a,b](f \oplus g) = [af, bg]$;

(4) $[e_1, e_2] = 1$ *for the embeddings of any biproduct.*

Lemma 2.2.18. *For any objects A, B, and C the morphism*

$$\alpha = ((p_1, p_1p_2), p_2p_2) = [e_1e_1, [e_1e_2, e_2]] \colon A \oplus (B \oplus C) \to (A \oplus B) \oplus C$$

exists. Moreover, α is a natural isomorphism.

Proof. Consider the following diagram:

$$\begin{tikzcd} A \oplus B \arrow[r, shift left, "1 \oplus e_1"] & A \oplus (B \oplus C) \arrow[l, shift left, "1 \oplus p_1"] \arrow[r, shift right, "p_2p_2"'] & C \arrow[l, shift right, "e_2e_2"'] \end{tikzcd}$$

We have $(1 \oplus p_1)(1 \oplus e_1) = 1_A \oplus 1_B = 1_{A\oplus B}$ and $p_2p_2e_2e_2 = 1_C$. Further

$$\begin{aligned} &\mathsf{Ker}(1 \oplus p_1) \\ &= p_1^{-1}0 \wedge p_2^{-1}p_1^{-1}0 \\ &= e_2 1 \wedge p_2^{-1}p_1^{-1}0 \\ &= e_2e_2^{-1}p_2^{-1}p_1^{-1}0 \\ &= e_2p_1^{-1}0 \\ &= e_2e_2 1 = \mathsf{Im}(e_2e_2), \end{aligned} \qquad \begin{aligned} &\mathsf{Im}(1 \oplus e_1) \\ &= e_1 1 \vee e_2e_1 1 \\ &= p_2^{-1}0 \vee e_2e_1 1 \\ &= p_2^{-1}p_2e_2e_1 1 \\ &= p_2^{-1}e_1 1 \\ &= p_2^{-1}p_2^{-1}0 = \mathsf{Ker}(p_2p_2). \end{aligned}$$

Thus the above diagram is a split product, thus a biproduct. By Theorem 2.2.9 (selecting $f = 1_{A\oplus B}$ and $g = 1_C$) there is morphism $\alpha \colon A \oplus (B \oplus C) \to (A \oplus B) \oplus C$ such that

$$p_1\alpha = (1 \oplus p_1) \quad \text{and} \quad p_2\alpha = p_2p_2$$

By the same theorem, α is furthermore an isomorphism. We have

$$\alpha = (p_1\alpha, p_2\alpha) = (1 \oplus p_1, p_2p_2) = ((p_1, p_1p_2), p_2p_2).$$

We also have

$$p_1\alpha e_1 = (1 \oplus p_1)e_1 = e_1 = p_1e_1e_1 \quad \text{and} \quad p_2\alpha e_1 = p_2p_2e_1 = 0 = p_2e_1e_1.$$

Thus $\alpha e_1 = e_1 e_1$. And also

$$p_1 \alpha e_2 = (1 \oplus p_1) e_2 = e_2 p_1 = p_1(e_2 \oplus 1) \quad \text{and} \quad p_2 \alpha e_2 = p_2 p_2 e_2 = p_2 = p_2(e_2 \oplus 1).$$

Thus $\alpha e_2 = e_2 \oplus 1$. Consequently

$$\alpha = [e_1 e_1, [e_2 \oplus 1]] = [e_1 e_1, [e_1 e_2, e_2]].$$

To verify naturality, we must show that the following diagram commutes for any f, g, and h:

$$\begin{tikzcd} A \oplus (B \oplus C) \arrow[r, "\alpha"] \arrow[d, "f \oplus (g \oplus h)"] & (A \oplus B) \oplus C \arrow[d, "(f \oplus g) \oplus h"] \\ X \oplus (Y \oplus Z) \arrow[r, "\alpha"] & (X \oplus Y) \oplus Z \end{tikzcd}$$

It does indeed commute:

$$\begin{aligned}
&\alpha(f \oplus (g \oplus h)) \\
=& ((p_1, p_1 p_2), p_2 p_2)(f \oplus (g \oplus h)) \\
=& ((p_1(f \oplus (g \oplus h)), p_1 p_2(f \oplus (g \oplus h))), p_2 p_2(f \oplus (g \oplus h))) \\
=& ((f p_1, p_1(g \oplus h)p_2), p_2(g \oplus h)p_2) \\
=& ((f p_1, g p_1 p_2), h p_2 p_2) \\
=& ((f \oplus g)(p_1, p_1 p_2), h p_2 p_2) \\
=& ((f \oplus g) \oplus h)((p_1, p_1 p_2), p_2 p_2) \\
=& ((f \oplus g) \oplus h)\alpha.
\end{aligned}$$

□

Theorem 2.2.19. *Any noetherian form $\mathbb{C}$ with biproducts forms a monoidal category*

$$\langle \mathbb{C}, \oplus, 0, \alpha, p_2^{0\oplus -}, p_1^{-\oplus 0} \rangle.$$

Proof. Corollary 2.2.11 shows that $\oplus$ forms a functor from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{C}$.

We already know that having biproducts forces pointedness, thus the zero object 0 exists.

For any object A, $p_2^{0\oplus A} \colon 0 \oplus A \to A$ is a natural isomorphism: It is a projection, since it is a split epi. Also,

$$p_2^{-1} 0 = e_1 1 = e_1 0 = 0.$$

Thus it is also an embedding, hence an isomorphism. For naturality, take any $f \colon A \to X$. We must show that the diagram

$$\begin{tikzcd} 0 \oplus A \arrow[r, "p_2"] \arrow[d, "1 \oplus f"] & A \arrow[d, "f"] \\ 0 \oplus X \arrow[r, "p_2"] & X \end{tikzcd}$$

commutes. Indeed it does by the definition of $1 \oplus f$. Similarly, $p_1^{A\oplus 0}\colon A \oplus 0 \to A$ is a natural isomorphism for any object A.

What is still left to do, is to prove that two certain diagrams commute. The first diagram is commutative, for any objects A and C:

$$\begin{tikzcd} A \oplus (0 \oplus C) \arrow[r, "\alpha"] \arrow[d, "1 \oplus p_2"] & (A \oplus 0) \oplus C \arrow[d, "p_1 \oplus 1"] \\ A \oplus C \arrow[r, "1"] & A \oplus C \end{tikzcd}$$

This indeed commutes:

$$\begin{aligned} (p_1 \oplus 1)\alpha &= (p_1 \oplus 1)((p_1, p_1p_2), p_2p_2) = (p_1(p_1, p_1p_2), 1p_2p_2) \\ &= (p_1, p_2p_2) = (1 \oplus p_2)(p_1, p_2) = 1 \oplus p_2. \end{aligned}$$

The second diagram is, for any A, B, C, and D:

$$\begin{tikzcd} A \oplus (B \oplus (C \oplus D)) \arrow[r, "\alpha"] \arrow[d, "1 \oplus \alpha"] & (A \oplus B) \oplus (C \oplus D) \arrow[r, "\alpha"] & ((A \oplus B) \oplus C) \oplus D \\ A \oplus ((B \oplus C) \oplus D) \arrow[rr, "\alpha"] & & (A \oplus (B \oplus C)) \oplus D \arrow[u, "\alpha \oplus 1"] \end{tikzcd}$$

This also commutes:

$$\begin{aligned} &(\alpha \oplus 1)\alpha(1 \oplus \alpha) \\ =\,&(\alpha p_1, p_2)\alpha(1 \oplus \alpha) \\ =\,&(((p_1, p_1p_2), p_2p_2)p_1, p_2)\alpha(1 \oplus \alpha) \\ =\,&(((p_1p_1, p_1p_2p_1), p_2p_2p_1), p_2)\alpha(1 \oplus \alpha) \\ =\,&(((p_1p_1\alpha, p_1p_2p_1\alpha), p_2p_2p_1\alpha), p_2\alpha)(1 \oplus \alpha) \\ =\,&(((p_1, p_1p_1p_2), p_2p_1p_2), p_2p_2)(1 \oplus \alpha) \\ =\,&(((p_1(1 \oplus \alpha), p_1p_1p_2(1 \oplus \alpha)), p_2p_1p_2(1 \oplus \alpha)), p_2p_2(1 \oplus \alpha)) \\ =\,&(((p_1, p_1p_1\alpha p_2), p_2p_1\alpha p_2), p_2\alpha p_2) \\ =\,&(((p_1, p_1p_2), p_1p_2p_2), p_2p_2p_2) \\ =\,&((p_1\alpha, p_1p_2\alpha), p_2p_2\alpha) \\ =\,&((p_1, p_1p_2), p_2p_2)\alpha \\ =\,&\alpha\alpha \end{aligned}$$

□

Remark 2.2.20. The monoidal structure given by biproducts is in fact a 'monoidal sum structure' in the sense of [13]. This follows from Theorem 2.2.19 and Proposition 2.2.7.

2.3 Commutators

Throughout this section we are working in a noetherian form with biproducts.

2.3.1 The general theory

Definition 2.3.1. For an object G and conormal subobjects X and Y, the *commutator* $[X, Y]_G$ is defined as follows: If $(C, e\colon G \to C, m\colon X \oplus Y \to C)$ is a colimit of $\mathsf{C}_{X\oplus Y}(\iota_X, \iota_Y)$, then $[X, Y]_G = \mathsf{Ker}e$.

The commutator $[1, 1]_G$ will be denoted by $[G, G]_G$ instead.

Proposition 2.3.2. *For any $f\colon A \to W$ and $g\colon B \to W$, if $(C, e\colon W \to C, m\colon A \oplus B \to C)$ is a colimit of $\mathsf{C}_{A\oplus B}(f, g)$, then $\mathsf{Ker}e = [\mathsf{Im}f, \mathsf{Im}g]_W$.*

Proof. Let $\mathsf{Im}f = X$ and $\mathsf{Im}g = Y$. Factorize $f = \iota_X a$ and $g = \iota_Y b$, where a and b are projections of the kernels of f and g respectively. Consider the following diagram, where (L, d, n) is a colimit of $\mathsf{C}_{X\oplus Y}(\iota_X, \iota_Y)$, and (C, e, m) is a colimit of $\mathsf{C}_{A\oplus B}(f, g)$

$$\begin{tikzcd}
 & & W \arrow[dl, "d"'] \arrow[dr, "e"] & & \\
 & L & & C & \\
X \arrow[uurr, bend left, "\iota_X"] \arrow[rr, "e_1"] & & X \oplus Y \arrow[ul, "n"] \arrow[ur, dashed, "m'"'] & & Y \arrow[uull, bend right, "\iota_Y"'] \arrow[ll, "e_2"'] \\
A \arrow[u, "a"] \arrow[rr, "e_1"'] & & A \oplus B \arrow[u, "a \oplus b"] \arrow[uur, bend right] & & B \arrow[u, "b"'] \arrow[ll, "e_2"]
\end{tikzcd}$$

The unlabelled arrow $A \oplus B \to C$ is m.

Since $(L, d, n(a \oplus b))$ forms a cocone of $\mathsf{C}_{A\oplus B}(f, g)$, $\mathsf{Ker}e \le \mathsf{Ker}d$. Notice, by Proposition 2.2.12 (5),

$$\mathsf{Ker}(a \oplus b) = p_1^{-1}\mathsf{Ker}a \wedge p_2^{-1}\mathsf{Ker}b = e_1\mathsf{Ker}a \vee e_2\mathsf{Ker}b.$$

Using this, we have

$$\begin{aligned} m(a \oplus b)^{-1}0 = m(e_1 a^{-1}0 \vee e_2 b^{-1}0) &= m e_1 a^{-1}0 \vee m e_2 b^{-1} \\ &= efa^{-1}0 \vee egb^{-1}0 = 0. \end{aligned}$$

So $\mathsf{Ker}m \ge \mathsf{Ker}a \oplus b$. Since $a \oplus b$ is a projection, there is a unique $m'\colon X \oplus Y \to C$ such that $m'(a \oplus b) = m$. Then we have

$$m' e_1 a = m'(a \oplus b)e_1 = m e_1 = ef = e\iota_X a.$$

Since a is a projection, it is an epi, thus $m'e_1 = e\iota_X$. Similarly $m'e_2 = e\iota_Y$. Thus (C, e, m') is a cocone of $\mathsf{C}_{X\oplus Y}(\iota_X, \iota_Y)$. Consequently $\mathsf{Ker}d \le \mathsf{Ker}e$, and thus $\mathsf{Ker}e = \mathsf{Ker}d = [X, Y]_W$. □

We establish some basic properties.

Proposition 2.3.3. *If A, B, X, and Y are conormal subobjects of G, with $X \leq A$ and $Y \leq B$, then*

$$[X, Y] \leq [A, B].$$

Proof. Since $X \leq A$, there is a unique k such that $\iota_A k = \iota_X$. Similarly, there is a unique l such that $\iota_B l = \iota_Y$. Suppose $(C, e\colon G \to C, m\colon A \oplus B \to C)$ is a colimit of $\mathsf{C}_{A\oplus B}(\iota_A, \iota_B)$. Then $(C, e\colon G \to C, m(k \oplus l)\colon X \oplus Y \to C)$ is a cocone of $\mathsf{C}_{X\oplus Y}(\iota_X, \iota_Y)$. Thus $[X, Y] \leq \mathsf{Ker} e = [A, B]$. □

Proposition 2.3.4. *For any conormal subobjects X and Y of G, we have*

$$[X, Y] \leq N,$$

for any N normal subobject containing X. In particular, if there is a smallest normal subobject $\overline{X}$ containing X, then $[X, Y] \leq \overline{X}$.

Proof. We have that $(G/N, \pi_N\colon G \to G/N, \pi_N \iota_Y p_2\colon X \oplus Y \to G/N)$ is a cocone of $\mathsf{C}_{X\oplus Y}(\iota_X, \iota_Y)$. Consequently $N = \mathsf{Ker}\pi_N \geq [X, Y]$. □

Proposition 2.3.5. *For any morphism $f\colon G \to H$ and conormal subobjects X and Y of G, we have*

(1) $f[X, Y] \leq [fX, fY]$;

(2) $[fX, fY] \leq N$ for any normal subobject $N \geq f[X, Y]$.

Proof. (1) Suppose $(C, e\colon H \to C, m\colon X \oplus Y \to C)$ is a colimit of $\mathsf{C}(f\iota_X, f\iota_Y)$. Then (L, ef, m) is a cocone of $\mathsf{C}(\iota_X, \iota_Y)$. Consequently

$$[X, Y] \leq f^{-1}e^{-1}0 = f^{-1}[fX, fY] \quad \Rightarrow \quad f[X, Y] \leq [fX, fY].$$

(2) Suppose $(C, e\colon G \to C, m\colon X \oplus Y \to C)$ is a colimit of $\mathsf{C}(\iota_X, \iota_Y)$. Since $f[X, Y] \leq N$, we have $[X, Y] \leq f^{-1}N$, and so $\mathsf{Ker} e \leq \mathsf{Ker}\pi_N f$. Thus there is an h such that $he = \pi_N f$. One can then readily observe that $(H/N, \pi_N, hm)$ is a cocone of $\mathsf{C}(f\iota_X, f\iota_Y)$. From this it follows that $[fX, fY] \leq N$. □

Corollary 2.3.6. *We have the following immediate consequences:*

(1) if there is a smallest normal subobject $\overline{f[X, Y]}$ containing $f[X, Y]$, then

$$\overline{f[X, Y]} = [fX, fY];$$

(2) if f is a projection, then $f[X, Y] = [fX, fY]$.

Proof. (1) Since $[fX, fY]$ is normal which contains $f[X, Y]$, we have

$$[fX, fY] \leq \overline{f[X, Y]} \leq [fX, fY],$$

from which equality follows.

(2) If f is a projection, then $f[X, Y]$ is normal. Then from (1) the result follows. □

Here is another corollary.

Corollary 2.3.7. *If any commutator of any two conormal subobjects is their meet, then any direct image preserves meets of conormal subobjects and any conormal subobject is normal.*

Proof. The last part is clear: take any conormal subobject X, then $[X, X] = X \wedge X = X$, and consequently X is normal.

Take any two conormal subobjects X and Y of the same object A. It is sufficient to prove that their meet is preserved under embeddings and projections. Let $d\colon A \to B$ be an embedding. Then we have

$$d(\iota_X 1 \wedge \iota_Y 1) = d(\iota_X \iota_X^{-1} \iota_Y 1) = d\iota_X \iota_X^{-1} d^{-1} d\iota_Y 1 = d\iota_X 1 \wedge d\iota_Y 1.$$

If $d\colon A \to B$ is a projection, we use the corollary above:

$$d(X \wedge Y) = d[X, Y] = [dX, dY] = dX \wedge dY.$$

□

Proposition 2.3.8. *If direct images preserve meets of conormal subobjects, then the commutator of any two conormal subobjects contains their meet.*

Proof. For conormal subobjects X and Y of A, let $(C, e\colon A \to C, m\colon X \oplus Y \to C)$ be the colimit of $\mathsf{C}(\iota_X, \iota_Y)$. We have

$$e(X \wedge Y) = eX \wedge eY = e\iota_X 1 \wedge e\iota_Y 1 = me_1 1 \wedge me_2 1 = m(e_1 1 \wedge e_2 1) = m0 = 0.$$

Thus $X \wedge Y \leq \mathsf{Ker}e = [X, Y]$. □

Putting the last proposition and corollary together, we get:

Proposition 2.3.9. *The commutator of any conormal subobjects X and Y is $X \wedge Y$ if and only if the direct images preserves the meet of conormal subobjects and all conormal subobjects are normal.*

Proof. The one direction is given by the corollary above.

For the converse, by Proposition 2.3.4, the commutator of X and Y is contained in $X \wedge Y$. And also by the proposition above, $X \wedge Y$ is contained in the commutator. Thus the commutator is $X \wedge Y$. □

One could use the above proposition in the following way to determine whether something is a biproduct.

Theorem 2.3.10. *Consider a noetherian form, in which direct images preserve meets of conormal subobjects, and suppose that split products exists. Consider any objects A and B, and their split product G, and two morphisms $f\colon A \to W$ and $g\colon B \to W$. If there is a cocone $(C, e\colon W \to C, m\colon G \to C)$ of $\mathsf{C}(f, g)$, where e is a projection of $\mathsf{Im}f \wedge \mathsf{Im}g$, then (C, e, m) is a colimit of $C(f, g)$.*

Proof. To show that (C, e, m) is a colimit of $\mathsf{C}(f, g)$, we are going to use Proposition 2.2.8. First, it is given that e is a projection. Consider any cocone (D, d, n) over $\mathsf{C}(f, g)$. We have

$$d(f1 \wedge g1) = df1 \wedge dg1 = ne_1 1 \wedge ne_2 1 = n(e_1 1 \wedge e_2 1) = n0 = 0.$$

So $\mathsf{Ker} d \geq \mathsf{Ker} e$. Thus by Proposition 2.2.8, (C, e, m) is a colimit. □

Commutators in biproducts can be computed component-wise as in:

Proposition 2.3.11. *If A, C are conormal subobjects of G, and B, D are conormal subobjects of H, then*

$$\begin{aligned} [e_1 A \vee e_2 B, e_1 C \vee e_2 D]_{G\oplus H} &= [e_1 A, e_1 C]_{G\oplus H} \vee [e_2 B, e_2 D]_{G\oplus H} \\ &= e_1 [A, C]_G \vee e_2 [B, D]_H. \end{aligned}$$

Proof. Since $[A, C]_G$ is normal, so is $e_1[A, C]_G$ by Proposition 2.2.12 (3), thus by Corollary 2.3.6 (1) we have $e_1[A, C]_G = [e_1 A, e_1 C]_{G\oplus H}$. Similarly $e_2[B, D]_H = [e_2 B, e_2 D]_{G\oplus H}$. By Proposition 2.2.12 (2) both $e_1 A \vee e_2 B$ and $e_1 C \vee e_2 D$ are conormal. Further, from Proposition 2.3.3 it follows that

$$[e_1 A \vee e_2 B, e_1 C \vee e_2 D]_{G\oplus H} \geq [e_1 A, e_1 C]_{G\oplus H} \vee [e_2 B, e_2 D]_{G\oplus H}.$$

Notice that

$$A \oplus B \underset{p_1 \oplus p_1}{\overset{e_1 \oplus e_1}{\rightleftarrows}} (A \oplus C) \oplus (B \oplus D) \underset{p_2 \oplus p_2}{\overset{e_2 \oplus e_2}{\leftrightarrows}} C \oplus D$$

Is a split product by Theorem 2.2.9 and Proposition 2.2.12(5), thus a biproduct of $A \oplus B$ and $C \oplus D$. Suppose $(L_1, d_1 \colon G \to L_1, n_1 \colon A \oplus C \to L_1)$ is a colimit of $\mathsf{C}(\iota_A, \iota_C)$, and $(L_2, d_2 \colon H \to L_2, n_2 \colon B \oplus D \to L_2)$ is a colimit of $\mathsf{C}(\iota_B, \iota_D)$. Then $(L_1 \oplus L_2, d_1 \oplus d_2, n_1 \oplus n_2)$ is a cocone of $\mathsf{C}_{(A\oplus C)\oplus(B\oplus D)}(\iota_A \oplus \iota_B, \iota_C \oplus \iota_D)$. Consequently

$$\begin{aligned} [e_1 A \vee e_2 B, e_1 C \vee e_2 D] &= [\mathsf{Im}(\iota_A \oplus \iota_B), \mathsf{Im}(\iota_C \oplus \iota_D)] \\ &\leq \mathsf{Ker}(d_1 \oplus d_2) \\ &= e_1 d_1^{-1} 0 \vee e_2 d_2^{-1} 0 = e_1 [A, C] \vee e_2 [B, D]. \end{aligned}$$

Thus the result is true. □

Corollary 2.3.12. *For objects A and B, we have*

$$[A \oplus B, A \oplus B]_{A\oplus B} = e_1 [A, A]_A \vee e_2 [B, B]_B.$$

Proof. We have

$$[A \oplus B, A \oplus B] = [e_1 A \vee e_2 B, e_1 A \vee e_2 B] = e_1 [A, A] \vee e_2 [B, B].$$

□

2.3.2 Trivial commutators

Lemma 2.3.13. *For any pair of morphisms $f\colon A \to W$ and $g\colon B \to W$, if $[\mathsf{Im}f, \mathsf{Im}g] = 0$, then there exists a unique $h\colon A \oplus B \to W$ such that $he_1 = f$ and $he_2 = g$; that is, $[f, g]$ exists. Conversely, if $[f, g]$ exists, then $[\mathsf{Im}f, \mathsf{Im}g] = 0$.*

Proof. Suppose $(C, e\colon W \to C, m\colon A \oplus B \to C)$ is a colimit of $\mathsf{C}(f, g)$. Suppose $[\mathsf{Im}f, \mathsf{Im}g] = 0$. Then $\mathsf{Ker}e = 0$ by Proposition 2.3.2, and since e is a projection by Proposition 2.2.6, e is an isomorphism. Then the required h is $e^{-1}m$. The converse also follows easily from Proposition 2.3.2. □

Proposition 2.3.14. *For any object G and conormal subobjects X and Y such that $X \vee Y = 1$, if $[X, Y] = 0$, then both X and Y are normal subobjects of G.*

Proof. Since $[\mathsf{Im}\iota_X, \mathsf{Im}\iota_Y] = [X, Y] = 0$, by Lemma 2.3.13 there is a morphism

$$h\colon X \oplus Y \to G$$

such that $he_1 = \iota_X$ and $he_2 = \iota_Y$. Notice that h is a projection, since

$$h1 = h(e_1 1 \vee e_2 1) = he_1 1 \vee he_2 1 = X \vee Y = 1.$$

Since $e_1 1$ and $e_2 1$ are normal subobjects of $X \oplus Y$, $he_1 1$ and $he_2 1$ are normal subobjects of G, that is, X and Y are normal subobjects of G. □

Usually in group theory, a normal subgroup X of G is defined to be a subgroup such that for any $g \in G$, $gXg^{-1} \subseteq X$, or equivalently $[X, G] \leq X$. The above result allows to prove the same here:

Corollary 2.3.15. *For any object G and conormal subobject X, X is a normal subobject if and only if $[X, G] \leq X$.*

Proof. By Proposition 2.3.4, if X is normal then $[X, G] \leq X$.

For the converse, consider the projection $p = \pi_{[X,G]}$. We have $[pX, pG] = p[X, G] = 0$, and $pX \vee pG = 1$, and pX and pG are conormal subobjects. Thus by Proposition 2.3.14, pX is a normal subobject of $G/[X, G]$. Since X contains the kernel of p, $X = p^{-1}pX$. Consequently X is normal as well. □

Remark 2.3.16. If we apply this corollary to semi-abelian categories seen as noetherian forms, we recover the main result Theorem 6.3 of [22].

One can readily observe that $[e_1 1, e_2 1] = 0$ for the embeddings of any biproduct. Another way to recognize whether an object is a biproduct of two subobjects:

Theorem 2.3.17. *If object G has conormal subobjects A and B such that*

- $A \vee B = 1$,
- $A \wedge B = 0$,
- $[A, B]_G = 0$,

then $G \cong A \oplus B$.

Proof. By Proposition 2.3.14, the first and the last points implies that both A and B are normal subobjects. Then the result follows from Corollary 2.2.14. □

2.3.3 Commutative objects

Definition 2.3.18. An object A is said to be *commutative* when $[A, A]_A = 0$.

From previous results on commutators, we have the following list of basic properties of commutative objects:

- for projection $p\colon A \to B$, if A is commutative, then so is B;
- for embedding $e\colon B \to A$, if A is commutative, then so is B;
- any conormal subobject of a commutative object is normal;
- $A \oplus B$ is commutative if and only if both A and B are commutative.

Theorem 2.3.19. *For any noetherian form $\mathbb{C}$ with biproducts, the full subcategory $\mathbb{A}$ of all commutative objects is a reflective subcategory.*

Proof. Take any object G in $\mathbb{C}$. We have

$$[G/[G,G], G/[G,G]] = [\pi_{[G,G]}G, \pi_{[G,G]}G] = \pi_{[G,G]}[G,G] = 0.$$

Thus $G/[G,G]$ is commutative. Take any morphism $f\colon G \to A$, where A is commutative. Then we have

$$f[G,G] \leq [fG, fG] = 0.$$

Thus $\mathsf{Ker}f \geq [G,G]$, and thus there is a unique $h\colon G/[G,G] \to A$ such that $f = h\pi_{[G,G]}$. Consequently, $\mathbb{A}$ is a full reflective subcategory of $\mathbb{C}$. □

There is another way of getting this full subcategory of commutative objects:

Theorem 2.3.20. *The internal monoids (with respect to $\oplus$) are exactly the commutative objects. The internal monoid structure is uniquely determined on every commutative object. Further any morphism between commutative objects preserves the monoid structures.*

Proof. Suppose $(M, m\colon M \times M \to M, u\colon 0 \to M)$ is an internal monoid. Then in particular the following diagram commutes

$$\begin{tikzcd}
0 \oplus M \arrow[r, "u \oplus 1"] \arrow[dr, "p_2^{0\oplus M}"'] & M \oplus M \arrow[d, "m"] \\
& M
\end{tikzcd}$$

We have

$$me_2^{M\oplus M} = m(u \oplus 1)e_2^{0\oplus M} = p_2^{0\oplus M}e_2^{0\oplus M} = 1_M.$$

Similarly we get that $me_1^{M\oplus M} = 1_M$. Thus $(M, 1_M\colon M \to M, m\colon M\oplus M \to M)$ is a cocone of $\mathsf{C}_{M\oplus M}(1_M, 1_M)$. Since it is a cocone with least possible kernel, by Proposition 2.2.8 it is a colimit, and thus $[M, M] = [\mathsf{Im}1_M, \mathsf{Im}1_M] \leq \mathsf{Ker}1_M = 0$. So M is commutative. Notice that we were forced to have $m = [1, 1]$, so if an object is an internal monoid, it is uniquely so.

Conversely, take any commutative object M. By Lemma 2.3.13 there is a unique morphism $m = [1, 1]\colon M \oplus M \to M$ such that $me_1 = 1 = me_2$. There is also a unique morphism $u\colon 0 \to M$. The following diagram commutes:

$$\begin{tikzcd}
0\oplus M \arrow[r, "u\oplus 1"] \arrow[dr, "p_2^{0\oplus M}"'] & M\oplus M \arrow[d, "m"] & M\oplus 0 \arrow[l, "1\oplus u"'] \arrow[dl, "p_1^{M\oplus 0}"] \\
 & M &
\end{tikzcd}$$

since

$$m(u\oplus 1) = [1, 1](u\oplus 1) = [u, 1] = p_2^{0\oplus M},$$

and similarly, the other triangle commutes. Further, the following diagram also commutes:

$$\begin{tikzcd}
M\oplus(M\oplus M) \arrow[r, "\alpha"] \arrow[d, "1\oplus m"'] & (M\oplus M)\oplus M \arrow[r, "m\oplus 1"] & M\oplus M \arrow[d, "m"] \\
M\oplus M \arrow[rr, "m"] & & M
\end{tikzcd}$$

Recalling from Section 2.2.3, $\alpha = [e_1e_1, [e_1e_2, e_2]]$. We have

$$\begin{aligned}
&m(m\oplus 1)\alpha \\
=\,&m(m\oplus 1)[e_1e_1, [e_1e_2, e_2]] \\
=\,&m[(m\oplus 1)e_1e_1, [(m\oplus 1)e_1e_2, (m\oplus 1)e_2]] \\
=\,&m[e_1me_1, [e_1me_2, e_2]] \\
=\,&m[e_1, [e_1, e_2]] \\
=\,&m[e_1, 1] \\
=\,&[me_1, m] \\
=\,&[me_1, me_2m] \\
=\,&m[e_1, e_2m] \\
=\,&m(1\oplus m).
\end{aligned}$$

For the morphism part: take any $f\colon A \to B$ between two commutative objects A and B. As demonstrated before $(A, [1_A, 1_A], u_A)$ and $(B, [1_B, 1_B], u_B)$ are the unique internal monoid structures on A and B, respectively. Trivially $fu_A = u_B$, since 0 is an initial object. Also

$$[1_B, 1_B](f \oplus f) = [f, f] = f[1_A, 1_A].$$

Thus f is an internal monoid morphism from A to B. □

Remark 2.3.21. Commutative objects defined above are precisely the indiscrete objects in the sense of Definition 2.8.1 of [13]. Although we have included the proof of Theorem 2.3.20, it is actually a simple corollary of Theorem 2.8.2 of [13]. Furthermore, it is easy to show that biproducts give a symmetric monoidal structure and hence, by Theorem 2.8.3 of [13], the unique internal monoid structure on each commutative object is an internal commutative monoid structure, which answers the question posed to the author by Tim Van der Linden.

The result below generalizes the fact that split extensions of abelian groups are products.

Proposition 2.3.22. *For any commutative objects A, X, and B, if the diagram*

$$\begin{tikzcd} A \arrow[r, "g"] & X \arrow[r, shift left, "f"] & B \arrow[l, dashed, shift left, "s"] \end{tikzcd}$$

satisfies $\mathsf{Ker} f = \mathsf{Im} g$*, g is an embedding and f is a projection, and $fs = 1_B$, then $X \cong A \oplus B$.*

Proof. The image of s is normal, since the image of s is conormal and X is commutative. Further, the kernel of f is conormal, since it is the image of g. Thus, by Corollary 2.2.15, we have

$$X \cong \mathsf{Ker} f \oplus \mathsf{Im} s = A \oplus B.$$

□

Chapter 3

Studying a fixed noetherian form

3.1 Introduction

In this chapter, we will first explore how basic notions in a category interact with the language of a noetherian form. After that, we show that a category $\mathbb{C}$ is semi-abelian if and only if there is a noetherian form over $\mathbb{C}$ such that the inverse images of conormal subobjects are conormal. This is an improvement of a known characterization which is given in terms of all subobjects being conormal. As a corollary, we obtain yet another characterization theorem: a category $\mathbb{C}$ with products and coproducts is semi-abelian if and only if there is a noetherian form over $\mathbb{C}$ where the biproducts are the products.

3.2 General observations in noetherian forms

Throughout this section we are working in a fixed noetherian form, unless stated otherwise.

3.2.1 Observations in connection to limits

Definition 3.2.1. If there is a largest conormal subobject contained in X, it will be denoted by $\underline{X}$. We will say that $\underline{X}$ exist when such a largest conormal subobject exists. The dual concept, that of a smallest normal subobject containing X, will be denoted by $\overline{X}$.

Proposition 3.2.2. *In a noetherian form, the diagram, where m is an embedding,*

$$\begin{tikzcd} P \arrow[r, dashed, "n"] \arrow[d, dashed, "g"] & A \arrow[d, "f"] \\ C \arrow[r, "m"] & B \end{tikzcd}$$

has a pullback if and only if there is a largest conormal subobject $\underline{f^{-1}m1}$ contained in $f^{-1}m1$. The pullback is given by (g, n) where n is the embedding of $\underline{f^{-1}m1}$.

Proof. Suppose the pullback of m and f exists. Suppose the following is a pullback square:

$$\begin{tikzcd}
P \arrow[r, "n"] \arrow[d, "g"] & A \arrow[d, "f"] \\
C \arrow[r, "m"] & B
\end{tikzcd}$$

We have

$$n1 \leq f^{-1}fn1 = f^{-1}mg1 \leq f^{-1}m1.$$

So $n1$ is a conormal subobject contained in $f^{-1}m1$. Suppose C is a conormal subobject contained in $f^{-1}m1$. Let ι_C be the embedding of C. Then

$$f\iota_C 1 \leq ff^{-1}m1 \leq m1.$$

Since m is an embedding, there is a unique u such that $mu = f\iota_C$. Since g and n is the pullback, there is in particular an h such that $nh = \iota_C$. Consequently $C \leq \mathsf{Im}n$, and thus $\mathsf{Im}n$ is the largest conormal subobject contained in $f^{-1}m1$ and n is its embedding.

For the converse, suppose there is a largest conormal subobject $\underline{f^{-1}m1}$ contained in $f^{-1}m1$. Let n be the embedding of it. We have

$$fn1 = f\underline{f^{-1}m1} \leq ff^{-1}m1 \leq m1.$$

Thus there is a unique g such that $mg = fn$. Consider the diagram

$$\begin{tikzcd}
W \arrow[dr, dashed, "h"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& P \arrow[r, "n"] \arrow[d, "g"] & A \arrow[d, "f"] \\
& C \arrow[r, "m"] & B
\end{tikzcd}$$

where $fu = mv$. We have

$$u1 \leq f^{-1}fu1 = f^{-1}mv1 \leq f^{-1}m1.$$

Thus $u1 \leq \underline{f^{-1}m1}$. Since n is the embedding of $\underline{f^{-1}m1}$, there is a unique h such that $nh = u$. We have

$$mgh = fnh = fu = mv.$$

And since m is a monomorphism, $gh = v$. And so g and n is the pullback of m and f. $\square$

As will be shown in the next chapter, if there is a noetherian form over a pointed category, then there is a noetherian form over the same category where the zero object has exactly one subobject (Corollary 4.3.6). So without loss of generality, we will assume that a noetherian form over a pointed category is such that the zero object has exactly one element. Alternatively, one could assume that all bottom subobjects 0 are conormal; this will force the zero object to have exactly one element.

Proposition 3.2.3. *In a noetherian form over a pointed category, the zero objects are exactly those objects which have one subobject. Further, the zero morphisms are exactly those morphisms with kernel being* 1*, or equivalently, with image being* 0*.*

Proof. By assumption, any zero object has exactly one subobject. Further, consider any object with one subobject. There is a morphism from the zero object to that object. Since both the domain and codomain have only one subobject, that morphism is forced to be an isomorphism.

If $f\colon G \to H$ is a zero morphism, then it factors through a zero object, and consequently $f1 = 0$ and $f^{-1}0 = 1$. If f has image 0, then it has to factor through the embedding ι_0 of the bottom subobject of its codomain. The domain of ι_0 is a zero object, since it only has one subobject. Thus f is a zero morphism. □

From Proposition 3.2.3 together with Proposition 3.2.2, it follows that

Corollary 3.2.4. *In a noetherian form over a pointed category,* f *has a categorical kernel if and only if there is a largest conormal subobject* $\underline{f^{-1}0}$ *contained in* $f^{-1}0$*. In either case, the embedding of* $\underline{f^{-1}0}$ *is the categorical kernel of* f*.*

Proposition 3.2.5. *Any noetherian form over a pointed category with pullbacks along embeddings and with finite products is finitely complete.*

Proof. Any split monomorphism is an embedding. From this and the fact that pullbacks along split monomorphisms and finite products implies finitely complete, the lemma follows. □

Proposition 3.2.6. *If there is a noetherian form over a category with products, such that the bottom subobjects are all conormal, then the category has an initial object.*

Proof. Take any object X, and let $\iota\colon I \to X$ be the embedding of the bottom subobject 0. Notice that $\mathsf{sub}I$ has exactly one element. Consider any morphism $f\colon I \to I$. Then both $(1,1),(1,f)\colon I \to I \times I$ are embeddings of the bottom subobject 0 of $I \times I$. So there exists a unique h such that $(1,f) = (1,1)h = (h,h)$. Thus $1 = h = f$, and so there is a unique morphism $I \to I$. Consider any object A. Suppose $j\colon J \to I \times A$ is the embedding of the bottom subobject 0 of $I \times A$. Then $\pi_1 j$ is trivially an isomorphism, for both its domain and codomain has one element. So $\pi_2 j(\pi_1 j)^{-1}$ is a morphism from I to A, which is then trivially an embedding of the bottom subobject of 0 of A. If $f\colon I \to A$ is another morphism, it is also an embedding of 0. Thus there exists a morphism $h\colon I \to I$ such that $i = fh$. But since there is exactly one morphism $I \to I$, $h = 1$. Therefore I is an initial object. □

For the rest of this subsection we are concerned with noetherian forms satisfying

(N) for any subobject there is a smallest normal subobject containing that subobject.

This used to be a consequence of an older version of Axiom 3 on noetherian forms. The older version stated that projections of any subobject must exists, not just the normal ones. Then, for any subobject, the kernel of its projection is the smallest normal subobject containing it.

Lemma 3.2.7. *In a noetherian form satisfying (N), the meet of any two normal subobjects is again normal.*

Proof. Consider any object X and normal subobjects A and B. Then we have

$$A \wedge B \leq \overline{A \wedge B} \leq A, B,$$

from which the lemma follows. □

Proposition 3.2.8. *Suppose there is a noetherian form, which satisfies (N), over a category with products. In this noetherian form, the projections of any categorical product $A \times B$ satisfies* $\mathsf{Ker}\pi_1 \wedge \mathsf{Ker}\pi_2 = 0$.

Proof. Let $N = \pi_1^{-1}0 \wedge \pi_2^{-1}0$. Consider the diagram, where π_N is a projection of N:

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_N"] & B \\
& (A \times B)/N \arrow[ul, dashed, "q_1"] \arrow[ur, dashed, "q_2"'] &
\end{tikzcd}$$

Since $\pi_I N = 0$ there is a unique morphism q_1 such that $q_1\pi_N = \pi_1$. Similarly, there is a unique morphism q_2 such that $q_2\pi_N = \pi_2$. We have

$$(q_1, q_2)\pi_N = (q_1\pi_N, q_2\pi_N) = (\pi_1, \pi_2) = 1.$$

Thus π_N is a split mono, and hence an isomorphism. Thus $\mathsf{Ker}p_1 \wedge \mathsf{Ker}p_2 = 0$. □

Proposition 3.2.9. *If there is a noetherian form, which satisfies (N) and all bottom subobjects are conormal, over a category with binary products, then the category is finitely complete and pointed.*

Proof. First to prove pointedness: Take any object X, and let $T = X/X$ be the domain of the projection of the top subobject of X. It is readily observed that T only has one subobject. From Proposition 3.2.8, for the product $T \times T$, we have

$$0 = \pi_1^{-1}0 \wedge \pi_2^{-1}0 = \pi_1^{-1}1 \wedge \pi_2^{-1}1 = 1 \wedge 1 = 1.$$

Thus $T \times T$ also has exactly one subobject. In particular, π_1 is an isomorphism. Consider any morphism $f\colon T \to T$. Then $\pi_1(1, f) = 1 = \pi_1(1, 1)$, and since π_1 is an isomorphism, $(1, f) = (1, 1)$, so $1 = f$. Thus there is a unique morphism $T \to T$, namely the identity morphism. Consider any object Y. If there are two morphisms $u, v\colon T \to Y$, then both are embeddings of the bottom subobject 0. Thus there is an $h\colon T \to T$ such that $uh = v$. But since we know that $h = 1_T$, $u = v$. So there is at most one morphism $T \to Y$. Dually, there is at most one morphism $Y \to T$. Consider the product $T \times Y$. Let $m\colon I \to T \times Y$ be an embedding of the bottom subobject 0. Then $\pi_1 m$ is an isomorphism, since both its domain and codomain only have one subobject. So we can construct a morphism $i = \pi_2 m(\pi_1 m)^{-1}\colon T \to Y$. Further, for the projection of the top subobject of Y, $\pi_Y\colon Y \to Y/Y$, $\pi_Y i$ is an isomorphism, so $(\pi_Y i)^{-1}\pi_Y\colon Y \to T$ is a morphism. Thus T is a zero object.

From the assumption that for any subobject there is a largest conormal subobject contained in it, by Proposition 3.2.2 the category has pullbacks. Thus it is finitely complete. □

Theorem 3.2.10. *If there is a noetherian form, which satisfies (N) and its dual, over a category with binary products and coproducts, and if all normal subobjects are conormal, then binary products are biproducts (in the sense of the previous chapter) with embeddings* $(1, 0)$ *and* $(0, 1)$.

Proof. The category is pointed, finitely complete and finitely cocomplete by Proposition 3.2.9 and its dual.

For any objects A and B, consider the product $A \times B$ with the four morphisms, where $p_1 = \pi_1$, $p_2 = \pi_2$, $e_1 = (1, 0)$, and $e_2 = (0, 1)$.

$$
\begin{tikzcd}
A \arrow[r, shift left, "e_1"] & A \times B \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & B \arrow[l, shift right, "e_2"']
\end{tikzcd}
$$

Clearly we have $p_1 e_1 = 1$ and $p_2 e_2 = 1$. Also, e_1 is an embedding such that $p_2 e_1 = 0$. Let $k\colon K \to A \times B$ be an embedding of $p_2^{-1}0$, which is conormal by assumption. Then we have

$$k = (p_1, p_2)k = (p_1 k, p_2 k) = (p_1 k, 0) = (1, 0)p_1 k = e_1 p_1 k.$$

Consequently we have

$$p_2^{-1}0 = k1 = e_1 p_1 k1 \leq e_1 1 \leq p_2^{-1}0.$$

Thus $p_2^{-1}0 = e_1 1$, and similarly $p_1^{-1}0 = e_2 1$. So $A \times B$ is a split product of A and B. Since the category is finitely complete and cocomplete, $A \times B$ is a biproduct of A and B. □

3.2.2 Relating projections to different types of epimorphisms

Recall that a morphism f in a category is called a *strong epimorphism* if for any commutative diagram,

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "u"'] & {} \arrow[d, "v"] \arrow[dl, dotted, "h"'] \\ {} \arrow[r, "m"'] & {} \end{tikzcd}$$

where m is a monomorphism, there is a unique h making the two triangles commute, that is $hf = u$ and $mh = v$.

Proposition 3.2.11. *Any strong epimorphism in a noetherian form is a projection.*

Proof. Consider any strong epimorphism f. It factors as $f = me$, where e is a projection of the kernel of f and m is an embedding of the image of f. We have the following diagram

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "e"'] & {} \arrow[d, "1"] \arrow[dl, dotted, "h"'] \\ {} \arrow[r, "m"'] & {} \end{tikzcd}$$

Since f is a strong epimorphism, there is an h such that $hf = e$ and $mh = 1$. From $hf = e$ it follows that h is a projection, and from $mh = 1$ it follows that h is an embedding. Thus h is an isomorphism, and hence f is a projection. □

Corollary 3.2.12. *Any regular epi in a noetherian form is a projection.*

Proof. Since any regular epimorphism is a strong epimorphism, it follows from the previous proposition. □

Recall than a *normal* epimorphism is a morphism which is the categorical cokernel of some morphism.

Proposition 3.2.13. *In a noetherian form over a pointed category (assuming that the zero object has exactly one subobject), the following are equivalent for any projection p which has a categorical kernel:*

(1) p *is a normal epimorphism;*

(2) $\overline{(\underline{p^{-1}0})} = p^{-1}0.$

Proof. Suppose projection p is a normal epimorphism. Then p is the cokernel of its kernel. By Corollary 3.2.4, the categorical kernel of p is the embedding of $\underline{p^{-1}0}$. By the dual of Corollary 3.2.4, the categorical cokernel of the embedding of $\underline{p^{-1}0}$ is the projection of $\overline{(\underline{p^{-1}0})}$, which is p. So $\overline{(\underline{p^{-1}0})} = p^{-1}0$.

Conversely, suppose $\overline{(\underline{p^{-1}0})} = p^{-1}0$. Then from Corollary 3.2.4 the cokernel of the embedding of $\underline{p^{-1}0}$ is the projection of $\overline{(\underline{p^{-1}0})}$, which is p. Thus p is a normal epimorphism. $\square$

Corollary 3.2.14. *In a noetherian form over a pointed category, in which normal subobjects are conormal, the normal epimorphisms are exactly the projections with conormal kernel. Further, the embeddings are exactly the monomorphisms.*

Proof. Suppose p is a normal epimorphism. Since it is in particular a strong epimorphism, by Proposition 3.2.11 it is a projection.

Conversely, take any projection p. Then, since $p^{-1}0$ is conormal, $\overline{(\underline{p^{-1}0})} = \overline{p^{-1}0} = p^{-1}0$. Thus by the previous proposition p is a normal epimorphism.

For the second part, we already know that any embedding is a monomorphism. Consider any monomorphism $m\colon A \to B$. Since normal subobjects are conormal, by Corollary 3.2.4, m has a kernel $k\colon K \to A$ which is the embedding of $m^{-1}0$. So $mk = 0 = m0$, and so $k = 0$. Consequently the image of k is the bottom subobject, but is also equal to $m^{-1}0$. Hence m is an embedding. $\square$

3.3 Semi-abelian and related categories

From results in [14] (see also [17]), it follows that with ordinary categorical subobjects a semi-abelian category constitutes a noetherian form. Moreover, from [17] we know that any noetherian form in which any subobject is conormal is a semi-abelian category. Here we are going to prove a seemingly stronger result that any noetherian form in which inverse images of conormal subobjects are conormal, is semi-abelian, provided that it is pointed and has products and coproducts.

The following lemma is a version of the short five lemma in a noetherian form.

Lemma 3.3.1. *In a noetherian form, consider the following commutative diagram*

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[d, "v"] \\
K' \arrow[r, "k'"] & A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

where k and k' are embeddings, and f and f' are projections, and $k1 = f^{-1}0$ and $k'1 = f'^{-1}0$. If u and v are projections, then so is w. Dually, if u and v are embeddings, then so is w.

Proof. Suppose u and v are projections. We have

$$\begin{aligned} & f'w1 = vf1 = 1 \\ \Rightarrow\, & f'^{-1}0 \vee w1 = 1 \\ \Rightarrow\, & k'1 \vee w1 = 1 \\ \Rightarrow\, & k'u1 \vee w1 = 1 \\ \Rightarrow\, & wk1 \vee w1 = 1 \\ \Rightarrow\, & w1 = 1. \end{aligned}$$

□

Theorem 3.3.2. *Any noetherian form with zero object, in which all normal subobjects are conormal, is protomodular.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"] & A \arrow[r, "f"] \arrow[d, "w"] & B \arrow[l, bend left, "s"] \arrow[d, "v"] \\ K' \arrow[r, "k'"] & A' \arrow[r, "f'"] & B' \arrow[l, bend left, "s'"] \end{tikzcd}$$

where k is the kernel of f, and k' is the kernel of f', and $fs = 1$ and $f's' = 1$. So k and k' are readily embeddings, and f and f' are projections. By Corollary 3.2.4, $k1 = f^{-1}0$ and $k'0 = f'^{-1}0$. Then by Lemma 3.3.1, it follows that if u and v are isomorphisms, so is w. Thus the category is protomodular. □

The lemma below is well known.

Lemma 3.3.3. *In a pointed category with products, for any product $A \times B$, $(1, 0)$ is the kernel of π_2 and $(0, 1)$ is the kernel of π_1.*

Proof. Suppose $\pi_2 k = 0$. Then $k = (\pi_1 k, 0) = (1, 0)(\pi_1 k)$. Since $(1, 0)$ is a mono, $\pi_1 k$ is the unique morphism such that $k = (1, 0)\pi_1 k$. Since also $\pi_2(1, 0) = 0$, $(1, 0)$ is the kernel of π_2. Similarly, $(0, 1)$ is the kernel of π_1. □

Lemma 3.3.4. *In any noetherian form with a zero object and with products, in which all normal subobjects are conormal, for any binary product, we have*

- $\mathsf{Im}(1, 0) = \mathsf{Ker}\pi_2$ *and* $\mathsf{Im}(0, 1) = \mathsf{Ker}\pi_1$.
- $\mathsf{Im}(1, 0) \vee \mathsf{Im}(0, 1) = 1$ *and* $\mathsf{Ker}\pi_1 \wedge \mathsf{Ker}\pi_2 = 0$.
- *For any f and g,*

$$\mathsf{Ker} f \times g = \pi_1^{-1}\mathsf{Ker} f \wedge \pi_2^{-1}\mathsf{Ker} g \quad \textit{and} \quad \mathsf{Im} f \times g = (1, 0)\mathsf{Im} f \vee (0, 1)\mathsf{Im} g.$$

Proof. The first part of proof follows from the above lemma and Corollary 3.2.4. For the second part, we have

$$(1,0)1 \vee (0,1)1 = (1,0)1 \vee \pi_1^{-1}0 = \pi_1^{-1}\pi_1(1,0)1 = \pi_1^{-1}1 = 1,$$

$$\pi_1^{-1}0 \wedge \pi_2^{-1}0 = \pi_1 0 \wedge (1,0)1 = (1,0)(1,0)^{-1}\pi_1^{-1}0 = (1,0)0 = 0.$$

For the last part, we have:

$$\begin{aligned}(f\times g)^{-1}0 = (f\times g)^{-1}(\pi_1^{-1}0 \wedge \pi_2^{-1}0) =& (f\times g)^{-1}\pi_1^{-1}0 \wedge (f\times g)^{-1}\pi_2^{-1}0 \\ =& \pi_1^{-1}f^{-1}0 \wedge \pi_2^{-1}g^{-1}0.\end{aligned}$$

Notice that $(f\times g)(1,0) = (1,0)f$ and $(f\times g)(0,1) = (0,1)g$. This can be demonstrated by composing with π_1 and π_2. Then

$$\begin{aligned}(f\times g)1 = (f\times g)((1,0)1 \vee (0,1)1) &= (f\times g)(1,0)1 \vee (f\times g)(0,1)1 \\ &= (1,0)f1 \vee (0,1)g1.\end{aligned}$$

□

Theorem 3.3.5. *Any noetherian form with a zero object and with products, in which inverse images of conormal subobjects are conormal and all the bottom subobjects are conormal, is a regular category.*

Proof. By Lemma 3.2.5, it is finitely complete. Further, regular epimorphisms are exactly normal epimorphisms (since by Theorem 3.3.2, the category is protomodular, which is also pointed). So by Lemma 3.2.14, the regular epimorphisms are exactly the projections.

Take the kernel pair (K, k_1, k_2) of morphism $f\colon A \to B$. Consider the factorization $f = ep$ where p is a projection and e is an embedding. Then $pk_1 = pk_2$ as well. If $pu = pv$, then $fu = fv$, so there is a unique h then such that $k_1 h = u$ and $k_2 h = v$. Thus (K, k_1, k_2) is also the kernel pair of projection p. Since p is a regular epi, it is the coequalizer of its kernel pair. Thus coequalizers of kernel pairs exist.

To show that regular epimorphisms are pullback stable, consider any regular epimorphism $p\colon B \to C$ and arbitrary morphism $f\colon A \to C$. Their pullback can be constructed as follows:

$$\begin{tikzcd}
E \arrow[dr, dashed, "e"] \arrow[drr, bend left, "\pi_2 e"] \arrow[ddr, bend right, "\pi_1 e"'] & & \\
& A\times B \arrow[r, dashed, "\pi_2"] \arrow[d, dashed, "\pi_1"'] & B \arrow[d, "p"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

where e is the equalizer of $f\pi_1$ and $p\pi_2$. So we need to prove that $\pi_1 e1 = 1$. We have that e is the embedding of $(f\pi_1, p\pi_2)^{-1}(1,1)1 = (f\times p)^{-1}(1,1)1$, since e can

can be constructed as the part of the pullback of $(f\pi_1, p\pi_2)$ and $(1_C, 1_C)$. We have, using Proposition 1.4.1,

$$\begin{aligned} f(\pi_1 e1) = \ & f\pi_1(f\times p)^{-1}(1,1)1 \\ &= \pi_1(f\times p)(f\times p)^{-1}(1,1)1 \\ &= \pi_1((f\times p)1 \wedge (1,1)1) \\ &= \pi_1(((1,0)f1 \vee (0,1)p1) \wedge (1,1)1) \\ &= \pi_1(((1,0)f1 \vee (0,1)1) \wedge (1,1)1) \\ &= \pi_1(((1,0)f1 \vee \pi_1^{-1}0) \wedge (1,1)1) \\ &= \pi_1((1,0)f1 \vee \pi_1^{-1}0) \wedge \pi_1(1,1)1) \\ &= \pi_1((1,0)f1 = f1 \end{aligned}$$

We also have, making again use of Proposition 1.4.1,

$$\begin{aligned} \pi_1 e1 &= \pi_1(f\times p)^{-1}(1,1)1 \\ &\geq \pi_1(f\times p)^{-1}0 \\ &= \pi_1(\pi_1^{-1}f^{-1}0 \wedge \pi_2^{-1}g^{-1}0) \\ &= \pi_1(\pi_1^{-1}f^{-1}0) \wedge \pi_1\pi_2^{-1}g^{-1}0 \\ &= f^{-1}0 \wedge \pi_1\pi_2^{-1}g^{-1}0 \\ &\geq f^{-1}0 \wedge \pi_1\pi_2^{-1}0 \\ &= f^{-1}0 \wedge \pi_1(1,0)1 \\ &= f^{-1}0 \end{aligned}$$

Putting these two calculations together, we get $\pi_1 e1 = 1$. □

Theorem 3.3.6. *In any noetherian form with a zero object and products, in which all normal subobjects are conormal, any reflexive relation (R, d_1, d_2, s) on any object X is effective.*

Proof. Since $d_1 s = 1 = d_2 s$, both d_1 and d_2 are projections. Since $\overline{d_1 d_2^{-1}0} = d_1 d_2^{-1}0$, the pushout

$$\begin{array}{ccc} R & \xrightarrow{d_2} & X \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow[f]{} & Y \end{array}$$

exists by the dual of Lemma 3.2.2, where f is the projection of $d_1 d_2^{-1}0$. Notice that $f = fd_1 s = f'd_2 s = f'$. Also notice that

$$\begin{aligned} d_2 d_1^{-1}0 = d_2(d_1^{-1}0 \vee d_2^{-1}0) &= d_2(d_1^{-1}d_1 d_2^{-1}0) = d_2 d_1^{-1} f^{-1}0 \\ &= d_2 d_2^{-1} f^{-1}0 = f^{-1}0 = d_1 d_2^{-1}0. \end{aligned}$$

Further notice that $\pi_1(d_1,d_2)d_1^{-1}0 = 0$, thus $\pi_1^{-1}0 \geq (d_1,d_2)d_1^{-1}0$ and similarly $\pi_2^{-1}0 \geq (d_1,d_2)d_2^{-1}0$. From all this, we have, together with the restricted modular law,

$$\begin{aligned}
(f\times f)^{-1}0 &= \pi_1^{-1}f^{-1}0 \wedge \pi_2^{-1}f^{-1}0\\
&= \pi_1^{-1}d_1d_2^{-1}0 \wedge \pi_2^{-1}d_2d_1^{-1}0\\
&= \pi_1^{-1}\pi_1(d_1,d_2)d_2^{-1}0 \wedge \pi_2^{-1}\pi_2(d_1,d_2)d_1^{-1}\\
&= (\pi_1^{-1}0 \vee (d_1,d_2)d_2^{-1}0) \wedge (\pi_2^{-1}0 \vee (d_1,d_2)d_1^{-1}0)\\
&= ((\pi_1^{-1}0 \vee (d_1,d_2)d_2^{-1}0) \wedge \pi_2^{-1}0) \vee (d_1,d_2)d_1^{-1}0\\
&= ((\pi_1^{-1}0 \wedge \pi_2^{-1}0) \vee (d_1,d_2)d_2^{-1}0 \vee (d_1,d_2)d_1^{-1}0\\
&= (d_1,d_2)d_1^{-1}0 \vee (d_1,d_2)d_2^{-1}0\\
&\leq (d_1,d_2)1.
\end{aligned}$$

We further have,

$$\begin{aligned}
&1 = fd_11\\
\Rightarrow\ &(1,1)1 = (1,1)fd_11 = (fd_1,fd_1)1 = (fd_1,fd_2)1 = (f\times f)(d_1,d_2)1\\
\Rightarrow\ &(f\times f)^{-1}(1,1)1 = (d_1,d_2)1 \vee (f\times f)^{-1}0 = (d_1,d_2)1.
\end{aligned}$$

Since R is a relation, (d_1,d_2) is a monomorphism. Since the category is pointed and normal subobjects are conormal, (d_1,d_2) is an embedding. From the above calculations, (d_1,d_2) is the embedding of $(f\times f)^{-1}(1,1)1$, thus (d_1,d_2) is the equalizer of $f\pi_1$ and $f\pi_2$. Thus $(R,\pi_1(d_1,d_2),\pi_2(d_1,d_2))$ is the pullback of f and f, that is, (R,d_1,d_2) is the kernel pair of f. □

Putting everything together, we get:

Theorem 3.3.7. *Any noetherian form with zero object, products and coproducts, in which inverse images of conormal subobjects are conormal and* 0 *is conormal, is semi-abelian.*

3.4 Comparison of biproducts with products and coproducts

Proposition 3.4.1. *Consider any noetherian form that has biproducts, (categorical) products and coproducts. For any pair of objects A and B, the canonical morphism $I\colon A+B\to A\times B$ (that is $\pi_iI\iota_j=\delta_{i,j}$) factors as*

$$\begin{tikzcd}
A+B \arrow[r, "e"'] \arrow[rr, bend left, "I"] & A\oplus B \arrow[r, "m"'] & A\times B
\end{tikzcd}$$

where e is a projection and m is an embedding, such that $e\iota_1 = e_1$, $e\iota_2 = e_2$, $p_1 = \pi_1 m$, and $p_2 = \pi_2 m$.

Proof. Let e be the unique morphism $A+B \to A \oplus B$ such that $e\iota_i = e_i$ for $i = 1, 2$. Then

$$1 = e_1 1 \vee e_2 1 = e\iota_1 1 \vee e\iota_2 1 = e(\iota_1 1 \vee \iota_2 1) \leq e1 \leq 1.$$

Thus e is a projection.

Dually, the unique morphism $m\colon A \oplus B \to A \times B$ such that $\pi_i m = p_i$ for $i = 1, 2$, is an embedding. Furthermore, we have

$$\pi_i m e \iota_j = p_i e_j = \delta_{i,j}.$$

Thus me is the canonical morphism. □

Theorem 3.4.2. *For any noetherian form $\mathbb{C}$ with biproducts and (categorical) products, the following are equivalent*

(1) For any two objects, their biproduct and product coincide.

(2) Inverse images of conormal subobjects are conormal.

(3) Normal subobjects are conormal.

(4) For any product, $(1, 0)$ and $(0, 1)$ are jointly extremal epimorphic.

Whenever any of the above holds, $\mathbb{C}$ is protomodular and Barr exact.

Proof. (1) ⇒ (2): Take a morphism $f\colon A \to B$ and a conormal subobject X of B. Consider the biproduct $A \oplus B$. Since it is a product, there is a morphism $h\colon A \to A \oplus B$ such that $p_1 h = 1$ and $p_2 h = f$. By Proposition 2.2.12(4), $p_2^{-1}X$ is conormal. Since h is a split mono, it is an embedding, from which it follows that

$$f^{-1}X = h^{-1}p_2^{-1}X$$

is conormal.

(2) ⇒ (3): Since biproducts exist, all bottom subobjects 0 are conormal, from which it follows that all normal subobjects are conormal.

(3) ⇒ (4): Consider any product $A \times B$. Take any monomorphism $f\colon W \to A \times B$ such that $(1, 0)$ and $(0, 1)$ factor through f, that is $(1, 0) = fa$ and $(0, 1) = fb$, for some morphisms a and b. By Lemma 3.2.14, f is an embedding. Further by Lemma 3.3.4, we have

$$f1 \geq fa1 \vee fb1 = (1, 0)1 \vee (0, 1)1 = 1.$$

Thus f is an isomorphism.

(4) ⇒ (1): Consider the morphism $m = (p_1, p_2)\colon A \oplus B \to A \times B$. Since $me_1 = (1, 0)$ and $me_2 = (0, 1)$, and m is a monomorphism, m is an isomorphism.

For the last part, if either of the above points hold, then by Theorems 3.3.2, 3.3.5, and 3.3.6, $\mathbb{C}$ is protomodular and Barr exact. □

Corollary 3.4.3. *Any noetherian form with a zero object, biproducts, products and coproducts is semi-abelian if and only if the inverse image of any conormal subobject is conormal.*

Chapter 4

General methods for building noetherian forms

4.1 Introduction

This chapter mainly deals with constructing new forms out of existing forms, with the goal to find new examples of noetherian forms. We will first look at some categorical constructions we can apply to functors. Some constructions preserve noetherian forms, like the product $F \times G$ of two noetherian forms F and G is again noetherian. Other constructions, such as pulling back, do not preserve noetherian forms. In some of these cases we identify sufficient conditions which allow preservation. Another construction is with the help of closure operators: from a given form and a closure operator on the subobjects, one could construct the form of closed subobjects. Under some mild conditions, this construction preserves noetherian forms. The construction of the form of subobjects from a form using closure operators, could be thought of as "throwing away" unnecessary subobjects to perhaps get a simpler form.

Note that when we are working with multiple forms and we want to make clear that a subobject is a subobject with respect to a form, say F, we will call it an F-subobject. And, to make clear what operations belong to what form, we will use a superscript (usually the symbol of the form) over the operations.

4.2 Categorical constructions

4.2.1 Dual categories

The dual of a form $F\colon \mathbb{B} \to \mathbb{C}$ is another form $F^{\mathrm{op}}\colon \mathbb{B}^{\mathrm{op}} \to \mathbb{C}^{\mathrm{op}}$. Since all of our axioms are self-dual, there is not much else to say.

4.2.2 Product of forms

Consider two forms $F\colon \mathbb{B} \to \mathbb{C}$ and $G\colon \mathbb{B}' \to \mathbb{C}'$. Their product is the product of functors: $F \times G\colon \mathbb{B} \times \mathbb{B}' \to \mathbb{C} \to \mathbb{C}'$, $(A, A') \mapsto (FA, GA')$ and $(f, f') \mapsto (Ff, Gf')$. Here the relation $\leq_{(f,g)}$ for a morphism (f, f') in $\mathbb{C} \times \mathbb{C}'$ defined as, for any $F \times G$-subobjects (A, A') and (B, B'),

$$(A, A') \leq_{(f,f')} (B, B') \iff A \leq^F_f B \text{ and } A' \leq^G_{f'} B'.$$

From this observation, it follows that the form $F \times G$ is orean if and only if both F and G are orean. Similarly $F \times G$ is noetherian if and only if both F and G are noetherian.

4.2.3 The diagonal of pullback of forms

Consider any two forms $F\colon \mathbb{A} \to \mathbb{C}$ and $G\colon \mathbb{B} \to \mathbb{C}$ over the same category $\mathbb{C}$, and consider their pullback

$$\begin{tikzcd} \mathbb{P} \arrow[r, "Q"] \arrow[d, "P"'] & \mathbb{B} \arrow[d, "G"] \\ \mathbb{A} \arrow[r, "F"] & \mathbb{C} \end{tikzcd}$$

Let H denote the diagonal of the pullback. That is, $H = GQ = FP$.

The category $\mathbb{P}$ consists of pairs of morphisms (f, g), where f is a morphism in $\mathbb{A}$ and g is a morphism in $\mathbb{B}$ such that $Ff = Gg$. The functors P and Q simply project each pair to the suitable component. In particular for any object $(A, B) \in \mathbb{P}$, $H(A, B) = X$ if and only if both $FA = X$ and $GB = X$; in other words

$$\mathsf{sub}^H X = \mathsf{sub}^F X \times \mathsf{sub}^G X.$$

Also, for any morphism $f\colon X \to Y$ in $\mathbb{C}$, and $(A, C) \in \mathsf{sub}^H X$ and $(B, D) \in \mathsf{sub}^H Y$, we have $(A, C) \leq^H_f (B, D)$ if and only if there is an arrow $(u, v)\colon (A, C) \to (B, D)$ in $\mathbb{P}$ such that $H(u, v) = f$. Further $H(u, v) = f$ if and only if $Fu = f$ and $Gv = f$. From this observation, we get

$$(A, C) \leq^H_f (B, D) \iff A \leq^F_f B \text{ and } C \leq^G_f D.$$

From this, it readily follows that:

Proposition 4.2.1. *H is orean if and only if both F and G are orean. And if that is the case, for any object X and H-subobjects (A, A') and (B, B') of X, we have*

- $0^H_X = (0^F_X, 0^G_X)$ *and* $1^H_X = (1^F_X, 1^G_X)$;
- $(A, A') \wedge^H (B, B') = (A \wedge^F B, A' \wedge^G B')$ *and* $(A, A') \vee^H (B, B') = (A \vee^F B, A' \vee^G B')$.

And further, for any morphism $f\colon X \to Y$ and H-subobjects (A, A') of X and (B, B') of Y, we have

- $f \cdot^H (A, A') = (f \cdot^F A, f \cdot^G A')$;
- $(A, A') \cdot^H f = (A \cdot^F f, A' \cdot^G f)$.

Throughout the rest of this section, we will assume that F, G and H are all orean.

For axiom (N2), if both F and G satisfy (N2), then H need not satisfy (N2). A counter-example of this is: Let $\mathbf{2}$ be the category with two objects X and Y and a single non-identity arrow $f\colon X \to Y$. Construct the unique noetherian form F over $\mathbf{2}$ where X has one subobject and Y has two subobjects, and construct the unique noetherian form G over $\mathbf{2}$ where X has two subobjects and Y has one subobject. Then f is strictly an embedding according F, while f is strictly a projection according to G. If H satisfied (N2), then the projections are exactly those with image being 1, and the embeddings those with kernel being 0. According to this, f is neither an H-embedding nor an H-projection. But also f cannot be factored into two non-identity arrows. This contradicts that H satisfies (N2).

From this counter-example, it seems necessary to require that a morphism ι is an F-embedding if and only if ι is a G-embedding, and dually, π is an F-projection if and only if π is a G-projection. This condition is also sufficient to deduce that H satisfies (N2).

Proposition 4.2.2. *Suppose both F and G satisfies (N2), and a morphism is an F-embedding/projection if and only if it is a G-embedding/projection. Then H satisfies (N2).*

Proof. Consider any morphism f. The morphism f factors as $f = mp$, where p is an F-projection and m is an F-embedding. Since p is also a G-projection, its G-image is 1. So

$$\mathsf{Im}^H m = (\mathsf{Im}^F m, \mathsf{Im}^G m) = (\mathsf{Im}^F f, \mathsf{Im}^G f) = \mathsf{Im}^H f.$$

Consider any morphism g such that $\mathsf{Im}^H g \leq \mathsf{Im}^H m$. We then also have $\mathsf{Im}^F g \leq \mathsf{Im}^F m$. Thus there exists a unique h such that $g = mh$. And so m is an embedding of $\mathsf{Im}^H f$. Dually, p is a projection of $\mathsf{Ker}^H f$. Hence H satisfies (N2). $\square$

I'm unsure whether H satisfies (N2) implies that both F and G satisfy (N2), but have a feeling that it is not the case. For (N3), this direction is possible.

Proposition 4.2.3. *If H satisfies (N3), then both F and G satisfy (N3).*

Proof. Suppose H satisfies (N3). Consider any two conormal F-subobjects A and A'. Then there are morphisms f and g such that $\mathsf{Im}^F f = A$ and $\mathsf{Im}^F g = A'$. So then $(A, \mathsf{Im}^G f)$ and $(A', \mathsf{Im}^G g)$ are conormal H-subobjects. Thus $(A \wedge^F A', \mathsf{Im}^G f \wedge^G \mathsf{Im}^G g)$ is conormal as well, and so there is a morphism h with that H-subobject being its H-image. Consequently $\mathsf{Im}^F h = A \wedge A'$, and so is a conormal F-subobject. Dually, the join of normal F-subobjects is normal. So F satisfies (N3). Similarly G also satisfies (N3). $\square$

I'm unsure whether it is true that if both F and G satisfies (N3), then H satisfies (N3). However, under the same assumptions on F and G as in Proposition 4.2.2, H satisfies (N3).

Proposition 4.2.4. *If F and G satisfy both (N2) and (N3), and any F-embedding / projection is also a G-embedding/projection, then H satisfies (N3).*

Proof. Consider any two conormal H-subobjects (A, A') and (B, B') of the same object. By Proposition 4.2.2, H satisfies (N2). So there are H-embeddings α of (A, A') and β of (B, B'). Then α is an F-embedding of A and β is an F-embedding of B. Since F satisfies (N2) and (N3), the pullback of α and β exists. Denote the diagonal of that pullback by γ. The F-image of γ is $A \wedge^F B$. For similar reasons the G-image of γ is $A \wedge^G B$. Consequently the H-image of γ is the meet of (A, A') and (B, B'), and so their meet is conormal in H.

With dual argument we can show that the join of normal H-subobjects is normal. Thus H satisfies (N3). □

Note that if we pick F and G to be the same noetherian form, then, from what we have shown, their pullback H will also be noetherian. This not only gives us a way to generate more noetherian forms out of an existing one over the same category, but also shows that there is not a unique noetherian form over a category.

4.2.4 Pullback of forms along a functor

For a form F and a functor U, with domain the same category, construct their pullback:

$$\begin{tikzcd} \mathbb{B}' \arrow[r, "G"] \arrow[d, "V"'] & \mathbb{C}' \arrow[d, "U"] \\ \mathbb{B} \arrow[r, "F"] & \mathbb{C} \end{tikzcd}$$

In this subsection we are interested in G when F is a form, whereas in the previous subsection we were interested in the diagonal when F and U are forms. Just like in the previous subsection, we are going to assume that $\mathbb{B}'$ is the category whose morphisms are pairs of morphisms (k, f), where k is a morphism of $\mathbb{B}$ and f is a morphism of $\mathbb{C}'$ such that $Fk = Uf$.

Proposition 4.2.5. *If F is a form, then so is G. Further, for any $f\colon X \to Y$ in $\mathbb{C}'$ and G-subobjects (A, X) of X and (B, Y) of Y, we have*

$$(A, X) \leq_f^G (B, Y) \iff A \leq_{Uf}^F B.$$

Proof. Consider any two parallel morphisms (k, f) and (l, g) in $\mathbb{B}'$ such that

$$G(k, f) = G(l, g).$$

That is, $f = g$. We have

$$Fk = FV(k,f) = UG(k,f) = UG(l,f) = FV(l,f) = Fl.$$

So $k = l$, since F is faithful, and so $(k,f) = (l,g)$. Thus G is faithful.

Consider any isomorphism (k,f) in $\mathbb{B}'$ such that $G(k,f) = 1$, that is $f = 1$. Since $(k,1)$ is an isomorphism, $V(k,1) = k$ is also an isomorphism. We have

$$Fk = FV(k,1) = UG(k,1) = U1 = 1.$$

Thus $k = 1$. So $(k,f) = (1,1) = 1$. Thus G is amnestic.

For the second part, consider any morphism $f\colon X \to Y$ in $\mathbb{C}'$ and G-subobjects (A,X) of X and (B,Y) of Y. Suppose $(A,X) \leq^G_f (B,Y)$. That is, there exist some morphism $(k,f)\colon (A,X) \to (B,Y)$ in $\mathbb{B}'$ such that $G(k,f) = f$. Applying V, we get a morphism $k\colon A \to B$ in $\mathbb{B}$ such that

$$Fk = FV(k,f) = UG(k,f) = Uf.$$

And so $A \leq^F_{Uf} B$.

Conversely, suppose $A \leq^F_{Uf} B$. So there is a $k\colon A \to B$ such that $Fk = Uf$. Because of $Fk = Uf$, $(k,f)\colon (A,X) \to (B,Y)$ is a morphism in $\mathbb{B}'$. Furthermore, $G(k,f) = f$. That is $(A,X) \leq^G_f (B,Y)$. □

Observe that for the special case of $f = 1_X$, we see that $(A,X) \leq^G (B,X)$ if and only if $A \leq^F B$.

Proposition 4.2.6. *If F is orean, then so is G. In particular, for any object $X \in \mathbb{C}'$,*

- *$(0,X)$ and $(1,X)$ are the bottom and top G-subobjects of X, where 0 and 1 are the bottom and top F-subobject of UX;*
- *for any two G-subobjects (A,X) and (B,X) of X, we have*

$$(A,X) \wedge^G (B,X) = (A \wedge^F B, X) \quad \textit{and} \quad (A,X) \vee^G (B,X) = (A \vee^F B, X).$$

Finally, for any morphism $f\colon X \to Y$ in $\mathbb{C}'$ and G-subobjects (A,X) of X and (B,Y) of Y, we have

- *$f \cdot^G (A,X) = (Uf \cdot^F A, Y)$, and*
- *$(B,Y) \cdot^G f = (B \cdot^F Uf, X)$.*

Proof. First, consider any two composable morphism $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathbb{C}'$, and G-subobjects (A,X) of X and (C,Z) of Z such that $(A,X) \leq^G_{gf} (C,Z)$. Then, by Proposition 4.2.5, $A \leq^F_{U(gf)} C$. Since F is orean, there is an F-subobject B of UY such that

$$A \leq^F_{Uf} B \leq^F_{Ug} C.$$

Since $FB = UY$, (B, Y) is in $\mathbb{B}'$ and is further a G-subobject of Y. Thus, again by Proposition 4.2.5, we have

$$(A, X) \leq^G_f (B, Y) \leq^G_g (C, Z),$$

thus demonstrating the first condition on orean forms.

To verify the other conditions of orean forms, consider any morphism $f\colon X \to Y$, and consider any finite set S of G-subobjects of X, and create the set

$$\{(B, Y) \in \mathsf{sub}^G Y \mid \forall_{(A,X)\in S}((A, X) \leq^G_f (B, Y))\}.$$

By assumption the set $\{B \in \mathsf{sub}^F UY \mid \forall_{(A,X)\in S}(A \leq^F_{Uf} B)\}$ has a minimum element N. By the previous proposition, (N, Y) is in the set above. For any element (B, Y) in the set above, we have $A \leq^F_{Uf} B$ for any $(A, X) \in S$, thus $N \leq^F B$, and so $(N, Y) \leq^G (B, Y)$, which means that (N, Y) is the minimum element of the above set. Consider any finite set T of G-subobjects of Y, and create the set

$$\{(A, X) \in \mathsf{sub}^G X \mid \forall_{(B,Y)\in T}((A, X) \leq^G_f (B, Y))\}.$$

By dual arguments, (M, X) is the maximum element of the above set, where M is the maximum element of $\mathsf{sub}^F X$ such that $M \leq^F_{Uf} B$ for all $(B, Y) \in T$.

Thus G is also orean. Those points listed are now readily observable. $\square$

From here on, both F and G are assumed to be orean.

A further basic result to observe, is that for any morphism $f\colon X \to Y$ in $\mathbb{C}'$, we have

(1) $\mathsf{Ker}^G f = (\mathsf{Ker}^F Uf, X)$, and

(2) $\mathsf{Im}^G f = (\mathsf{Im}^F Uf, Y)$.

From this observation, together with the above proposition, we immediately have:

Proposition 4.2.7. *If F satisfies (N1), then so does G.*

Axioms (N2) and (N3) do not carry so easily over from F to G, as it did with (N1). Also, no sufficient and useful conditions were found such that those axioms would carry over. Though one useful observation (following from how images and kernels are computed in G) is perhaps that if both F and G satisfies (N2), then a morphism f in $\mathbb{C}'$ is a G-embedding/projection if and only if Uf is an F-embedding/projection.

4.2.5 Functor categories

Recall that a category is called *small* when both the collection of objects and the collection of morphisms are (small) sets. Further, recall that for a categories $\mathbb{C}$ and category $\mathbb{A}$, the *functor category* $\mathbb{B}^{\mathbb{A}}$ has as objects all functors $\mathbb{A} \to \mathbb{B}$ and as morphisms natural transformations between these functors.

Consider any form $F\colon \mathbb{B} \to \mathbb{C}$ and small category $\mathbb{A}$. Create a new functor

$$F^{\mathbb{A}}\colon \mathbb{B}^{\mathbb{A}} \to \mathbb{C}^{\mathbb{A}}, A \mapsto F \circ A, \alpha \mapsto F\alpha,$$

where by $F\alpha$ we mean the natural transformation with components $(F\alpha)_i = F(\alpha_i)$ for any object $i \in \mathbb{A}$. This new functor $F^{\mathbb{A}}$ is faithful, for take any two natural transformations $\alpha, \beta\colon A \dashrightarrow B$ in $\mathbb{B}^{\mathbb{A}}$ such that $F\alpha = F\beta$. So for any object $i \in \mathbb{A}$, $F\alpha_i = F\beta_i$. Since F if faithful, $\alpha_i = \beta_i$ for all $i \in \mathbb{A}$, thus $\alpha = \beta$. It is also amnestic, for take any natural isomorphism α in $\mathbb{B}^{\mathbb{A}}$ such that $F\alpha = 1$ (the identity natural transformation). So then $F\alpha_i = 1$ for any object $i \in \mathbb{A}$. Thus $\alpha_i = 1$, since F is amnestic and α_i is an isomorphism, and thus α is an identity transformation. In summary, $F^{\mathbb{A}}$ is again a form.

The following proposition describes that the new form $F^{\mathbb{A}}$ is related component-wise in some sense to the old form F.

Proposition 4.2.8. *For any natural transformation $\alpha\colon X \dashrightarrow Y$ in $\mathbb{C}^{\mathbb{A}}$, and subobjects $A \in \mathsf{sub}X$ and $B \in \mathsf{sub}Y$,*

$$A \leq_{\alpha}^{F^{\mathbb{A}}} B \quad \Leftrightarrow \quad \forall_{i\in\mathbb{A}}(Ai \leq_{\alpha_i}^{F} Bi).$$

Proof. Notice that $A \leq_{\alpha}^{F^{\mathbb{A}}} B$ if and only if there is a natural transformation $\kappa\colon A \to B$ in $\mathbb{B}^{\mathbb{A}}$ such $F\kappa = \alpha$. So for any $i \in \mathbb{A}$, there is a morphism $k_i\colon Ai \to Bi$ such that $Fk_i = \alpha_i\colon Xi \to Yi$. That is, $Ai \leq_{\alpha_i}^{F} Bi$.

Conversely, suppose there is a natural transformation $\alpha\colon X \dashrightarrow Y$ in $\mathbb{C}^{\mathbb{A}}$ such that for all $i \in \mathbb{A}$, $Ai \leq_{\alpha_i}^{F} Bi$. Then for each $i \in \mathbb{A}$, there is a $\kappa_i\colon Ai \to Bi$ such that $F\kappa_i = \alpha_i$. These family of maps κ forms a natural transformation from A to B: consider any morphism $f\colon i \to j$ in $\mathbb{A}$, then, since α is a natural transformation,

$$F(B(f)\kappa_i) = FB(f)F\kappa_i = Y(f)\alpha_i = \alpha_j X(f) = F\kappa_j FA(f) = F(\kappa_j A(f)),$$

and so $B(f)\kappa_i = \kappa_j A(f)$. Also $F\kappa = \alpha$, hence $A \leq_{\alpha}^{F^{\mathbb{A}}} B$. □

The poset of $F^{\mathbb{A}}$-subobjects of a functor $X\colon \mathbb{A} \to \mathbb{C}$ is almost "component-wise". At least to check whether two subobject of X are equal, one only needs to check whether they are equal on objects.

Lemma 4.2.9. *For any functor $X\colon \mathbb{A} \to \mathbb{B}$, and any family of objects $\{D_i \in \mathsf{sub}^F X(i) \mid i \in A\}$ such that for any morphism $f\colon i \to j$ in $\mathbb{A}$, $D_i \leq_{Xf}^{F} D_j$, there is a unique functor $D \in \mathsf{sub}^{F^{\mathbb{A}}} X$ such that $\forall_{i\in\mathbb{A}}(Di = D_i)$. Equivalently, the above construction defines a bijection*

$$\mathsf{sub}^{F^{\mathbb{A}}} X \approx \{(D_i)_{i\in\mathbb{A}} \in \prod_{i\in\mathbb{A}} \mathsf{sub}^F X(i \mid \forall_{f\colon i\to j \text{ in } \mathbb{A}}(D_i \leq_{Xf} D_j)\}.$$

Proof. Suppose we have such a family of objects $\{D_i \in \mathsf{sub}^F X(i) \mid i \in \mathbb{A}\}$. Define a function $D\colon \mathbb{A} \to \mathbb{B}$ such that $Di = D_i$. For any $f\colon i \to j$ in $\mathbb{A}$, since $Di \leq_{Xf} Dj$, there is a unique $k\colon Di \to Dj$ such that $Fk = Xf$. Let Df be this unique k. This readily makes D a functor such that $FD = X$. Thus D is the unique $F^{\mathbb{A}}$-subobject of X such that $Di = D_j$.

For the inverse of this construction, any $F^{\mathbb{A}}$-subobject D of X produces a family $(Di)_{i\in\mathbb{A}}$ such that $Di \in \mathsf{sub}^F X(i)$ for any $i \in \mathbb{A}$. Further, since for any $f\colon i \to j$ in $\mathbb{A}$, $FDf = Xf$, we have $Di \leq^F_f Dj$. $\square$

Since the relation $\leq_\alpha$ is component-wise, it is then not to surprising that if F is orean, then so is $F^{\mathbb{A}}$, and all the arising structure is component-wise.

Proposition 4.2.10. *If F is orean, then so is $F^{\mathbb{A}}$. In this case, for any functor X in $\mathbb{C}^{\mathbb{A}}$,*

- *the bottom element of $\mathsf{sub}^{F^{\mathbb{A}}} X$ is the unique $F^{\mathbb{A}}$-subobject $0_X\colon \mathbb{A} \to \mathbb{B}$ of X such that $0_X(i)$ is the bottom element of $\mathsf{sub}^F X(i)$ for all $i \in \mathbb{A}$, and*
- *the top element of $\mathsf{sub}^{F^{\mathbb{A}}} X$ is the unique $F^{\mathbb{A}}$-subobject $1_X\colon \mathbb{A} \to \mathbb{B}$ of X such that $1_X(i)$ is the top element of $\mathsf{sub}^F X(i)$ for all $i \in \mathbb{A}$.*

For any two $F^{\mathbb{A}}$-subobject A and B of X

- *their join $A \vee B$ is the unique $F^{\mathbb{A}}$-subobject of X such that $(A \vee^{F^{\mathbb{A}}} B)(i) = A(i) \vee^F B(i)$, for all $i \in \mathbb{A}$, and*
- *their meet $A \wedge B$ is the unique $F^{\mathbb{A}}$-subobject of X such that $(A \wedge^{F^{\mathbb{A}}} B)(i) = A(i) \wedge^F B(i)$ for all $i \in \mathbb{A}$.*

Lastly, for any natural transformation $\alpha\colon X \stackrel{\cdot}{\longrightarrow} Y$ in $\mathbb{C}^{\mathbb{A}}$ and $F^{\mathbb{A}}$-subobjects A of X and B of Y, we have

- *$\alpha \cdot^{F^{\mathbb{A}}} A$ is the unique $F^{\mathbb{A}}$-subobject of Y such that $(\alpha A)(i) = \alpha_i \cdot^F A(i)$ for all $i \in \mathbb{A}$, and*
- *$B \cdot^{F^{\mathbb{A}}} \alpha$ is the unique $F^{\mathbb{A}}$-subobject of X such that $(B \cdot^{F^{\mathbb{A}}} \alpha)(i) = B(i) \cdot^F \beta_i$ for all $i \in \mathbb{A}$.*

Proof. Suppose F is orean. To prove that $F^{\mathbb{A}}$ is orean, is to prove all those listed points. If those listed family of objects could be made into suitable functors, then by Proposition 4.2.8 and Lemma 4.2.9 we are done. Lemma 4.2.9 will be used throughout this proof.

For the families of subobjects $(0_X(i))_{i\in\mathbb{A}}$ and $(1_X(i))_{i\in\mathbb{A}}$, we have for any $f\colon i \to j$ in $\mathbb{A}$,

$$0_X(i) \leq^F_{Xf} 0_X(j) \quad \text{and} \quad 1_X(i) \leq^F_{Xf} 1_X(j),$$

since F is orean, and thus they can be made into $F^{\mathbb{A}}$-subobjects of X.

For the $F^{\mathbb{A}}$-subobjects A and B of X, we have, for any $f\colon i \to j$ in $\mathbb{A}$,

$$X(f) \cdot^F (Ai \vee^F Bi) = X(f) \cdot^F Ai \vee^F X(f) \cdot^F Bi \leq^F_{Xj} Aj \vee^F Bj.$$

Thus $Ai \vee^F Bi \leq^{F^{\mathbb{A}}}_{Xf} Aj \vee^F Bj$, and so it can be made into an $F^{\mathbb{A}}$-subobject of X. Dually, $Ai \wedge^F Bi$ can also be made into an $F^{\mathbb{A}}$-subobject of X.

Lastly, consider any natural transformation $\alpha \colon X \dashrightarrow Y$ and $F^{\mathbb{A}}$-subobject A of X. Take any $f \colon i \to j$ in $\mathbb{A}$. We have, using the fact that α is natural,

$$Y(f) \cdot^F \alpha_i \cdot^F Ai = \alpha_j \cdot^F X(f) \cdot^F A(i) \leq^F_{Y(j)} \alpha_j \cdot^F A(j).$$

Thus the family $(\alpha_i \cdot^F Ai)_{i \in \mathbb{A}}$ can be made into an $F^{\mathbb{A}}$-subobject of Y. Dually, the family $B(i) \cdot^F \alpha_i$ can be made into an $F^{\mathbb{A}}$-subobject of X. □

The axioms of a noetherian form are also satisfied by $F^{\mathbb{A}}$, provided that F satisfies them. This will be demonstrated in the next few propositions.

Proposition 4.2.11. *If F satisfies (N1), then $F^{\mathbb{A}}$ satisfies (N1).*

Proof. Since meets and joins, and direct and inverse images are component-wise, it is clear. □

Proposition 4.2.12. *If F satisfies (N2), then $F^{\mathbb{A}}$ satisfies (N2). In particular, Any morphism $\alpha \colon X \dashrightarrow Y$ in $\mathbb{C}^{\mathbb{A}}$ factorizes as $\pi\iota$, where π_i is a projection of the kernel of α_i and ι_i is an embedding of the image of α_i, for any $i \in \mathbb{A}$.*

Proof. Since F satisfies (N2), any α_i factorizes as $\pi_i\iota_i$ where π_i is a projection of the kernel and ι_i an embedding of the image. Denote the codomain of π_i by $Z(i)$. For any $f \colon i \to j$ in $\mathbb{A}$, we have the following commutative diagram

$$\begin{tikzcd}
Xi \arrow[r, "\pi_i"] \arrow[d, "Xf"] & Zi \arrow[r, "\iota_i"] \arrow[d, dashed, "Zf"] & Yi \arrow[d, "Yf"] \\
Xj \arrow[r, "\pi_j"] & Zj \arrow[r, "\iota_j"] & Yj
\end{tikzcd}$$

Notice that

$$0 \cdot^F \pi_j \cdot^F X(f) = 0 \cdot^F \alpha_j \cdot^F \cdot^F X(f) = 0 \cdot^F Y(f) \cdot^F \alpha_i \geq^F 0 \cdot \alpha_i = 0 \cdot^F \pi_i.$$

Since π_i is a projection, there is a unique $Z(f)$ such that $Z(f)\pi_i = \pi_j X(f)$. This definition of $Z(f)$ makes Z a functor $\mathbb{A} \to \mathbb{C}$ and makes π into a natural transformation $X \dashrightarrow Z$. Since π_i is also in particular an epimorphism, one could deduce that $\iota \colon Z \dashrightarrow Y$ is also a natural transformation. Further $\alpha = \iota\pi$. Since direct and inverse images are component-wise, the kernel of α is the kernel of π and the image of α is the image of ι.

Suppose $\eta \colon X \dashrightarrow W$ has kernel containing the kernel of α. Then the kernel of η_i contains the kernel of α_i for every $i \in \mathbb{A}$ and so there is a unique κ_i such that $\kappa_i\pi_i = \eta_i$. This family $(\kappa_i)_{i \in \mathbb{A}}$ forms a natural transformation $\kappa \colon Z \dashrightarrow W$, and unique such that $\kappa\pi = \eta$. Thus π is a projection of the kernel of α. Dually, ι is an embedding of the image of α. □

From the above proof we can extract that an $F^{\mathbb{A}}$-subobject A of X is conormal/normal if and only if $A(i)$ is a conormal/normal F-subobject of $X(i)$ for every $i \in \mathbb{A}$.

Proposition 4.2.13. *If F satisfies (N3), then $F^{\mathbb{A}}$ satisfies (N3).*

Proof. This follows immediately from the above observation. □

The remainder of this subsection is dedicated to showing that if F admits biproducts, then so does $F^{\mathbb{A}}$. Biproducts will turn out to be component-wise. It is well known that if a category $\mathbb{C}$ has a limit/colimit of some form then $\mathbb{C}^{\mathbb{A}}$ also has a limit/colimit of the same form. A proof of this well known fact will be shown in the lemma below, where for each object i in $\mathbb{A}$, we let $E_i\colon \mathbb{C}^{\mathbb{A}} \to \mathbb{C}$ denote the "evaluation functor" (that is, $E_i\alpha = \alpha_i$ for any natural transformation α in the functor category).

Lemma 4.2.14. *Consider any functor category $\mathbb{C}^{\mathbb{A}}$ and consider any functor*

$$J\colon \mathbb{D} \to \mathbb{C}^{\mathbb{A}}.$$

If for every object i in $\mathbb{A}$, E_iJ has a limit

$$(L_i, (p^i_d\colon L_i \to Jd(i))_{d\in\mathbb{D}}),$$

then J has a limit $(L, (p_d\colon L \to Jd)_{d\in\mathbb{D}})$, where $L(i) = L_i$ and for every $d \in \mathbb{D}$, $(p_d)_i = p^i_d)$. Dually, if for every object i in $\mathbb{A}$, E_iJ has a colimit

$$(C_i, (e^i_d\colon Jd(i) \to C_i)_{d\in\mathbb{D}}),$$

then J has a colimit $(C, (e_d\colon Jd \to C))_{d\in\mathbb{D}})$, where $C(i) = C_i$ and for every $d \in \mathbb{D}$, $(e_d)_b = e^i_d)$.

Proof. Suppose that for every object i in $\mathbb{A}$, E_iJ has a limit

$$(L_i, (p^i_d\colon L_i \to Jd(i))_{d\in\mathbb{D}}).$$

Define $L\colon \mathbb{A} \to \mathbb{C}$ by $L(i) = L_i$ for every $i \in \mathbb{A}$. For any $f\colon i \to j$ in $\mathbb{A}$,

$$(L_i, ((Jd)f \circ p^i_d)_{d\in\mathbb{D}})$$

forms a cone over E_jJ. To verify this, take any $h\colon d \to e$ in $\mathbb{D}$. Then $J(h)\colon J(d) \overset{\cdot}{\longrightarrow} J(e)$ is a natural transformation. So then for $f\colon i \to j$ in $\mathbb{A}$, we have

$$J(h)_j \circ J(d)f \circ p^i_d = J(e)f \circ J(h)_i \circ p^i_d = J(e)f \circ p^i_e.$$

Thus there is a unique arrow $f'\colon L_i \to L_j$ such that for every $d \in \mathbb{D}$,

$$p^j_d f' = (Jd)f \circ p^i_d.$$

Define $L(f)$ to be this f'. This definition makes L a functor, and for every $d \in \mathbb{D}$, $p_d\colon L \overset{\cdot}{\longrightarrow} Jd$, where $(p_d)_i = p^i_d$ for every $i \in \mathbb{A}$, is a natural transformation.

Moreover we have for any $h\colon d \to e$ in $\mathbb{D}$, $(Jh)p_d = p_e$. So $(L, (p_d)_{d\in\mathbb{D}})$ is a cone over J.

Consider any cone $(K, (q_d)_{d\in\mathbb{D}})$ over J. Then for every $i \in \mathbb{A}$, $(Ki, ((q_d)_i)_{d\in\mathbb{D}})$ forms a cone over E_iJ. Thus there exists a unique $h_i\colon Ki \to Li$ such that, for every $d \in \mathbb{D}$, $(p_d)_i h_i = (q_d)_i$. To deduce that L is a limit of J, it would be sufficient to show that this $h\colon K \overset{\cdot}{\longrightarrow} L$ is a natural transformation. For a fixed $j \in \mathbb{A}$, the family $((p_d)_j)_{d\in\mathbb{D}}$ is jointly monic. For any $f\colon i \to j$ in $\mathbb{A}$, we have

$$(p_d)_j \circ Lf \circ h_i = (Jd)f \circ (p_d)_i \circ h_i = (Jd)f \circ (q_d)_i = (q_d)_j \circ (Kf) = (p_d)_j \circ h_j \circ (Kf).$$

Thus $(Lf)h_i = h_j(Kf)$. So $h\colon K \overset{\cdot}{\longrightarrow} L$ is a unique natural transformation such that $p_d h = q_d$, for all $d \in \mathbb{D}$. □

Definition 4.2.15. Suppose F admits biproducts. For functors $X, Y\colon \mathbb{A} \to \mathbb{C}$ define $X \oplus Y\colon \mathbb{A} \to \mathbb{C}$ as

$$(X \oplus Y)(i) = X(i) \oplus Y(i)$$

for any $i \in \mathbb{A}$, and for any $f\colon i \to j$ in $\mathbb{B}$

$$(X \oplus Y)(f) = X(f) \oplus Y(f).$$

That is, $A \oplus B$ is the composite of the functors

$$\mathbb{A} \xrightarrow{\Delta} \mathbb{A} \times \mathbb{A} \xrightarrow{X \times Y} \mathbb{C} \times \mathbb{C} \xrightarrow{\oplus} \mathbb{C}$$

Lemma 4.2.16. *For functors X and Y, the following are natural transformations:*

(1) $e_1\colon X \to X \oplus Y$, *where* $(e_1)_i = e_1^{Xi\oplus Yi}$;

(2) $e_2\colon Y \to X \oplus Y$, *where* $(e_2)_i = e_2^{Xi\oplus Yi}$;

(3) $p_1\colon X \oplus Y \to X$, *where* $(p_1)_i = p_1^{Xi\oplus Yi}$;

(4) $p_2\colon X \oplus Y \to Y$, *where* $(p_2)_i = p_2^{Xi\oplus Yi}$.

That is, the ith components of e_1, e_2, p_1 and p_2 are the respective embeddings and projections of the biproduct $Xi \oplus Yi$.

Proof. This follows immediately from the definition of biproducts of morphisms. □

Theorem 4.2.17. *Suppose F admits biproducts. For any two functors $X, Y\colon \mathbb{A} \to \mathbb{C}$, $X \oplus Y$ together with those natural transformations e_1, e_2, p_1, and p_2 of the previous lemma, is the biproduct of X and Y in $\mathbb{C}^{\mathbb{A}}$.*

Moreover, commutators are computed point-wise as well. That is, for any conormal $F^{\mathbb{A}}$-subobjects A and B of functor X,

$$[A, B]_i = [Ai, Bi].$$

Proof. The "limit condition" and the "colimit condition" follows immediately from Lemma 4.2.14. Since kernels and images are computed component-wise, we have $p_1e_1 = 1_X$, $p_2e_2 = 1_Y$, $\mathsf{Ker}p_2 = \mathsf{Im}e_1$, and $\mathsf{Ker}p_1 = \mathsf{Im}e_2$. Thus $X \oplus Y$ is the biproduct of X and Y.

The statement of commutators also follows immediately from Lemma 4.2.14, that limits are computed component-wise, and the fact that kernels are computed component-wise. □

4.3 Closure operators

Recall the following notion from [23]:

Definition 4.3.1. A *closure operator* on a form F over a category $\mathbb{C}$ assigns to each F-subobject S of an object $X \in \mathbb{C}$ an F-subobject $C(S)$, such that

- $A \leq_f B \implies C(A) \leq_f C(B)$, for any $f\colon X \to Y$ and F-subobjects A of X and B of Y, and
- C is *extensive*, that is $S \leq_X C(S)$ for any object X and F-subobject of X.

C is said to be an *idempotent* closure operator when in addition $C(C(S)) = C(S)$ holds.

Given a closure operator C, by selecting the *closed* F-subobjects, that is subobjects S satisfying $S = C(S)$, we get a new form F^C. For this new form, for any $f\colon X \to Y$ in $\mathbb{C}$ and F^C-subobjects A of X and B of Y, we have

$$A \leq_f^{F^C} B \iff A \leq_f^F B.$$

Because of this, we could drop the superscript of $\leq_f$ for the rest of this section.

Closure operators preserve orean forms in the following sense.

Theorem 4.3.2. *For an orean form F and any idempotent closure operator C on F, the form F^C is also orean. In particular, for any $X \in \mathbb{C}$, we have*

- $1^{F^C} = 1^F$, *and* $0^{F^C} = C(0^F)$;
- $A \wedge^{F^C} B = A \wedge^F B$, *and* $A \vee^{F^C} B = C(A \vee^F B)$, *for any two F^C-subobjects A and B of X.*

And for any morphism $f\colon X \to Y$ in $\mathbb{C}$ and F^C-subobjects A of X and B of Y, we have

- $B \cdot^{F^C} f = B \cdot^F f$, *and*
- $f \cdot^{F^C} A = C(f \cdot^F A)$.

Proof. Consider any morphism $f\colon X \to Y$ in $\mathbb{C}$ and finite subset $S \subseteq \mathsf{sub}^{F^C} X$. We have to show that the set

$$\{B \in \mathsf{sub}^{F^C} Y \mid \forall_{A \in S}(A \leq_f B)\}$$

has a minimum element. We do know that the set of F-subobjects B of Y such that $\forall_{A \in S}(A \leq_f B)$ has a minimum element N. Notice that for any element B in the set above, we have $N \leq B$. Consequently, $C(N) \leq C(B) = B$. Since $C(N)$ is also in the above set, $C(N)$ is the desired minimum element.

Now consider any finite subset $T \subseteq \mathsf{sub}^{F^C} Y$. We have to show that the set

$$\{A \in \mathsf{sub}^{F^C} X \mid \forall_{B \in T}(A \leq_f B)\}$$

has a maximum element. We know that the set of F-subobjects A of X such that $\forall_{B \in T}(A \leq_f B)$ has a maximum element M. Notice that for any element A in the above set, we have $A \leq M$. Consequently, $A = C(A) \leq C(M)$. Since $C(M)$ is in the above set, $C(M)$ is the desired maximum. Further, since M is the maximum such F-subobject and $C(M)$ is also an F-subobject, we have

$$M \leq C(M) \leq M.$$

Thus $M = C(M)$. $\square$

It is not always true that F^C is noetherian when F is noetherian. A simple counter example is: take the two element category with exactly one unique non-identity arrow $f\colon X \to Y$. There is a unique (up to isomorphism) noetherian form, where the subobject lattice of X only has one object, and the subobject lattice of Y has two subobjects, 0^Y the bottom and 1^Y the top. The function defined on the subobject lattices as $C(0^Y) = 1^Y$, and fixes all other subobjects is a closure operator. However, in the form of closed subobjects, both X and Y has exactly one subobject, forcing f to be an isomorphism (if F^C were to be a noetherian form), which is not the case.

Under some conditions on the noetherian form F and the closure operator C, one could conclude that F^C is noetherian, as demonstrated below. Before we do that, notice the following

Lemma 4.3.3. *For any two F-subobjects A and B of the same object, we have*

$$C(A \vee^F B) = C(A) \vee^{F^C} C(B).$$

Proof. We have

$$\begin{aligned} C(A) \vee^{F^C} C(B) &= C(C(A) \vee^F C(B)) \\ &\geq C(A \vee^F B) \\ &\geq C(A), C(B). \end{aligned}$$

From that, the lemma follows. $\square$

Theorem 4.3.4. *Suppose F is noetherian. If*

- *all conormal subobjects are closed,*
- *$C(f \cdot^F A) \cdot^F f = C((f \cdot^F A) \cdot^F f)$ holds for every morphism $f\colon X \to Y$ and F-subobject A of X, and*
- *$C(N) \leq C(K)$ implies $N \leq K$, for any two normal F-subobjects,*

then F^C is noetherian. In particular, embeddings and projections of F^C is the same as for F. And the normal F^C-subobjects are of the form $C(N)$ where N is a normal F-subobject.

Proof. Notice that the second point implies that $\mathsf{Ker}^{F^C} f = C(\mathsf{Ker}^F f)$. To verify (N1), take any morphism $f\colon X \to Y$ and F^C-subobjects A of X and B of Y. We have

$$\begin{aligned}
(f \cdot^{F^C} A) \cdot^{F^C} f &= C(f \cdot^F A) \cdot^F f \\
&= C((f \cdot^F A) \cdot^F f) \\
&= C(A \vee^F \mathsf{Ker}^F f) \\
&= C(A) \vee^{F^C} C(\mathsf{Ker}^F f) \\
&= A \vee^{F^C} \mathsf{Ker}^{F^C} f.
\end{aligned}$$

For the other part of (N1), we have

$$\begin{aligned}
f \cdot^{F^C} (B \cdot^{F^C} f) &= C(f \cdot^F (B \cdot^F f)) \\
&= C(B \wedge^F \mathsf{Im}^F f) \\
&= C(B \wedge^{F^C} \mathsf{Im}^{F^C} f) \\
&= B \wedge^{F^C} \mathsf{Im}^{F^C} f.
\end{aligned}$$

To verify (N2), it is sufficient to show that the embeddings and projections of F^C are the same as embeddings and projections of F. The embeddings are the same for both forms, since images are closed. Take any F-projection $p\colon X \to Y$. Consider any morphism f whose F^C-kernel is greater than the F^C-kernel of p. That is $\mathsf{Ker}^{F^C} p \leq \mathsf{Ker}^{F^C} f$, equivalently $C(\mathsf{Ker}^F p) \leq C(\mathsf{Ker}^F f)$. The third point implies that $\mathsf{Ker}^F p \leq \mathsf{Ker}^F f$. And thus f factors uniquely through p, demonstrating that p is also a projection in F^C. Conversely, suppose p is a projection in F^C. Consider any f such that $\mathsf{Ker}^F p \leq \mathsf{Ker}^F f$. Then, $C(\mathsf{Ker}^F p) \leq C(\mathsf{Ker}^F f)$; equivalently, the F^C-kernel of f contains the F^C-kernel of p. Thus f factors uniquely through p, so any projection in F^C is a projection in F.

Since all F-images are closed, an F^C-subobject is conormal in F^C if and only if it is conormal in F. It then follows that the meet of conormal F^C-subobjects is again conormal. From the second point it follows that normal F^C-subobjects are of the form $C(N)$, where N is a normal F-subobject. So for any two normal F^C-subobjects $C(N)$ and $C(M)$, we have, using the lemma just before the theorem, $C(N) \vee^{F^C} C(M) = C(N \vee^F M)$. Since $N \vee^F M$ is normal in F, $C(N) \vee^{F^C} C(M)$ is normal F^C. This shows (N3). □

The following corollary is just for interest sake, but could be of some use finding a noetherian form over a given category (limiting the forms that needs to be considered).

Corollary 4.3.5. *If there is a noetherian form F over a category $\mathbb{C}$ with an initial object I, where I has exactly one normal subobject, then there is a noetherian form over $\mathbb{C}$ where I has exactly one subobject.*

Proof. For every object $X \in \mathbb{C}$ there is a unique morphism $i_X : I \to X$. Denote the images of these morphisms by I_X. Clearly, for any subobject A of X, $A \vee I_X \geq A$ and $(A \vee I_X) \vee I_X = A \vee I_X$. Further, notice for any morphism $f : X \to Y$, $f \cdot I_X = I_Y$. So for any subobjects A of X and B of Y, we have

$$\begin{aligned} A \leq_f B &\Rightarrow f \cdot A \leq B \\ &\Rightarrow f \cdot I_X \vee f \cdot A \leq B \vee I_Y \\ &\Rightarrow f \cdot (I_X \vee A) \leq B \vee I_Y \\ &\Rightarrow I_X \vee A \leq_f B \vee I_Y. \end{aligned}$$

So $C(A) = A \vee I_X$ forms an idempotent closure operator on F. To verify that F^C is again noetherian, we only need to check the conditions listed in the theorem above. The first point is satisfied, since I_X is the least conormal F-subobject of X, for any object X. To show this, take any morphism f into X. Then f factors through i_X, since I is an initial object. Thus the image of i_X is less than the image of f. For the second point, consider any morphism $f : X \to Y$ and any F-subobject A of X. We have,

$$\begin{aligned} C(f \cdot A) \cdot f &= (f \cdot A \vee I_Y) \cdot f \\ &= (f \cdot A \vee f \cdot I_X) \cdot f \\ &= (f \cdot (A \vee I_X)) \cdot f \\ &= A \vee I_X \vee \mathsf{Ker} f \\ &= (f \cdot A) \cdot f \vee I_X \\ &= C((f \cdot A) \cdot f). \end{aligned}$$

Lastly, for the third point, consider any two normal subobjects N and K of the same object X. By assumption, 0 is the unique normal subobject of I, so for any normal subobject M of X, we have

$$M \wedge I_X = i_X i_X^{-1} M = i_X 0 = 0.$$

This, together with the restricted modular law, gives

$$(N \vee I_X) \wedge (N \vee K) = N \vee (I_X \wedge (N \vee K)) = N \vee 0 = N,$$

and similarly $(K \vee I_X) \wedge (N \vee K) = K$. So

$$C(N) \leq C(K) \Rightarrow C(N) \wedge (N \vee K) \leq C(K) \wedge (N \vee K) \Rightarrow N \leq K.$$

So by the above theorem F^C will be a noetherian form. For this new noetherian form, I has exactly one F^C-subobject. □

Corollary 4.3.6. *If there is a noetherian form over a pointed category $\mathbb{C}$, then there is a noetherian form over $\mathbb{C}$ in which the zero object has exactly one subobject.*

Proof. Denote the zero object by 0. Suppose there is a projection $f\colon 0 \to X$. Since 0 is in particular a terminal object, there is a $g\colon X \to 0$. Composing f and g, we get $gf\colon 0 \to 0$, forcing $gf = 1$. Consequently f is also an embedding, thus an isomorphism. Thus the only normal subobject of 0, is the kernel of the identity. And now from the previous corollary the result follows. □

4.4 A general construction

In this section we are going to construct a noetherian form out of two forms F_s and F_e over the same category $\mathbb{C}$. This construction here is a generalization of the construction Professor Zurab Janelidze and the author used to create a noetherian form over the category of sets, where F_s is the form of "subobjects" and F_e is the form of "quotient objects". It was generalized in such a way that it also gives sufficient conditions when the pullback of two orean forms is a noetherian form. In particular, from this general construction it follows that the diagonal of the pullback of a noetherian form along itself is again a noetherian form.

In this section, instead of using superscript F_s or F_e to denote from which form the operations are, we are instead going to simply use superscript s or superscript e respectively. We require that these two forms, F_s and F_e satisfies the following requirements:

	F_s	F_e
Orean	×	×
$f \cdot (B \cdot f) = B \wedge \mathsf{Im} f$	×	×
$(f \cdot A) \cdot f = A \vee \mathsf{Ker} f$		×
Meet of conormal subobjects are conormal	×	
Join of normal subobjects are normal		×

Let H be the pullback of the forms F_s and F_e. As explained in Subsection 4.2.3,

- H-subobjects of X are pairs (A, E) where A is an F_s-subobject of X, and E is an F_e-subobject of X;
- Order, meets, joins, direct and inverse images are all component-wise.

Again, from Subsection 4.2.3, H is orean.

Lastly, suppose there is an idempotent closure operator C on H, where the closure of (A, E) is $(A * E, E)$, and $*$ has the following properties:

- $A \leq A * E$;
- $(A * E) * E = A * E$;
- $(A, E) \leq_f (B, D) \quad \Rightarrow \quad (A * E, E) \leq_f (B * D, D)$;

- $A * 0 = A$.

The first three points are equivalent to C being an idempotent closure operator.

Let $F = H^C$ be the form of closed subobjects of H.

This new form F need not be a noetherian form. How we will proceed, is to add sufficient assumptions on F_s, F_e and $*$ to make F satisfy each of the axioms of a noetherian form.

First of all, we notice that we can easily compute the F-kernel of a morphism. This observation will be useful for other manipulations.

Lemma 4.4.1. *For any $f\colon X \to Y$ in $\mathbb{C}$,*

$$\mathsf{Ker}^F f = (\mathsf{Ker}^s f, \mathsf{Ker}^e f).$$

Proof. We have

$$\mathsf{Ker}^F f = (0^s * 0^e, 0^e) \cdot^F f = (0^s, 0^e) \cdot^F f = (0^s \cdot^s f, 0^e \cdot^e f) = (\mathsf{Ker}^s f, \mathsf{Ker}^e f).$$

□

Stated differently, normal subobjects are closed. Normal subobjects being closed is in fact equivalent to the bottom subobject being closed. The above shows the one direction. For the other direction, $(0^s, 0^e)$ is normal, since it is the kernel of the identity morphism, thus it is closed.

We are systematically going to add sufficient assumptions so that this form F will be a noetherian form.

Assumption 1. Let $f\colon X \to Y$ be any morphism in $\mathbb{C}$. For any F-subobject (A, E) above the F-kernel of f, we have

$$((f \cdot^s A) * (f \cdot^e E)) \cdot^s f = A * E.$$

Proposition 4.4.2. *Under Assumption 1, F satisfies the identity*

$$(f \cdot^F (A, E)) \cdot^F f = (A, E) \vee^F \mathsf{Ker}^F f.$$

Proof. Suppose Assumption 1 is satisfied. For any morphism f and F-subobject (A, E) above the kernel of f. We have

$$\begin{aligned}
(f \cdot^F (A, E)) \cdot^F f &= ((f \cdot^s A) * (f \cdot^e E), f \cdot^e E)) \cdot^F f \\
&= (((f \cdot^s A) * (f \cdot^e E)) \cdot^s f, (f \cdot^e E) \cdot^e f) \\
&= (A * E, (f \cdot^e E) \cdot^e f) \\
&= (A * E, E) \\
&= (A, E).
\end{aligned}$$

And so that identity is satisfied. □

Assumption 2. For any morphism $f\colon X \to Y$, we have

$$\mathsf{Im}^s f * \mathsf{Im}^e f = \mathsf{Im}^s f.$$

Stated differently, the above assumption says that conormal H-subobjects are closed.

Proposition 4.4.3. *Under Assumption 2, F satisfies the identity*

$$f \cdot^F ((A, E) \cdot^F f) = (A, E) \wedge^F \mathsf{Im}^F f.$$

Proof. Suppose Assumption 2 holds. For any morphism $f\colon X \to Y$ and F-subobject (B, D) below the image of f, we have

$$\begin{aligned} f \cdot^F ((B, D) \cdot^F f) &= f \cdot^F (B \cdot^s f, D \cdot^e f) \\ &= ((f \cdot^s (B \cdot^s f)) * (f \cdot^e (D \cdot^e f)), f \cdot^e (D \cdot^e f)) \\ &= ((B \wedge^s \mathsf{Im}^s f) * (D \wedge^e \mathsf{Im}^e f), D \wedge^e \mathsf{Im}^e f) \\ &= (B * D, D) \\ &= (B, D). \end{aligned}$$

And so that identity is satisfied. □

So Assumptions 1 and 2 together gives that F satisfies (N1).

Assumption 3. If two morphisms f and g have the same F_s-image, then they have the same F_e-image. Moreover, the map which assigns to each F_s-image $\mathsf{Im}^s f$ the F_e-image $\mathsf{Im}^e f$, preserves meets.

Assumption 4. If two morphisms f and g have the same F_e-kernel, then they have the same F_s-kernel. Moreover, the map which assigns to each F_e-kernel $\mathsf{Ker}^e f$ the F_s-kernel $\mathsf{Ker}^s f$ preserves joins.

Assumption 5. Any morphism f in $\mathbb{C}$ factors as $f = mp$ where m is an F_s-embedding of $\mathsf{Im}^s f$ and p is an F_e-projection of $\mathsf{Ker}^e f$.

Proposition 4.4.4. *Under all the assumptions, (N2) is satisfied.*

Proof. By Assumption 5, any morphism $f\colon X \to Y$ factorizes as $f = mp$, where p is a projection of $\mathsf{Ker}^e f$ and m is an embedding of $\mathsf{Im}^s f$. We have the following by Assumption 3:

$$\mathsf{Im}^F m = (\mathsf{Im}^s m, \mathsf{Im}^e m) = (\mathsf{Im}^s f, \mathsf{Im}^e f) = \mathsf{Im}^F f.$$

Suppose $g\colon X' \to Y$ has image smaller than $\mathsf{Im} f$, then in particular $\mathsf{Im}^s g \leq \mathsf{Im}^s f$, and so it must factor through m uniquely. Thus m is an F-embedding of $\mathsf{Im}^F f$. Similarly, p is an F-projection of $\mathsf{Ker}^F f$. □

Proposition 4.4.5. *Under all the assumptions, (N3) is satisfied.*

Proof. Consider two images $\mathsf{Im}^F f$ and $\mathsf{Im}^F g$, where f and g have the same codomain. Their meet is

$$(\mathsf{Im}^s f \wedge^s \mathsf{Im}^s g, \mathsf{Im}^e f \wedge^e \mathsf{Im}^e g).$$

Since the meet of two F_s-images is again an F_s-image, we have

$$\mathsf{Im}^s f \wedge^s \mathsf{Im}^s g = \mathsf{Im}^s h$$

for some h. By Assumption 3, we have $\mathsf{Im}^e f \wedge \mathsf{Im}^e g = \mathsf{Im}^e h$. Consequently $\mathsf{Im}^F f \wedge^F \mathsf{Im}^F g = \mathsf{Im}^F h$.

For the other part, consider two kernels $\mathsf{Ker}^F f$ and $\mathsf{Ker}^F g$, where f and g have a common domain. Their join is

$$\mathsf{Ker}^F f \vee^F \mathsf{Ker}^F g = ((\mathsf{Ker}^s f \vee^s \mathsf{Ker}^s g) * (\mathsf{Ker}^e f \vee^e \mathsf{Ker}^e g), \mathsf{Ker}^e f \vee^e \mathsf{Ker}^e g).$$

Since F_e satisfies the join part of $(N3)$, there is a morphism h such that $\mathsf{Ker}_e f \vee \mathsf{Ker}^e g = \mathsf{Ker}^e h$. Then by Assumption 4, $\mathsf{Ker}^s f \vee^s \mathsf{Ker}^s g = \mathsf{Ker}^s h$. From this, we have

$$\mathsf{Ker}^F f \vee^F \mathsf{Ker}^F g = (\mathsf{Ker}^s h * \mathsf{Ker}^e h, \mathsf{Ker}^e h) = (\mathsf{Ker}^s h, \mathsf{Ker}^e h) = \mathsf{Ker}^F h.$$

□

Chapter 5

Concrete noetherian forms

5.1 Introduction

In this chapter we give concrete examples and counterexamples of noetherian forms. The most surprising example is given by the category of sets, where the image of a function $f\colon X \to Y$ is the usual image $\{fx \mid x \in X\}$ and the kernel of f is the kernel relation $\{(x,y) \mid fx = fy\}$, and the "subobject" lattices are combinations of the subset and partition lattices. By pulling back this form along a forgetful functor from any variety of universal algebras, we still get a noetherian form. Hence, any variety gives rise to a noetherian form. In this noetherian form, subalgebras and congruences are the conormal and normal subobjects, respectively.

5.2 Sets and varieties

5.2.1 A noetherian form over the category of sets

To show that the category of sets can be seen as a noetherian form, we are going to use the construction in Section 4.4 of the previous chapter.

Let F_s be the form of subsets. That is, for any set X, $\mathsf{sub}^s X$ is the lattice of subsets under the subset-inclusion relation. And the direct and inverse images are as usual. Let F_e be the form of equivalence relations. That is, for any set X, $\mathsf{sub}^e X$ is the lattice of equivalence relations on X under subset-inclusion. For any morphism $f\colon X \to Y$ and equivalence relations R on X and S on Y, the direct and inverse images of f is defined as follows, where D denotes the discrete equivalence relation and I denotes the indiscrete equivalence relation:

$$f_*R = \{(fx, fy) \mid xRy\} \cup D_Y \quad \text{and} \quad f^*S = \{(x,y) \in X \times X \mid f(x)Sf(y)\}.$$

Both forms are orean, but neither is noetherian. For the form F_s, the following simple observation shows why it cannot be noetherian: the F_s-kernel of any function is the empty set, so by (N1) we have for any function f and subset A,

$$f^{-1}fA = A \vee \emptyset = A.$$

This we know is not true in sets. Or equivalently: if F_s were to be noetherian, then all functions have to be embeddings, since their kernels are trivial. But any embedding is a monomorphism, and so all functions are injective, which is again a contradiction. The simplest observation showing that F_e is not noetherian is the following: there is exactly one equivalence relation on $\emptyset$ and on $\{*\}$; consequently the unique function $\emptyset \to \{*\}$ is both a projection and an embedding, thus an isomorphism/bijection, assuming F_e is noetherian.

Both F_s and F_e satisfy the identity $f \cdot (B \cdot f) = B \wedge \mathsf{Im} f$. Only F_e satisfies the identity $(f \cdot A) \cdot f = A \vee \mathsf{Ker} f$. Further, the meet of conormal F_s-subobjects are conormal again (since all F_s-subobjects are conormal), and the join of normal F_e-subobjects are again normal (since all F_e-subobjects are normal). So the forms F_s and F_e satisfy the initial listed conditions of the first table in Subsection 4.4 of the previous chapter. Let H denote the pullback of F_s along F_e, and let $*$ be defined as

$$A * E = \pi_E^{-1} \pi_E A,$$

for any subset A of X and equivalence relation E on X, where $\pi_E \colon X \to X/E$ is the map sending each element x of X to its equivalence class under E. This is readily extensive and idempotent. Also $A * D = A$, where D is the discrete equivalence relation on X. Further, consider any map $f \colon X \to Y$, and H-subobjects (A, R) of X and (B, S) of Y such that $(A, R) \leq_f (B, S)$. So in particular $R \leq^e S \cdot^e f$. Consequently there is a $g \colon X/R \to X/S$ defined by $g[x] = [fx]$, such that $g\pi_R = \pi_S f$. Also, $fA \leq^s B$. We have

$$\begin{aligned} A * R &= \pi_R^{-1} \pi_R A \\ &\leq \pi_R^{-1} g^{-1} g \pi_R A \\ &= f^{-1} \pi_S^{-1} \pi_S f A \\ &\leq f^{-1} \pi_S^{-1} \pi_S B \\ &= f^{-1} B * S. \end{aligned}$$

And so $(A*R, R) \leq_f^m (B*S, S)$, where $(A, R) \leq_f^H (B, S)$. So C defined by $C(A, R) = C(A * R, R)$ is an idempotent closure operator for which H-subobjects of the form $(A, 0)$ are closed.

Let F be the form of all closed H-subobjects. Now we are systematically going to demonstrate the assumptions, thus concluding that F is a noetherian form over the category of sets.

Proposition 5.2.1 (Assumption 1). *For any map $f \colon X \to Y$ and any F-subobject (A, E) above the F-kernel of f, we have*

$$f^{-1}(fA * f_* E) = A * E.$$

Proof. Notice that there is a map $g \colon X/E \to Y/(f_*E), [x] \mapsto [fx]$, which is well-defined by the definition of f_*E. For this new map g, we have $g\pi_E = \pi_{f_*E} f$. Further notice that g is injective: if $g[x] = g[y]$, then $[fx] = [fy]$. So $f(x) f_*(E) f(y)$. Then there are $a, b \in X$ such that aEb and $fa = fx$ and $fb = fy$. Since E contains the kernel relation of f, aEx and bEy, and so xEy which means g is injective. We have

$$\begin{aligned} f^{-1}(fA * f_*E) &= f^{-1}\pi_{f_*E}^{-1}\pi_{f_*E}fA \\ &= \pi_E^{-1}g^{-1}g\pi_E A \\ &= \pi_E^{-1}\pi_E A \\ &= A * E. \end{aligned}$$

□

It will be useful to know that

$$\mathsf{Im}^e f = (\mathsf{Im}^s f)^2 \cup D,$$

for any map f, where D is the discrete equivalence relation on the codomain of f. Checking the next two assumptions are straightforward after this observation.

Proposition 5.2.2 (Assumption 2)**.** *For any map $f\colon X \to Y$, we have*

$$\mathsf{Im}^s f * \mathsf{Im}^e f = \mathsf{Im}^s f.$$

Proposition 5.2.3 (Assumption 3)**.** *If two maps have the same F_s-image, then they have the same F_e-image. Moreover, the map which assigns to each F_s-image the F_e-image preserves meets.*

Keeping in mind that the F_s-kernel of any map f is the empty set, the next assumption follows just as easily as the previous two.

Proposition 5.2.4 (Assumption 4)**.** *If two maps have the same F_e-kernel, then they have the same F_s-kernel. Moreover, the map which assigns to each F_e-kernel the F_s-kernel preserves joins.*

Proposition 5.2.5 (Assumption 5)**.** *Any map $f\colon X \to Y$ factors as $f = mp$, where p is an F_e-projection of the F_e-kernel of f and m is an F_s-embedding of the F_s-image of f.*

Proof. The F_e-projection p of the F_e-kernel of f is nothing but a surjection with kernel relation the same as the kernel relation of f. And the F_s-embedding m of the F_s-image of f is nothing but an injection whose (usual) image is the same as the (usual) image of f. Any map f decomposes into such a surjection followed by an injection. □

So by the results is Subsection 4.4, F is a noetherian form over the category of sets.

5.2.2 Exploring noetherian forms over the category of sets

Here we will make general observations for any noetherian form over the category of sets. The aim is to develop enough results so that we can show that the noetherian form $F_{\mathbf{Set}}$ constructed in the next subsection is a minimum (in some sense) form over the category of sets. Throughout this section we are working in a fixed noetherian form over sets, unless stated otherwise.

We first start with a trivial observation.

Proposition 5.2.6. *In any noetherian form over the category of sets*

- *the embeddings are exactly the injections, and*
- *the projections are exactly the surjections.*

Proof. Any embedding is a monomorphism, thus an injection. Similarly, any projection is an epimorphism, thus a surjection.

Conversely, consider any injection f. By Axiom 4, it must factor as $f = mp$ where p is a projection and m an embedding. But then p is an injection as well, thus a bijection/isomorphism, forcing f to be an embedding. Similarly, we can show that the surjections are projections. $\square$

Conormal subobjects corresponds to embeddings, and by the above, embeddings corresponds to subsets. Therefore we will represent the conormal subobjects by the usual image of their corresponding embeddings. Similarly, normal subobjects correspond to equivalence relations. Therefore we will represent any normal subobject by the kernel relation of its corresponding projection. In particular, by this representation, for any map f, $\mathsf{Im} f$ is its usual image, and $\mathsf{Ker} f$ is its kernel relation.

Notation. The order relation on subobjects of the noetherian form will be denoted by $\leq$, and the usual subset relation by $\subseteq$. The direct image and inverse image of a map f will be denoted by f_* and f^* respectively. The usual direct and inverse image of subsets or of relations will be denoted by f and f^{-1} respectively.

We first explore the relationship between the ordering, and direct and inverse images in the noetherian form, with the ordering, and direct and inverse images of subsets and equivalence relations.

Proposition 5.2.7. *For any set X, we have*

- *$A \leq B$ if and only if $A \subseteq B$, for any two conormal subobjects of X, and*
- *$R \leq S$ if and only if $R \subseteq S$ (considered as subsets of $X \times X$), for any two normal subobjects of X.*

Proof. The corresponding embeddings of A and B are the the inclusions $\iota_A \colon A \to X$ and $\iota_B \colon B \to X$ respectively. We have the following:

$$A \leq B \Leftrightarrow \exists_h(\iota_A = \iota_B h) \Leftrightarrow A \subseteq B.$$

Similarly, let $\pi_R\colon X \to X/R$ and $\pi_S\colon X \to X/S$ be surjections with respective kernel relations R and S. These surjections correspond to the normal subobjects R and S respectively. We similarly have,

$$R \leq S \Leftrightarrow \exists_h(h\iota_R = \iota_S) \Leftrightarrow R \subseteq S.$$

□

Corollary 5.2.8. *For any set X, the top subobject is the subset X and the bottom subobject is the discrete equivalence relation D on X.*

Proof. The top subobject is conormal, and the largest conormal subobject is the subset X.

The bottom subobject is normal, and the smallest normal subobject is the discrete equivalence relation D on X. □

Proposition 5.2.9. *For any set X, we have, where $\wedge$ and $\vee$ denote the meet and join in the noetherian form,*

- *$A \wedge B = A \cap B$ for any conormal subobjects A and B;*
- *$R \vee S$ is the smallest equivalence relation containing both R and S.*

Proof. Consider any two conormal subobjects A and B. Their embeddings will be injections with usual images being A and B. The image of the diagonal of the pullback of these two maps is $A \wedge B$. But the usual image of this image is also $A \cap B$.

Similarly, for any two normal subobjects R and S, the diagonal of the pushout of their respective projections will have noetherian kernel $R \vee S$ and will also have as kernel relation the smallest equivalence relation containing both R and S. □

Proposition 5.2.10. *For any map $f\colon X \to Y$, and subset A of X, $f_*X = fX$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "g"'] & X \arrow[r, "f"] & Y \\ & fA \arrow[ur, "m"'] & \end{tikzcd}$$

where g is a restriction of f and m the inclusion of fA, and ι_A is the inclusion of A. We then have

$$f_*A = f_*\iota_*1 = m_*g_*1 = m_*1 = fA.$$

□

Proposition 5.2.11. *For any map $f\colon X \to Y$ and conormal subobject A of X, we have*

- $\underline{f^*A} = f^{-1}A$;
- $f^*A = f^{-1}A \vee \mathsf{Ker} f$.

Proof. For the first point, the pullback of the inclusion $\iota_A\colon A \to B$ and $f\colon X \to Y$ exists. The pullback is

$$\begin{tikzcd} f^{-1}A \arrow[r, "\iota_{f^{-1}A}"] \arrow[d, "g"] & X \arrow[d, "f"] \\ A \arrow[r, "\iota_A"] & Y \end{tikzcd} \tag{5.1}$$

where g is defined by $a \mapsto fa$. By Proposition 3.2.2, $\iota_{f^{-1}A}$ is an embedding of the largest conormal subobject $\underline{f^*A}$ contained in f^*A. From this observation the first point follows.

The second point is true for any embedding, since the kernel of embeddings are 0 and inverse images of conormal subobjects along embeddings are conormal. Consider a projection f. The pullback of ι_A and f is given by Diagram 5.1. Since f is a surjection/projection, so is g. We have, using Proposition 5.2.10,

$$f^*A = f^*\iota_A g(1) = f^*\iota_A g(f^{-1}A) = f^*f(\iota_{f^{-1}A}(f^{-1}A)) = f^*f_*(f^{-1}A) = f^{-1}A \vee \mathsf{Ker} f.$$

To show this then for an arbitrary f, factorize f into projection-embedding $f = mp$. Then

$$f^*A = p^*m^*A = p^*(m^{-1}A) = p^{-1}m^{-1}A \vee \mathsf{Ker} p = f^{-1}A \vee \mathsf{Ker} f.$$

$\square$

Corollary 5.2.12. *For any conormal subobject A of X, and any normal subobject R, we have*

- $\underline{A \vee R} = \pi_R^{-1}\pi_R A = \{x \in X \mid \exists_{a\in X}(xRa)\}$;
- $A \vee R = \pi_R^{-1}\pi_R X \vee R$.

Proof. For simplicity, denote π_R by p. By the previous proposition, we have

$$\underline{A \vee R} = \underline{p^*pA} = p^{-1}pA, \quad \text{and also}$$

$$A \vee R = p^*p_*A = p^*pA = p^{-1}pA \vee \mathsf{Ker} p.$$

$\square$

For any equivalence relation S on Y and map $f\colon X \to Y$, $f^{-1}S$ is defined to be $x(f^{-1}S)y$ if and only if $f(x)Sf(y)$. This makes $f^{-1}S$ an equivalence relation. Further, for any equivalence relation R on X, fR is the smallest equivalence relation containing $\{(fx, fy) \mid xRy\}$. Notice that this definition of direct image of an equivalence relation makes the following diagram a pushout:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\pi_R"] & Y \arrow[d, "p"] \\
X/R \arrow[r, "q"] & Y/fR
\end{tikzcd}$$

where p is a projection of fR and q is the map sending $[a]$ to $[fa]$.

Proposition 5.2.13. *For any map $f\colon X \to Y$ and normal subobject R of Y, $f^*R = f^{-1}R$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[r, "\pi_R"] & Y/R \\
& X/f^{-1}R \arrow[ur, "m"'] &
\end{tikzcd}$$

where g is the projection of $f^{-1}R$ (that is, a surjection with kernel relation $f^{-1}R$) and m is defined by $m[x] = [fx]$. The map m is well-defined, since $x(f^{-1}R)y$ if and only if $f(x)Rf(y)$. For the same reason, m is injective as well, thus an embedding. We have

$$f^*R = f^*\pi^*0 = g^*m^*0 = g^*0 = f^{-1}R.$$

□

Proposition 5.2.14. *For any morphism $f\colon X \to Y$ and normal subobject R of X, we have*

- $\overline{f_*R} = fR;$
- $f_*R = fR \wedge \mathsf{Im} f.$

Proof. For the first point, consider the pushout of the projection $\pi_R\colon X \to X/R$ and $f\colon X \to Y$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi_R"] & Y \arrow[d, "p"] \\
X/R \arrow[r, "q"] & Y/fR
\end{tikzcd} \tag{5.2}$$

where p is the projection of fR and $q[x] = [fx]$. By the dual of Proposition 3.2.2, p is the projection of $\overline{f_*R}$. From this observation, the first point follows.

The second point is true for any projection. Suppose f is an embedding. The pushout of π_R and f is given by Diagram 5.2. Since f is an embedding, so will q be (if $X = \emptyset$, then $q = f$; if $X \neq \emptyset$, then f is a split mono and consequently q is a split mono; in either case q is an embedding). We have, using Proposition 5.2.13,

$$f_*R = f_*\pi_R^{-1}q^{-1}0 = f_*f^{-1}p^{-1}0 = f_*f^{-1}fR = f_*f^*fR = fR \wedge \mathsf{Im}f.$$

To show this then for an arbitrary f, factorize f as a projection followed by an embedding, $f = mp$. Then

$$f_*R = m_*p_*R = m_*(pR) = mpR \wedge \mathsf{Im}m = fR \wedge \mathsf{Im}f.$$

□

Corollary 5.2.15. *For any normal subobject R and conormal subobject A of X, we have*

- $\overline{A \wedge R} = \iota_A\iota_A^{-1}R = (R \cap A^2) \cup D$;
- $A \wedge R = A \wedge \iota_A\iota_A^{-1}R$.

Proof. Let e denote the embedding of A, instead of ι_A. Then, using the previous proposition,

$$\overline{A \wedge R} = \overline{e_*e^{-1}R} = ee^{-1}R, \text{ and also}$$

$$A \wedge R = e_*e^{-1}R = ee^{-1}R \wedge \mathsf{Im}e = A \wedge ee^{-1}R.$$

□

With Corollaries 5.2.12 and 5.2.15 we reach the goal of this subsection; these results are needed to show that the noetherian form in the next subsection is a minimum one. The rest of this subsection consists of a handful of general results on noetherian forms over the category of sets, just for general interest. They could, however, be used to quickly determine that a form is not noetherian over sets.

Proposition 5.2.16. *For any set X and conormal subobject A and normal subobject R, $R \wedge A = 0$ if and only if none of the elements of A are related by R.*

Proof. If none of the elements of A is R-related, then $\iota_A\iota_A^{-1}R = D$. Then by Corollary 5.2.15, $A \wedge R = A \wedge D = D = 0$.

Suppose now that $A \wedge R = 0$. Suppose two different elements $a, b \in A$ are R-related, that is, aRb. Since $\{a\}$ and $\{a, b\}$ are contained in the same equivalence class under R, by Corollary 5.2.12, they have the same join with R. Also, they have the same meet with R, both meets being 0. This is a contradiction by the restricted modular law:

$$\{a\} = (\{a, b\} \wedge R) \vee \{a\} = \{a, b\} \wedge (R \vee \{a\}) = \{a, b\} \wedge (R \vee \{a, b\}) = \{a, b\}.$$

□

A different, and perhaps more direct, proof of the above is to work with the fact that $A \wedge R = 0$ exactly when $\pi_R \iota_A$ is an embedding.

Proposition 5.2.17. *For any set X and non-empty conormal subobject A and normal subobject R, $A \vee R = 1$ if and only if for every $x \in X$ there is an $a \in A$ such that xRa.*

Proof. From Corollary 5.2.12, the backwards direction is clear.

For the forward direction, suppose there is an $x \in X$ such that no $a \in A$ is R-related to x. Take any $a \in A$, and let S be the smallest equivalence relation containing R such that xSa. Then $R \vee X = 1 = S \vee X$, and by Corollary 5.2.15, $R \wedge A = S \wedge A$. But that contradicts the restricted modular law. $\square$

Proposition 5.2.18. *We cannot have that inverse images of conormal subobjects are conormal and at the same time direct images of normal subobjects are normal. Stated differently, it is impossible to have $f^*A = f^{-1}A$ and $f_*R = fR$, for any map f and conormal subobject A of its codomain and normal subobject R of its domain.*

Proof. Suppose it was the case. Take set $A = \{a, b, c, d\}$, and subset $\{a\}$ and equivalence relation $R = \{\{a, b\}, \{c, d\}\}$ (defined in terms of its equivalence classes). We have

$$(R \vee \{a\}) \wedge R = \pi_R^{-1}\pi_R\{a\} \wedge R = \{a, b\} \wedge R = \iota_{\{a,b\}}\iota_{\{a,b\}}^{-1}R = R \cap \{a, b\}^2 \cup D \neq R.$$

Thus contradicting that $(\mathsf{sub}A, \wedge, \vee)$ is a lattice. $\square$

5.2.3 A minimum noetherian form over the category of sets

Consider the noetherian form F over sets which was constructed in the Subsection 5.2.1. With the help of the following closure operator, we will construct a minimum noetherian form over the category of sets.

Proposition 5.2.19. *Consider any F-subobject (A, R). Let R' denote the smallest equivalence relation containing R in which A is a subset of an equivalence class of R'. Then*

$$C(A, R) = (A, R')$$

defines an idempotent closure operator on F.

Proof. The C is readily extensive and idempotent. For any morphism $f \colon X \to Y$ and F-subobjects (A, R) of X and (B, S) of Y, suppose that $(A, R) \leq_f (B, S)$. In particular we have

$$R \leq f^*S \leq f^*S'.$$

Further, since B is an equivalence class of S', $f^{-1}B$ is an equivalence class of f^*S', and $f^{-1}B$ containing A. Thus, by definition, $R' \leq f^*S'$. Consequently $(A, R') \leq_f (B, S')$. So C is an idempotent closure operator. $\square$

To construct that R', simply merge together all the equivalence classes contained in A (A is a union of equivalence classes of R) and define an equivalence relation from knowing these equivalence classes. A simple way of finding R' is: $R' = A^2 \cup R$.

From this, we can construct a new form F^C. This new form is noetherian as well.

Theorem 5.2.20. *The form $F_{\mathbf{Set}} = F^C$ over the category of sets, for which subobjects of a set X are pairs (A, R), where R is an equivalence relation on X and A is either an equivalence class of R or the empty set $A = \emptyset$, is a noetherian form.*

Proof. We are going to prove this theorem using Theorem 4.3.4.

Since any conormal F-subobject is of the form $(A, A^2 \cup D)$, where D is the discrete equivalence relation, all conormal subobjects are closed. Also, any normal F-subobject is of the form $(\emptyset, R)$, so is also closed. Further, for any map $f\colon X \to Y$ and F-subobject (B, S) of Y, we have

$$C(B, S) \cdot^F f = (B, S') \cdot^F f = (f^{-1}B, f^*S') = C(f^{-1}B, f^*S) = C((B, S) \cdot^F f).$$

The second last equality follows from the observation that $(f^*S)' = f^*S'$. Thus by Theorem 4.3.4, $F_{\mathbf{Set}}$ is also a noetherian form over the category of sets. $\square$

A significant feature of this new form, is:

Proposition 5.2.21. *Any $F_{\mathbf{Set}}$-subobject is equal to a join of a conormal and a normal subobject.*

Proof. Consider any set X and $F_{\mathbf{Set}}$-subobject (A, R). The pair $(A, A^2 \cup D)$ is a conormal $F_{\mathbf{Set}}$-subobject and the pair $(\emptyset, R)$ is a normal $F_{\mathbf{Set}}$-subobject. The join of these two pairs gives the original $F_{\mathbf{Set}}$-subobject back. $\square$

The above proof shows that any $F_{\mathbf{Set}}$-subobject is expressible as a join $A \vee R$, where R is an equivalence relation and A is either an equivalence class of R or is the empty subset. The proposition below will show that in an arbitrary noetherian form over the category of sets, such joins are always distinct. Being more specific: in any noetherian form G over the category of sets, we have normal subobjects (equivalence relations), conormal subobjects (subsets), and joins, so we have the following subset of $\mathsf{sub}^G X$, for any set X,

$$\{A \vee R \,|\, R \text{ is an equivalence relation on } X, \text{ and } \\ A \text{ is either an equivalence class of } R \text{ or empty}\}.$$

The proposition below shows that all those joins in the set above are distinct. And, the proposition above showed that the above set is exactly the elements of the subobject lattice of $\mathsf{sub}^{F_{\mathbf{Set}}} X$. So $\mathsf{sub}^{F_{\mathbf{Set}}} X$ can be seen as a subset of $\mathsf{sub}^G X$, or even a sub-poset by the corollary below, for any set X and any noetherian form G. In that sense $F_{\mathbf{Set}}$ is a minimum form over the category of sets.

Proposition 5.2.22. *Consider any noetherian form over the category of sets. For any object X and conormal subobjects A and B of X and normal subobjects R and S of X, where*

- *A is an equivalence class of R or is empty, and*
- *B is an equivalence class of S or is empty,*

we have

$$A \vee R = B \vee S \quad \Longrightarrow \quad A = B \quad \text{and} \quad R = S.$$

Proof. By Corollary 5.2.12, $\underline{A \vee R} = A$ and $\underline{B \vee S} = B$. And so, if $A \vee R = B \vee S$, then $A = B$. In particular, A is also an equivalence class of both R and S and thus also an equivalence class of $R \vee S$, or A is empty. In either case, the inverse images of both R and $R \vee S$ under ι_A go to the indiscrete equivalence relation on A, and thus we have

$$\iota_A \iota_A^{-1} R = \iota_A \iota_A^{-1}(R \vee S).$$

Further, by Corollary 5.2.15, we have

$$A \wedge R = A \wedge \iota_A \iota_A^{-1} R = A \wedge \iota_A \iota_A^{-1}(R \vee S) = A \wedge (R \vee S).$$

By the restricted modular law, we have

$$\begin{aligned} R &= R \vee (A \wedge R) \\ &= R \vee (A \wedge (R \vee S)) \\ &= (R \vee A) \wedge (R \vee S) \\ &= (A \vee R \vee S) \wedge (R \vee S) \\ &= R \vee S. \end{aligned}$$

The second last line is because $A \vee R = A \vee S$. And so $S \leq R$. Similarly, $R \leq S$. And thus $R = S$. □

Corollary 5.2.23. *Consider any noetherian form over the category of sets. For any object X and conormal subobject A and B of X and normal subobjects R and S of X, where*

- *A is an equivalence class of R or is empty, and*
- *B is an equivalence class of S or is empty,*

we have

$$A \vee R \leq B \vee S \quad \Longleftrightarrow \quad A \leq B \quad \text{and} \quad R \leq S.$$

Proof. If $A \vee R \leq B \vee S$, then by applying Corollary 5.2.12, we get $A \leq B$. So we have

$$A \vee R \leq B \vee R \leq B \vee S.$$

By applying Corollary 5.2.12 again, we get

$$B \leq \pi_R^{-1}\pi_R B \leq \pi_S^{-1}\pi_S B = B.$$

Thus either B is empty, or B is a union of equivalence classes of R and therefore B is an equivalence class $R \vee S$. Since $B \vee R \leq B \vee S$, we have

$$B \vee (R \vee S) = B \vee S,$$

so by the proposition above, $R \vee S = S$. That is, $R \leq S$.

The converse is clear. □

5.2.4 Varieties

Consider the category of any variety, and the noetherian form $F_{\mathbf{Set}} \colon \mathbb{B} \to \mathbf{Set}$ over the category of sets. Construct the pullback of $F_{\mathbf{Set}}$ along the forgetful functor $U \colon \mathbb{V} \to \mathbf{Set}$:

$$\begin{tikzcd} \mathbb{P} \arrow[r, "F_{\mathbb{V}}"] \arrow[d, "V"'] & \mathbb{V} \arrow[d, "U"] \\ \mathbb{B} \arrow[r, "F_{\mathbf{Set}}"] & \mathbf{Set} \end{tikzcd}$$

Just as in Subsection 4.2.4, we will assume that $\mathbb{P}$ is the category where morphisms are pairs (k, f), where k is in $\mathbb{B}$ and f is in $\mathbb{V}$ such that $F_{\mathbf{Set}}k = Uf$. Further, the subobjects of $X \in \mathbb{V}$ are pairs (A, X), where A is an $F_{\mathbf{Set}}$-subobject of UX, and all the operations are component-wise. Further, the direct image and inverse images of morphism $f \colon X \to Y$ is computed as follows:

$$f \cdot^{F}_{\mathbb{V}} (A, X) = (Uf \cdot^{F}_{\mathbf{Set}}, Y) \quad \text{and} \quad (B, Y) \cdot^{F}_{\mathbb{V}} = (B \cdot^{F}_{\mathbf{Set}}, X).$$

Proposition 5.2.24. *The functor $F_{\mathbb{V}}$ is an orean form which satisfies (N1).*

The above is just a restatement of Proposition 4.2.7 in this context.

Proposition 5.2.25. *The orean form $F_{\mathbb{V}}$ satisfies (N2).*

Proof. Consider any morphism $f \colon X \to Y$ in $\mathbb{V}$. Factorizes f as $f = mp$, where $m \colon I \to Y$ is an injective morphism and $p \colon X \to I$ a surjective morphism. The $F_{\mathbb{V}}$-image of m is the $F_{\mathbb{V}}$-image of f, and the $F_{\mathbb{V}}$-kernel of p is the $F_{\mathbb{V}}$-kernel of f.

We are first going to show that p is a projection. Consider any morphism $g \colon X \to Z$ whose kernel is above the kernel of p. So the $F_{\mathbf{Set}}$-kernel of Ug is above the $F_{\mathbf{Set}}$-kernel of Up. Since Up is an $F_{\mathbf{Set}}$-projection, there is a unique map h such that

$hU(p) = U(g)$. But $h\colon UI \to UZ$ is also a morphism. A simple proof of this is, consider any natural number n and any n-ary term t and n elements $a_1, \ldots, a_n$ of I. Then there are $b_1, \ldots, b_n$ in X such that $pb_1 = a_1, \ldots, pb_n = a_n$. We have

$$\begin{aligned} ht(a_1, \ldots, a_n) &= ht(pb_1, \ldots, pb_n) \\ &= hpt(b_1, \ldots, b_n) \\ &= gt(b_1, \ldots, b_n) \\ &= t(gb_1, \ldots gb_n) \\ &= t(hpb_1, \ldots, hpb_n) \\ &= t(ha_1, \ldots, ha_n). \end{aligned}$$

Thus p is a projection of the kernel of f.

For showing that m is an embedding of the image of f, consider any map $g\colon Z \to Y$ whose $F_{\mathbb{V}}$-image is below the $F_{\mathbb{V}}$-image of m. Then the $F_{\mathbf{Set}}$-image of Ug is below the $F_{\mathbf{Set}}$-image of Um. Since Um is an $F_{\mathbf{Set}}$-embedding, there is a unique map $h\colon UZ \to UI$ such that $U(m)h = U(g)$. This h is also a morphism. To check this, consider any natural number n and any n-ary term t and n elements $a_1 \ldots a_n$ of Z. We have

$$mht(a_1, \ldots, a_n) = t(mha_1, \ldots, mha_n) = mt(ha_1, \ldots, ha_n).$$

And since m is an injection, $ht(a_1, \ldots, a_n) = t(ha_1, \ldots ha_n)$. Thus m is an embedding of the image of f. □

Proposition 5.2.26. *The orean form $F_{\mathbb{V}}$ satisfies (N3).*

Proof. It is sufficient to check that the intersection of any two subalgebras is again a subalgebra, and that the join of two congruences as equivalence relations is again a congruence. Both of these are well-known. □

The arguments above still work if we started with an arbitrary noetherian form F over sets instead of $F_{\mathbf{Set}}$; then the pullback G of F along the forgetful functor U is still a noetherian form.

5.3 Grandis exact categories

From the results obtained in [16], it follows that any Grandis exact category is a noetherian form. In this subsection we show this on two specific examples, in order to provide further concrete illustrations of the axioms of a noetherian form. Both of these examples serve to show that biproducts may exist in a noetherian form that admits neither usual categorical products, nor coproducts.

In this section we are relying on the equivalent definition of a noetherian form given at the end of Subsection 1.3.

5.3.1 Sets and partial bijections

The objects are sets. The morphisms from set A to set B are triples (X, Y, f), where $X \subseteq A$ and $Y \subseteq B$ and $f\colon X \to Y$ is a bijection. The composite of

$$A \xrightarrow{(X,Y,f)} B \xrightarrow{(U,V,g)} C$$

is $(f^{-1}(Y \cap U), g(Y \cap U), k)$, where $k(x) = gf(x)$ for any $x \in f^{-1}(Y \cap U)$. That is you suitably restrict f and g and then compose them. This forms a category.

Lemma 5.3.1. *Composition is associative, and for any object A, $1'_A = (A, A, 1_A)$ is the identity maps.*

Proof. Readily, $1'_A$ is indeed the identity map of A. For associativity of composition, consider the following three composable morphisms

$$A \xrightarrow{f'=(X,Y,f)} B \xrightarrow{g'=(U,V,g)} C \xrightarrow{h'=(R,S,h)} D$$

The first component of $(h'g')f'$ is

$$f^{-1}(Y \cap g^{-1}(V \cap R)),$$

and the first component of $h'(g'f')$ is

$$\begin{aligned} f^{-1}g^{-1}(R \cap g(U \cap Y)) &= f^{-1}g^{-1}((R \cap V) \cap g(U \cap Y)) \\ &= f^{-1}(g^{-1}(R \cap V) \cap g^{-1}g(U \cap Y)) \\ &= f^{-1}(g^{-1}(V \cap R) \cap U \cap Y) \\ &= f^{-1}(Y \cap g^{-1}(V \cap R)). \end{aligned}$$

So their first components are the same. Further, for any element in the first component x, the third component of $h'(g'f')$ at x is $h(gf)x$ and the third component of $(h'g')f'$ at x is $(hg)fx$. Thus, since both third components are bijections, they are equal, and hence $h'(g'f') = (h'g')f'$.

It is clear that $1'_A = (A, A, 1_A)$ is the identity morphism for any object A. □

The subobjects of a set A are the subsets of A, and order is given by subset inclusion. Thus $\mathsf{sub}A$ is a bounded lattice for every object A.

For any morphism $f' = (X, Y, f)\colon A \to B$ the direct image of $U \subseteq A$ is

$$f'_*U = f(U \cap X)$$

and the inverse image of $V \subseteq B$ is

$$f'^*V = f^{-1}(V \cap Y) \cup A \backslash X.$$

Proposition 5.3.2. *Axiom 1 is satisfied.*

Proof. From the observations before, it forms a category.

To show that direct and inverse image forms a monotone Galois connection, take any morphism $f' = (X, Y, f)\colon A \to B$ and subobjects $U \subseteq A$ and $V \subseteq B$. We have

$$\begin{aligned}
& f'_* U \subseteq V \\
\iff & f(U \cap X) \subseteq V \\
\iff & f(U \cap X) \subseteq V \cap Y \\
\iff & U \cap X \subseteq f^{-1}(V \cap Y) \\
\iff & U \cap X \subseteq f^{-1}(V \cap Y) \cup A \backslash X = f'^* V.
\end{aligned}$$

For the identity morphism $(A, A, 1_A)\colon A \to A$, readily those direct and inverse image maps are identity maps as well. For any two composable morphisms $f' = (X, Y, f)\colon A \to B$ and $g' = (U, V, g)\colon B \to C$, we have for any subobjects $R \subseteq A$,

$$\begin{aligned}
(g'f')_* R &= gf(R \cap f^{-1}(Y \cap U)) \\
&= gf(R \cap X \cap f^{-1}(Y \cap U)) \\
&= g(f(R \cap X) \cap f(f^{-1}(Y \cap U))) \\
&= g(f(R \cap X) \cap Y \cap U) \\
&= g(f(R \cap X) \cap U) \\
&= g'_* f(R \cap X) = g'_* f'_* R.
\end{aligned}$$

Further, for any subobjects $R \subseteq A$ and $S \subseteq C$, we have

$$\begin{aligned}
& R \leq (gf)^* S \\
\iff & (gf)_* R \leq S \\
\iff & g_* f_* R \leq S \\
\iff & f_* R \leq g^* S \\
\iff & R \leq f^* g^* S
\end{aligned}$$

By taking suitable choices for R, we will get that $(gf)^* S = f^* g^* S$. $\square$

Notice that for any morphism $f' = (X, Y, f)\colon A \to B$, we have

- $\mathsf{Ker} f' = A \backslash X$, and
- $\mathsf{Im} f' = Y$.

Proposition 5.3.3. *Axiom 2 is satisfied.*

Proof. Consider any morphism $f' = (X, Y, f)\colon A \to B$ and subobjects $U \subseteq A$ and

$V \subseteq B$. We have

$$\begin{aligned}
f'^{*}f'_{*}U &= f'^{*}f(U \cap X) \\
&= f^{-1}(f(U \cap X) \cap Y) \cup A\backslash X \\
&= f^{-1}f(U \cap X) \cup A\backslash X \\
&= (U \cap X) \cup A\backslash X \\
&= U \cup A\backslash X \\
&= U \cup \mathsf{Ker} f',
\end{aligned}$$

and

$$\begin{aligned}
f'_{*}f'^{*}V &= f((f'^{*}V) \cap X) \\
&= f((f^{-1}(V \cap Y) \cup A\backslash X) \cap X) \\
&= f(f^{-1}(Y \cap V) \cap X) \\
&= ff^{-1}(Y \cap V) = V \cap Y \\
&= V \cap \mathsf{Im} f'.
\end{aligned}$$

□

Notice that all subobjects are both normal and conormal: For a subobject X of an object A, X is the image of the morphism $(X, X, 1_X)\colon X \to A$, and X is the kernel of the morphism $(A\backslash X, A\backslash X, 1_{A\backslash X})\colon A \to A\backslash X$. Those morphisms are also a respective embedding and projection of X, as shown in the proof below.

Proposition 5.3.4. *Axiom 3 is satisfied. In particular, for any set A and subobject X, the embedding of X is $\iota_X = (X, X, 1)\colon X \to A$ and the projection $\pi_X = (A\backslash X, A\backslash X, 1)\colon A \to A\backslash X$.*

Proof. Consider any morphism $g' = (U, V, g)\colon B \to A$ whose image is below X, that is $V \subseteq X$. Suppose there is a morphism $h' = (R, S, h)\colon B \to X$ such that $\iota_X h' = g'$. Then, in particular, $1_X(S \cap X) = V$ and $h^{-1}(S \cap X) = U$. But $S \subseteq X$, which forces $S = V$. Further, for any $v \in V$, $1_X^{-1}h^{-1}(v) = g^{-1}(v)$, and since both are bijections with codomains V, $R = U$ and $h = g$. Furthermore, $(U, V, g)\colon B \to X$ is indeed a morphism, since $V \subseteq X$ and $U \subseteq B$. This establishes that ι_X is an embedding of X.

Now, consider any morphism $g' = (U, V, g)\colon A \to B$ whose kernel is above X, that is $\mathsf{Ker} g' = A\backslash U \supseteq X$. Suppose there is a morphism $h' = (R, S, h)\colon A\backslash X \to B$ such that $h'\pi_X = g'$. Then $R = 1_{A\backslash X}^{-1}(R \cap A\backslash X) = U$. And for any $u \in U$, $h(u) = h(1_{A\backslash X}u) = gu$, and since they are both bijections with the same domain, $S = V$ and $h = g$. Furthermore $(U, V, g)\colon A\backslash X \to B$ is indeed a morphism, since $U \subseteq A\backslash X$ and $V \subseteq B$. This establishes that π_X is a projection of X. □

Proposition 5.3.5. *Axiom 4 is satisfied.*

Proof. Consider any morphism $f' = (X, Y, f)\colon A \to B$. First observe that

$$(X, Y, f)\colon X \to Y$$

is an isomorphism, with inverse (X, Y, f^{-1}). The projection of the kernel $A\backslash X$ is $(X, X, 1)\colon A \to X$ and the embedding of the image Y of f is $(Y, Y, 1)\colon Y \to B$. Composing these morphisms

$$(Y, Y, 1)(X, Y, f)(X, X, 1),$$

gives our starting morphism. Thus any morphism factorizes as the projection of the kernel followed by an isomorphism followed by the embedding of the kernel. $\square$

Proposition 5.3.6. *Axiom 5 is satisfied.*

Proof. Axiom 5 is trivially satisfied, since all subobjects are both normal and conormal. $\square$

The rest of this section is to show that biproducts exists and to compute what commutators and cocommutators are.

Interestingly, we have that direct and inverse images both preserve meets and joins.

Proposition 5.3.7. *For any morphism $f' = (X, Y, f)\colon A \to B$ and subobjects U, V of A and subobjects R, S of B, we have*

$$f'_*(U \cap V) = f'_*U \cap f'_*V \quad \text{and} \quad f'^*(R \cup S) = f'^*R \cup f'^*S.$$

Proof. Since $f\colon X \to Y$ is a bijection, the direct and inverse images both preserve $\cap$ and $\cup$. Using this observation, we have

$$f'_*(U \cap V) = f'(X \cap U \cap V) = f'(X \cap U) \cap f'(X \cap V) = f'_*U \cap f'_*V.$$

And also

$$f'^*(R \cup S) = f^{-1}(Y \cap (R \cup S)) \cup A\backslash X = (f^{-1}(Y \cap R) \cup f^{-1}(Y \cap S)) \cup A\backslash X = f'^*R \cup f'^*S.$$

$\square$

Then, by Proposition 2.3.9, if biproducts exists then the commutators has to be intersection and cocommutators union.

For any two sets A and B, we will use $A \times \{1\} \cup B \times \{2\}$ for their disjoint union $A \sqcup B$. The map $e_1\colon A \to A \times \{1\}$ is defined by $a \mapsto (a, 1)$. It's inverse is p_1. Similarly $e_2\colon B \to B \times \{2\}$ and p_2 are defined. For simplicity, we denote $A \times \{1\}$ by A' and $B \times \{2\}$ by B'.

Proposition 5.3.8. *For any two sets A and B, the biproduct of A and B is*

$$\begin{tikzcd} A \arrow[r, shift left, "{(A, A', e_1)}"] & A \sqcup B \arrow[l, shift left, "{(A', A, p_1)}"] \arrow[r, shift right, "{(B', B, p_2)}"'] & B \arrow[l, shift right, "{(B, B', e_2)}"'] \end{tikzcd}$$

Further, for any two subobjects X and Y of W, their commutator is $[X, Y] = X \cap Y$ and their commutator is $(X, Y) = X \cup Y$.

Proof. Immediately from the definitions of these maps, we have $\mathsf{Im} e'_1 = \mathsf{Ker} p'_2$ and $\mathsf{Im} e'_2 = \mathsf{Ker} p'_1$, $p'_1 e'_1 = 1'_A$ and $p'_2 e'_2 = 1'_B$.

Consider any two morphisms $f' = (U, F, f)\colon A \to W$ and $g' = (V, G, g)\colon B \to W$. Let

$$e' = (W \backslash (F \cap G), W \backslash (F \cap G), 1)\colon W \to W \backslash (F \cap G), \text{ and}$$

$$m' = (e_1 f^{-1}(F \backslash G) \cup e_2 g^{-1}(G \backslash F), F \Delta G, m)\colon A \sqcup B \to W \backslash (F \cap G),$$

where $m(a, 1) = fa$ and $m(b, 2) = gb$, and $F\Delta G$ is the disjoint union of F and G. This forms a cocone $(W \backslash (F \cap G), e, m)$ of $\mathsf{C}(f', g')$. By Theorem 2.3.10, this is then the colimit of $\mathsf{C}(f', g')$.

Now, consider any $f' = (U, F, f)\colon W \to A$ and $g' = (V, G, g)\colon W \to B$. Let

$$m' = (W \backslash (U \cap V), W \backslash (U \cap V), 1)\colon W \backslash (U \cap V) \to W, \text{ and}$$

$$d' = (U \Delta V, p_1^{-1}(f(U \backslash V)) \cup p_1^{-1}(g(V \backslash U)), h)\colon W \backslash (U \cap V) \to A \sqcup B,$$

where h is defined by: if $u \in U$, then $hu = (fu, 1)$ and if $v \in V$, then $hv = (gv, 2)$. Readily $(W\backslash, m', d')$ forms a cone of $\mathsf{L}(f', g')$. Further, m is an embedding of

$$W \backslash (U \cap V) = W \backslash U \cup W \backslash V = \mathsf{Ker} f' \cup \mathsf{Ker} g'.$$

And so by the dual of Theorem 2.3.10, it is a limit of $\mathsf{L}(f', g')$. □

5.3.2 Bounded modular lattices and modular connections

Here the objects are bounded modular lattices. Morphisms are so-called modular connections, as defined in [10]. A morphism here is a pair of functions $f = (f_*, f^*)\colon X \to Y$ where both $f_*\colon A \to B$ and $f^*\colon B \to A$ are order preserving functions, which satisfies the following condition: for any $x \in X$ and $y \in Y$

$$f^* f_* x = x \vee f^* 0 \quad \text{and} \quad f_* f^* y = y \wedge f_* 1.$$

A consequence is that f_* and f^* forms a monotone Galois connection. Composition of these morphisms is just composition term-wise, as in

$$g \circ f = (g_* f_*, f^* g^*).$$

This composition is well-defined, since for any x in the domain of f, we have

$$\begin{aligned}
(gf)^*(gf)_* x &= f^* g^* g_* f_* x \\
&= f^*(f_* x \vee g^* 0) \\
&= f^*(f_* 1 \wedge (f_* x \vee g^* 0)) \\
&= f^*(f_* x \vee (f_* 1 \wedge g^* 0)) \\
&= f^*(f_* x \vee f_* f^* g^* 0) \\
&= f^* f_*(x \vee f^* g^* 0) \\
&= x \vee f^* g^* 0 \vee f^* 0 \\
&= x \vee f^* g^* 0 \\
&= x \vee (g \circ f)^* 0.
\end{aligned}$$

And by dual arguments, we will get $(gf)_*(gf)^*x = x \wedge (gf)_*1$. Evidently the identity map for object A is $1_A = (1_A, 1_A)\colon A \to A$. Isomorphisms are exactly those maps f for which both f_* and f^* are isomorphisms.

For any bounded modular lattice A, the subobjects are just the elements of A, and the order relation on the subobjects is as in the lattice. For a morphism $f = (f_*, f^*)\colon A \to B$ the direct image of $a \in A$ under f is given by f_*a and the inverse image of $b \in B$ is given by f^*b.

The first two axioms are immediately true by definition.

Proposition 5.3.9. *Axiom 1 is satisfied.*

Proposition 5.3.10. *Axiom 2 is satisfied.*

Note that for any element $x \in X$, we have a morphism $\iota_x = (\iota_*, \iota^*) \downarrow x \to X$, where $\iota_* y = y$ and $\iota^* y = y \wedge x$. The image of ι_x is x, thus x is conormal. Further, we also have a morphism $\pi_x\colon X \to \uparrow x$, where $\pi_* y = y \vee x$ and $\pi^* y = y$. The kernel of π_x is x, thus x is normal as well.

Proposition 5.3.11. *Axiom 3 is satisfied. In particular, the embedding of x is ι_x and the projection π_x as described above.*

Proof. Suppose $f = (f_*, f^*)\colon Y \to X$ is a modular connection with $\mathsf{Im} f \leq x$. If there is a morphism $h\colon Y \to \downarrow x$ such that $\iota h = f$, then it forces $h_* a = \iota_* h_* a = f_* a$, for any $a \in Y$ and $h^* b = h^* \iota^* b = f^* b$ for any $b \in \downarrow x$. But h so defined is indeed a well-defined modular connection, since $\mathsf{Im} f \leq x$. Thus ι_x is the embedding of x.

By dual arguments, π_x is the projection of x. □

Proposition 5.3.12. *Axiom 4 is satisfied.*

Proof. Consider any modular connection $f = (f_*, f^*)\colon X \to Y$. The projection of the kernel of f is $\pi\colon X \to \uparrow f^*0$ and the embedding of the image of f is $\iota\colon \downarrow f_*1 \to Y$. Define an $h = (h_*, h^*)\colon \uparrow f^*0 \to \downarrow f_*1$ by $h_* a = f_* a$ and $h^* b = f^* b$. This h is an isomorphism. By composing π and h and ι, we get f. □

Proposition 5.3.13. *Axiom 5 is satisfied.*

Proof. It is trivially satisfied, since all subobjects are both normal and conormal. □

We could instead have worked with bounded distributive lattices instead of modular lattices, and the above construction and proofs are still valid.

For modular lattices as well as distributive lattices,

$$\begin{tikzcd} X \arrow[r, shift left, "e_1"] & X \times Y \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & Y \arrow[l, shift right, "e_2"'] \end{tikzcd} \tag{5.3}$$

where the maps are defined as

- $(e_1)_* x = (x, 0)$ and $e_1^*(x, y) = x$;

- $(e_2)_*y = (0, y)$ and $e_2^*(x, y) = y$;
- $(p_1)_*(x, y) = x$ and $p_1^*x = (x, 1)$;
- $(p_2)_*(x, y) = y$ and $p_2^*y = (1, y)$;

forms a split product. Thus, if biproducts exists, then this should be the biproduct. However, this is not the case for modular lattices. It is still true for distributive lattices.

Proposition 5.3.14. *The noetherian form of bounded modular lattices and modular connections does not have biproducts.*

Proof. Consider any $f\colon A \to W$ and $g\colon B \to W$. By Proposition 2.3.4, we have that $[\mathsf{Im}f, \mathsf{Im}g] \leq \mathsf{Im}f \wedge \mathsf{Im}g$. Let (C, e, m) be the colimit of $\mathsf{C}(f, g)$. Let $a = \mathsf{Im}f$ and $b = \mathsf{Im}g$. For $D = \mathsf{Im}f \wedge \mathsf{Im}g$ let $d\colon W \to\uparrow D, x \mapsto x \vee D$ be the projection of D. There is a cocone (D, d, n) over $\mathsf{C}(f, g)$. Indeed, since $\mathsf{Ker}e \leq \mathsf{Ker}d$, there is an $h\colon C \to D$ such that $d = he$; let $n = hm$. Notice that

$$\begin{aligned}
n_*1 &= n_*((e_1)_*1 \vee (e_2)_*1) = n_*(e_1)_*1 \vee n_*(e_2)_*1 \\
&= d_*f_*1 \vee d_*g_*1 = d_*a \vee d_*b \\
&= a \vee b \vee (a \wedge b) = a \vee b.
\end{aligned}$$

Notice that for any $(x, y) \in X \times Y$,

$$(x, y) = (e_1)_*(e_1)^*(x, y) \vee (e_1)_*(e_1)^*(x, y).$$

This allows us to compute n^*z, for any $z \in\uparrow(a \wedge b)$:

$$n^*z = (e_1)_*(e_1)^*n^*z \vee (e_1)_*(e_1)^*n^*z = (e_1)_*f^*d^*z \vee (e_2)_*g^*d^*z = (f^*z, g^*z).$$

Further, we have for any $z \in\uparrow(a \wedge b)$,

$$\begin{aligned}
z \wedge (a \vee b) &= n_*n^*z \\
&= n_*(f^*z, g^*z) \\
&= n_*(e_1)_*f^*z \vee n_*(e_2)_*g^*z \\
&= d_*f_*f^*z \vee d_*g_*g^*z \\
&= d_*(z \wedge a) \vee d_*(z \wedge b) \\
&= (z \wedge a) \vee (z \wedge b) \vee (a \wedge b) \\
&= (z \wedge a) \vee (z \wedge b).
\end{aligned}$$

Since we selected arbitrary bounded modular lattices and elements, any bounded modular lattice should then satisfy this condition

$$z \wedge (a \vee b) = (z \wedge a) \vee (z \wedge b), \text{ for any } z \geq (a \wedge b).$$

This is a contradiction. Thus biproducts do not exist. $\square$

For the distributive case, the following will be useful. This came from [10], page 52, (1.66).

Lemma 5.3.15. *For any modular connection $f = (f_*, f^*)\colon X \to Y$, f_* preserves meets and f^* preserves joins.*

Proof. For any two elements $x, y \in X$, we have

$$\begin{aligned} f_*(x \wedge y) &= f_*((x \wedge y) \vee f^*0) = f_*((x \vee f^*0) \wedge (y \vee f^*0)) \\ &= f_*(f^*f_*x \wedge f^*f_*y) = f_*f^*(f_*x \wedge f_*y) \\ &= f_*x \wedge f_*y \wedge f_*1 = f_*x \wedge f_*y. \end{aligned}$$

With the dual argument, f^* preserves joins. □

A consequence of the above lemma, is that if biproducts does exist (which we will show), the commutator of x and y is their meet and their cocommutator is their join.

Proposition 5.3.16. *The noetherian form of bounded distributive lattices and modular connections does have biproducts.*

Proof. Consider the split product $X \times Y$ as defined above. Consider any pair of modular connections $f\colon X \to W$ and $g\colon Y \to W$. Let $e\colon W \to \uparrow(\mathsf{Im}f \wedge \mathsf{Im}g)$ be the projection of $\mathsf{Im}f \wedge \mathsf{Im}g$. Define the modular connection $m\colon X \times Y \to \uparrow(\mathsf{Im}f \wedge \mathsf{Im}g)$ by

- $m_*(x, y) = f_*x \vee g_*y \vee (\mathsf{Im}f \wedge \mathsf{Im}g) = (f_*x \vee g_*1) \wedge (f_*1 \vee g_*y)$, and
- $m^*a = (f^*a, g^*a)$.

This is indeed a modular connection, since it is order preserving, and

$$\begin{aligned} m^*m_*(x, y) &= m^*((f_*x \vee g_*1) \wedge (f_*1 \vee g_*y)) \\ &= (f^*((f_*x \vee g_*1) \wedge (f_*1 \vee g_*y)), g^*((f_*x \vee g_*1) \wedge (f_*1 \vee g_*y))) \\ &= (f^*(f_*x \vee g_*1), g^*(f_*1 \vee g_*y)) \\ &= (x \vee f^*0 \vee f^*g_*1, y \vee g^*0 \vee g^*f_*1) \\ &= (x, y) \vee (f^*g_*1, g^*f_*1) \\ &= (x, y) \vee (f^*(f_*1 \wedge g_*1), g^*(f_*1 \wedge g_*1)) \\ &= (x, y) \vee m^*(\mathsf{Im}f \wedge \mathsf{Im}g) \\ &= (x, y) \vee \mathsf{Ker}m, \end{aligned}$$

and

$$\begin{aligned} m_*m^*a &= m_*(f^*a, g^*a) \\ &= f_*f^*a \vee g_*g^*a \vee (\mathsf{Im}f \wedge \mathsf{Im}g)) \\ &= (a \wedge \mathsf{Im}f) \vee (a \wedge \mathsf{Im}g) \vee (\mathsf{Im}f \wedge \mathsf{Im}g) \\ &= (a \wedge (\mathsf{Im}f \vee \mathsf{Im}g)) \vee (\mathsf{Im}f \wedge \mathsf{Im}g) \\ &= (a \vee (\mathsf{Im}f \wedge \mathsf{Im}g)) \wedge (\mathsf{Im}f \vee \mathsf{Im}g) \\ &= a \wedge (\mathsf{Im}f \vee \mathsf{Im}g) \\ &= a \wedge \mathsf{Im}m. \end{aligned}$$

$(\uparrow(\mathsf{Im}f \wedge \mathsf{Im}g), e, m)$ is a cocone, since

$$m_*(e_1)_*x = m_*(x, 0) = f_*x \vee (\mathsf{Im}f \wedge \mathsf{Im}g) = e_*f_*x$$

and similarly $m_*(e_2)_*y = e_*g_*y$.

Since $\mathsf{Ker}e = \mathsf{Im}f \wedge \mathsf{Im}g$, that cocone is a colimit of $\mathsf{C}(f, g)$ by Theorem 2.3.10.

Further, consider any (new) pair of modular connections $f \colon W \to X$ and $g \colon W \to Y$. Let $n \colon \downarrow(\mathsf{Ker}f \wedge \mathsf{Ker}g)$ be the embedding of $\mathsf{Ker}f \wedge \mathsf{Ker}g$. Define $d \colon \downarrow(\mathsf{Ker}f \wedge \mathsf{Ker}g)$ by

- $d_*a = (f_*a, g_*a)$, and
- $d^*(x, y) = f^*x \wedge g^*y \wedge (\mathsf{Ker}f \vee \mathsf{Ker}g)$.

By dual arguments one can show that d is a modular connection and that $(\downarrow(\mathsf{Ker}f \vee \mathsf{Ker}g), n, d)$ is a limit of $\mathsf{L}(f, g)$. □

5.4 Posets seen as categories

In this section, as in the previous section, we will use the equivalent definition of a noetherian form.

We will assume that the order is defined as $a \leq b$ if and only if there is an $a \to b$.

Proposition 5.4.1. *Any preorder, where for any object a, $\downarrow a$ is a bounded lattice, can be made into a noetherian form where every morphism is an embedding and for any object a, $\mathsf{sub}(a) = \downarrow a$.*

Proof. Let the subobject lattices of any object a be $\downarrow a$. Consider any morphism $f \colon a \to b$ (so in particular $a \leq b$). For any subobject x of a and subobject y of b, define the direct image of f as $f_*x = x$, and define the inverse image as $f^*y = y \wedge a$. With this, Axiom 1 and 2 are readily satisfied. For Axiom 3, notice that all subobjects are conormal, and only 0 is normal. For object a, the projection of 0 is just 1_A. The embedding for $x \in \mathsf{sub}(a) = \downarrow a$, is the unique map $x \to a$. Axiom 4 is trivially satisfied, since every map is an embedding. Likewise, Axiom 5 is trivially satisfied since only 0 is normal, while all subobjects are conormal. □

Corollary 5.4.2. *Any finite poset with meets can be made into a noetherian form.*

Proof. Since $\downarrow$ is a finite poset with meets, it also has joins, thus is a lattice. Then the result follows from the above proposition. □

Corollary 5.4.3. *Any lattice can be made into a noetherian form.*

Non-trivial lattices provide a class of examples of noetherian forms, where both products and coproducts exist, but split products (and thus biproducts) do not.

From the proposition above the category with exactly one non-identity arrow f can be made into a noetherian form where f is strictly an embedding (since that category is a lattice), but dually can be made into a noetherian form where f is strictly a projection. This is an example where a category can be made into a noetherian form in two different ways. But more importantly, it shows that for a noetherian form, one cannot always deduce from categorical properties whether a morphism is an embedding or projection for a given form.

There are examples of (finite) preorders that cannot be made into a noetherian form, for example:

$$\begin{tikzcd} & {} \arrow[dl, "a"'] \arrow[d, "b"] \arrow[dr, "c"] & \\ {} & {} & {} \\ & {} \arrow[ul, "x"] \arrow[u, "y"] \arrow[ur, "z"'] & \end{tikzcd}$$

for, since no arrow decomposes into two non-identity arrows, every labelled arrow should either be an embedding or a projection (cannot be both, since they are not isomorphisms). Since the pushout of a and b does not exists, at least a or b must be an embedding. Continuing this argument, we will get that at least two of the arrows a, b, and c are embeddings. Similarly at least two of the arrows x, y, and z are embeddings. But then there are among the pairs (a, x), (b, y), and (c, z) is a pair where both are embeddings. But since their pullbacks do not exist, it is impossible.

It can be argued out that the same diagram without c and z cannot be made into a finite noetherian form (subobject lattices are finite). Following the arguments in the previous paragraph, the only possibility left where it could possibly be a noetherian form, without loss of generality, is when a and y are embeddings, and b and x are projections. Since the direct image of any embedding is a injection of subobject lattices, and the direct image of any projection is a surjection of subobject lattices, and the lattices are finite, we have

$$|\mathsf{dom}a| < |\mathsf{cod}a| = |\mathsf{cod}x| < |\mathsf{dom}x| = |\mathsf{dom}y| < |\mathsf{cod}y| = |\mathsf{cod}b| < |\mathsf{dom}b| = |\mathsf{dom}a|.$$

But that is a contradiction.

Notice first that for the category with just two parallel arrows, if it were to be a noetherian form, then the one is a projection and the other an embedding (since pullbacks and pushouts do not exist). Consider the category

$$\begin{tikzcd} & C \arrow[d, shift left] \arrow[d, shift right] \\ A \arrow[r, shift left] \arrow[r, shift right] & B \end{tikzcd}$$

As discussed, at least one of each pair of parallel arrows should be an embedding. But then that is a contradiction, since pullbacks of embeddings along embeddings exists in any noetherian form. This gives another category that cannot be made into a noetherian form, which is smaller (in terms of amount of objects and arrows) than our previous counter example.

Chapter 6

Ranked complemented and ranked Boolean lattices

6.1 Introduction

In this chapter we present a further development of the theory presented in [27].

The present chapter fits better in the research area of lattice theory, with applications in the study of subgroup lattices, rather than in categorical algebra, to which the rest of thesis belongs. Although we do not make use much of the existing work from the literature on subgroup lattices, it is worth mentioning that [26] contains a large body of research in this area (there are, however, interesting contributions to this field not covered in [26], such as [29]).

The following definition is taken from [27]:

Definition 6.1.1. A lattice L is said to be 0-*complemented* if L consists of only one element. For $n \geqslant 1$, a lattice L is said to be n-*complemented* if it is bounded below and for any $x \in L$, there exists $y \in L$, called an n-*complement* of x, such that

- $x \wedge y$ is the bottom element of the lattice, and
- $\uparrow(x \vee y)$ is $(n-1)$-complemented.

The *rank of complementedness* $\mathsf{rc}(L)$ of L is the smallest natural number m for which L is m-complemented.

We recover usual notions of complement and complemented lattice by setting $n = 1$. It is a simple result that the subgroup lattice $\mathsf{sub}(G)$ of a finite abelian group G is complemented if and only if every element in $\mathsf{sub}(G)$ has square-free order. The question of generalizing this result to a characterization of finite abelian groups whose elements have $(n+1)$-free order (the order is not divisible by $(n+1)$-th power of any prime) is what led the me to the notion above. Such groups turn out to be precisely those whose subgroup lattices are n-complemented in the above sense, as proved in my honors project. However, the proof in my honors project was done only using group theory. Professor While writing up my honors project for publication during my PhD, it was pointed out to me that that proof does not seem to rely much on group theory. Exploring this led me to new work and results. One of the (new) main results is the following, from which the classification of which finite abelian groups have an n-complemented subgroup lattice follows as a corollary.

Theorem 6.1.2. *For any modular lattice L having finite height and any natural number n, the following conditions are equivalent:*

(i) *L is n-complemented.*

(ii) *L contains a chain $0 = a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n = 1$ such that each interval $[a_i, a_{i+1}]$ is complemented.*

(iii) *For every chain $a_0 < a_1 < \cdots < a_m$ in L where each a_i is meet-irreducible, we have $m \leqslant n$.*

In the subgroup lattice of a finite abelian group, we always have $\uparrow a \wedge b$ is n-complemented if both $\uparrow a$ and $\uparrow b$ are n-complemented. To explore when the same holds for an arbitrary modular lattice, is what led be to the theorem above.

One of the other main results in my honors project was that for finite distributive lattices a lattice is n-complemented if and only if it is n-Boolean in the following sense

Definition 6.1.3. A lattice L is said to be *n-Boolean* if it is bounded if it contains a chain of elements

$$0 = a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n = 1$$

such that each interval $[a_i, a_{i+1}]$ is a Boolean algebra. The *rank of Booleanness* $\mathsf{rb}(L)$ of L is the smallest natural number m for which L is m-complemented.

That finite distributive n-complemented lattices and n-Boolean lattices coincide also follows as a corollary of the above theorem.

It turns out that the subgroup lattice of a finite abelian group is n-Boolean if and only if the order of the group itself is n-free. This was known in my honors project as well, but also originally had a group-theoretic proof. Similar as for the n-complemented case, understanding what the rank of Booleanness of $\uparrow a \wedge b$ is provided that we know what the rank of Booleanness of $\uparrow a$ and $\uparrow b$ is, is useful for understanding the original group-theoretic proof of classifying which finite abelian groups have n-Boolean subgroup lattices.

Almost all of the results in this chapter are new. Including exploring the connection of the notion of a pseudo-complement and the notion of an n-complement.

6.2 Ranked complementedness

6.2.1 General lattices

The definition of an n-complemented lattice requires that the lattice must have a bottom element 0. Moreover, it can be deduced from the definition that it has the top element 1 as well.

Proposition 6.2.1. *For any natural number $n \geqslant 1$, if an element of L has an n-complement, then L is a bounded lattice. Consequently, any n-complemented lattice is bounded.*

Proof. The proof is by induction. Consider a lattice in which there are some elements x and y which are 1-complements of each other. Thus $\uparrow(x \vee y)$ is 0-complemented, from which it follows that $x \vee y$ is the top element. Further, $x \wedge y$ is the bottom element, thus the lattice is bounded. Suppose the statement is true for n. Consider any lattice L in which there is an element x that has an $(n+1)$-complement y. Then $x \wedge y$ is the bottom element, thus L is bounded below. Further $\uparrow (x \vee y)$ is n-complemented, thus bounded by the induction hypothesis. Consequently L is bounded above, and therefore bounded. □

It is not difficult to see that an n-complement of an element x_1 in a (bounded) lattice is an element y_1 such that the following holds:

$$\begin{aligned} 0 &= x_1 \wedge y_1 \\ \& \ \forall_{x_2 \geqslant x_1 \vee y_1} \exists_{y_2} [x_1 \vee y_1 &= x_2 \wedge y_2 \\ &\vdots \\ \& \ \forall_{x_n \geqslant x_{n-1} \vee y_{n-1}} \exists_{y_n} [x_{n-1} \vee y_{n-1} &= x_n \wedge y_n \\ \& \ \ x_n \vee y_n &= 1] \ldots] \end{aligned}$$

This allows us to draw the following (almost) consecutive conclusions:

- The trivial lattice is n-complemented for every $n \geqslant 0$.
- For any $n \geqslant 1$ and lattice L, an element $x \in L$ is an n-complement of 0 if and only if $\uparrow x$ is $(n-1)$-complemented
- 1 is an n-complement of 0, for any $n \geqslant 1$.
- In any lattice, 0 is the unique n-complement of 1, for each $n \geqslant 1$.
- For each $n \geqslant 1$, every n-complement of an element $x \in L$ is also an m-complement of x for every $m > n$.
- For each $n \geqslant 0$, every n-complemented lattice is m-complemented for every $m > n$.

In Table 6.1, we classify all lattices having at most 6 elements according to their rank of complementedness.

It would be interesting to know how many (isomorphism classes) of finite lattices of given size and given rank of complementedness there are. Note that the problem of counting of lattices of a given size is open. Table 6.2 summarizes these numbers for the lattices in Table 6.1.

In some cases there are results that help to compute the rank of complementedness easier. Most of these results are for modular lattices having finite height, but Lemma 6.2.2 is for general lattices.

Table 6.1: This table displays lattices having 6 or less elements, together with the rank of complementedness indicated above each lattice.

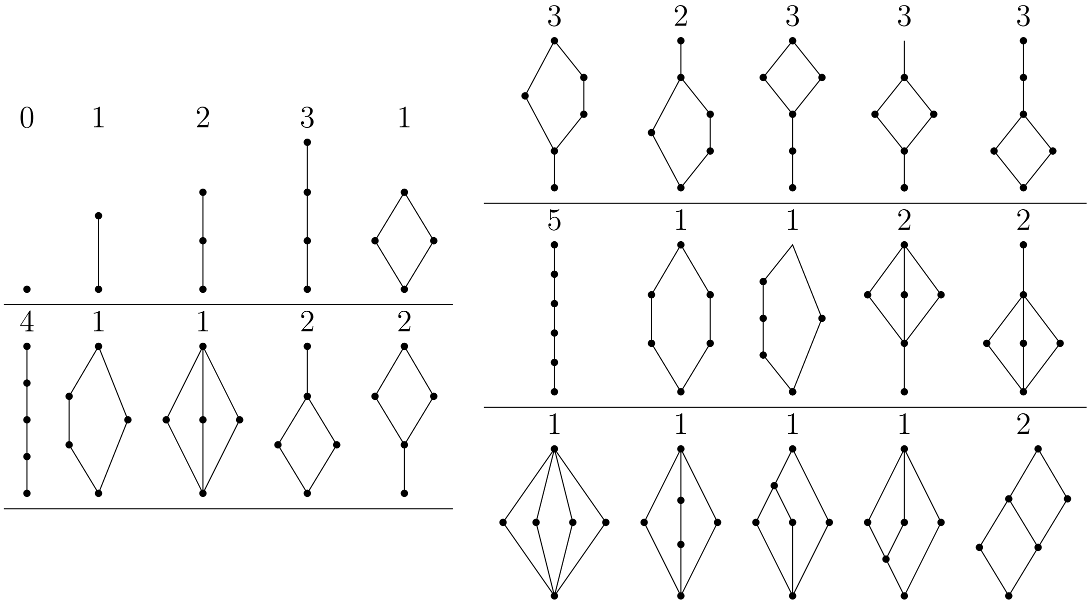


Table 6.2: The entry in the m-th row and n-th column is the number of (isomorphism classes) of lattices having m elements and the rank of complementedness being n.

	0	1	2	3	4	5
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	1	0	0	0
4	0	1	0	1	0	0
5	0	2	2	0	1	0
6	0	6	4	4	0	1

Recall that in a bounded above lattice, an element (different from the top element) is called a *coatom* if there is no element strictly between it and the top element. A lattice is *coatomic* if every element is below some coatomic element.

Lemma 6.2.2. *For any coatomic lattice L having a unique coatomic element c, we have that $\mathsf{rc}(L)$ exists if and only if $\mathsf{rc}(\downarrow c)$ exists. In either case, we have*

$$\mathsf{rc}(L) = \mathsf{rc}(\downarrow c) + 1.$$

Proof. Both directions will be proved by induction.

Consider any coatomic lattice L with a unique coatomic element c such that L is 1-complemented. So c must have a complement d. Either $d = 1$ or $d \leq c$ (since the lattice is coatomic and c is unique coatomic element). Since $c \vee d = 1 \neq c$, we must have that $d = 1$. But then $c = c \wedge d = 0$. Consequently $\downarrow c$ is 0-complemented. Suppose for any coatomic $(n+1)$-complemented L having a unique coatomic element c, we have that $\downarrow c$ is n-complemented. Consider any coatomic $(n+2)$-complemented lattice L having a unique coatomic element c. For 0 in $\downarrow c$, c is an $(n+1)$-complement. For $0 \neq x \in \downarrow c$, it has an $(n+2)$-complement y in L. Since $x \neq 0$, $y \neq 1$, thus $y \leqslant c$. Since $\uparrow(x \vee y)$ is $(n+1)$-complemented in L, by the induction hypothesis $[x \vee y, c]$ is n-complemented. Thus y is an $(n+1)$-complement of x in $\downarrow c$. Therefore $\downarrow c$ is $(n+1)$-complemented. So by mathematical induction, if $\mathsf{rc}(L)$ exists, then $\mathsf{rc}(\downarrow c)$ exists and $\mathsf{rc}(L) \geqslant \mathsf{rc}(\downarrow c) + 1$.

Consider any coatomic lattice L with a unique coatomic element c such that $\downarrow c$ is 0-complemented. Then $c = 0$, and consequently L is 1-complemented. Suppose for any coatomic lattice L with a unique coatomic element c, if $\downarrow c$ is n-complemented, then L is $(n+1)$-complemented. Consider any coatomic lattice L with a unique coatomic element c such that $\downarrow c$ is $(n+1)$-complemented. For 1 in L, 0 is an $(n+2)$-complement. Take any $1 \neq x \in L$. Then $x \leq c$, and so x has an $(n+1)$-complement y in $\downarrow c$. We have $x \wedge y = 0$ and $[x \vee y, c]$ is n-complemented. By the induction hypothesis, $\uparrow x \vee y$ is $(n+1)$-complemented. Consequently L is $(n+2)$-complemented. So by mathematical induction, if $\mathsf{rc}(\downarrow c)$ exists, then $\mathsf{rc}(L)$ exists and $\mathsf{rc}(L) \leqslant \mathsf{rc}(\downarrow c) + 1$. $\square$

The dual of this result is not true, as shown by the lattice in the top left corner in the second column of Table 1.

Finite lattices are obviously not the only ones that have rank of complementedness. The Boolean algebra of subsets of any set is a complemented lattice and so has rank of complementedness equal to 1. In general it is not clear which lattices have rank of complementedness. However, as we will now see, there is an easy sufficient condition. A lattice is said to have *finite height* when it does not contain infinite chain, and moreover, there is a natural number n (called the height of the lattice) for which there exists a chain of elements

$$0 = a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n = 1$$

in the lattice, such that there is not a longer chain in the lattice.

Proposition 6.2.3. *Every lattice L with finite height n has rank of complementedness and $\mathsf{rc}(L) \leqslant n$.*

Proof. For $n = 0$ it is clear. Suppose it is true for some natural number n. Take any lattice with height $n+1$. For any $a \in L\backslash\{0\}$, the height of $\uparrow a$ is at most n, thus $\uparrow a$ is n-complemented. Consequently, 0 is an $(n+1)$-complement of a. Thus L is $(n+1)$-complemented, and so $\mathsf{rc}(L) \leqslant n+1$. □

The bound specified in the proposition above can be reached only for chains, as shown by the proposition below.

Proposition 6.2.4. *For any lattice L having finite height, the following are equivalent:*

(a) L is a chain;

(b) L has $\mathsf{rc}(L)+1$ many elements;

(c) The height of L is equal to $\mathsf{rc}(L)$;

(d) For all $x, y \in L$ such that $x < y$, $\mathsf{rc}(\uparrow x) > \mathsf{rc}(\uparrow y)$.

Proof. The proof of the equivalences is by induction on the height of L. If the height of L is 0, then the proposition is readily true. Suppose the proposition is true for any lattice L of height less or equal to n. Consider any lattice L with height $n+1$.

(a) $\Rightarrow$ (b): Let c be a coatomic element. Then by the induction hypothesis $\downarrow c$ has $\mathsf{rc}(\downarrow c)+1$ elements. Since L is a finite chain, c is the unique coatomic element, so then by Proposition 6.2.2, $\mathsf{rc}(L) = \mathsf{rc}(\downarrow c)+1$. Since L has one more element than $\downarrow c$, L has $\mathsf{rc}(L)+1$ elements.

(b) $\Rightarrow$ (c): If L has $\mathsf{rc}(L)+1$ elements, then the height of L is bounded by $\mathsf{rc}(L)$. But by Proposition 6.2.3, the height of L bounds $\mathsf{rc}(L)$. Thus $\mathsf{rc}(L)$ is the height of L.

(c) $\Rightarrow$ (d): By assumption $\mathsf{rc}(\uparrow 0)$ is equal to the height of $\uparrow 0$. For any $x > 0$, the height of $\uparrow x$ is less than $n+1$ and for any $a, b \in \uparrow x$, we still have $a > b \Rightarrow \mathsf{rc}(\uparrow a) < \mathsf{rc}(\uparrow b)$. So by the induction hypothesis, $\mathsf{rc}(\uparrow x)$ is equal to the height of $\uparrow x$. From these observations, (d) immediately follows.

(d) $\Rightarrow$ (a): From (d) and the fact that the height of L is $n+1$, it follows that $\mathsf{rc}(L) \geqslant n+1$. If $n = 0$, then the lattice is of height 1, and consequently L is forced to be the two element chain. So suppose $n \geqslant 1$. For any element x which is neither atomic nor 0, $\uparrow x$ has height less or equal to $n-1$, consequently $\uparrow x$ is $(n-1)$-complemented and thus 0 is an n-complement of x. For 0, 1 is an n-complement. Take any atomic elements a and b. If $a \neq b$, then $a \wedge b = 0$ and $\uparrow a \vee b$ is $(n-1)$-complemented (since its height is bounded by $n-1$). Consequently L is an n-complemented lattice, contradicting $\mathsf{rc}(L) \geqslant n+1$. Thus L has a unique atomic element a. Further, by the induction hypotheses it follows that $\uparrow a$ is a chain. Thus L is a chain. □

The next two propositions show that the class of n-complemented lattices is closed under homomorphic images and products.

Proposition 6.2.5. *Surjective lattice homomorphisms preserve n-complements, for $n \geqslant 1$. Consequently, images of n-complemented lattices are n-complemented.*

Proof. For $n = 1$, consider any surjective lattice homomorphism $f\colon L \to M$. Take any $x \in L$ such that x has a 1-complement $y \in L$. Then $fx \wedge fy = f(x \wedge y) = 0$ and $fx \vee fy = f(x \vee y) = 1$. So fy is a 1-complement of fx. Thus 1-complements are preserved. Suppose n-complements are preserved under any surjective lattice homomorphism. Consider any surjective lattice homomorphism $f\colon L \to M$, and take any $x \in L$ such that it has an $(n+1)$-complement y. We have $fx \wedge fy = f(x \wedge y) = 0$. Also

$$\uparrow(fx \vee fy) = \uparrow f(x \vee y) = f(\uparrow x \vee y).$$

The restriction of f to $\uparrow x \vee y \to f(\uparrow x \vee y)$ is a surjection. Since by assumption n-complements are preserved and $\uparrow x \vee y$ is n-complemented, $\uparrow fx \vee fy$ is n-complemented. Hence fy is an $(n+1)$-complement of fx. ☐

Proposition 6.2.6. *For any family of lattices $(L_i)_{i\in I}$, the product $L = \prod_{i\in I} L_i$ is n-complemented if and only if all the factors are n-complemented. Consequently*

$$\mathsf{rc}\left(\prod_{i\in I} L_i\right) = \mathsf{max}(\mathsf{rc}(L_i))_{i\in I}.$$

Proof. If the product L is n-complemented, then for each $i \in I$, L_i is n-complemented by Proposition 6.2.5, since $\pi_i\colon L \to L_i$ is a surjective lattice homomorphism.

The converse we prove by induction. A product of 0-complemented lattices (trivial lattices), is still a 0-complemented lattice. Suppose that a product of n-complemented lattices is an n-complemented lattice. Consider any family of $(n+1)$-complemented lattices $(L_i)_{i\in I}$. Take any element $a = (a_i)_{i\in I} \in L = \prod_{i\in I} L_i$. Let $b = (b_i)_{i\in I}$ be an element of L such that for every i, b_i is an $(n+1)$-complement of a_i in L_i. Then we have for any $i \in I$

$$(a \wedge b)_i = a_i \wedge b_i = 0.$$

So $a \wedge b = 0$. Also,

$$\uparrow(a \vee b) = \prod_{i\in I} \uparrow(a_i \vee b_i).$$

Since for each $i \in I$, $\uparrow(a_i \vee b_i)$ is n-complemented, $\uparrow(a \vee b)$ is n-complemented as well. Thus L is $(n+1)$-complemented.

The last part of the proposition can be readily observed from the first part. ☐

6.2.2 Modular lattices

Even though any finite lattice has a rank of complementedness, finding this rank is not always so straightforward. However, for finite modular lattices there are easier ways to determine the rank of complementedness, namely with the help of the results in Theorem 6.1.2. Another useful result is that in any finite modular lattice, for any two elements a and b we have

$$\mathsf{rc}(\uparrow a \wedge b) = \max\{\mathsf{rc}(\uparrow a), \mathsf{rc}(\uparrow b)\}.$$

This subsection is dedicated to prove these stated results, and also other similar results.

Definition 6.2.7. A lattice is *relatively* n-complemented if any interval in it is n-complemented.

The result below will be used often throughout this subsection and the next. It generalizes the fact that all complemented modular lattices are relatively complemented (see for example Theorem 6.1 in [3]).

Lemma 6.2.8. *Any modular n-complemented lattice is relatively n-complemented.*

Proof. The result is true for $n = 0$. Suppose it is true for n. Consider any modular $(n+1)$-complemented lattice L and $a \leqslant b$ in L. Consider any $x \in [a, b]$. Let y be an $(n+1)$-complement of x in L. The claim is that $(y \wedge b) \vee a$ is an $(n+1)$-complement of x in $[a, b]$: We have

$$x \wedge ((y \wedge b) \vee a) = (x \wedge y \wedge b) \vee a = a.$$

Since $\uparrow x \vee y$ is n-complemented, by the induction hypothesis, $[x \vee y, (x \vee y) \vee b)]$ is n-complemented. By the diamond isomorphism theorem we have

$$[x \vee y, (x \vee y) \vee b] \cong [(x \vee y) \wedge b, b] = [x \vee ((y \wedge b) \vee a), b].$$

Thus $\uparrow x \vee ((y \wedge b) \vee a)$ is n-complemented in $[a, b]$, and thus $(y \wedge b) \vee a$ is an $(n+1)$-complement of x in $[a, b]$. □

Proposition 6.2.9. *In a modular lattice, if $\uparrow a$ and $\uparrow b$ are 1-complemented, then $\uparrow (a \wedge b)$ is 1-complemented.*

Proof. Take any $x \geqslant a \wedge b$. Let y be a 1-complement of $x \vee a$ in $\uparrow a$ and z be a 1-complement of $(x \wedge a) \vee b$ in $\uparrow b$. The claim is that $y \wedge z$ is a 1-complement of x in $\uparrow (a \wedge b)$. We have

$$a \wedge b \leqslant x \wedge (y \wedge z) = x \wedge (x \vee a) \wedge y \wedge z = x \wedge a \wedge z \leqslant a,$$

but also $x \wedge a \wedge z \leqslant ((x \wedge a) \vee b) \wedge z = b$. Consequently $x \wedge (y \wedge z) = a \wedge b$. We also have

$$x \vee (y \wedge z) = x \vee (x \wedge a) \vee (y \wedge z) = x \vee (y \wedge ((x \wedge a) \vee z)) = x \vee y = 1.$$

Thus $y \wedge z$ is a 1-complement of z in $\uparrow (a \wedge b)$. □

Corollary 6.2.10. *In a modular lattice L, if $[a, b]$ and $[c, d]$ are 1-complemented, then $[a \wedge c, b \wedge d]$ is 1-complemented.*

Proof. We have

$$[a, a \vee (b \wedge d)] \cong [a \wedge (b \wedge d), b \wedge d] = [a \wedge d, b \wedge d].$$

Since $[a, b]$ is a 1-complemented lattice, $[a, a \vee (b \wedge d)]$ is as well, and thus also $[a \wedge d, b \wedge d]$. By similar argument $[b \wedge c, b \wedge d]$ is 1-complemented. Differently stated, $\uparrow a \wedge d$ and $\uparrow c \wedge b$ are 1-complemented in $\downarrow b \wedge d$. Using the proposition above, we get $\uparrow a \wedge c$ is 1-complemented if $\downarrow b \wedge d$, which was to be proven. $\square$

Lemma 6.2.11. *Any modular lattice in which there exists a sequence of length $n > 0$,*

$$0 = a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n = 1$$

such that the interval between any two consecutive elements is 1-complemented, is an n-complemented lattice.

Proof. The lemma is true for $n = 1$. Suppose the lemma is true for $n > 0$. Take any modular lattice L in which there is such a sequence of length $n + 1$. Let $a_1 = t$. Take any $a \in L$. Let b be a complement of $a \wedge t$ in $\downarrow t$. We have

$$a \wedge b = a \wedge t \wedge b = 0.$$

Also

$$a \vee b \geqslant (a \wedge t) \vee b = t,$$

Consequently $\uparrow a \vee b$ is n-complemented by Lemma 6.2.8, since $\uparrow t$ is n-complemented by the induction hypothesis. Thus b is an $(n + 1)$-complement of a. Hence L is $(n + 1)$-complemented. $\square$

Recall that an element x of a complete lattice L is said to be *compact* when for any $X \subseteq L$, if $x \leq \bigvee X$, then $x \leq \bigvee Y$ for some finite subset Y of X. We call a lattice *compact* when every element is compact. Dually we have the notion of a *cocompact* lattice. One can readily observe that a lattice is cocompact if and only if for any x and subset X, if $x = \bigwedge X$ then $x = \bigwedge Y$ for some finite subset Y of X. Any finite lattice is in particular cocompact.

Theorem 6.2.12. *Any cocompact modular lattice is n-complemented if and only if there is a sequence of length n*

$$0 = a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n = 1$$

such that the interval between any two consecutive elements is 1-complemented. Further, for a and b in a cocompact modular lattice such that $\uparrow a$ and $\uparrow b$ are n-complemented, we have that $\uparrow a \wedge b$ is n-complemented.

Proof. The proof is by induction on n. The theorem statement is trivially true for $n = 0$. From Proposition 6.2.9 it follows that the theorem statement is true for $n = 1$. Suppose the theorem statement is true for $n \geqslant 1$.

Take any cocompact modular $(n+1)$-complemented lattice L. By the induction hypothesis and the fact that the lattice is cocompact, there is a least element t such that $\uparrow t$ is n-complemented. By the induction hypothesis again, there is a sequence $t, a_3, \ldots, a_{n+1} = 1$ such that the interval between any two consecutive elements is 1-complemented. To show that $\downarrow t$ is 1-complemented, take any $a \in \downarrow t$. Let b be an $(n+1)$-complement of a in L. We have

$$a \wedge (b \wedge t) = 0.$$

We also have

$$a \vee (b \wedge t) = (a \vee b) \wedge t = t,$$

since $\uparrow(a \vee b)$ is n-complemented, and t is the least such. Thus $\downarrow t$ is 1-complemented. Thus there is such a sequence of length $n+1$ as claimed in the theorem statement. Conversely, by Lemma 6.2.11, if there is such a sequence of length $n+1$ in a modular lattice, then that lattice is $(n+1)$-complemented. This shows the first part of the theorem statement for $n+1$.

For the last part: take any a and b in a cocompact modular lattice L such that $\uparrow a$ and $\uparrow b$ are $(n+1)$-complemented. So there are sequences

$$a_0 = a \leqslant a_1 \leqslant \cdots \leqslant a_{n+1} = 1 \quad \text{and} \quad b_0 = b \leqslant b_1 \leqslant \cdots \leqslant b_{n+1} = 1$$

where the interval between any two consecutive elements is 1-complemented. By Corollary 6.2.10

$$a_0 \wedge b_0 \leqslant a_1 \wedge b_1 \leqslant \cdots \leqslant a_{n+1} \wedge b_{n+1}$$

is again such sequence in $\uparrow a \wedge b$. Hence $\uparrow a \wedge b$ is $(n+1)$-complemented. □

Any lattice with finite height is cocompact, however we will see shortly that any cocompact modular lattice with a rank of complementedness has finite height.

Lemma 6.2.13. *Consider any modular complemented lattice. If $a_1 \leqslant a_2$, and b_1 is a complement of a_1, then there is a complement b_2 of a_2 such that $b_2 \leqslant b_1$.*

Proof. Let c be a complement of $a_2 \wedge b_1$, then $b_1 \wedge c$ will be a complement of a_2: $a_2 \wedge (b_1 \wedge c) = 0$. Further

$$a_2 \vee (b_1 \wedge c) = a_2 \vee (a_2 \wedge b_1) \vee (b_1 \wedge c) = a_2 \vee (b_1 \wedge ((a_2 \wedge b_1) \vee c)) = a_2 \vee b_1 \geqslant a_1 \vee b_1 = 1.$$

□

Recall, in an ordered set, b *covers* a (or a is *covered by* b) if $a < b$ and there is no element x such that $a < x < b$. Symbolically b covers a is denoted by $a \prec b$.

Proposition 6.2.14. *Any cocompact modular lattice L for which $\mathsf{rc}(L)$ exists, has finite height.*

Proof. First observe that for any element $x \neq 1$ in a cocompact lattice, there is a y such that $y \succ x$: consider the poset $X = \uparrow x \backslash \{x\}$. Any chain in X is bounded below in X (by cocompactness), thus by Zorn's Lemma there is a minimum y in X.

If $\mathsf{rc}(L) = 0$, then the lattice trivially has finite height. Suppose $\mathsf{rc}(L) = 1$. If L does not have finite height, then one can construct a sequence $a_1 \prec a_2 \prec \dots$. Let b_1 be a complement of a_1, and inductively by the previous lemma let b_{i+1} be a complement of a_{i+1} such that $b_i \leqslant b_{i+1}$, for $i \geqslant 1$. Since the lattice is cocompact, this sequence $b_1 \geqslant b_2 \geqslant \dots$ must eventually be constant, say for N, $b_N = b_{N+1} = b_{N+2} \dots$. Then we have

$$(a_N \vee b_N) \wedge a_{N+1} = a_{N+1} \neq a_N = a_N \vee (b_N \wedge a_{N+1}),$$

contradicting modularity. Thus if $\mathsf{rc}(L) = 1$ and it is cocompact and modular, it has finite height.

For $\mathsf{rc}(L) \geqslant 2$, use Theorem 6.2.12 to get a sequence of elements $0 = a_0 \leqslant a_1 \leqslant \dots \leqslant a_{\mathsf{rc}(L)} = 1$ for which the interval between any two consecutive elements is complemented. So each such interval has finite height, and thus the modular lattice L itself has finite height. □

The above proposition and theorem establishes point (ii) of Theorem 6.1.2. The following lemma is used to prove point (iii) of Theorem 6.1.2.

Lemma 6.2.15. *In a modular lattice, if $a \prec b$, $\mathsf{rc}(\uparrow b)$ exists and a is meet irreducible, then*

$$\mathsf{rc}(\uparrow a) = \mathsf{rc}(\uparrow b) + 1.$$

Proof. The proof is by induction on $\mathsf{rc}(\uparrow b)$. If $\mathsf{rc}(\uparrow b) = 0$, then $\uparrow a$ is just a two element chain, for which we have $\mathsf{rc}(\uparrow a) = 1 = \mathsf{rc}(\uparrow b) + 1$. Suppose for $\mathsf{rc}(\uparrow b) = n$ the lemma is true. Consider the case when $\mathsf{rc}(\uparrow b) = n+1$. Take any $x \in \uparrow a$. If $x = a$, then 1 is an $(n+2)$-complement of x in $\uparrow a$. So suppose $x \neq a$. Then $x \geqslant b$, thus by Lemma 6.2.8 $\uparrow x$ is $(n+1)$-complemented, thus a is an $(n+2)$-complement of $x \neq a$ in $\uparrow a$. Thus $\uparrow a$ is $(n+2)$-complemented. If $\uparrow a$ is $(n+1)$-complemented, the only $(n+1)$-complement of b in $\uparrow a$ is a, which then implies that $\uparrow b$ is n-complemented. This contradicts that $\mathsf{rc}(\uparrow b) = n + 1$. Thus $\mathsf{rc}(\uparrow a) = n + 2 = \mathsf{rc}(\uparrow b) + 1$. □

Corollary 6.2.16. *In a modular lattice having finite height, if a and b are meet irreducible elements such that $a < b$, then $\mathsf{rc}(\uparrow a) > \mathsf{rc}(\uparrow b)$.*

Proof. Since the lattice is of finite height, there is an element c covering a. From the lemma we have $\mathsf{rc}(\uparrow a) = \mathsf{rc}(\uparrow c) + 1$. Since $b > a$ and a is meet irreducible, $b \geqslant c$. Since the lattice is modular, we have

$$\mathsf{rc}(\uparrow b) \leqslant \mathsf{rc}(\uparrow c) < \mathsf{rc}(\uparrow a).$$

□

Theorem 6.2.17. *Any modular lattice L with finite height is n-complemented if and only if every chain of meet irreducible elements has length at most n, for $n \geqslant 0$. Equivalently,* $\mathsf{rc}(L)$ *is equal to the length of a longest chain of meet irreducible elements.*

Proof. Suppose L is n-complemented. By the corollary, it inductively follows that for any chain of meet irreducible elements, $x_0 < \cdots < x_m$, $\mathsf{rc}(\uparrow x_0) \geqslant m$. And since the lattice is n-complemented, $m \leqslant n$, which shows the one direction.

The converse is proven by induction on n. For $n = 0$, suppose the length of every chain of meet irreducible elements is bounded by 0, that is, consists of exactly one element. The top element 1 is always meet irreducible. If it contains more elements, then there is an element x which is covered by 1. But then x is also meet irreducible and $x < 1$ forms a chain of meet irreducible elements of length 1, which is a contradiction. Thus the lattice only contains 1 element, and consequently is 0-complemented. Suppose that if the length of every chain of meet irreducible elements is bounded by n, the lattice is n-complement, for any modular lattice having finite height. Consider any modular lattice L having finite height, such that the length of every chain of meet irreducible elements is bounded by $n + 1$. Take any meet irreducible element x in L. If $x = 1$, then $\uparrow x$ is $(n + 1)$-complemented. If $x \neq 1$, then there is an element $y \succ x$. By assumption on L, the length of any chain of meet irreducible elements in $\uparrow y$ is bounded by n, thus $\uparrow y$ is n-complemented by the induction hypothesis. Then by Lemma 6.2.15 $\uparrow x$ is $(n+1)$-complemented. Since the lattice is of finite height, 0 is equal to some finite meet of meet irreducible elements $x_1, \ldots, x_m$. As already shown $\uparrow x_i$ is $(n + 1)1$-complemented for every $1 \leqslant i \leqslant n$, thus by Theorem 6.2.12 $\uparrow 0 = L$ is $(n + 1)$-complemented. □

Theorem 6.2.12 together with Theorem 6.2.17 gives Theorem 6.1.2 stated in the Introduction.

Notice that from Theorem 6.2.12 it follows that a modular lattice with finite height is n-complemented if and only if its dual is n-complemented. Consequently, the dual of Theorem 6.2.17 is true: Any modular lattice with finite height is n-complemented if and only if any chain of join irreducible elements has length at most n.

By Theorem 6.2.12, if a lattice is n-complemented, then there is a sequence

$$a_0 < a_1 < \cdots < a_n$$

such that the interval between any two consecutive elements is complemented. In the proof that sequence was constructed inductively by finding minimum t such that $\uparrow t$ is $(n - 1)$-complemented, and then set $a_1 = t$. With the help of the above Theorem, there is a more straight forward way to construct a sequence, by use of the following proposition:

Proposition 6.2.18. *In any modular lattice L with finite height and* $\mathsf{rc}(L) = n+1$, *for the set of coatomic elements C, we have* $\mathsf{rc}(\uparrow \bigwedge C) = 1$ *and* $\mathsf{rc}(\downarrow \bigwedge C) = n$.

Proof. From the Proposition 6.2.9, we get $\mathsf{rc}(\uparrow \bigwedge C) = 1$. From Theorem 6.2.12 there is a sequence $0 = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{n+1} = 1$ such that the interval between any two consecutive elements is a complemented lattice. So $\mathsf{rc}(\uparrow x_n) = 1$. From Theorem 6.2.17, the meet irreducible elements in $\uparrow x_n$ are exactly 1 and the coatomic elements above x_n. Since L has finite height, x_n is equal a meet of meet irreducible elements, consequently $x_n \geqslant \bigwedge C$. Since $\downarrow x_n$ is n-complemented, so is $\downarrow \bigwedge C$. This forces $\mathsf{rc}(\downarrow \bigwedge C) = n$, else by Theorem 6.2.12 it would contradict $\mathsf{rc}(L) = n+1$. $\square$

The algorithm works as follows now, to construct the sequence $a_0 < \cdots < a_n$: Let $a_n = 1$, and let a_i be the meet of all coatomic elements in $\downarrow a_{i+1}$, for $i < n$. One could prove that this sequence and the sequence constructed in the proof of Theorem 6.2.12 are the same sequences.

The rest of this subsection is dedicated to further results of when $\uparrow a \wedge b$ is n-complemented in a modular lattice provided $\uparrow a$ and $\uparrow b$ are n-complemented.

Proposition 6.2.19. *In any modular lattice, if $\uparrow a$ is a chain, and $\uparrow a$ and $\uparrow b$ are n-complemented, then $\uparrow(a \wedge b)$ is n-complemented.*

Proof. We prove this by induction on n. For $n = 0$ it is true. Suppose it is true for n. Take any modular lattice and elements a and b such that $\uparrow a$ is a chain, and $\uparrow a$ and $\uparrow b$ are $(n+1)$-complemented. Take any $x \in\uparrow(a \wedge b)$. We will split it up into two cases: whether $(x \wedge b) \leqslant a$ or not, and in each case construct an $(n+1)$-complement for x.

Suppose $(x \wedge b) \leqslant a$. Let y be an $(n+1)$-complement of $x \vee b$ in $\uparrow b$. We have

$$x \wedge y \leqslant (x \vee b) \wedge y = b.$$

Consequently, we have as well

$$x \wedge y = x \wedge y \wedge b = x \wedge b \leqslant a.$$

Thus $x \wedge y \leqslant a \wedge b$, and thus $x \wedge y = a \wedge b$. By choice of y, $\uparrow(x \vee y) =\uparrow(x \vee y \vee b)$ is n-complemented. Thus y in an $(n+1)$-complement of x in $\uparrow a \wedge b$.

Suppose $(x \wedge b) \nleqslant a$. Let $x_1 = b \vee (x \wedge a)$, and let y_1 be an $(n+1)$-complement of x_1 in $\uparrow b$. Let $y = y_1 \wedge a$. We have

$$a \wedge b \leqslant x \wedge y_1 \wedge a \leqslant ((x \wedge a) \vee b) \wedge y_1 = b.$$

But $x \wedge y_1 \wedge a \leqslant a$ as well. Thus $x \wedge y = a \wedge b$. Since $(x \wedge b) \vee a \neq a$, $\uparrow((x \wedge b) \vee a)$ is an n-complemented chain, since $\uparrow a$ is a chain which is $(n+1)$-complemented. Thus

$$\uparrow(x_1 \vee y_1) \wedge (a \vee (b \wedge x))$$

is n-complemented by the induction hypothesis. Notice the following

$$\begin{aligned} x \vee y &= x \vee (y_1 \wedge a) \\ &\geqslant (x \wedge a) \vee (y_1 \wedge a) \vee (b \wedge x) \\ &= (((x \wedge a) \vee y_1) \wedge a) \vee (b \wedge x) \\ &= (x \wedge a) \vee y_1) \wedge (a \vee (b \wedge x)) \\ &= (b \vee (x \wedge a) \vee y_1) \wedge (a \vee (b \wedge x)) \\ &= (x_1 \vee y_1) \wedge (a \vee (b \wedge x)). \end{aligned}$$

Consequently, $\uparrow(x \vee y)$ is n-complemented by Lemma 6.2.8. Thus y is an $(n+1)$-complement of x in $a \wedge b$. ☐

I do not know whether it is true in any modular lattice (not having finite height), that if $\uparrow a$ and $\uparrow b$ are n-complemented, then $\uparrow a \wedge b$ is n-complemented. However in the distributive case, it is very straightforward.

Proposition 6.2.20. *Let L be a distributive lattice. If $\uparrow a$ and $\uparrow b$ are n-complemented, then $\uparrow a \wedge b$ is n-complemented.*

Proof. The result is true for $n = 0$. Suppose it is true for n. Consider any distributive lattice and elements a and b such that $\uparrow a$ and $\uparrow b$ are $(n+1)$-complemented. Take any $x \in \uparrow a \wedge b$. Let y_a be an $(n+1)$-complement of $x \vee a$ in $\uparrow a$, and y_b be an $(n+1)$-complement of $x \vee b$ in $\uparrow b$. We have

$$x \wedge (y_a \wedge y_b) = (x \vee (a \wedge b)) \wedge y_a \wedge y_b = (x \vee a) \wedge (x \vee b) \wedge y_a \wedge y_b = a \wedge b.$$

Further,

$$\uparrow x \vee (y_a \wedge y_b) = \uparrow(x \vee y_a) \wedge (x \vee y_b) = \uparrow((x \vee a) \vee y_a) \wedge ((x \vee b) \vee y_b).$$

By the induction hypothesis the above is n-complemented. Thus $y_a \wedge y_b$ is an $(n+1)$-complement of x in $\uparrow a \wedge b$. ☐

6.2.3 Subgroup lattices of finite abelian groups

We will classify all finite abelian groups whose lattices of subgroups are n-complemented (Theorem 6.2.25). Recall the following well-known result (see e.g. [21]):

Theorem 6.2.21. *For any G is a finite abelian group, there exists a sequence $p_1, \ldots, p_m$ of (not necessarily distinct) prime numbers, a sequence $a_1, \ldots, a_m$ of natural numbers, such that there is an isomorphism of groups*

$$G \cong \prod_{i=1}^{m} \mathbb{Z}_{p_i^{a_i}}.$$

Notice the following:

Lemma 6.2.22. *If G is a finite abelian group, then all elements of G have n-free order if and only if $G \cong \Pi_{i=1}^{m} \mathbb{Z}_{{p_i}^{a_i}}$, where for all i, $a_i < n$ and p_i is prime.*

By Proposition 6.2.5, we have:

Proposition 6.2.23. *For any surjective group homomorphism $f\colon A \to B$ between two finite abelian groups, if $\mathsf{sub}A$ is n-complemented, then so is $\mathsf{sub}B$.*

Proposition 6.2.24. *For any two finite abelian groups A and B, $\mathsf{sub}(A \times B)$ is n-complemented if and only if both $\mathsf{sub}A$ and $\mathsf{sub}B$ are n-complemented.*

Proof. By the previous proposition, if $\mathsf{sub}(A \times B)$ is n-complemented, then both $\mathsf{sub}A$ and $\mathsf{sub}B$ are n-complemented. Conversely, suppose both $\mathsf{sub}A$ and $\mathsf{sub}B$ are n-complemented. Then the upsets $\uparrow A \times 0$ and $\uparrow 0 \times B$ in $\mathsf{sub}(A \times B)$ are isomorphic to $\mathsf{sub}B$ and $\mathsf{sub}A$, respectively, and so they are n-complemented. Consequently, by Theorem 6.2.12, $\mathsf{sub}(A \times B)$ is n-complemented. □

Theorem 6.2.25. *If G is a finite abelian group, then $\mathsf{sub}(G)$ is n-complemented if and only if all elements of G have $(n+1)$-free order. Equivalently,*

$$\mathsf{rc}\left(\prod_{i=1}^{m} \mathbb{Z}_{p_i^{a_i}}\right) = \mathsf{max}\{a_1, \ldots, a_m\},$$

where $p_1, \ldots, p_m$ are arbitrary prime numbers and $a_1, \ldots, a_m$ are arbitrary natural numbers.

Proof. Notice that for a prime p and a natural number a, we have $\mathsf{rc}(\mathsf{sub}(\mathbb{Z}_{p^a})) = a$, since $\mathsf{sub}(\mathbb{Z}_{p^a})$ is a chain with $a+1$ elements. Let

$$G \cong \prod_{i=1}^{m} \mathbb{Z}_{p_i^{a_i}},$$

where p's and a's are as in the statement of the theorem. Then

$$\begin{aligned} & \text{all elements of } G \text{ have } (n+1)\text{-free order} \\ \Leftrightarrow\ & \mathsf{rc}(\mathsf{sub}(\mathbb{Z}_{p_i^{a_i}})) = a_i \leqslant n \text{ for each } i \text{ (Lemma 6.2.22)} \\ \Leftrightarrow\ & \mathsf{sub}G \text{ is } n\text{-complemented (Proposition 6.2.24).} \end{aligned}$$

The second statement in the theorem readily follows. □

We can give an alternative proof of the above theorem by using the dual of Theorem 6.2.17. First notice the following:

Lemma 6.2.26. *For any finite abelian group A, $X \in \mathsf{sub}A$ is join irreducible if and only if X is a cyclic group with prime power order.*

Proof. If X is a cyclic group with prime power order, then X is join irreducible. If X is not cyclic with prime power order, then X decomposes (properly) into a product, and consequently X is not join irreducible. □

The alternative proof of Theorem 6.2.25:

Proof. Take any finite abelian group $A = \prod_{i=1}^{m} \mathbb{Z}_{p_i^{a_i}}$. Let a be the maximum among $a_1, \ldots, a_m$, and let p be its corresponding prime. Then

$$p^a\mathbb{Z}_{p^a}, p^{a-1}\mathbb{Z}_{p^a}, \ldots, \mathbb{Z}_{p^a}$$

is a chain of length a of join irreducible elements. Consider any other chain of join irreducible elements, and let X denote the top element of this chain. Then $X \cong \mathbb{Z}_{q^b}$, for some prime q. Since all elements of A have $(a+1)$-free, the generator of X must have $(a+1)$-free order as well. So $b \leqslant a$. Therefore the length of that chain of join irreducible elements is bounded by a. Consequently by the dual of Theorem 6.2.17

$$\mathsf{rc}\left(\prod_{i=1}^{m} \mathbb{Z}_{p_i^{a_i}}\right) = \mathsf{max}\{a_1, \ldots, a_n\},$$

which was to be demonstrated. □

6.2.4 Set of complements

Definition 6.2.27. In a lattice L, for $n \geqslant 1$, the set of n-complements of an element a of L is denoted by $C_n(a)$.

Recall that a *convex* subset S of a lattice is a subset such that $[a, b] \subseteq S$ whenever $a, b \in S$ and $a \leqslant b$.

Proposition 6.2.28. *In any modular lattice L, $C_{n+1}(a)$ is convex and relatively n-complemented for any $a \in L$, for every $n \geqslant 0$.*

Proof. Suppose $x \leqslant y \leqslant z$ and $x, z \in C_{n+1}(a)$. Then

$$0 \leqslant y \wedge a \leqslant z \wedge a = 0.$$

Also

$$a \vee y \in \uparrow (a \vee x),$$

Thus $\uparrow a \vee y$ is n-complemented. Consequently $y \in C_{n+1}(a)$, and therefore $C_{n+1}(a)$ is convex.

Take any interval $[x, y]$ in $C_{n+1}(a)$. We have

$$[x, y] = [y \wedge (a \vee x), y] \cong [a \vee x, y \vee (a \vee x)].$$

Since $\uparrow a \vee x$ is relatively n-complemented, $[x, y]$ is also n-complemented. □

Proposition 6.2.29. *In any distributive lattice L, $C_n(a)$ is closed under meets and joins for any $a \in L$ and $n \geqslant 1$.*

Proof. Take any $x, y \in C_n(a)$. We have $(x \wedge y) \wedge a = 0$. Also

$$\uparrow a \vee (x \wedge y) = \uparrow (a \vee x) \wedge (a \vee y),$$

which is $(n-1)$-complemented. Thus $x \wedge y \in C_n(a)$.

We also have $(x \vee y) \wedge a = (x \wedge a) \vee (y \wedge a) = 0$ and $\uparrow a \vee (x \vee y)$ is $(n-1)$-complemented. Thus $x \vee y \in C_n(a)$. □

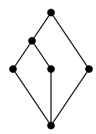
Proposition 6.2.30. *In any n-complemented lattice L, if a has a pseudocomplement a^*, then a^* is the largest n-complement of a for any $a \in L$ and $n \geqslant 1$.*

Proof. Let x be an n-complement of $a \vee a^*$. Then we have

$$0 = (a \vee a^*) \wedge x \geqslant (a \wedge x) \vee (a^* \wedge x) \geqslant 0.$$

Thus $a \wedge x = 0$, and thus $x \leqslant a^*$. We have that $\uparrow a \vee a^* \vee x = \uparrow a \vee a^*$ is $(n-1)$-complemented. Since $a \wedge a^* = 0$, a^* is indeed an n-complement of a, and consequently the largest one. □

Note that a lattice can be n-complemented such that $C_n(a)$ has a top element, for every element a, without the lattice being pseudocomplemented, for example the following complement lattice:



But the next theorem shows that if the lattice is further modular, then it is pseudocomplemented. For the theorem, we require the following lemma.

Lemma 6.2.31. *In any modular n-complemented lattice, if $a \wedge x = 0$ then there is a $y \in C_n(a)$ such that $x \leqslant y$, for any $n \geqslant 1$.*

Proof. Let b be an n-complement of $(a \vee x)$. So $\uparrow a \vee (x \vee b)$ is $(n-1)$-complemented. For checking that the meet is trivial, we have

$$a \wedge (x \vee b) = a \wedge (a \vee x) \wedge (x \vee b) = a \wedge (x \vee ((a \vee x) \wedge b)) = a \wedge x = 0.$$

Thus $x \vee b$ is an n-complement of a above x. □

Theorem 6.2.32. *In any modular n-complemented lattice L, a has a pseudocomplement if and only if $C_n(a)$ has a top element, for any $a \in L$ and $n \geqslant 1$.*

Proof. The one direction is given by Proposition 6.2.30. For the other direction, suppose that $C_n(a)$ has a top element a'. If $a \wedge x = 0$, then by the previous lemma there is a $y \in C_n(a)$ such that $x \leqslant y$. But also $y \leqslant a'$, thus $x \leqslant a'$. Hence a' is the pseudocomplement of a. □

So in the modular case, the set of n-complements of an element a can determine the pseudocomplement. In the distributive case, we have a sort of converse, that the pseudocomplement can describe the set of n-complements.

Theorem 6.2.33. *In any distributive n-complemented lattice L, for $n \geqslant 1$, in which there is minimum element t such that $\uparrow t$ is $(n-1)$-complemented (equivalently t is the bottom element of $C_n(0)$), if a has a pseudocomplement a^*, then $C_n(a) = [t \wedge a^*, a^*]$ for any $a \in L$.*

Proof. We already know that a^* is the top element of $C_n(a)$. For any $x \in C_n(a)$, $\uparrow a \vee x$ is $(n-1)$-complemented, thus $a \vee x \geqslant t$. We further have

$$x = (x \wedge a^*) \vee (a \wedge a^*) = (x \vee a) \wedge a^* \geqslant t \wedge a^*.$$

Also, $\uparrow a \vee (t \wedge a^*) = \uparrow (a \vee t) \wedge (a \vee a^*)$ is $(n-1)$-complemented. So $t \wedge a^*$ is the bottom element of $C_n(a)$. Since $C_n(a)$ is convex, the result follows. □

Recall that in any pseudocomplemented lattice, $(x \vee y)^* = x^* \wedge y^*$ for any $x, y \in L$ (see [3] Theorem 7.1. for example).

Corollary 6.2.34. *In any distributive pseudo- and n-complemented lattice, for $n \geqslant 1$, in which $C_n(0)$ has a bottom element t,*

$$C_n(a) \wedge C_n(b) = \{x \wedge y \mid x \in C_n(a) \text{ and } y \in C_n(b)\} = C_n(a \vee b),$$

for any $a, b \in L$.

Proof. We have

$$\begin{aligned} C_n(a) \wedge C_n(b) &= [t \wedge a^*, a^*] \wedge [t \wedge b^*, b^*] = [t \wedge a^* \wedge b^*, a^* \wedge b^*] \\ &= [t \wedge (a \vee b)^*, (a \vee b)^*] = C_n(a \vee b). \end{aligned}$$

□

Proposition 6.2.35. *Consider any distributive pseudo- and n-complemented lattice L, for $n \geqslant 1$, in which $C_n(0)$ has bottom element t. For $a, b \in L$, if $a \leqslant b$, then there is a split epimorphism from $C_n(a)$ to $C_n(b)$.*

Proof. From Theorem 6.2.33, we have $C_n(a) = [t \wedge a^*, a^*]$ and $C_n(b) = [t \wedge b^*, b^*]$. Define

$$f\colon C_n(a) \to C_n(b), x \mapsto x \wedge b^*, \quad \text{and} \quad g\colon C_n(b) \to C_n(a), y \mapsto y \vee (t \wedge a^*).$$

Since $a \leqslant b$, $a^* \geqslant b^*$, and consequently these maps are well-defined. Since the lattice is distributive, these maps are lattice homomorphisms. Furthermore

$$fg(y) = f(y \vee (t \wedge a^*)) = (y \vee (t \wedge a^*)) \wedge b^* = y \vee (t \wedge a^* \wedge b^*) = y \vee (t \wedge b^*) = y.$$

Thus f is a split epimorphism from $C_n(a)$ to $C_n(b)$. □

6.3 Ranked Booleanness

6.3.1 The general theory

This subsection is just a collection of some general known results of n-Boolean lattices.

We can extend Proposition 6.2.4 as follows, whose proof is readily clear:

Proposition 6.3.1. *For a lattice L having $n+1$ elements, where $n \geq 1$, the following conditions are equivalent:*

(a) L *is a chain.*

(b) $\mathsf{rb}(L) = n$.

We also have analogous results about surjective lattice homomorphisms and products:

Proposition 6.3.2. *Surjective lattice homomorphisms preserves Boolean sequences. Consequently, the homomorphic image of an n-Boolean lattice is an n-Boolean lattice.*

Proof. Take a surjective lattice homomorphism $f\colon L \to M$, where L is an n-Boolean lattice. Suppose $a_0, \ldots, a_n$ is a Boolean sequence in L. For every i, $[fa_i, fa_{i+1}]$ is a Boolean algebra, since it is an image of a distributive 1-complemented lattice. Thus $0 = fa_0, \ldots, fa_n = 1$ is a Boolean sequence of length $n+1$ in M, thus M is also n-Boolean. □

Proposition 6.3.3. *If $(L_i)_{i\in I}$ is a family of lattices, then the product $L = \prod_{i\in I} L_i$ is an n-Boolean lattice if and only if all the factors are n-Boolean lattices. Consequently*

$$\mathsf{rb}\left(\prod_{i\in I} L_i\right) = \max(\mathsf{rb}(L_i))_{i\in I}.$$

Proof. If the product L is an n-Boolean lattice, then so is every factor, since for each i, $\pi_i\colon L \to L_i$ is a surjective lattice morphism. For the converse, notice that for $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ in L.

$$[x, y] = \prod_{i\in I} [x_i, y_i].$$

If for each i, $[x_i, y_i]$ is a Boolean algebra, then $[x, y]$ is also a Boolean algebra. Let

$$a_{0,j}, a_{1,j}, \ldots a_{n,j}$$

be a Boolean sequence of rank n in L_j, for $i \in I$. Define $a_i = (a_{i,j})_{j\in I}$. Then $a_0, a_1, \ldots, a_n$ is a Boolean sequence of rank n in L. □

The lemma below could be used to show that the set of n-Boolean sequences (chains of length n such the interval between any two consecutive elements are Boolean algebras) forms a lattice. Further, the lemma will also be used to show that, the same as for n-complemented lattices, that modular n-Boolean lattices are "relatively n-Boolean" (that is, any interval is n-Boolean).

Lemma 6.3.4. *In a modular lattice, for any $x \leq y$, if $[x, y]$ is a Boolean algebra, then $[x \vee a, y \vee a]$ is also a Boolean algebra for any a. Dually $[x \wedge a, y \wedge a]$ is also a Boolean algebra.*

Proof. We have

$$[y \wedge (x \vee a), y] \cong [x \vee a, y \vee (x \vee a)] = [x \vee a, y \vee a].$$

Since the left interval is an interval of a Boolean algebra $[x, y]$, itself is a Boolean algebra, thus $[x \vee a, y \vee a]$ is a Boolean algebra. □

Proposition 6.3.5. *In a modular n-Boolean lattice, for any pair of elements $a \leq b$, $[a, b]$ is also n-Boolean.*

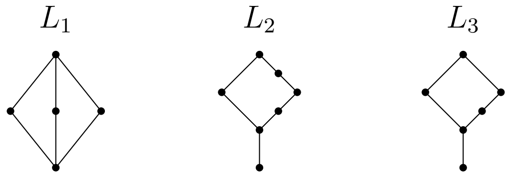
Proof. Take any Boolean sequence of rank n: $x_0, x_1, \ldots x_n$. Then by the above lemma, the following will be a Boolean sequence of rank n in $[a, b]$:

$$(x_0 \vee a) \wedge b, (x_1 \vee a) \wedge b, \ldots (x_n \vee a) \wedge b.$$

□

6.3.2 Comparing rank of complementedness and Booleanness

The rank of complementedness and Booleanness does not always coincide. For example consider the following lattices:



We have $\mathsf{rc}(L_1) = 1 < 2 = \mathsf{rb}(L_1)$, $\mathsf{rc}(L_2) = 4 > 3 = \mathsf{rb}(L_2)$, and $\mathsf{rc}(L_3) = 3 = 3 = \mathsf{rb}(L_3)$.

For the modular case, we always have $\mathsf{rc}(L) \leq \mathsf{rb}(L)$.

Theorem 6.3.6. *If L is a modular n-Boolean lattice, then L is n-complemented. That is $\mathsf{rc}(L) \leq \mathsf{rb}(L)$.*

Proof. This follows immediately from Lemma 6.2.11. □

If the lattice is distributive with finite height then we have equality.

Theorem 6.3.7. *If L is a distributive lattice having finite height, then L is n-complemented if and only if L is n-Boolean. Equivalently, $\mathsf{rc}(L) = \mathsf{rb}(L)$.*

Proof. This follows immediately from Theorem 6.2.12. □

6.3.3 Relatively prime elements in lattices

In Subsection 6.2.2 we studied under what conditions is $\uparrow a \wedge b$ in a modular lattice n-complemented provided that both $\uparrow a$ and $\uparrow b$ are n-complemented. This subsection is dedicated to understand and explore the same question, but for n-Boolean lattices. Results obtained surrounding this question will be useful in particular to classify which finite abelian groups have an n-Boolean subgroup lattice. Without any further restrictions, $\mathsf{rb}(\uparrow a \wedge \uparrow b) \leq \mathsf{rb}(\uparrow a) + \mathsf{rb}(\uparrow b)$ as shown below.

Proposition 6.3.8. *In a modular lattice, if $\uparrow a$ and $\uparrow b$ are n- and m-Boolean respectively, then $\uparrow a \wedge b$ is $(n+m)$-Boolean.*

Proof. Let $a_0, a_1, \ldots a_n$ and $b_0, b_1, \ldots b_m$ be Boolean sequences of rank n and m in $\uparrow a$ and $\uparrow b$ respectively. Consider the sequence:

$$a_0 \wedge b_0, a_1 \wedge b_0, \ldots a_n \wedge b_0 = b_0, b_1, \ldots b_m.$$

Since the lattice is modular, we have

$$[a_i \wedge b, a_{i+1} \wedge b] = [a_i \wedge (a_{i+1} \wedge b), a_{i+1} \wedge b] \cong [a_i, a_i \vee (a_{i+1} \wedge b)].$$

The last interval is an interval in the Boolean algebra $[a_i, a_{i+1}]$, thus itself is a Boolean algebra. Thus $[a_i \wedge b, a_{i+1} \wedge b]$ is a Boolean algebra, and thus the above sequence is a Boolean sequence of rank $n + m$. □

A sufficient condition that $\uparrow a \wedge b$ is always n-Boolean provided $\uparrow a$ and $\uparrow b$ are both n-Boolean, is when a and b are so called "relatively prime". This condition in particular is good enough to help classify which finite abelian groups have a subgroup lattice which is n-Boolean.

Definition 6.3.9. Two elements a and b of lattice L are called *relatively prime* in L if for every $s \in\uparrow(a \wedge b)$ there exists unique $x \in\uparrow a$ and $y \in\uparrow b$ such that $s = x \wedge y$.

The motivation of the definition came from the following observation:

Proposition 6.3.10. *For finite groups A and B, the following are equivalent:*

- *$A \times 0$ and $0 \times B$ are relatively prime in $\mathsf{sub}(A \times B)$,*
- *the order of A is relatively prime to the order of B.*

Proof. Suppose the orders of A and B are relatively prime. Take any subgroup $S \in\uparrow((A \times 0) \wedge (0 \times B)) = \mathsf{sub}(A \times B)$. Take any element $(a, b) \in S$. Let the order of a and b be n and m respectively. So n and m are relatively prime, so by Bézout's identity there exist u and v such that $un + vm = 1$. So $vm \cdot (a, b) \in S$. We have

$$vm \cdot (a, b) = (vm \cdot a, 0) = (vm \cdot a + un \cdot a, 0) = ((vm + un) \cdot a, 0) = (a, 0).$$

So $(a, 0) \in S$. Similarly $(0, b) \in S$. Consequently

$$S = \pi_1 S \times \pi_2 S = A \times \pi_2 S \wedge \pi_1 S \times B.$$

Also $A \times \pi_2 S$ and $\pi_2 S \times B$ is readily the unique pair of subgroups above $A \times 0$ and $0 \times B$ respectively such that their meet is S. Thus $A \times 0$ and $0 \times B$ are relatively prime.

Suppose the orders of A and B are not relatively prime. Let prime p be a common divisor of the orders of A and B. Then by First Sylow Theorem (See for example [6], Theorem 36.8) there are elements $x \in A$ and $y \in B$ both having order p. Let $S = \langle (x, y) \rangle$, therefore the order of S is p. Suppose there are subgroups $Y \geq A \times 0$ and $X \geq 0 \times B$ such that $X \wedge Y = S$. So $(x, y) \in X$, thus $(x, 0) \in X$. Since also $(x, 0) \in Y$, $(x, 0) \in S$, and consequently the order of S is p^2 which is a contradiction. Thus $A \times 0$ and $0 \times B$ cannot be relatively prime. $\square$

The following proposition states that in Definition 6.3.9 rewriting s as a meet of x and y is an isomorphism of lattice in some sense. It also gives an alternative method of checking whether two elements are relatively prime, which is sometimes more convenient.

Proposition 6.3.11. *For any two elements a and b of the same lattice, a and b are relatively prime if and only if*

$$h\colon \uparrow a \times \uparrow b \longrightarrow \uparrow(a \wedge b), \quad (x, y) \mapsto x \wedge y$$

is an isomorphism of lattices, whose inverse is $g\colon s \mapsto (s \vee a, s \vee b)$.

Proof. If it is an isomorphism, then a and b are evidently relatively prime.

Suppose a and b are relatively prime. Take any $s \in \uparrow(a \wedge b)$. Then $s = x \wedge y$, for unique $x \in \uparrow a$ and $y \in \uparrow b$. We have

$$s \leq (s \vee a) \wedge (s \vee b) \leq x \wedge y = s.$$

Consequently, by uniqueness of x and y, $x = s \vee a$ and $y = s \vee b$. From this observation it follows that h and g are inverses (as functions) of each other. It only remains to show that either one is a morphism of lattices. From the definition of g, g is clearly join preserving. For the meet preserving part: take any $s, r \in \uparrow(a \wedge b)$. We have $((s \vee a) \wedge (r \vee a)) \wedge ((s \vee b) \wedge (r \vee b)) = s \wedge r$. By uniqueness again, $(s \wedge r) \vee a = (s \vee a) \wedge (r \vee a)$ and $(s \wedge r) \vee b = (s \vee b) \wedge (r \vee b)$. Thus g preserves meets, thus g, and therefore h as well, is a lattice isomorphism. □

Proposition 6.3.12. *A lattice is bounded above if and only if it has a pair of relatively prime elements.*

Proof. If it is bounded above, then the top element is trivially relatively prime to itself.

Suppose it has relatively prime elements a and b. If $t \geq a \vee b$, then

$$a \vee b = (a \vee b) \wedge (a \vee b) = (a \vee b) \wedge t.$$

By uniqueness $t = a \vee b$. Further, for any x in the lattice, we have

$$x \leq (a \vee b) \vee x = a \vee b.$$

Thus the lattice is bounded above with top element $a \vee b$. □

Theorem 6.3.13. *In a distributive lattice, the pair a and b is relatively prime if and only if $a \vee b = 1$.*

Proof. We already have shown that any two relatively prime elements have join 1.

Suppose we have two elements a and b such that $a \vee b = 1$. Note that for any $x \geq a$ and $y \geq b$, $x \vee y = 1$. Define the functions h and g as in the statement of the above proposition. We have for any $s \in \uparrow(a \wedge b)$.

$$hgs = (s \vee a) \wedge (s \vee b) = s \vee (a \wedge b) = s.$$

Also, for any $x \in \uparrow a$ and $y \in \uparrow b$, we have

$$gh(x, y) = ((x \wedge y) \vee a, (x \wedge y) \vee b) = (x \wedge (a \vee y), y \wedge (b \vee x)) = (x, y).$$

So h and g are bijective and inverses of each other. Furthermore, since the lattice is distributive, both h and g are lattice morphisms, thus isomorpmisms. Thus, from Proposition 6.3.11, the pair a and b is relatively prime. □

The corollary below even further motivates why elements satisfying the condition of Definition 6.3.9 are called "relatively prime".

Corollary 6.3.14. *In the dual of the divisibility lattice of $\mathbb{N}$, the pair a and b is relatively prime if and only if* $\mathsf{gcd}(a,b) = 1$ *(that is, they are relatively prime in the classical sense).*

The following proposition is just for interest sake, how relatively prime elements give an alternative description of distributive lattices.

Proposition 6.3.15. *For any lattice L, the following are equivalent:*

- *L is distributive;*
- *for every $x \in L$, we have $a, b \in \downarrow x$ are relatively prime in $\downarrow x$ if and only if $a \vee b = x$.*

Proof. The first point implies the second point by Theorem 6.3.13. Suppose the second point. Suppose for $x, y, z \in L$ we have

$$x \wedge y = x \wedge z \quad \text{and} \quad x \vee y = x \vee z.$$

So in $\downarrow x \vee y$, x and z are relatively prime. We have $y \geq x \wedge z$, so there is an $a \geq x$ and $b \geq z$ such that $y = a \wedge b$. We have

$$x \wedge z = x \wedge y = x \wedge a \wedge b = x \wedge b.$$

By uniqueness, $z = b$, thus $y = a \wedge z$. Similarly we get that $z = c \wedge y$ for unique $x \leq c \leq x \vee y$. Consequently

$$y = a \wedge z = a \wedge c \wedge y,$$

Thus $y \leq a \wedge c \leq c$, and thus $z = c \wedge y = y$. Therefore the lattice is distributive. $\square$

Proposition 6.3.16. *If a and b are relatively prime in L, and $a \leq c \leq e$ and $b \leq d \leq f$, then*

$$[c, e] \times [d, f] \longrightarrow [c \wedge d, e \wedge f], \quad (x, y) \mapsto x \wedge y$$

is an isomorphism.

Proof. The map is well-defined. Furthermore, since a and b are relatively prime, this map is an injective lattice morphism; it is the restriction of h in Proposition 6.3.11. Take any $s \in [c \wedge d, e \wedge f]$. Then $s = x \wedge y$ for unique $x \geq a$ and $y \geq b$. But since $c \wedge d \leq x \wedge y \leq e \wedge f$, we have $c \leq x \leq e$ and $d \leq y \leq f$ (by applying the inverse of h). Thus the map is also surjective, thus an isomorphism as stated. $\square$

Theorem 6.3.17. *For any two relatively prime elements a and b in a lattice L, if $\uparrow a$ and $\uparrow b$ are n-Boolean, then $\uparrow (a \wedge b)$ is n-Boolean as well.*

Proof. Let $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ be Boolean sequences of length n in $\uparrow a$ and $\uparrow b$ respectively. The claim is that $a_0 \wedge b_0, \ldots, a_n \wedge b_n$ is a Boolean sequence of length n in $\uparrow a \wedge b$.

Take any $0 \leq i < n$. By Proposition 6.3.16 the following are isomorphic:

$$[a_i, a_{i+1}] \times [b_i, b_{i+1}] \cong [a_i \wedge b_i, a_{i+1} \wedge b_{i+1}].$$

Since both factors on the left are a Boolean algebras, the interval on the right is a Boolean algebra. Thus $\uparrow a \wedge b$ is n-Boolean. $\square$

6.3.4 Application to finite abelian groups

From Proposition 6.3.8, we have the following corollary.

Corollary 6.3.18. *If A and B are finite abelian groups such that $\mathsf{sub}(A)$ and $\mathsf{sub}(B)$ are a-Boolean and b-Boolean respectively, then $\mathsf{sub}(A \times B)$ is $(a+b)$-Boolean.*

Lemma 6.3.19. *For a finite abelian group G, $\mathsf{sub}(G)$ is a Boolean algebra if and only if it is cyclic with square-free order.*

Proof. A finite abelian group's subgroup lattice is distributive if and only if it is cyclic (see for example Corolalry 1.2.4 in [26]). From this together with the subgroup lattice of a finite abelian group is complemented if and only if all elements have square-free order, the result follows. □

Lemma 6.3.20. *If A and B are finite abelian groups whose orders are relatively prime, and $\mathsf{sub}(A)$ and $\mathsf{sub}(B)$ are n-Boolean and m-Boolean respectively, then $A \times B$ is $\mathsf{max}\{n,m\}$-Boolean.*

Proof. This follows from Proposition 6.3.10 and Theorem 6.3.17. □

Theorem 6.3.21. *If G is a finite abelian group, then $\mathsf{sub}(G)$ is n-Boolean if and only if the order of G is $(n+1)$-free.*

Proof. If $\mathsf{sub}(G)$ is 0-Boolean, then G is the trivial group, and thus the order of G is 1-free. Suppose that for any finite abelian group G and $k \leqslant n$, $\mathsf{sub}(G)$ is k-Boolean implies the order of G is $(k+1)$-free. Take any finite abelian group G such that $\mathsf{sub}(G)$ is $(n+1)$-Boolean. Take a Boolean sequence $(G_0, \dots, G_{n+1})$, and let k be the largest integer such that $G_k \neq G$. Then $\mathsf{sub}(G_k)$ is k-Boolean, and thus the order of G_k is $(k+1)$-free, thus also $(n+1)$-free. We have $|G| = |G/G_k||G_k|$. Since $\mathsf{sub}(G/G_k)$ is a Boolean algebra, $|G/G_k|$ is just a product of distinct primes, therefore the order of G is $(n+2)$-free. So by mathematical induction, the one direction is true.

Conversely, suppose the order of G is $(n+1)$-free. We have

$$G \cong \prod_{i=1}^{m} \prod_{j=1}^{k_i} \mathbb{Z}_{p_i^{a_{i,j}}},$$

where for each i, p_i is a prime different from the previous primes. Let, for each i,

$$p^{a_i} = \left| \prod_{j=1}^{k_i} \mathbb{Z}_{p_i^{a_{i,j}}} \right|.$$

Then, for each i, $\mathsf{sub}(\prod_{j=1}^{k_i} \mathbb{Z}_{p_i^{a_{i,j}}})$ is a_i-Boolean. And thus $\mathsf{sub}(\prod_{i=1}^{m} \prod_{j=1}^{k_i} \mathbb{Z}_{p_i^{a_{i,j}}})$ is $\mathsf{max}\{a_1, \dots, a_m\}$-Boolean. Since the order of G is $(n+1)$-free, for every i, $a_i \leqslant n$. And so $\mathsf{max}\{a_1, \dots, a_m\} \leqslant n$, and thus $\mathsf{sub}(G)$ is n-Boolean. □

Corollary 6.3.22. *Consider any group morphism $f\colon G \to H$. We have the following:*

- *If f is surjective and $\mathsf{sub}G$ is n-Boolean, then so is $\mathsf{sub}H$;*
- *if f is injective and $\mathsf{sub}H$ is n-Boolean, then so is $\mathsf{sub}G$.*

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