Incompressible flow with variations in density

by

Avotra Elie Rakotoarisoa

Thesis presented in partial fulfilment of the requirements for the degree of Masters of Science in Applied Mathematics in the Faculty of Science at Stellenbosch University

Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.

Supervisors: Dr G.P.J. Diedericks
Dr M.F. Maritz

December 2018

Declaration

By submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Signature: ....
Avotra Elie Rakotoarisoa

Date: ....December 2018..........

Abstract

This study involves the investigation of incompressible flow with variable density. The fact that variable density does not necessarily imply that the flow is compressible, may require some clarification. An attempt is made in this thesis to clarify this ambiguity by investigating examples of incompressible flow with density that varies with pressure, temperature and salinity.

In order to investigate incompressible flow with variations in density, the conditions of incompressibility that will simplify the continuity equation are determined by using scaling analysis. The Boussinesq approximation as well as the hydrostatic approximation is then applied to simplify the momentum equations of incompressible fluid flow with variations in density. Depth-averaging is also used to re-derive the shallow water equations, also with variable density.

A numerical method for solving the one-dimensional shallow water equations (suggested by Benkaldoun and Saiëd) is then reviewed. It is also implemented and applied to solve some typical examples in order to illustrate the behaviour of the flow under the assumptions of incompressible flow with density that varies with temperature and salinity.

The main results of this study can be summarized as follows: The scaling analysis serves to explain in a systematic way some conditions of incompressible flow, such as that the speed of sound must be large compared to the flow velocity, and that the diffusion of heat and salt should be negligible. Next, the solution of the one-dimensional shallow water equations, using the stated numerical method, yields qualitatively expected results.

Uittreksel

Hierdie studie behels ’n ondersoek na onsamedrukbare vloei met veranderlike digtheid. Die feit dat veranderlike digtheid nie noodwendig beteken dat die vloei samedrukbaar is nie, mag ’n verduideliking verg. ’n Poging om hierdie oënskynlike dubbelsinnigheid uit te klaar word in hierdie tesis aangewend deur voorbeelde van onsamedrukbare vloei wat met druk, temperatuur en soutgehalte verander, te ondersoek.

Ten einde onsamedrukbare vloei met veranderlike digtheid te ondersoek, is die voorwaardes van onsamedrukbaarheid wat tot vereenvoudiging in die kontinuïteitsvergelyking lei, deur skaal-analise vasgestel. Die Boussinesq benadering sowel as die hidrostatiese benadering word dan toegepas om die momentumvergelykings vir onsamedrukbare vloei met veranderlike digtheid, te vereenvoudig. Diepte-gemiddeldes word ook gebruik om die vlak-water-vergelykings weer te herlei, hier ook met veranderlike digtheid.

’n Numeriese metode om die vlak-water-vergelykings op te los (voorgestel deur Benkaldoun en Saiëd) word hersien. Dit word ook geïmplementeer en aangewend omtipiese voorbeelde op te los waar die gedrag van vloei onder die aannames van onsamedrukbaarheid met digtheid wat verander met temperatuur en soutgehalte, geïllustreer word.

Die hoof resultate van die studie kan as volg opgesom word: Die skaalanalise dien goed om die voorwaardes van onsamedrukbare vloei in ’n sistematiese manier te verduidelik, byvoorbeeld dat die spoed van klank groot moet wees in vergelyking met die vloeisnelheid, en dat die diffusie van hitte en sout weglaatbaar moet wees. Verder toon die oplossing van die gemelde numeriese metode kwalitatief verwagte resultate.

Acknowledgements

My sincere gratitude goes to my supervisors Dr G. P. J. Diedericks and Dr M.F. Maritz for their presence, guidance and close collaboration throughout this study. I am very thankful to them for their invaluable help and suggestions. I will never forget all the advice and lessons that you gave me both on the professional and personal level. Furthermore, many thanks also to the Stellenbosh University and the African Institute for Mathematical Sciences (AIMS) since this study would not have been possible without the financial support from them. I am also grateful to all of those who helped me during this project, including Prof. Francois Smit, Dr Marèt Cloete and Ms Sharon Fortuin. Finally, I would like to express my profound gratitude to my family and friends, Mom, Tanjona, Brice, Bernoulli, Ando sy Ony ary Vahatra for their unconditional love and support.

Contents

List of Figures

Nomenclature

Symbol	Description
$\underline{R}$	Spacial coordinates
$\underline{R}_0$	Material coordinates
$\underline{r}$	Spatial position vector
$\underline{t_0}$	Initial time
$\underline{u}$	Flow velocity vector
$\underline{u}_l$	Flow velocity vector in Lagrangian description
$\underline{f}$	Arbitrary vector function
ϕ	Potential scalar
$\underline{\psi}$	Potential vector
T	Temperature
S	Salinity
Cl	Chlorinity
p	Pressure
m	Mass
ρ	Density
V	Volume
F_{normal}	Force
e	Internal energy
h	The specific enthalpy

η	The specific entropy
c_p	The specific heat at constant pressure
c_v	The specific heat at volume pressure
c	Speed of sound
y_p	Changing position along y-axis
V_m	Material volume
A_m	Surface of the material volume
V_s	Fixed volume
A_s	Surface of the fixed volume
$\underline{J}_S$	Vector salt flux
$\underline{k}_S$	Diffusion coefficient of salt
$\underline{Q}_S$	Sources or/ and sinks of salt
$\underline{J}_T$	Vector heat flux
$\underline{k}_T$	Diffusion coefficient of heat
$\underline{Q}_T$	Sources or/ and sinks of heat
μ	Chemical potential
$\boldsymbol{\phi}$	Function of viscous dissipation
U	Horizontal characteristic flow velocity
W	Vertical characteristic flow velocity
L	Horizontal characteristic length
H	Vertical characteristic length
k	Spring constant
$\mathbf{U}$	Vector form of the conservative variables
$\mathbf{F}(\mathbf{U})$	Vector flux function
$\mathbf{Q}$	Vector source terms
$\mathcal{F}_{i+\frac{1}{2}}$	Numerical flux
$\mathcal{U}_i^n$	Approximate value of the average of $\mathbf{U}$ at time t^n

$\mathcal{Q}_i^n$	Approximate value of the average of $\mathbf{Q}$ at time t^n
τ	Parameter
X	Parametric representation of the characteristic curves

Chapter 1

Introduction

1.1 Introduction and problem statement

For different studies in fluid mechanics and various applications in Computational Fluid Dynamics, it is a common assumption to take the fluid density as constant. The latter usually implies that the continuity equation is reduced to the divergence of the flow velocity being equal to zero ($\underline{\nabla} \cdot \underline{u} = 0$). Although this equation is well known as the mathematical interpretation of incompressible flow (Panton, 2013), some ambiguity and maybe incorrect perception arise when it is stated that fluid density being constant implies incompressible flow. Indeed, it should be pointed out that we might have a variations in density but that the flow is incompressible. It is then our purpose to give more clarity and more understanding to incompressibility.

In this study, we are interested in studying incompressible flow with variations in density. In particular, we are interested in a situation where the fluid density is a function of the temperature and the salinity. Such situations can be encountered in the motion of water in the ocean. Indeed, underlying assumptions for taking the oceanic hydrodynamic flow as incompressible but allowing the fluid density to vary will be a center point of investigation here.

Through this investigation, in order to familiarized the reader with the physical and mathematical description of the fluid motion, an introduction to fluid mechanics, and especially the kinematics of fluid mechanics and thermodynamics of seawater is presented. Moreover, a structured technique of scaling analysis based on the nondimensionalization of the continuity equation is defined as the fundamental tool for the derivation of the conditions of incompressible flow. In addition, although the ocean is considered highly stratified (Dijkstra, 2008), a section about the depth-averaging of the governing equations is also presented in this study. Furthermore, an existing numerical method will be reviewed in order to solve the set of governing equations corresponding to the flow which satisfies the conditions of incompressibility with varying density.

1.2 Objectives

As discussed above, the aim of this study is to investigate incompressible flow with variations in density. In order to accomplish this goal, the following primary objectives shall be discussed in detail throughout this study:

- The first objective is to derive all the conditions that related to the flow being incompressible while the fluid density is considered as a function of pressure, temperature, and salinity. In other words, we would like to seek the conditions for which the material derivative of the fluid density, $\frac{D\rho}{Dt}$, is negligible compared to $\underline{\nabla} \cdot \underline{u}$.

- The second objective is to derive the governing equations which satisfy the conditions of incompressibility.

- The third objective is to review in detail the modified finite volume method of characteristics suggested by Benkhaldoun and Seaïd (2010) in order to solve the one-dimensional shallow water equations.

In order to accomplish the cited objectives above, this study has to follow various sub-objectives. The first sub-objective is to provide a comprehensive review of the kinematics of fluids and an introduction to thermodynamics concepts. The second sub-objective consists of the derivation of the continuity equation, the conservation of energy and salt in order to rewrite the continuity equation in expanded form and to apply the scaling analysis on it. The third sub-objective is the derivation of the Navier-Stokes equation associated with the Boussinesq approximation and the hydrostatic approximation. In addition, by doing the depth-average of the continuity equation and momentum equation, the one-dimensional shallow water equations will be derived from this study.

1.3 Background

It can be seen from the previous section that the main objectives of this study may be divided into two main parts. The first consists of deriving the conditions of incompressibility of flow, and the second consists of solving the one-dimensional shallow water equations using the modified finite volume method of characteristics. Taking the first objective, many authors have discussed incompressible fluids and flow in literature. As stated before, incompressibility is a well-used assumption for studying different types of flow. For instance, Bachelor (1967) discussed more the fluid properties rather than flow properties in terms of incompressibility. He stated that the incompressible flow results from the motion of an incompressible fluid. The latter means that the changes in the pressure does not affect the fluid density. On the other hand, according to Panton (2013), incompressible flow is more related to the flow velocity and the speed of sound, and it occurs when the flow velocity is much smaller compared to the velocity of sound. Furthermore, in a similar way as Panton (2013), according to LeBlond and Mysak (1981), incompressibility of flow is more related to the local speed of sound within a fluid. For instance, in the ocean dynamics, the speed of sound is often assumed to be infinite, then the fluid density cannot change along a streamline which implies it does not vary with respect to the pressure. In addition, some changes in the temperature and the salinity may also occur in the ocean in the absence of diffusion. As a consequence of these assumptions, the ocean model is defined to be incompressible, and the continuity equation is reduced to $\underline{\nabla} \cdot \underline{u} = 0$.

The second objective consists of solving the one-dimensional shallow water equations. The latter equations are well known as a set of hyperbolic partial differential equations and many approaches have been made in literature to solve such type of equations numerically. For example, the finite volume methods based on the approximate Riemann solver methods have been well used to solve these hyperbolic partial differential equations without taking into account the source terms (LeVeque, 2002). However, when the source terms are taken into account consideration, the approximate Riemann solver schemes are mostly susceptible to instability and produce non-physical oscillations (Benkhaldoun and Seaïd, 2010). In order to avoid such undesirable cases, various schemes have been developed in order to solve the water equations, for instance, the Roe's scheme (Roe, 1981), the Godunov-type finite volume scheme (Chippada S., 1998), TVD-MacCormack scheme (Chippada S., 1998), the ENO and WENO schemes for solving one-dimensional shallow water equations (Yulong Xing, 2005). In this study, the modified finite volume method of characteristics suggested by Benkhaldoun and Seaïd (2010) is used in order to solve the one-dimensional shallow water equations. Such a method is advantageous since it has been proven by Benkhaldoun and Seaïd (2010) that it is stable with the source terms, and it does not need the approximate Riemann solver to compute the numerical flux. Indeed, this numerical method consists of a two-stage numerical scheme to update the solution at each time step. It is called a *predictor-corrector stage*. The numerical flux is constructed in the predictor stage using the method of characteristics. The solutions of the shallow water equations are recovered using the corrector stage.

1.4 Outline of the thesis

The first chapter is devoted to the introduction and the problem statement of this thesis. Next, chapter 2 is devoted to the fundamental background in fluid mechanics and thermodynamics of seawater. In this case, the kinematics of fluids is described in detail with the notion of streamline and path-line. To continue, in chapter 3, the conditions for which the material derivative of the fluid density is negligible compared to the divergence of the flow velocity are derived. Furthermore, in chapter 4, the governing equations which satisfy $\rho = \rho(T, S)$ with the shallow water equations are derived. chapter 5 is devoted to the numerical solution of the one-dimensional shallow water equations that were derived in chapter 4, and the finite volume method (FVM) is used. In the last chapter, the summary and conclusions of this study are given as well as some recommendations for further work.

Chapter 2

Background of fluid mechanics

In this chapter, an overview of the basic concepts of fluid mechanics and thermodynamics especially for seawater are reviewed. In Section 2.1, a basic definition of a fluid is first given. In Section 2.2, the kinematics of fluids is discussed. In this case, a detailed discussion about the method of describing fluid motion is presented. In addition, the definitions of path-lines and streamlines are also given. Moreover, in Section 2.3, different types of flows are presented. And finally, Section 2.4 is devoted to the introduction of thermodynamic relationship between seawater and the equations of state.

2.1 Definition of fluids

A fluid is defined as a material which does not have a preferred shape and its elements can be rearranged without affecting its macroscopic properties. Both liquids and gases are treated as fluids and are distinguished from solids by this definition (Bachelor, 1967). Such fluids are illustrated in Figures 2.1a and 2.1b. Alternatively, a fluid is a substance that deforms continuously when subjected to external forces. Fluid mechanics is the mathematical description of the behaviour of fluids. Some assumptions are needed in order to process this mathematical description, and one of th e fundamental assumptions is to assume the treatment of a fluid as a continuum. This means that its macroscopic behaviour can be defined by being continuous in space. This continuum hypothesis can be made because the fluid is seen as being continuous with respect to the distances between its molecules. This hypothesis implies that we can study the various properties of fluids such as the density, the velocity, and the temperature in terms of a continuous function of position and time. In this case, these various properties above are continuous point functions, and are regarded as differentiable at all points in space and time (Whitaker, 1968).

(a) Liquid as an example of a fluid

(b) Smoke as an example of a fluid

Figure 2.1: Different types of fluid

2.2 Kinematics of fluids

In general, kinematics is defined as the study of motion without taking into account the causes which bring it about, for example, dynamical laws (Panton, 2013). In this chapter, we will be interested in the kinematics of continuous media. In fluid mechanics, the study of the kinematics of fluids is characterized by two types of approaches, namely the Lagrangian and the Eulerian approaches. Specifically, we will be interested in describing the displacement, the velocity, and the acceleration with these approaches.

2.2.1 Lagrangian description

2.2.1.1 Position

The Lagrangian description of motion is a method which describes the fluid motion and the properties of the fluid particles with regards to the coordinates where the motion of the fluid particles starts to occur. This approach is mostly used to study the kinematics of the mechanics of rigid bodies and solid particles (Whitaker, 1968).

In order to illustrate this description mathematically, consider an element of fluid which is set into motion with respect to spatial coordinates $\underline{R}$ and let t be the time. At the initial time $t = t_0$, the fluid particle is at the initial position with coordinates $x_0, y_0,$ and z_0, and is located at the position vector $\underline{R}_0 = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$. And after a certain amount of time $t > t_0$, that fluid particle is now located at its new position vector $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ such that

$$\begin{aligned} x &= x_0 + \int_0^t \frac{dx}{dt'} dt' \\ y &= y_0 + \int_0^t \frac{dy}{dt'} dt' \\ z &= z_0 + \int_0^t \frac{dz}{dt'} dt'. \end{aligned} \tag{2.2.1}$$

We may rewrite Equation (2.2.1) in terms of a vector as follows:

$$\underline{r} = \underline{R}_0 + \int_0^t \frac{d\underline{r}}{dt'} dt', \tag{2.2.2}$$

where $\underline{r}$ is called the spatial position vector since it locates the element of fluid in space at time t, while $\underline{R}_0$ is called the material position vector and it represents the coordinates used to identify a given fluid particle at its initial position at initial time $t = t_0$. Henceforth, the so-called *material coordinates* which is defined as the specific set of spatial coordinates where the fluid particles occupied at initial time $t = t_0$. Since only one, not two or more fluid particles can occupy the same place at time t, then the position vector $\underline{r}$ in Equation (2.2.1) can be rewritten as

$$\underline{r} = \underline{r}\left(\underline{R}_0, t\right). \tag{2.2.3}$$

According to the previous expression, it is then possible to define a vector function $\underline{f}$ that maps the initial coordinates of the fluid particle to its current configuration. Such function can be expressed as follows:

$$\underline{r}(t) = \underline{f}(\underline{R}_0, t), \tag{2.2.4}$$

where $\underline{f}$ is called a particle path function (Panton, 2013). The Lagrangian description of motion is illustrated in Figure 2.2 and its variables are defined by the initial position x_0, y_0 and z_0. Therefore, the Lagrangian variables can be derived from the spatial coordinates or *vice versa* using the vector function $\underline{f}$. This can be done using the fact that $\underline{f}$ is bijective, that is

$$\underline{R}_0 = \underline{f}^{-1}(\underline{r}, t),$$

where $\underline{f}^{-1}$ is the inverse of $\underline{f}$.

2.2.1.2 Velocity and acceleration

Let $\underline{u}_l$ be the velocity of the fluid particle in the Lagrangian description at time t. Here, the velocity of the fluid particle is measured with respect to its reference coordinates and time t. It is then noticed that in order to evaluate the time derivative of the position vector which is presented in Equation (2.2.1), the material coordinates is kept constant. We then have

$$\begin{aligned}
\underline{u}_l &= \left.\frac{\partial \underline{r}}{\partial t}\right|_{\underline{R}_0} \\
&= \left.\frac{\partial x(\underline{R}_0, t)}{\partial t}\right|_{\underline{R}_0} \underline{i} + \left.\frac{\partial y(\underline{R}_0, t)}{\partial t}\right|_{\underline{R}_0} \underline{j} + \left.\frac{\partial z(\underline{R}_0, t)}{\partial t}\right|_{\underline{R}_0} \underline{k} \\
&= u_0\,\underline{i} + v_0\,\underline{j} + w_0\,\underline{k}.
\end{aligned} \tag{2.2.5}$$

where the components of $\underline{u}_l$ in the Lagrangian description are simply the partial time derivatives to t. In the same way, the acceleration of the fluid particle in the Lagrangian description is given by

$$\underline{a}_l = \left.\frac{\partial \underline{u}_l}{\partial t}\right|_{\underline{R}_0}. \tag{2.2.6}$$

The fluid particle in the Lagrangian description is represented in Figure 2.2.

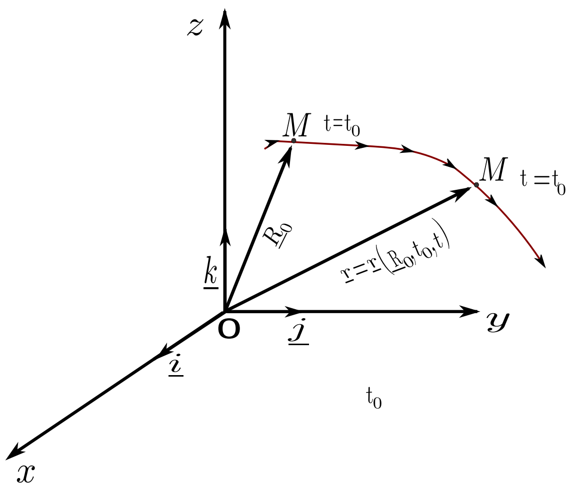

Figure 2.2: Lagrangian description of the flow

2.2.1.3 Path-line

A path-line is inseparable to the Lagrangian description since it represents a curve in space which describes the trajectory of an individual fluid particle (Whitaker, 1968). A real example of a path-line is the trajectory taken by one dust particle carried by the motion of the wind. It can then be described as a record of where the particle of the fluid has been. Therefore, a path-line is illustrated as a consequence of the Lagrangian description. An example of path-lines is illustrated in Figure 2.3.

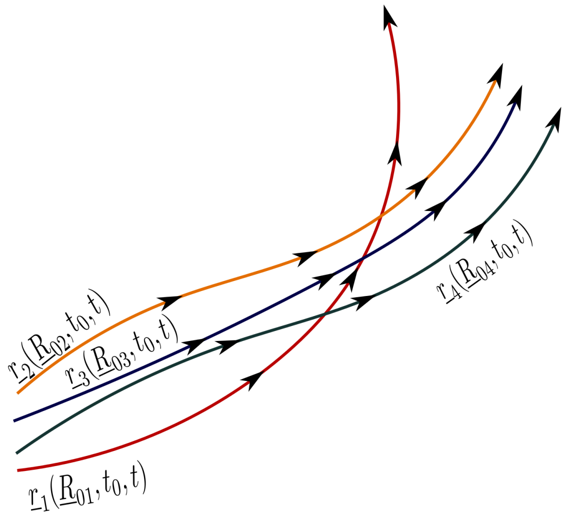

Figure 2.3: Illustration of path-lines

2.2.2 Eulerian description

Instead of tracking a fluid particle along a trajectory, one could do the measurement at a fixed point; then the fluid properties are measured at time t and location (x, y, z) using the spatial coordinates. In contrast to the Lagrangian description, the initial position of fluid motion will not be considered when the flow properties of fluid particle are investigated. The fluid particle is then located by the spatial position vector $\underline{r}$, such that

$$\underline{r} = \underline{r}\,(x, y, z, t)\,. \tag{2.2.7}$$

In this description, the fluid motion is characterized by a vector velocity field which is denoted by $\underline{u}$ in the Eulerian description. In this case, the Eulerian velocity is given by

$$\underline{u} = u\underline{i} + v\underline{j} + w\underline{k}, \tag{2.2.8}$$

where its components can be deduced from the Lagrangian velocity by transforming all Lagrangian variables to Eulerian variables, we then have

$$\underline{u} = \underline{u}_l(\underline{f}^{-1}(\underline{r}, t), t),$$

with f defined in Equation (2.2.4). The Eulerian description is illustrated in Figure 2.4,

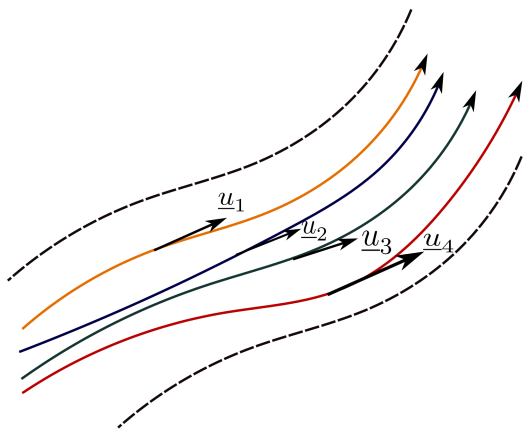

Figure 2.4: Eulerian description of the flow

Streamline

The notion of a streamline only makes sense under the Eulerian description and according to Whitaker (1968), a streamline is defined as a curve which is everywhere tangent to the Eulerian vector velocity of fluid particles. This curve is illustrated in Figure 2.5. A streamline is mathematically given by the following expression:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, \tag{2.2.9}$$

where u, v and w are the respective components of the vector field velocity.

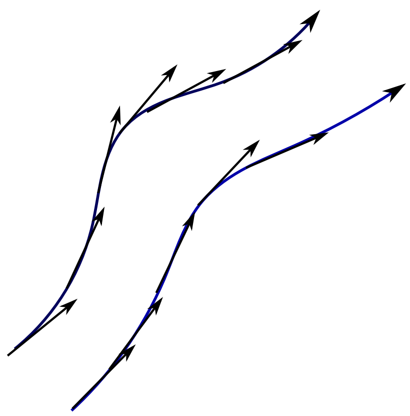

Figure 2.5: Illustration of two streamlines

2.2.3 Time derivatives

For an arbitrary scalar function $f(x, y, z, t)$ which is associated with a fluid particle, we can take its time derivative. Let us firstly suppose that the spatial coordinates are not taken into account, and that the function is only dependent on time, then the time derivative is given by

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}. \tag{2.2.10}$$

Let us now suppose that we need to take into consideration the point in space where the scalar function f is measured during the interval t to $t + \Delta t$. In this case the time derivative which is presented in Equation (2.2.10) is not well defined without some explanations. Consider first that the scalar function f is defined as a function of time t and the material coordinates $\underline{R}_0$ such that

$$\underline{f} = \underline{f}(\underline{R}_0, t).$$

In order to illustrate the time derivative of this function, we may imagine the following example: An observer is moving in the direction of the $y-$axis of a fixed frame of reference, and is continuously measuring how its position y_p is changing along the $y-$axis. This measurement can be thought of time derivative of its position and it is evaluated by taking the material coordinates constant. In this particular example, as the quantity can only be measured by the observer then such a time derivative is so-called a material derivative and denoted by

$$\frac{Dy_p}{Dt} = \left(\frac{dy_p}{dt}\right)_{\underline{R}_0} = \lim_{\Delta t \to 0} \left[\frac{y_p(t+\Delta t) - y_p(t)}{\Delta t}\right]_{\underline{R}_0}. \tag{2.2.11}$$

In contrast to the illustration above, if the real position of the fluid particles is unrelated to its initial position, then it is suitable to measure the properties of the scalar function at a fixed point in space. In this case where the function f is dependent on the current position, rather than we expressed it as function of the initial position, we have

$$\underline{f} = \underline{f}(\underline{r}, t). \tag{2.2.12}$$

In this case, evaluating the time derivative of the function f can be thought of as a measurement of the properties of succession of fluid particles passing through this fix point. The time derivative of the function f by taking the spatial coordinates constant is then given by

$$\frac{\partial \underline{f}}{\partial t} = \left(\frac{d\underline{f}}{dt}\right)_{\underline{r}} = \lim_{\Delta t \to 0} \left[\frac{\underline{f}(t+\Delta t) - \underline{f}(t)}{\Delta t}\right]_{\underline{r}}, \tag{2.2.13}$$

where $\dfrac{\partial \underline{f}}{\partial t}$ is called a partial derivative, or a local derivative.

Consider now the case when the variation of the spatial coordinates is needed when the scalar function $\underline{f}$ is measured. This case can be illustrated by considering the following situation:

Let us suppose that there is an observer sitting in a boat in a river which is moving with velocity $\underline{u}_l$ and he measures the variation of the temperature. Let T be the scalar function of the temperature such that

$$\begin{aligned} T &= T(\underline{r}, t) \\ &= T(x, y, z, t), \end{aligned} \tag{2.2.14}$$

and using the fact that the variation of the spatial coordinates is considered. It was seen before that the spatial coordinates are a function of material coordinates $\underline{R}_0$ and time t then

$$\begin{aligned} T &= T\left[\underline{r}(\underline{R}_0, t), t\right] \\ &= T\left[x(\underline{R}_0, t), y(\underline{R}_0, t), z(\underline{R}_0, t), t\right]. \end{aligned} \tag{2.2.15}$$

By keeping the material coordinates $\underline{R}_0$ constant, and taking the time derivative of the temperature, the following expression is obtained

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\left(\frac{dx}{dt}\right)_{\underline{R}_0} + \frac{\partial T}{\partial y}\left(\frac{dy}{dt}\right)_{\underline{R}_0} + \frac{\partial T}{\partial z}\left(\frac{dz}{dt}\right)_{\underline{R}_0}. \tag{2.2.16}$$

It is expected that the time derivatives of the spatial coordinates by taking the material coordinates constant are the components of the river velocity

$$\left(\frac{dx}{dt}\right)_{\underline{R}_0} = u_0, \left(\frac{dy}{dt}\right)_{\underline{R}_0} = v_0, \quad \text{and} \quad \left(\frac{dz}{dt}\right)_{\underline{R}_0} = w_0,$$

and the derivative of temperature T holding $\underline{r}$ constant is the partial derivative. In vector notation Equation (2.2.16) becomes

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underline{u}_l \cdot \underline{\nabla} T. \tag{2.2.17}$$

In the case where u_l can be transformed in terms of Eulerian variables, Equation (2.2.17) can be rewritten as

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underline{u} \cdot \underline{\nabla} T. \tag{2.2.18}$$

It is noted that Equation (2.2.18) is the most general type of time derivative that will be encountered. We have seen it for a scalar field but it can also be applied to a vector field. For example, we have the vector acceleration field, such that

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla}\, \underline{u}, \tag{2.2.19}$$

where $\underline{u}$ is the Eulerian velocity field.

Equation (2.2.19) is defined as the acceleration of fluid particles moving with the fluid motion. The first term on the right hand side of Equation (2.2.19) is defined as the time derivative of the velocity at a fixed point in space and is called the local acceleration, and the second is named the convective acceleration.

2.3 Types of flow

In this section, different kinds of flow will be briefly presented. A flowing fluid may fall into more than one of these flow categories.

2.3.1 Incompressible flow

An entire section will be devoted to investigate incompressible flow in the next chapter. Here, just a brief definition will be introduced. A flow is defined as incompressible when its velocity relative to its boundary is very small compared to the velocity of sound in the fluid (Pedlosky, 2013). Moreover, a well used conception of incompressible flow is related to the conservation of mass. In this case, the flow is defined as incompressible when the net rate of change of the mass of the fluid flowing from a particular point per unit of volume is zero. This can be interpreted as the divergence of the flow velocity being equal to zero,

$$\underline{\nabla} \cdot \underline{u} = 0, \tag{2.3.1}$$

where $\underline{u}$ is the flow velocity (Cohen et al., 2004).

2.3.2 Compressible flow

A fluid flow motion is defined as compressible as soon as its local velocity is close to the speed of sound. Using the Mach number which is the ratio of the flow velocity to the velocity of sound, we can characterise a fluid flow to be compressible or not. In this case, the flow is defined to be compressible when the numerical value of the Mach number is equal to or greater than one. A formal definition of the Mach number will be given in the next chapter.

Some examples of compressible flow are given in Figures 2.6 and 2.7. The first figure illustrates the flow around a bullet where the air density changes abruptly due to the fact that the velocity of the bullet is very high. This causes the formation of a wave pattern around the bullet. Figure 2.7 illustrates the same phenomenon. Here, the wave pattern around the supersonic jet is more visible and such waves are called shock waves. The latter is the result of motion which is faster than the local speed of sound in the fluid.

Figure 2.6: Bullet in compressible flow at Mach number $M = 1.1$ (Van Dyke, 1982)

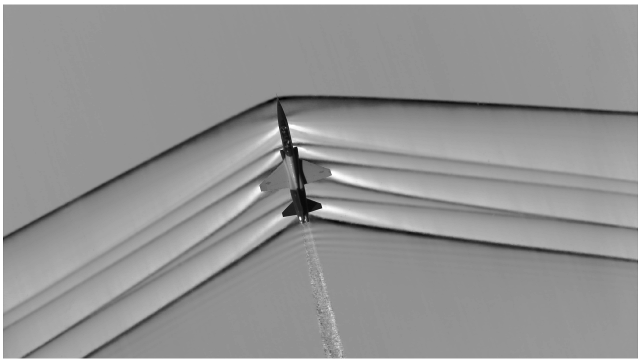

Figure 2.7: Supersonic jet flying at Mach number $M = 1.1$ (Van Dyke, 1982)

2.3.3 Laminar flow

The flow is defined as laminar if the fluid particles appear to move in smooth paths. Such flow is characterized by the fluid motion in thin layers on the top of each other. This implies that the flow velocity remains constant for each point along the streamline. Figure 2.8 shows one example of laminar flow around an obstacle.

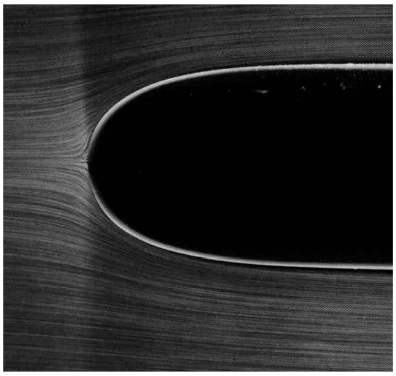

Figure 2.8: This figure illustrates the motion of air around the ogive and this is characterized by a laminar flow pattern (Van Dyke, 1982)

2.3.4 Turbulent flow

In contrast to laminar flow, turbulent flow is characterized by an irregular or disordered motion (Bachelor, 1967). In order to characterize whether the flow is turbulent or not, the Reynolds number is commonly used. The latter will be defined in the next chapter. The turbulent flow is characterized by a higher value of the Reynolds number while laminar flow has lower values. Turbulent flow can be illustrated in Figure 2.9. It can be seen that the figure may be divided into three parts, the first part represents the laminar flow pattern which is the same as shown in the section about laminar flow. The second part represents the instability of laminar flow and its transition to turbulent flow, and the final part represents the turbulent flow.

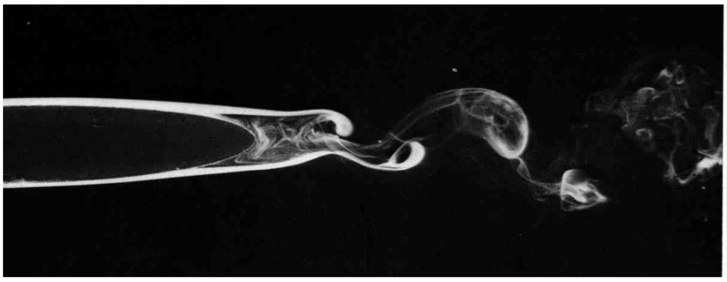

Figure 2.9: Illustration of laminar-separation-turbulent flow (Van Dyke, 1982)

2.3.5 Stratified flow

A stratified flow is a flow such that the fluid density varies in the vertical direction only. In these terms, the density changes with height, for instance, stratified flow occurs when warm water lies above cold water or when freshwater is above salt water.

2.3.6 Steady and unsteady flow

When the fluid motion does not change in time, it is called a steady flow, while unsteady is a flow that is dependent on time. In general, laminar flow may be steady or unsteady whereas turbulent flows are unsteady.

2.3.7 Rotational and irrotational flow

According to Cohen et al. (2004), a flow is defined as irrotational if the motion of the fluid particles is in pure translation. In other words, irrotational flow is characterized by motion of fluid along streamlines and does not rotate about its own center of gravity. Fluid particles do not rotate but are translating in a circular path are still considered irrotational. In this case, the flow is moving in circular path but has no vorticity. A flow is rotational when the fluid particles undergo rotation. This implies that the fluid particles would rotate as they translate along the streamlines. In order to illustrate these two types of flow mathematically, the flow velocity may be split into a rotation part by introducing a vector potential $\underline{\psi}$, and irrotational part by giving a scalar ϕ such that

$$\underline{u} = \underline{\nabla}\phi + \underline{\nabla} \times \underline{\psi}. \tag{2.3.2}$$

By taking the curl of $\underline{u}$, we obtain

$$\underline{\nabla} \times \underline{u} = \underline{\nabla} \times \underline{\nabla}\phi + \underline{\nabla} \times \left(\underline{\nabla} \times \underline{\psi}\right) = \underline{\nabla} \times \left(\underline{\nabla} \times \underline{\psi}\right), \tag{2.3.3}$$

where $\underline{\nabla} \times \underline{u}$ is called the vorticity. Then the flow is defined to be irrotational when the vorticity is equal to zero, in that case, the flow velocity is just expressed as gradient of the potential scalar ϕ, which means $\underline{u} = \underline{\nabla}\phi$. If the flow is rotational, then $\underline{u} = \underline{\nabla} \times \underline{\psi}$ (Cohen et al., 2004).

2.4 Review of thermodynamics

Without resorting to the details of the subject of thermodynamics, only a summary of fundamental ideas of thermodynamics will be given. Some useful terms from thermodynamics which will be used in the equation of state of the fluid density will be discussed in this section.

2.4.1 Introductory concepts

Similar to fluid mechanics and other engineering sciences, the subjects about thermodynamics are very useful in terms of mathematical and physical descriptions of properties intended to meet human needs. In thermodynamics, it is important to define the notion of a system which is defined as a domain where the study will be focused, for example, a free body or as complex as physical phenomenon. A system has an external which is called surrounding, and the separation which distinguishes the system from its external surroundings is called the boundary (Moran et al., 2010). In general, thermodynamics may be defined as the study of systems where an exchange of energy is involved. For example, for a given system, the energy can leave this system to another. The transfer of energy can be done only in two ways, as a transfer of heat by the variation of temperature or as a transfer by work.

2.4.2 Thermodynamic properties

In order to study a system thermodynamically, knowledge of its properties is required. Such properties are defined as macroscopic characteristics of a system which can be mass, volume, energy, density, and/or temperature. Furthermore, it is necessary to introduce the word *state* which is basically defined as the condition of a system as described by its properties. In addition, the thermodynamic properties can be distinguished by two classes, the first is extensive properties and the second is intensive properties. Basically, the extensive properties combine additively which means that for an overall system, its properties are the sum of the properties of the parts out of which the system is comprised, for example mass and volume density. In contrast to the first property, intensive properties are not additive, and their values are independent of the size of the system. Density and temperature are examples.

2.4.3 Thermodynamic variables and coefficients

A thermodynamic system may be characterized by several properties which can be thermodynamic variables or coefficients. In this paragraph, the definition of some of these properties will be given accordingly to their utility in the next chapter.

2.4.3.1 Pressure

The usual definition of pressure is force per area of surface. According to Moran et al. (2010), the expression of pressure is given by

$$p = \lim_{A \to A'} \left(\frac{F_{\text{normal}}}{A} \right), \tag{2.4.1}$$

where A' is the smallest area for which a definite value of the ratio exists (Moran et al., 2010). This formula means that for a given force on a surface, the pressure is obtained by dividing the force by the area of surface. The SI unit of pressure is $\mathrm{Pa} = \mathrm{N} \cdot \mathrm{m}^{-2}$.

2.4.3.2 Temperature

A fundamental understanding of temperature is not easy, but the concept of temperature can be understood by noting the interaction between two different isolated systems brought into contact. In order to illustrate this, let us consider two copper blocks, one hotter than the other. If the blocks were isolated from their surroundings and were brought into contact then there is an exchange of heat between the two blocks and after an amount of time when this exchange ceases, the two blocks are in thermal equilibrium. This phenomenon is so-called the heat transfer where the driving force is the difference of temperature. Then the temperature can be defined as a physical property that determines if the two systems are in thermal equilibrium or not (Brodkey and Hershey, 2003). In the SI system, the temperature can be measured using either the Kelvin scale, K, or using the Celsius scale, °C. One example of a physical property that may depend on temperature is the fluid density. For instance, for a well-chosen relationship that links the temperature and the pure water density, the profile of the water density as a function of temperature can be seen in Figure 2.10

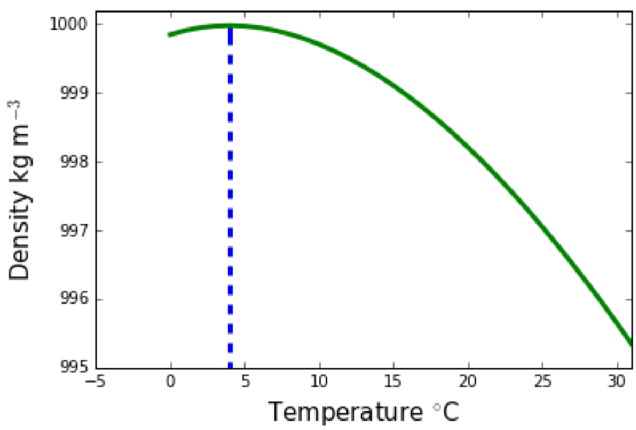

Figure 2.10: Density of pure water as a function of temperature

It can be seen from Figure 2.10 that at 4°C the density is at its maximum value $\rho = 1000\,\mathrm{kg} \cdot \mathrm{m}^{-3}$ whereas above and below that temperature the density is lower.

2.4.3.3 Salinity

Before giving a definition of salinity, let us first define seawater. It is the water from the sea or ocean which is a combination of fresh water and different types of salt. The quantity of salt in seawater is characterized by the salinity. The salinity can be defined as the total sum of solid material in grams contained in one kilogram of seawater when all the carbonates have been converted to oxide, all the bromine and iodine replaced by chlorine and all the organic material oxidized (Müller, 2006). The unit of salinity is parts per thousand, denoted by ppt or ‰. Furthermore, the salinity can be determined from the titration of one of the ion components of seawater which is chlorine. Therefore, the salinity can be expressed as a linear function of chlorinity Cl (amount of Chlorine in the water) such that we have the following expression:

$$S(‰) = 1.805\mathrm{Cl}(‰) + 0.03 \tag{2.4.2}$$

where this formula is valid for the range of salinities from 2.69 ppt to 40 ppt (Mamayev, 2010).

2.4.3.4 Density

The density is an intensive thermodynamic property, and is defined as the ratio of mass to volume. According to Moran et al. (2010), explicit expression of density is given by

$$\rho = \lim_{V \to V'} \left(\frac{m}{V} \right), \tag{2.4.3}$$

where V' is the smallest volume for which a definite value of the ratio exists (Moran et al., 2010). In the SI system, the unit of density is $\mathrm{kg} \cdot \mathrm{m}^{-3}$. We have the specific volume v which is also an intensive property and it is defined as the inverse of the density, which means the volume per unit of mass. The unit of specific volume in the SI system is $\mathrm{kg}^{-1} \cdot \mathrm{m}^3$. The density may vary from one position to another, for instance, in the ocean, it will be seen later that the changes in density are due to the variations of state variables such as the pressure, the temperature, as well as the salinity.

2.4.3.5 Additional thermodynamic variables

The term energy is one of the most important thermodynamic properties for a given system. There are several types of energy in the study of thermodynamics. Some will be defined here:

- The specific internal energy $e\,[\mathrm{m}^2 \cdot \mathrm{s}^{-2}]$, is defined as the total energy in a system per unit of volume which means the combination of the kinetic energy and the potential energy of molecules.

- The specific enthalpy $h = e + p\,v\left[\mathrm{J \cdot kg^{-1}}\right]$, which is merely defined as the sum of the specific internal energy and the product of the pressure p with the specific volume v.
- The specific entropy $\eta\left[\mathrm{J \cdot K^{-1} \cdot kg^{-1}}\right]$ is the entropy per unit of volume. The simplest way to understand the entropy is that it measures the disorder within a macroscopic system (Schmidt et al., 1984).

Furthermore, the term *specific heat* is one of the thermodynamic properties expressing how the energy changes with temperature. For the case of seawater, it depends on temperature, pressure and salinity. Then, it will be useful to introduce, the specific heat at constant pressure and salinity denoted by c_p, by definition

$$c_p = \left(\frac{\partial h}{\partial T}\right)_{p,S}. \tag{2.4.4}$$

It is also useful to define the specific heat at constant specific volume denoted by c_v. It is expressed as

$$c_v = \left(\frac{\partial e}{\partial T}\right)_{v,S}, \tag{2.4.5}$$

where v is the specific volume, e is the specific internal energy and h is the specific enthalpy. These may also be expressed as

$$c_v = T\left(\frac{\partial \eta}{\partial T}\right)_{v,S} \quad \text{and} \quad c_p = T\left(\frac{\partial \eta}{\partial T}\right)_{p,S}. \tag{2.4.6}$$

where η is defined as the specific entropy, p is the pressure, S is the salinity (Müller, 2006).

2.4.4 Thermodynamic variables and sound

The velocity of sound is an important variable in fluid dynamics as well as in thermodynamics, since it plays an important role in terms of qualification of types of flow. Let c be the commonly used symbol which indicates the speed of sound. Sound can be thought of as a propagation of a plane, weak, pressure disturbance in a fluid (Schmidt et al., 1984). Since, there is a link between the pressure and the fluid density, it will be expected that there is also an explicit relationship between sound velocity and the fluid density. In the case where the process is defined under a constant specific entropy and salinity, according to Müller (2006), the sound velocity is expressed by

$$\frac{1}{c^2} = \left(\frac{\partial \rho}{\partial p}\right)_{\eta,S}, \tag{2.4.7}$$

where p is the pressure, ρ is the density, η is the specific entropy. Now, if the temperature and the salinity are constant, then Equation (2.4.7) can expressed as :

$$\frac{1}{c^2}\frac{c_p}{c_v} = \left(\frac{\partial \rho}{\partial p}\right)_{T,S}. \tag{2.4.8}$$

In addition, for seawater the specific heats at constant pressure and at constant volume differ only by 1% in value from experimental results. Then, we may consider that the ratio of specific heats $\frac{c_p}{c_v}$ is approximately equal to 1 (Müller, 2006).

2.4.5 Equation of state of seawater

The concept of a state was briefly introduced in Section 2.4.2. In order to characterize the state of system, there is a choice of variables that contributes to the description of this system. For example, if seawater is taken as a thermodynamic system, then the common used set of variables is the pressure p, the temperature T, and the salinity S. We call these variables, the state variables, and the equation of state is related to the relationship between these state variables. For instance, by considering the seawater density ρ, the general form of the equation of state can be written as follows:

$$f(\rho, p, T, S) = 0, \tag{2.4.9}$$

which can be rewritten in a commonly used form in oceanography as:

$$\rho = \rho(p, T, S). \tag{2.4.10}$$

Equation (2.4.10) is a non-linear relation in p, T, and S and has no simple analytical form. The formulation is obtained from a standard oceanographic table or can be estimated by a best-fit polynomial which is valid over a restricted range of temperature and salinity. One of these relationships is the Eckart's formulation which was derived by Eckart (Mamayev, 2010). The formulation is based on empirical fits to data in temperature and salinity ranges: $0\,^\circ\mathrm{C} < T < 40\,^\circ\mathrm{C}$ and $0\,\mathrm{ppt} < S < 40\,\mathrm{ppt}$. Using the experimental data, the following relation has been constructed

$$\rho = \frac{1000P_0}{1779.5 + 11.25T - 0.0745T^2 - (3.80 + 0.01T)S + 0.6980P_0}, \tag{2.4.11}$$

where $P_0 = 5890 + 38T - 0.3745T^2 + 3S$, T is the temperature, and S is the salinity. It is noticed that the equation above may depends also on the pressure but as its effect is very small in our study, it will be neglected.

For the case of fresh water where $S \approx 0\,\mathrm{ppt}$, Eckart's formulation does not give the maximum value of the water density at the temperature of $4\,^\circ\mathrm{C}$ (Mamayev,

2010) as expected. This is an important physical aspect when doing simulations of the thermocline formation in lakes or stagnant basins. Therefore, an alternative formulation is needed for these cases of study. A well known formulation is the UNESCO formulation, which is known as the international equation of state for seawater. This formulation is given by

$$\begin{aligned}
&\rho = \rho_0 + A\,S + B\,S^{3/2} + C\,S^2, \\
&\\
&\rho_0 = 999.842594 + (6.793952 \times 10^{-2})\,T - (9.095290 \times 10^{-3})\,T^2 + \\
&\qquad (1.001685 \times 10^{-4})\,T^3 - (1.120083 \times 10^{-6})\,T^4 + (6.536332 \times 10^{-9})\,T^5, \\
&A = 8.24493 \times 10^{-1} - (4.0899 \times 10^{-3})\,T + (7.6438 \times 10^{-5})\,T^2 - \\
&\qquad (8.2467 \times 10^{-7})\,T^3 + (5.3875 \times 10^{-9})\,T^4, \\
&B = -5.72466 \times 10^{-3} + (1.0227 \times 10^{-4})\,T - (1.6546 \times 10^{-6})\,T^2, \\
&C = 4.8314, \times 10^{-4}
\end{aligned} \tag{2.4.12}$$

where S is the salinity and T is the temperature. This formulation is only valid for the ranges $0\,^{\circ}\mathrm{C} < T < 40\,^{\circ}\mathrm{C}$, and $0.5\,\mathrm{ppt} < S < 43\,\mathrm{ppt}$.

As the problems in seawater are time-dependent, the time derivatives are needed to be evaluated. Since we are studying fluid flow motion, the time derivatives have to be a derivative following the motion, i.e material derivative. By applying the chain rule, Equation (2.4.10) can be rewritten in terms of differential form with respect to time as follows:

$$\frac{D\rho}{Dt} = \left(\frac{\partial \rho}{\partial p}\right)_{T,S} \frac{Dp}{Dt} + \left(\frac{\partial \rho}{\partial T}\right)_{p,S} \frac{DT}{Dt} + \left(\frac{\partial \rho}{\partial S}\right)_{T,p} \frac{DS}{Dt}. \tag{2.4.13}$$

Following Müller (2006), the following notations can be introduced to simplify Equation (2.4.13):

- $\kappa = \dfrac{1}{\rho}\left(\dfrac{\partial \rho}{\partial p}\right)_{T,S}$ is the coefficient of isothermal compressibility and has unit Pa^{-1}.

- $\alpha = -\dfrac{1}{\rho}\left(\dfrac{\partial \rho}{\partial T}\right)_{p,S}$ is the coefficient of thermal expansion and has unit K^{-1}.

- $\beta = \dfrac{1}{\rho}\left(\dfrac{\partial \rho}{\partial S}\right)_{(T,p)}$ is the coefficient of haline contraction and has unit ppt^{-1}.

Then by substituting these coefficients into Equation (2.4.13), we have the following:

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \kappa\frac{Dp}{Dt} - \alpha\frac{DT}{Dt} + \beta\frac{DS}{Dt}. \tag{2.4.14}$$

In Equation (2.4.14), the coefficients κ, α and β are first defined as thermodynamic properties because, each of these properties is defined as partial derivative of other thermodynamic properties. These coefficients can be also viewed as fluid properties whereas the time derivative is a flow properties. For instance, the isothermal compressibility κ indicates the changes in the fluid density which will take place when pressure changes while the temperature and the salinity remain constant. Such changes are then defined regardless of whether or not this fluid is flowing since it is a thermodynamic characteristic of that fluid. This is a property of the fluid.

2.5 Summary

In summary, we have developed from this chapter the kinematics of fluids where we have seen two essential methods of describing the fluid motion. Both the Lagrangian and Eulerian methods of description have been mathematically formulated and explained. In addition, some of the fundamental concepts of thermodynamics of seawater have been discussed in this chapter. The main objective of this chapter was to give the reader an overview of some of the backgrounds that may be used throughout the future chapters. The next chapter will be focused on the first objective of this thesis which is to show the conditions for which $\underline{\nabla} \cdot \underline{u} = 0$ is satisfied.

Chapter 3

The continuity equation with variable density

The main objective of this chapter is to derive the conditions for an incompressible flow where the fluid density ρ depends on the pressure, the temperature, and the salinity. In order to achieve this goal, the conservation of mass, salt, and energy equation are firstly derived in Section 3.1 and then a developed form of the continuity equation is deduced from these conservative equations. In Section 3.2, some conditions of incompressibility from the literature are reviewed and cited. And finally, Section 3.4 is devoted to the presentation of the scaling analysis which is applied to the modified form of the continuity equation in order to obtain plausible conditions of incompressibility.

3.1 Balance equations

This section will be devoted to the derivation of the conservation equations that will be used later. In this case, the conservation of mass is the first equation that will be derived.

3.1.1 Conservation of mass

Consider any arbitrary material volume V_m bounded by a closed surface A_m moving along the fluid trajectory and has mass $M = \int_{V_m} \rho \, dV$ (see in Figure 3.1). The conservation of mass implies that the mass of the volume V_m remains constant in time when it moves with velocity $\underline{u}$ with the flow.

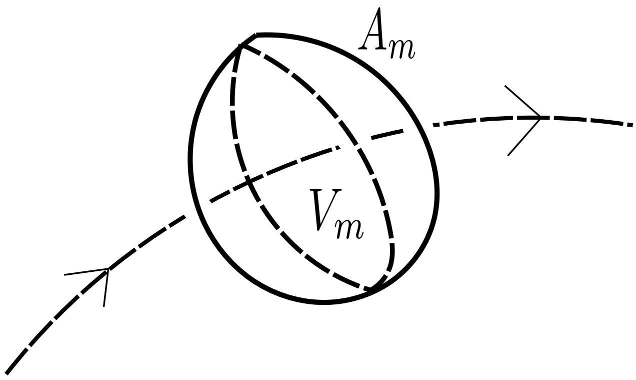

Figure 3.1: Material volume V_m

Mathematically, this can be presented as follows:

$$\frac{DM}{Dt} = \frac{D}{Dt}\int_{V_m} \rho dV = 0. \tag{3.1.1}$$

It is noticed that the material derivative is used here since we are studying the variation of the mass along the flow. Applying the Reynolds transport theorem (Whitaker, 1968), yields

$$\frac{D}{Dt}\int_{V_m} \rho dV = \int_{V_m} \frac{\partial \rho}{\partial t} dV + \oint_{A_m} \rho \underline{u} \cdot \underline{n}\, dA = 0, \tag{3.1.2}$$

where A_m is a closed surface surrounding the material volume V_m, and $\underline{n}$ is the outward-pointing normal to A_m. Applying the divergence theorem to the last term on the right hand side of Equation (3.1.2), the surface integral can be rewritten as a volume integral, then Equation (3.1.2) can be rewritten as follows:

$$\frac{D}{Dt}\int_{V_m} \rho dV = \int_{V_m} \frac{\partial \rho}{\partial t} dV + \int_{V_m} \underline{\nabla} \cdot (\rho \underline{u})\; dV \;=\; 0, \tag{3.1.3}$$

$$= \int_{V_m} \left(\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) \right) dV \;=\; 0. \tag{3.1.4}$$

Equation (3.1.4) can be interpreted that the changes in time of ρ within V_m are only due to the advective flux or transport flux $\rho \underline{u}$. After doing the manipulations as follows:

$$\int_{V_m} \left(\frac{\partial \rho}{\partial t} + \rho \underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla} \rho \right) dV \;=\; 0 \tag{3.1.5}$$

Using the definition of the material derivative yields:

$$\int_{V_m} \left[\frac{D\rho}{Dt} + \rho\, \underline{\nabla} \cdot \underline{u} \right] dV = 0. \tag{3.1.6}$$

Since Equation (3.1.6) is always true for any arbitrary volume V_m (Whitaker, 1968), the integrand must be zero. Therefore, the continuity equation is deduced and given by

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \underline{\nabla} \cdot \underline{u} = 0. \tag{3.1.7}$$

Furthermore, as the explicit expression of the time derivatives of temperature and salt are needed, the following paragraph will be devoted to these expressions.

3.1.2 Conservation of salt

Contrary to the previous statement, let us now consider the case where V_s is fixed in space and investigate the the rage of change of the salt density, denoted by ρS within V_s. According to Olbers et al. (2012), the rate of change of ρS within V_s can be defined by its transport through the surface A_s, and from the interior sources and sinks. The first is characterized by an outward transport of ρS through a surface element dA and by a non-advective vector flux $\underline{J}_S$ which may occur by diffusion of salt. Then, the total transport across A_s is given by

$$\oint_{A_s} (\rho S\underline{u} + \underline{J}_S) \cdot \underline{n}\, dA, \tag{3.1.8}$$

where $\rho S\underline{u}$ is the advective flux vector or transport flux vector, and $\underline{n}$ is an outward normal vector to A_s (Olbers et al., 2012). The second term is defined as the change due to the internal sources or sinks of ρS which may be salt sources or chemical reactions. Let Q_S be the combination of the internal sources or sinks within V_s which is expressed by

$$\int_{V_s} Q_s dV \tag{3.1.9}$$

As Equations (3.1.8) and (3.1.9) contribute to the change of ρS in time, the conservation equation of salt is then given by

$$\frac{\partial}{\partial t}\int_{V_s}(\rho S)dV + \oint_{A_s}(\rho S\underline{u} + \underline{J}_S)\cdot \underline{n} dA + \int_{V_s} Q_S dV = 0. \tag{3.1.10}$$

Since V_s is fixed in space, the first term in (3.1.10) can be rewritten as

$$\frac{\partial}{\partial t}\int_{V_s}(\rho S)dV = \int_{V_s}\frac{\partial}{\partial t}(\rho S)dV. \tag{3.1.11}$$

In addition, applying the divergence theorem to the second term in (3.1.10), yields

$$\int_{V_s}\left[\frac{\partial}{\partial t}(\rho S)dV + \underline{\nabla}\cdot(\rho S\underline{u} + \underline{J}_S) + Q_S\right] dV = 0, \tag{3.1.12}$$

where $\underline{J}_S$ is the vector salt flux, and Q_S is combination of all sources and sinks. Since Equation (3.1.12) is always true for any arbitrary fixed volume V_s, we have

$$\frac{\partial}{\partial t}(\rho S) + \underline{\nabla} \cdot (\rho S \underline{u} + \underline{J}_S) + Q_S = 0. \tag{3.1.13}$$

Applying the same rearrangement as we have done in Section 3.1.1, for Equation (3.1.6) and (3.1.7), it follows:

$$\frac{D}{Dt}(\rho S) + \rho S \underline{\nabla} \cdot \underline{u} = -\underline{\nabla} \cdot \underline{J}_S + Q_S. \tag{3.1.14}$$

Furthermore, Fick's law of diffusion for salt is given by

$$\underline{J}_S = -\rho k_S \underline{\nabla} S, \tag{3.1.15}$$

where k_S is the coefficient of diffusion (Bird et al., 2007). It is noticed that the origin of the negative sign in the Fick's law occurs since in the case of diffusion, the salt can disperse from higher to lower concentrations. Therefore, Equation (3.1.14) may be rearranged and rewritten as follows:

$$\rho \frac{DS}{Dt} = \underline{\nabla} \cdot (\rho k_S \underline{\nabla} S) + Q_S. \tag{3.1.16}$$

The latter equation is the general form of the conservation equation of salt. In addition, if we assume that all the combination of sources and sinks are negligible, the conservation equation of salt can be rewritten in the simplified form

$$\rho \frac{DS}{Dt} = \underline{\nabla} \cdot (\rho k_S \underline{\nabla} S). \tag{3.1.17}$$

Equation (3.1.17) is the equation which will be used later in the differential form of the continuity equation.

3.1.3 Conservation of energy (Heat)

It has been mentioned in the previous chapter, Subsection 2.4.1 that the transfer of energy can be interpreted as a transfer of heat. In a similar way to the previous subsection for the conservation equation of salt, let us investigate the change of heat denoted by, $\rho c_p T$, in time within the fixed volume V_s. It is well known that the heat can be transported by advection, diffusion and radiation within a fluid. Therefore, the total of the transport of heat through the surface A_s of the volume can be expressed as follows:

$$\oint_{A_m} (\rho c_p T \underline{u} + \underline{J}_T) \cdot \underline{n} dA, \tag{3.1.18}$$

where $\rho c_p T \underline{u}$ is the advective heat flux vector, and $\underline{J_T}$ is the diffusive heat flux vector (Olbers et al., 2012). In addition, if we take into a consideration the sources and sinks denoted by Q_T within the volume V_s, we have

$$\int_{V_s} Q_T dV. \tag{3.1.19}$$

As the change in time of $\rho c_p T$ within V_s is due to Equations (3.1.18) and (3.1.19), we have the following relationship:

$$\frac{\partial}{\partial t} \int_{V_s} \rho c_p T dV = - \oint_{A_m} (\rho c_p T \underline{u} + \underline{J_T}) \cdot \underline{n} dA + \int_{V_s} Q_T dV. \tag{3.1.20}$$

Considering that the volume V_s is fixed and using the same method as we have done for Equation (3.1.12), yields

$$\int_{V_s} \left(\frac{\partial}{\partial t} \rho c_p T + \underline{\nabla} \cdot (\rho c_p T \underline{u}) + \underline{\nabla} \cdot \underline{J_T} + Q_T \right) dV = 0. \tag{3.1.21}$$

Since again Equation (3.1.21) is valid for any arbitrary volume V_s then

$$\frac{\partial}{\partial t} (\rho c_p T) + \underline{\nabla} \cdot (\rho c_p T \underline{u}) + \underline{\nabla} \cdot \underline{J_T} + Q_T = 0. \tag{3.1.22}$$

Rearranging Equation (3.1.22) in the same way as we have done for Equation (3.1.12), yields

$$\frac{D}{Dt} (\rho c_p T) + \rho c_p T (\underline{\nabla} \cdot \underline{u}) + \underline{\nabla} \cdot \underline{J_T} + Q_T = 0. \tag{3.1.23}$$

Moreover, according to LeBlond and Mysak (1981), the conductive heat flux vector is given by

$$\underline{J_T} = -k_T \underline{\nabla} T, \tag{3.1.24}$$

where k_T is a heat conductivity. Using the assumption that sources and sinks of heat are negligible. Equation (3.1.23) can be rearrange as follows:

$$c_p T \left(\frac{D\rho}{Dt} + \rho (\underline{\nabla} \cdot \underline{u}) \right) + \rho \frac{D}{Dt} (c_p T) - \underline{\nabla} \cdot (k_T \underline{\nabla} T) = 0. \tag{3.1.25}$$

Using the continuity equation, the latter can be simplified as

$$\rho c_p \frac{DT}{Dt} = \underline{\nabla} \cdot (k_T \underline{\nabla} T). \tag{3.1.26}$$

The equation above is defined as a simplified form of the conservation equation of heat flux expressed in terms of variation in time of the temperature.

Furthermore, a more general equation of the variation in time of temperature can be deduced from the fundamental thermodynamic relation which is given by

$$de = T d\eta - p\, d \left(\frac{1}{\rho} \right) + \mu dS, \tag{3.1.27}$$

where e is the specific internal energy, η is the specific entropy, and μ is the chemical potential. Equation (3.1.27) can be physically interpreted as the changes in internal energy de is due to the amount of heat $Td\eta$, the work done on the body $-pd\left(\frac{1}{\rho}\right)$ and the chemical work due to the salinity μdS. In particular, according to Pedlosky (2013), we have

$$\frac{De}{Dt} = T\frac{D\eta}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right) + \mu\frac{DS}{Dt}. \tag{3.1.28}$$

The specific entropy η can be a function of pressure p, temperature T and salinity S. Then

$$\eta = \eta\left(p, T, S\right), \tag{3.1.29}$$

which can be rewritten in differential form in time as follows:

$$\frac{D\eta}{Dt} = \left(\frac{\partial\eta}{\partial p}\right)_{T,S}\frac{Dp}{Dt} + \left(\frac{\partial\eta}{\partial T}\right)_{p,S}\frac{DT}{Dt} + \left(\frac{\partial\eta}{\partial S}\right)_{T,p}\frac{DS}{Dt}. \tag{3.1.30}$$

Therefore, from Equation (3.1.30), we can deduce the expression of $\frac{D\eta}{Dt}$ in terms of the time derivative of p, T and S. In this case, we need to evaluate the coefficients of Equation (3.1.30) using some thermodynamics relations. Let us first start to evaluate $\left(\frac{\partial\eta}{\partial p}\right)_{T,S}$. From the Maxwell relation which, can be found in Moran et al. (2010), we have

$$\left(\frac{\partial\eta}{\partial p}\right)_{T,S} = -\left(\frac{\partial(\frac{1}{\rho})}{\partial T}\right)_{p,S} = \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_{p,S} = -\frac{\alpha}{\rho}, \tag{3.1.31}$$

where α is the coefficient of thermal expansion. Moreover, from Chapter 2, Section 2.4, we have

$$c_p = T\left(\frac{\partial\eta}{\partial T}\right)_{p,S},$$

and from Vallis (2017), we have

$$\left(\frac{\partial\eta}{\partial S}\right)_{T,p} = -\frac{\mu}{T}.$$

As a consequence of the thermodynamic relations above, Equation (3.1.28) can be rewritten as

$$\frac{D\eta}{Dt} = -\frac{\alpha}{\rho}\frac{Dp}{Dt} + \frac{c_p}{T}\frac{DT}{Dt} - \frac{\mu}{T}\frac{DS}{Dt}. \tag{3.1.32}$$

Now combining Equations (3.1.28) and (3.1.32) times temperature T, yields

$$\frac{De}{Dt} = -\frac{\alpha}{\rho}T\frac{Dp}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right) + c_p\frac{DT}{Dt}. \tag{3.1.33}$$

In addition, according to Pedlosky (2013), the changes in internal energy are linked to the first law of thermodynamics which is expressed by

$$\rho\frac{De}{Dt} = -p\rho\frac{D}{Dt}\left(\frac{1}{\rho}\right) + \underline{\nabla}\cdot\left(k_T\underline{\nabla}T\right) + \Phi + \rho Q_T, \tag{3.1.34}$$

where k_T is the thermal conductivity, Q_T is the combination of all sources or sinks, Φ is the function of viscous dissipation, p is the pressure, and α is coefficient of thermal expansion. Substituting Equation (3.1.34) into (3.1.33), yields

$$\rho c_p \frac{DT}{Dt} = \alpha T \frac{Dp}{Dt} + \underline{\nabla} \cdot (k_T \underline{\nabla} T) + \Phi + \rho Q_T. \tag{3.1.35}$$

In the case where it is assumed that there are no sources and sinks of heat, as well as neglecting the viscous dissipation function, $Q_T = 0$ and $\Phi = 0$. Therefore, by the conservation of energy, we have

$$\rho c_p \frac{DT}{Dt} = \underline{\nabla} \cdot (k_T \underline{\nabla} T) + \alpha T \frac{Dp}{Dt}. \tag{3.1.36}$$

The latter equation is the same as we have derived from Equation (3.1.26) for the conservation of heat, except that it contains the rate of change in time of the pressure. In the case where the diffusion of temperature is negligible, Equation (3.1.36) can be simplified to

$$\begin{aligned} \rho c_p \frac{DT}{Dt} &= \alpha T \frac{Dp}{Dt}, \\ \frac{DT}{Dt} &= \frac{\alpha T}{\rho c_p} \frac{Dp}{Dt}. \end{aligned} \tag{3.1.37}$$

Notice that from Müller (2006), the expression $\dfrac{\alpha T}{\rho c_p}$ is referred to as the adiabatic lapse rate which is denoted by Γ. This implies that Equation (3.1.37) can be rewritten as

$$\frac{DT}{Dt} = \Gamma \frac{Dp}{Dt}. \tag{3.1.38}$$

The latter equation states how changes in time of temperature will affect the pressure and *vice-versa*. However, a further investigation is needed to understand it. Equation (3.1.36) will be used in the continuity equation in order to find the condition of incompressibility of the flow.

3.2 Incompressible flow as defined in the literature

The notion of incompressibility is described in many sources but converges in one idea. The following two first concepts define the condition of incompressibility for a general case of fluid motion, whereas the last is related to the ocean modelling.

3.2.1 Concept 1 (Bachelor, 1967)

According to Bachelor (1967), incompressible flow has it source from the motion of an incompressible fluid. This implies that the concept of incompressible flow is related to the characteristics of the incompressible fluid. As it has been stated at Bachelor

(1967), a fluid is defined to be incompressible when its density does not affected by the changes in pressure. For example, gases may be considered to be incompressible if the variations of the pressure on it are very small. In addition, it has been pointed out that the fluid density may change with respect to the molecular diffusion of heat or solute. However, for the motion of an incompressible fluid, the effects of all different types of diffusions are negligible. As a result of these assumptions, Bachelor (1967) stated that for an incompressible fluid

$$\frac{D\rho}{Dt} = 0, \tag{3.2.1}$$

and the mass conservation becomes

$$\underline{\nabla} \cdot \underline{u} = 0. \tag{3.2.2}$$

Therefore, it is usually taken for flow of an incompressible fluid $\underline{\nabla} \cdot \underline{u} = 0$ (Bachelor, 1967).

3.2.2 Concept 2 (Panton, 2013)

Panton (2013) pointed out in an explicit way the difference between incompressible flow and an incompressible fluid. He stated that incompressible flow is a fluid mechanics term whereas incompressible fluid is a thermodynamic term. According to Panton (2013), the main condition for a flow to be called incompressible is that the Mach number tends to zero (M $\to 0$). In addition, the term incompressible flow means that the changes in time of the density of a fluid particle are negligible, and it is mathematically defined as

$$\frac{1}{\rho}\frac{D\rho}{Dt} = 0, \tag{3.2.3}$$

and the mass conservation becomes

$$\underline{\nabla} \cdot \underline{u} = 0. \tag{3.2.4}$$

3.2.3 Concept 3 (LeBlond and Mysak, 1981)

Combining equation (2.4.13) from Section 2.4.5 and Equation (3.1.7), it follows that

$$\left(\frac{\partial\rho}{\partial p}\right)_{T,S}\frac{Dp}{Dt} + \left(\frac{\partial\rho}{\partial T}\right)_{p,S}\frac{DT}{Dt} + \left(\frac{\partial\rho}{\partial S}\right)_{T,p}\frac{DS}{Dt} = \rho\underline{\nabla}\cdot\underline{u}. \tag{3.2.5}$$

In the ocean, by borrowing an idea from LeBlond and Mysak (1981), incompressible condition means local sound velocity c in the ocean is infinite. Since, from Subsection 2.4.4, Equation (2.4.7), we have

$$\left(\frac{\partial\rho}{\partial p}\right)_{\eta} = \frac{1}{c^2}. \tag{3.2.6}$$

Then incompressibility can be interpreted as

$$\left(\frac{\partial \rho}{\partial p}\right)_{\eta} = 0, \tag{3.2.7}$$

where η is the specific entropy. According to Müller (2006), we have a similar expression to Equation (3.2.6) in terms of temperature T and salinity S which is given by

$$\left(\frac{\partial \rho}{\partial p}\right)_{T,S} = \frac{c_p}{c_v}\frac{1}{c^2}, \tag{3.2.8}$$

where c_p and c_v are the specific heat capacity at constant pressure and constant volume, respectively. Furthermore, LeBlond and Mysak (1981) stated that if the ocean is assumed to be nondiffusive $k_T = k_S = 0$ then

$$\frac{DS}{Dt} = 0 \qquad \text{and} \qquad \frac{DT}{Dt} = 0. \tag{3.2.9}$$

As a consequence of these assumptions, LeBlond and Mysak (1981) stated that for an incompressible flow related to the ocean, neither changes in density with respect to pressure nor diffusion of salt and temperature take place, then the fluid density does not change along the flow trajectory:

$$\frac{D\rho}{Dt} = 0, \tag{3.2.10}$$

and the continuity equation can be reduced to

$$\underline{\nabla} \cdot \underline{u} = 0.$$

It can be seen from the three concepts above that the main idea of incompressibility of flow is that the fluid density remains constant along the trajectory of the flow. While for an incompressible fluid its density does not change with respect to the variation of pressure. Therefore, incompressible flow implies that the divergence of the flow velocity is zero and for an incompressible fluid, the local sound velocity is infinite.

The next section will be devoted to derive these conditions of incompressibility using the nondimensionalization technique.

3.3 Dimensionless numbers

In this paragraph, some useful dimensionless numbers will be defined. A dimensionless number is often used to classify the flow regimes, or, more generally, hydrodynamic processes.

3.3.1 Reynolds number

The Reynolds number is a dimensionless number which is defined as the ratio of the inertial forces to the viscous forces acting on a fluid (Bachelor, 1967). In addition, it was mentioned in the previous chapter, Section 2.3, that the Reynolds number is used to characterise if the flow is turbulent or laminar. Using the definition of the Reynolds number, its expression is given by

$$\mathrm{Re} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}} = \frac{\rho U L}{\mu}, \tag{3.3.1}$$

where ρ is the fluid density, L is the characteristic length of the domain, U is the flow velocity and μ is the dynamic viscosity.

3.3.2 Prandtl number

In the previous section, Subsection 3.1.3, it has been stated that heat can be transported by diffusion. The Prandtl number is a dimensionless quantity that relates the rate of diffusion of momentum compared to the rate of diffusion of heat for a particular fluid (Panton, 2013). Its expression is given by

$$\mathrm{Pr} = \frac{\text{Viscous diffusion rate}}{\text{Thermal diffusion rate}} = \frac{c_p \mu}{k_T}, \tag{3.3.2}$$

where c_p is defined as the specific heat capacity, k_T is the heat conductivity, μ is the dynamic viscosity.

3.3.3 Peclet number

The Peclet number is defined as the ratio of advection to the diffusion transport of matter or thermodynamic quantities of the fluid flow (Bird et al., 2007). It is therefore a dimensionless number which measures the strength of advection relative to the diffusion. For instance, for a salt transport, its expression is given by

$$\mathrm{Pe} = \frac{\text{Advective transport of salt}}{\text{Diffusive transport of salt}} = \frac{LU}{k_S}, \tag{3.3.3}$$

where k_S is defined as the diffusion coefficient of salt, L is the characteristic length of the domain, U is a characteristic flow velocity. Furthermore, as the heat flux can also be transported by advection, the Peclet number can be rewritten as follows:

$$\mathrm{Pe}_h = \mathrm{Pr} \cdot \mathrm{Re} = \frac{\rho c_p L U}{k_T}. \tag{3.3.4}$$

3.3.4 Mach number

As mentioned in the previous chapter, the Mach number is a dimensionless number defined as the ratio of the flow velocity to the velocity of sound in the fluid. Its expression is given by

$$\mathrm{M} = \frac{\text{Fluid velocity}}{\text{Sound velocity}} = \frac{U}{c}, \tag{3.3.5}$$

where c is the local velocity of sound in the fluid.

3.4 Conditions for incompressible flow using scaling analysis

3.4.1 Overview of the technique

In this chapter, the technique of scaling analysis is briefly illustrated through a simple physical process. According to Krantz (2007), *scaling analysis* involves a structured method in order to transform a quantity into a dimensionless form. This quantity may be defined as a dependent or an independent variable as well as its derivative in a set of governing equations for a physical process. After applying a scaling analysis to the set of governing equations, it turns out that all the measured quantities are removed and the set of dimensionless equations obtained are represented in a parametric representation. Such a representation permits us to compare various terms of the set of equations by order of magnitude. In this case, the set of dimensionless equations are only functions of the dimensionless variables as well as the dimensionless groups.

In order to illustrate the result from the scaling analysis for an arbitrary physical problem, let us consider an initial value problem of a simple mechanical vibration which is given by

$$m\frac{d^2x}{dt^2} + k\,x = 0; \qquad x(t_0 = 0) = L; \quad \left.\frac{dx}{dt}\right|_{t_0=0} = U, \tag{3.4.1}$$

where m defines the mass of an object, k defines the spring constant, and t_0 is the initial time. Without going into details, applying the nondimensionalization technique to Equation (3.4.1), yields

$$\frac{d^2x^*}{dt^{*2}} + x^* = 0; \qquad x^*(t_0 = 0) = 1; \quad \left.\frac{dx^*}{dt^*}\right|_{t_0=0} = \frac{U}{L}\sqrt{\frac{m}{k}}. \tag{3.4.2}$$

The equation above is the dimensionless form of Equation (3.4.1), and the variables with asterisks are dimensionless. Concerning the dimensionless variables, they are defined from the ratio of a dimensional variable to a constant expression which has the same dimensions as the variable (Panton, 2013). This implies that all of the dimensionless variables and their derivatives are of order one, $O(1)$. This means that in terms of magnitude, a particular dimensionless variable and its derivative are bounded between zero and approximately one (Krantz, 2007). For instance, the dimensionless variables that have been used to scale Equation (3.4.1) are defined as follows:

$$t^* = \frac{t}{\sqrt{\frac{m}{k}}}, \qquad x^* = \frac{x}{L},$$

where $\sqrt{\frac{m}{k}}$ is called the time scale and has the dimension time, and L is the characteristic length and has the dimension length. Moreover, the dimensionless groups can be thought of as a combination of the scale variables resulting from the nondimensionalization of the set of equations. The dimensionless groups can also be obtained from the combination of different variables using a dimensional analysis (Bachelor, 1967). For instance, different types of dimensionless groups have been introduced in the Section 3.3 as dimensionless numbers. One example of these dimensional groups is obtained here from Equation (3.4.2) and given by

$$\left.\frac{dx^*}{dt^*}\right|_{t_0=0} = \frac{U}{L}\sqrt{\frac{m}{k}}.$$

To summarise, the technique of scaling analysis is very useful to simplify a complex dimensional equation. The resulting coefficients which are contained in the dimensionless equation will then be used to compare which of the terms in the equations will be great or small with respect to each other. Therefore, the dimensionless equation can be reduced by removing its terms which have the smallest coefficients.

3.4.2 Stepwise procedure for the technique

Several processes are involved in the technique of scaling analysis, but only the essential points will be presented in terms of steps. In order to illustrate these steps, a simple example from Krantz (2007) will be taken. It consists of a steady state heat conductive problem.

Consider the steady state heat conduction in a homogeneous solid having a rectangular form and with constant physical properties. The solid has a length L in the x-direction and height H in the y-direction, and is infinite in the z-direction, as illustrated in Figure 3.2. In the figure $H < L$. The aim is to determine for which condition the lateral heat conduction can be ignored and the problem can be approximated as one-dimensional heat conduction in the y-direction.

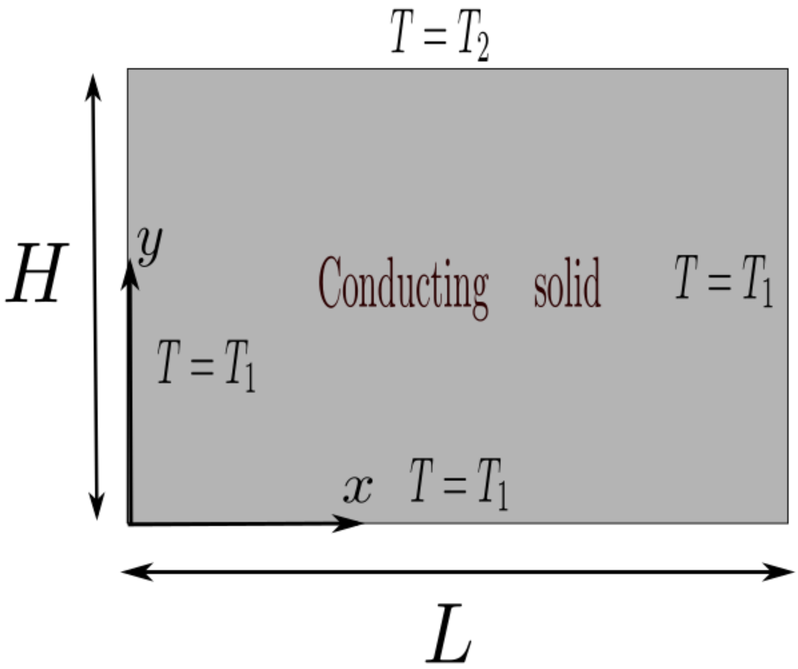

Figure 3.2: Steady two-dimensional heat conduction in a homogeneous solid

Step 1:

The first step in scaling analysis is to define or to introduce the set of governing equations with initial and boundary conditions (Krantz, 2007). In our case, we have the heat conduction equation which is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{3.4.3}$$

$$T = T_1 \quad \text{at} \quad x = 0; \qquad T = T_1 \quad \text{at} \quad y = 0 \tag{3.4.4}$$

$$T = T_1 \quad \text{at} \quad x = L; \qquad T = T_2 \quad \text{at} \quad y = H \tag{3.4.5}$$

Step 2:

The second step involves the introduction of the unspecified dimensionless variables for all dependent and independent variables, and their derivatives. Basically, they are defined as the ratio of the dimensional variable, q, relative to the unspecified reference variable, q_r, to the unspecified scale variable, q_s. Such expression can be presented as follows:

$$q^* \equiv \frac{q - q_r}{q_s}, \tag{3.4.6}$$

where q^* is the dimensionless variable. According to Panton (2013), the reference variable q_r is defined as the absolute value of the variable important to the problem. For instance, for a heat transfer, the differences in temperature are important; therefore, the reference should be a value at some point in the domain. In general, it is usually taken as zero for the other variables. On the other hand, the scale variable q_s is often used to measure or compare the changes of the dimensional variable (Panton, 2013). In our case, since the temperature is not usually referenced to zero at any of the boundaries,

$$T^* = \frac{T - T_r}{T_s}; \qquad y^* = \frac{y}{y_s}; \qquad x^* = \frac{x}{x_s} \tag{3.4.7}$$

Step 3:

Substituting the expression the of dimensional variables from Equation (3.4.7) into Equations (3.4.3) to (3.4.5) yields

$$\frac{y_s^2}{x_s^2}\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} = 0 \tag{3.4.8}$$

$$T^* = \frac{T_1 - T_r}{T_s} \quad \text{at} \quad x_s x^* = 0; \qquad T^* = \frac{T_1 - T_r}{T_s} \quad \text{at} \quad y_s y^* = 0 \tag{3.4.9}$$

$$T^* = \frac{T_1 - T_r}{T_s} \quad \text{at} \quad x_s x^* = L; \qquad T^* = \frac{T_2 - T_r}{T_s} \quad \text{at} \quad y_s y^* = H. \tag{3.4.10}$$

Step 4:

In this step, the unspecified reference and scale variables will be determined, and this may be done using the given boundary or initial conditions. In addition, it has been stated from the previous subsection that the dimensionless variables should be bounded between zero and approximately one. Therefore, we have from the second member on the right hand side of Equation (3.4.9)

$$T^* = 0 \quad \Rightarrow \quad \frac{T_1 - T_r}{T_s} = 0 \quad \Rightarrow \quad T_r = T_1, \tag{3.4.11}$$

and in the same way, from Equation (3.4.10), we have

$$T^* = 1 \quad \Rightarrow \quad \frac{T_2 - T_1}{T_s} = 1 \quad \Rightarrow \quad T_s = T_2 - T_1. \tag{3.4.12}$$

Moreover, we have

$$\frac{x_s}{L} = 1 \quad \Rightarrow \quad x_s = L \qquad \text{and} \qquad \frac{y_s}{H} = 1 \quad \Rightarrow \quad y_s = H. \tag{3.4.13}$$

Step 5:

As a consequence of the previous step, this step consists of rewriting the dimensionless equations in terms of the new scale and the reference variables. In our case we have, using the expression of x_s and y_s from Equation (3.4.13) into (3.4.8) yields

$$\frac{H^2}{L^2}\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} = 0. \tag{3.4.14}$$

By substituting the expression of T_r and T_s from Equation (3.4.11) and (3.4.12) into (3.4.9) and (3.4.10) gives

$$T^* = 0 \quad \text{at} \quad Lx^* = 0; \qquad T^* = 0 \quad \text{at} \quad Hy^* = 0 \tag{3.4.15}$$

$$T^* = 0 \quad \text{at} \quad Lx^* = L; \qquad T^* = 1 \quad \text{at} \quad Hy^* = H, \tag{3.4.16}$$

which can be condensed as

$$T^*(0, y^*) = T^*(x^*, 0) = T^*(1, y^*) = 0 \quad \text{and} \quad T^*(x^*, 1) = 1. \tag{3.4.17}$$

Step 6:

This final step is about the investigation of the dimensionless equation that we have previously derived. A reduction of the equation may occur by neglecting some terms which are considered to be very small compared to each other. In this case, Equation (3.4.14) indicates that its first term will drop out from the governing equations if the following condition is satisfied

$$\frac{H^2}{L^2} \lll 1. \tag{3.4.18}$$

It must be noted that this condition is in terms of dimensionless group but has its physical significance which is the ratio of the magnitude of the lateral heat conduction to the vertical heat conduction. Then, Equation (3.4.14) is reduced to

$$\frac{\partial^2 T^*}{\partial y^{*2}} = 0, \tag{3.4.19}$$

and have a solution as

$$T^* = y^* \tag{3.4.20}$$

This result can be interpreted that the effect of the lateral heat-conduction can be ignored if the criterion in Equation (3.4.18) is satisfied. More details about the particular example can be found in Krantz (2007). The main objective of this subsection is to give an overview of the structured method of the scaling analysis. In the following subsection this method will be used to nondimensionalize the continuity equation.

3.4.3 Nondimensionalization of the continuity equation

In order to find all the plausible conditions to quantify the incompressibility of a flow, the technique of nondimensionalization will be applied to the continuity equation. The purpose is then to find under which conditions $\dfrac{D\rho}{Dt} = 0$. By doing so, all the steps that have been mentioned above will be presented as follows:

Step 1: This consists of writing the dimensional equation in the same way as it has been done in the previous subsection. Here, the governing equation is the continuity equation. From the previous chapter, Equation (2.4.14), we have

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \kappa\frac{Dp}{Dt} - \alpha\frac{DT}{Dt} + \beta\frac{DS}{Dt}, \tag{3.4.21}$$

where κ is the isothermal compressibility, α is the thermal expansion and β is the haline contraction. Therefore, the continuity equation becomes:

$$\kappa\frac{Dp}{Dt} - \alpha\frac{DT}{Dt} + \beta\frac{DS}{Dt} = -\underline{\nabla}\cdot\underline{u}. \tag{3.4.22}$$

Substituting each material derivative of the state variables by its expression which has been derived previously from Section 3.1, Equation (3.4.22) yields

$$\kappa\frac{Dp}{Dt} - \frac{\alpha}{\rho\, c_p}\left(k_T\underline{\nabla}^2 T + \alpha T\frac{Dp}{Dt}\right) + \frac{\beta k_S}{\rho}\underline{\nabla}\cdot(\rho\,\underline{\nabla}S) = -\underline{\nabla}\cdot\underline{u}, \tag{3.4.23}$$

where k_T and k_S are the heat conductivity and the salt diffusivity, respectively. They are taken as constant. In terms of spatial geometry, we consider the fact that the distance in the x, y-plane and the z-axis may be not the same. Then, the vector gradient $\underline{\nabla}$ is split into two parts, as horizontal $\underline{\nabla}_h$ and vertical $\dfrac{\partial}{\partial z}\underline{k}$. Therefore, Equation (3.4.23) can be rewritten as follows:

$$\begin{aligned}\left(\kappa - \frac{\alpha^2 T}{\rho\, c_p}\right)\left(\frac{\partial p}{\partial t} + \underline{u}_{\mathrm{h}}\cdot\underline{\nabla}_{\mathrm{h}}p + w\frac{\partial p}{\partial z}\right) - \frac{\alpha}{\rho\, c_p}k_T\left(\underline{\nabla}_{\mathrm{h}}^2 T + \frac{\partial^2 T}{\partial z^2}\right) + \\ \frac{\beta k_S}{\rho}\underline{\nabla}_{\mathrm{h}}\cdot(\rho\underline{\nabla}_{\mathrm{h}}S) + \frac{\beta k_S}{\rho}\frac{\partial}{\partial z}\left(\rho\frac{\partial S}{\partial z}\right) = -\underline{\nabla}\cdot\underline{u},\end{aligned} \tag{3.4.24}$$

where $\underline{\nabla}_{\mathrm{h}}$ is the horizontal gradient and $\underline{u}_h$ is the horizontal component of the vector velocity.

Step 2: This step consists of forming the dimensionless variables for the dependent and independent variables as well as their derivatives. In the same way as we have seen in the previous subsection, the reference variables are needed to be taken into account. But here, all the reference variables are naturally taken to be equal to zero. For the velocities and spatial coordinates this assumption is straightforward, but for the other variables, it is set to zero to simplify the problem. The dimensionless variables are then expressed as follows:

$$
\begin{aligned}
&x^* \equiv \frac{x}{x_s}; \qquad y^* \equiv \frac{y}{y_s}; \qquad z^* \equiv \frac{z}{z_s}; \qquad t^* \equiv \frac{t}{t_s}; \qquad p^* \equiv \frac{p}{p_s};\\
&T^* \equiv \frac{T}{T_s}; \quad S^* \equiv \frac{S}{S_s}; \quad u^* \equiv \frac{u}{u_s}; \qquad v^* \equiv \frac{v}{v_s}; \quad w^* \equiv \frac{w}{w_s};\\
&\frac{\partial^*}{\partial t^*} \equiv \frac{1}{t_s}\frac{\partial}{\partial t}; \qquad \frac{\partial^*}{\partial x^*} \equiv \frac{1}{x_s}\frac{\partial}{\partial x}; \qquad \frac{\partial^*}{\partial y^*} \equiv \frac{1}{y_s}\frac{\partial}{\partial y}; \qquad \frac{\partial^*}{\partial z^*} \equiv \frac{1}{z_s}\frac{\partial}{\partial z};\\
&\kappa^* \equiv \frac{\kappa}{\kappa_s}; \qquad {c_p}^* \equiv \frac{c_p}{{c_p}_s}; \qquad \beta^* \equiv \frac{\beta}{\beta_s}; \qquad \rho^* \equiv \frac{\rho}{\rho_s}; \qquad \alpha^* \equiv \frac{\alpha}{\alpha_s}.
\end{aligned}
\tag{3.4.25}
$$

Step 3: When all the dimensionless variables are presented, in this step, all the variables in Equation (3.4.24) will be substituted by their corresponding dimensionless variables. In this case, Equation (3.4.24) becomes

$$
\begin{aligned}
&\left(\kappa_s\kappa^* - \frac{\alpha_s^2{\alpha^*}^2\,T_s}{{c_p}_s{c_p}^*\,\rho_s}\frac{T^*}{\rho^*}\right)\left(\frac{p_s}{t_s}\frac{\partial p^*}{\partial t^*} + \frac{p_s v_s}{x_s}u^*\frac{\partial p^*}{\partial x^*} + \frac{p_s v_s}{y_s}v^*\frac{\partial p^*}{\partial y^*} + \frac{p_s w_s}{z_s}w^*\frac{\partial p^*}{\partial z^*}\right) -\\
&\frac{\alpha_s\alpha^*\,k_T}{\rho_s\,{c_p}_s\,\rho^* c_p^*}\left(\frac{T_s}{x_s^2}\frac{\partial^2 T^*}{\partial {x^*}^2} + \frac{T_s}{y_s^2}\frac{\partial^2 T^*}{\partial {y^*}^2} + \frac{T_s}{z_s^2}\frac{\partial^2 T^*}{\partial {z^*}^2}\right) + \beta_s\rho_s\beta^* k_S\left(\frac{S_s}{x_s^2}\frac{\partial}{\partial x^*}\left(\rho^*\frac{\partial S^*}{\partial x^*}\right) +\right.\\
&\left.\frac{S_s}{y_s^2}\frac{\partial}{\partial y^*}\left(\rho^*\frac{\partial S^*}{\partial y^*}\right) + \frac{S_s}{z_s^2}\frac{\partial}{\partial z^*}\left(\rho^*\frac{\partial S^*}{\partial z^*}\right)\right) = -\left(\frac{v_s}{x_s}\frac{\partial u^*}{\partial x^*} + \frac{v_s}{x_s}\frac{\partial v^*}{\partial y^*} + \frac{w_s}{z_s}\frac{\partial w^*}{\partial z^*}\right).
\end{aligned}
\tag{3.4.26}
$$

Step 4: Since all the reference variables are already set to zero, it now remains to determine all the scales. In order to find these scales, it has been seen from the previous subsection that initial and boundary conditions are needed. In this case such conditions are unspecified, however it is our interest to study different types of physical processes which occur in the sea or ocean. As a consequence of that, the flows which our study are related involves the flow problems in the ocean [1]. For instance, the flow in a coastal estuary or an open channel flows. With the fact that the initial and the boundary conditions are not given, the scales which are well used to scale the Navier-Stokes equations will directly be given and used. It is expected that such scales will work for different types of flow related to the ocean. We then have the following scales for each variable:

- The characteristic vertical and horizontal length scale are defined by H and L.

[1] Large scale domain with the approximation that the turbulence phenomena are neglected.

- The horizontal velocity scale is denoted by U in the horizontal, and W for the vertical.
- The characteristic time scale is characterized by an advective time scale given by $\dfrac{L}{U}$.
- Moreover, the pressure scale is given by $\rho_0 U^2$, and for the temperature and the salinity, the scales are defined as the difference of temperature and salinity in the seawater denoted by ΔT and ΔS (Dijkstra, 2008).
- And finally, for the thermodynamic properties, the scales are just taken as constant values with subscript 0. This implies that the density scale is a constant density ρ_0, for κ, it is κ_0 and so on (Panton, 2013).

Step 5: Applying these scales variables to Equation (3.4.26), yields

$$\left(\kappa_0\,\kappa^* - \frac{\alpha_0^2\alpha^{*2}}{c_{p_0}c_p{}^*}\frac{\Delta T}{\rho_0}\frac{T^*}{\rho^*}\right)\left(\frac{\rho_0 U^3}{L}\frac{\partial p^*}{\partial t^*} + \frac{\rho_0 U^3}{L}u^*\frac{\partial p^*}{\partial x^*} + \frac{\rho_0 U^3}{L}v^*\frac{\partial p^*}{\partial y^*} + W\frac{\rho_0 U^2}{H}w^*\frac{\partial p^*}{\partial z^*}\right) -$$
$$\frac{\alpha_0\,k_T\,\alpha^*}{\rho_0\,c_{p_0}c_p{}^*\;\rho^*}\left(\frac{\Delta T}{L^2}\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\Delta T}{L^2}\frac{\partial^2 T^*}{\partial y^{*2}} + \frac{\Delta T}{H^2}\frac{\partial^2 T^*}{\partial z^{*2}}\right) + \beta_0\rho_0\,\beta^* k_S\left[\frac{\Delta S}{L^2}\frac{\partial}{\partial x^*}\left(\rho^*\frac{\partial S^*}{\partial x^*}\right) +\right.$$
$$\left.\frac{\Delta S}{L^2}\frac{\partial}{\partial y^*}\left(\rho^*\frac{\partial S^*}{\partial y^*}\right) + \frac{\Delta S}{H^2}\frac{\partial}{\partial z^*}\left(\rho^*\frac{\partial S^*}{\partial z^*}\right)\right] = -\left(\frac{U}{L}\frac{\partial u^*}{\partial x^*} + \frac{U}{L}\frac{\partial v^*}{\partial y^*} + \frac{W}{H}\frac{\partial w^*}{\partial z^*}\right),$$
$$(3.4.27)$$

where ΔT and ΔS are the temperature and the salinity scales. Doing some rearrangements to Equation (3.4.27) yields

$$\frac{\rho_0 U^3}{L}\left(\kappa_0\,\kappa^* - \frac{\alpha_0^2\,s\,\Delta T}{c_{p_0}\;\rho_0}\frac{\alpha^{*2}T^*}{c_p{}^*\rho^*}\right)\left(\frac{\partial p^*}{\partial t^*} + u^*\frac{\partial p^*}{\partial x^*} + v^*\frac{\partial p^*}{\partial y^*} + \frac{L\,W}{H\,U}w^*\frac{\partial p^*}{\partial z^*}\right) -$$
$$\frac{\Delta T}{L^2}\frac{\alpha_0\,k_T}{\rho_0\,c_{p_0}}\frac{\alpha^*}{c_p{}^*\;\rho^*}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} + \frac{L^2}{H^2}\frac{\partial^2 T^*}{\partial z^{*2}}\right) + \beta_0\rho_0\frac{\Delta S}{L^2}\beta^* k_S\left(\frac{\partial}{\partial x^*}\left(\rho^*\frac{\partial S^*}{\partial x^*}\right) +\right.$$
$$\left.\frac{\partial}{\partial y^*}\left(\rho^*\frac{\partial S^*}{\partial y^*}\right) + \frac{L^2}{H^2}\frac{\partial}{\partial z^*}\left(\rho^*\frac{\partial S^*}{\partial z^*}\right)\right) = -\frac{U}{L}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{W\,L}{H\,U}\frac{\partial w^*}{\partial z^*}\right).$$
$$(3.4.28)$$

Equation (3.4.28) can be simplified by cancelling $\frac{U}{L}$ on the right hand side and using the fact that $\underline{u}^*{}_h \cdot \underline{\nabla}^*{}_{\mathrm{h}} p = u^* \frac{\partial p^*}{\partial x^*} + v^* \frac{\partial p^*}{\partial y^*}$ and $\underline{\nabla}^{*2}_{\mathrm{h}} = \frac{\partial^{2*}}{\partial x^{*2}} + \frac{\partial^{2*}}{\partial y^{*2}}$ which gives

$$\rho_0 U^2 \left(\kappa_0\, \kappa^* - \frac{\alpha_0^2\, \Delta T}{c_{p0}\, \rho_0} \frac{\alpha^{*2} T^*}{\rho^* c_p{}^*} \right) \left(\frac{\partial p^*}{\partial t^*} + \underline{u}^*{}_h \cdot \underline{\nabla}^*{}_{\mathrm{h}} p + \frac{L\, W}{H\, U} w^* \frac{\partial p^*}{\partial z^*} \right) -$$
$$\frac{\Delta T}{L\, U} \frac{\alpha_0\, k_T}{\rho_0\, c_{p0}} \frac{\alpha^*}{c_p{}^*\, \rho^*} \left(\underline{\nabla}^{*2}_{\mathrm{h}} T^* + \frac{L^2}{H^2} \frac{\partial^2 T^*}{\partial z^{*2}} \right) + \beta_0 \rho_0 \frac{\Delta S}{L\, U} \beta^* k_S \left[\underline{\nabla}^*_h \cdot (\rho \underline{\nabla}^*_h S) + \right.$$
$$\left. \frac{L^2}{H^2} \frac{\partial}{\partial z^*} \left(\rho^* \frac{\partial S^*}{\partial z^*} \right) \right] = - \left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{W\, L}{H\, U} \frac{\partial w^*}{\partial z^*} \right). \tag{3.4.29}$$

In addition, it has been shown from Panton (2013) that $\kappa_0 \rho_0 = \frac{c_{p0}}{c_{v0}} \frac{1}{c^2} = \frac{\gamma}{c^2}$, where c is the velocity of sound and $\gamma = \frac{c_{p0}}{c_{v0}}$ is the ratio of the heat capacities. Substituting this expression into (3.4.29), yields

$$\gamma \frac{U^2}{c^2} \left(\kappa^* - \frac{\alpha_0^2\, \Delta T}{c_{p0} \kappa_0\, \rho_0} \frac{\alpha^{*2} T^*}{\rho^* c_p{}^*} \right) \left(\frac{\partial p^*}{\partial t^*} + \underline{u}^*{}_h \cdot \underline{\nabla}^*{}_{\mathrm{h}} p + \frac{L\, W}{H\, U} w^* \frac{\partial p^*}{\partial z^*} \right) -$$
$$\frac{\Delta T}{L\, U} \frac{\alpha_0\, k_T}{\rho_0\, c_{p0}} \frac{\alpha^*}{c_p{}^*\, \rho^*} \underline{\nabla}^{*2}_{\mathrm{h}} T^* - \frac{L^2}{H^2} \frac{\Delta T}{L\, U} \frac{\alpha_0\, k_T}{\rho_0\, c_{p0}} \frac{\alpha^*}{c_p{}^*\, \rho^*} \frac{\partial^2 T^*}{\partial z^{*2}} + \beta_0 \rho_0\, k_S \frac{\Delta S}{L\, U} \beta^* \underline{\nabla}^*_{\mathrm{h}} \cdot (\rho \underline{\nabla}^*_{\mathrm{h}} S) +$$
$$\beta_0 \rho_0\, k_S \frac{\Delta S}{L\, U} \frac{L^2}{H^2} \beta^* \frac{\partial}{\partial z^*} \left(\rho^* \frac{\partial S^*}{\partial z^*} \right) = - \left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{W\, L}{H\, U} \frac{\partial w^*}{\partial z^*} \right). \tag{3.4.30}$$

It can be seen from this equation that the first condition that we need to state for this particular problem is $\frac{W\, L}{H\, U} = 1$. This is the first requirement in order to cast the divergence of the flow velocity in a dimensionless form. As a consequence of that, we have

$$\frac{L^2}{H^2} \frac{1}{LU} = \frac{L}{H^2\, U} = \frac{1}{W\, H}. \tag{3.4.31}$$

Arranging Equation (3.4.30) using this previous assumption, and Equation (3.4.31) yields

$$\gamma \frac{U^2}{c^2}\left(\kappa^* - \frac{\alpha_0^2\,\Delta T}{c_{p0}\kappa_0\,\rho_0}\frac{\alpha^{*2}T^*}{\rho^* c_p{}^*}\right)\left(\frac{\partial p^*}{\partial t^*} + \underline{u}^*{}_h\cdot\underline{\nabla}^*{}_{\mathrm{h}}p + w^*\frac{\partial p^*}{\partial z^*}\right) - \frac{\Delta T}{L\,U}\frac{\alpha_0\,k_T}{\rho_0\,c_{p0}}\frac{\alpha^*}{c_p{}^*\,\rho^*}\underline{\nabla}^{*2}_{\mathrm{h}}T^* -$$

$$\frac{\Delta T}{W\,H}\frac{\alpha_0\,k_T}{\rho_0\,c_{p0}}\frac{\alpha^*}{c_p{}^*\,\rho^*}\frac{\partial^2 T^*}{\partial z^{*2}} + \beta_0\rho_0\frac{\Delta S}{L\,U}\beta^* k_S\,\underline{\nabla}^*_{\mathrm{h}}\cdot(\rho\underline{\nabla}^*_{\mathrm{h}}S) + \beta_0\rho_0\frac{\Delta S}{H\,W}\beta^* k_S\,\frac{\partial}{\partial z^*}\left(\rho^*\frac{\partial S^*}{\partial z^*}\right) =$$

$$-\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right). \tag{3.4.32}$$

Furthermore, some of the coefficients in Equation (3.4.32) can be rewritten in terms of dimensionless number which have been presented in Section 3.3. Such substitution is presented as follows:

$$\frac{U}{c} = \mathrm{M} \tag{3.4.33}$$

$$\frac{\rho_0 c_{p0} L U}{k_T} = \mathrm{Pe}_h \tag{3.4.34}$$

$$\frac{\rho_0 c_{p0} W H}{k_T} = \widetilde{\mathrm{Pe}}_h \tag{3.4.35}$$

$$\frac{LU}{k_S} = \mathrm{Pe} \tag{3.4.36}$$

$$\frac{WH}{k_S} = \widetilde{\mathrm{Pe}}, \tag{3.4.37}$$

where $\sim$ indicates the value of the relevant Peclet number in the vertical direction. Therefore, Equation (3.4.32) becomes

$$\gamma \mathrm{M}^2\left(\kappa^* - \frac{\alpha_0^2\,\Delta T}{c_{p0}\kappa_0\,\rho_0}\frac{\alpha^{*2}T^*}{\rho^* c_p{}^*}\right)\left(\frac{\partial p^*}{\partial t^*} + \underline{u}^*{}_h\cdot\underline{\nabla}^*{}_{\mathrm{h}}p + w^*\frac{\partial p^*}{\partial z^*}\right) -$$

$$\frac{\Delta T\alpha_0}{\mathrm{Pe}_h}\frac{\alpha^*}{c_p{}^*\,\rho^*}\underline{\nabla}^{*2}_{\mathrm{h}}T^* - \frac{\Delta T\alpha_0}{\widetilde{\mathrm{Pe}}_h}\frac{\alpha^*}{c_p{}^*\,\rho^*}\frac{\partial^2 T^*}{\partial z^{*2}} + \frac{\Delta S\,\beta_0\rho_0}{\mathrm{Pe}}\beta^*\,\underline{\nabla}_{\mathrm{h}}\cdot(\rho\underline{\nabla}_{\mathrm{h}}S) +$$

$$\frac{\beta_0\Delta S\rho_0}{\widetilde{\mathrm{Pe}}}\beta^*\,\frac{\partial}{\partial z^*}\left(\rho^*\frac{\partial S^*}{\partial z^*}\right) = -\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right). \tag{3.4.38}$$

Step 6: Let us now consider Equation (3.4.38) and see how it may be simplified using the physical interpretation of the dimensionless numbers or groups. If the Mach number is very small which means $\mathrm{M} \lll 1$ then the term which contains the Mach number may safely be neglected as α^2 is already very small. This simplification is considered as the main condition for flow to be incompressible (Panton, 2013). In addition, if the Peclet number, for heat transfer, and for the transport of salt, are large enough so that $\dfrac{\alpha_0 \Delta T}{\widetilde{\mathrm{Pe}}_h} \lll 1$, $\dfrac{\beta_0 \Delta S}{\widetilde{\mathrm{Pe}}} \lll 1$ and $\dfrac{\alpha_0 \Delta T}{\mathrm{Pe}_h} \lll 1$, $\dfrac{\beta_0 \Delta S}{\mathrm{Pe}} \lll 1$, then all of the terms which contain the Peclet numbers can also be neglected. Furthermore, the numerical values of α_0 and β_0 are already very small as well as the temperature and salinity scale. If these conditions hold, then Equation (3.4.38) can be rewritten as

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0. \tag{3.4.39}$$

Using the expression of the dimensionless variables, Equation (3.4.39) can be rewritten as

$$\frac{L}{U}\frac{\partial u}{\partial x} + \frac{L}{U}\frac{\partial v}{\partial y} + \frac{H}{W}\frac{\partial w}{\partial z} = 0. \tag{3.4.40}$$

Since $\dfrac{W\,L}{H\,U} = 1$, then Equation (3.4.40) becomes

$$\begin{aligned} \frac{L}{U}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) &= 0, \\ \underline{\nabla} \cdot \underline{u} &= 0. \end{aligned} \tag{3.4.41}$$

This equation is the mathematical interpretation of a flow being incompressible.

3.4.4 Conditions for incompressible flow

In summary, the conditions for flow to be incompressible in the case where $\rho = \rho(p, T, S)$ are given as follows:

- $\mathrm{M}^2\gamma \lll 1$: The first condition required is then the Mach number should be very low. This first implies that the fluid velocity is very small compared to the speed of sound. Since the square of the speed of sound is inversely proportional to the changes of the fluid density with respect to the pressure. Then if the Mach number is very small the speed of sound goes to infinite then the changes of the fluid density with respect to the pressure becomes negligible. As a consequence of that, for an incompressible flow, the change in the pressure does not affect the fluid density along the trajectory of the fluid particle. This condition has been justified by Panton (2013).
- $\dfrac{\alpha_0 \Delta T}{\mathrm{Pe}_h} \lll 1$: In addition, if the density is a function of temperature, the low Mach number is not the only requirement. An alternative requirement is that

the heat is not transported by diffusion, but only by advection. Then the thermal diffusion is negligible.

- $\dfrac{\beta_0 \Delta S}{\mathrm{Pe}} \lll 1$: Similarly to the previous case, if the density varies with respect to the salinity. It is also required that the salt is only transported by advection, then there is no molecular salt diffusion.

3.5 Summary

In summary, the balance equations which are related to the state variables, and the structured technique of scaling analysis have been presented in detail throughout this chapter. Moreover, some examples of the dimensionless number have been given in this chapter. It has been found that incompressibility of flow means a very low Mach number and a very high Peclet number. Such results can be interpreted in the way that the motion of the fluid occurs under low velocity and the temperature and salt are transported only by advection. Therefore, if such conditions are satisfied then the rate of change of the fluid density along the trajectory is equal to zero, $D\rho/Dt = 0$, and then the continuity equation is reduced to $\underline{\nabla} \cdot \underline{u} = 0$.
In the following chapters, we shall no longer use the fact the fluid density depends on pressure p, but it that depends only on the temperature T and the salinity S.

Chapter 4

Variable density in the Navier-Stokes equation

This chapter aims to derive the governing equations which satisfy $\rho = \rho(T, S)$ and some results from Chapter 3. In order to achieve this aim, the stress equations of motion are first derived in Section 4.1 followed by the derivation of the Navier-Stokes equation in Section 4.2. Furthermore, according to the geometrical and the physical properties of the domain where the flow occurs, some approximations are presented in Section 4.3 and Section 4.4, for instance, the Boussinesq approximation and the hydrostatic approximation. Finally, with the fact that some stratifications might also occur in the ocean, the shallow water equations are then derived in the last section of this chapter. The latter equations are widely used for different types of flows in the ocean.

4.1 The stress equations of motion

According to Lerner and Trigg (2005), the equations of motion are defined as a set of equations that describe the behaviour of a physical system as a function of its motion and time. In Chapter 1, we have introduced the equation of motion where it occurs without taking into consideration the forces which may cause this motion. In this case, we need to consider the dynamic description of the motion. Newtonian dynamics is the fundamental study of the dynamics of a particle or the body according to Newton's second law (Whitaker, 1968). For a motion of a continuous body, the linear momentum principle which is so-called Euler's first law of mechanics is more suitable to derive the equation of motion. Euler's first law of mechanics says that the time rate of change of the momentum is equal to the total forces acting on the body or the system (Whitaker, 1968).

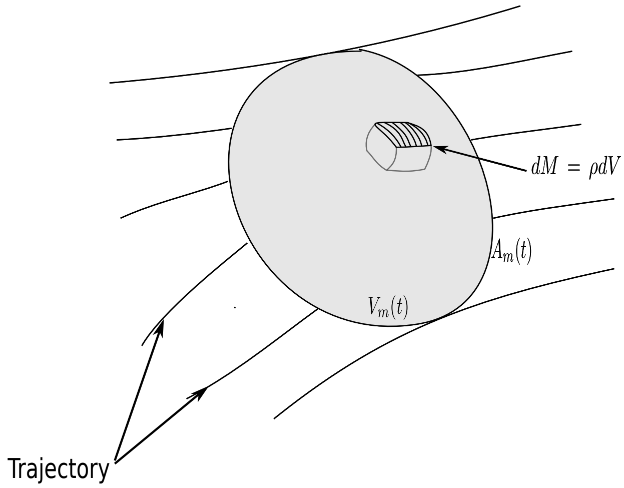

Figure 4.1: Material volume $V_m(t)$

In order to derive the equation of fluid motion, Euler's law can be developed by considering a material volume of fluid $V_m(t)$ which may be continuously changing shape in time t. In addition, it does not exchange mass with its neighbours and its surrounding surface is denoted by $A_m(t)$ (see Figure 4.1). Contained in $V_m(t)$, is an elementary volume dV which has a differential mass dM and is given by

$$dM = \rho\, dV,$$

where ρ is the fluid density field which is a continuous function of space and time. Since the momentum is defined as the product of the mass and the fluid velocity, for an element of mass dM, its expression is given by

$$\underline{u}\, dM = \rho\, \underline{u}\, dV, \tag{4.1.1}$$

where $\underline{u}$ is the fluid velocity field.
Integrating Equation (4.1.1) over the control volume gives:

$$\int_{V_m(t)} \underline{u}\, dM = \int_{V_m(t)} \rho\, \underline{u}\, dV. \tag{4.1.2}$$

As stated by the linear momentum principle, the time rate of change of the momentum is equal to the total forces acting on the body or the system, we then have

$$\frac{D}{Dt} \int_{V_m(t)} \rho\, \underline{u}\, dV = \sum \underline{F}_{\text{app}}, \tag{4.1.3}$$

where $\sum \underline{F}_{\text{app}}$ represents all the applied forces acting on the material volume of the fluid (Lerner and Trigg, 2005). These forces consist of body forces and surface forces. In fluid mechanics, the body force that is usually exerted on the material volume is just the gravitation force, and can be expressed as

$$\underline{F}_{\text{body}} = \int_{V_m(t)} \rho \underline{g} \, dV.$$

The surface forces exerted by the surrounding neighbourhood of the material volume is given by

$$\underline{F}_{\text{surf}} = \int_{A_m(t)} \underline{\underline{T}} \cdot \underline{n} \, dA,$$

where $\underline{\underline{T}}$ is defined as the stress *tensor* [1] and $\underline{n}$ is the normal vector to the surface. Therefore, Equation (4.1.3) can be rewritten as follows:

$$\frac{D}{Dt} \int_{V_m(t)} \rho \underline{u} \, dV = \int_{V_m(t)} \rho \underline{g} \, dV + \int_{A_m(t)} \underline{\underline{T}} \cdot \underline{n} \, dA. \tag{4.1.4}$$

In order to set Equation (4.1.4) in a simple equation under one integral, the Reynolds transport theorem will be applied to the term on the left hand side of this equation, and the divergence theorem to the second term on the right hand side.

According to Whitaker (1968), the Reynolds transport theorem tells us that if ϕ is an arbitrary scalar field, and ρ is the fluid density, the material derivative of the integral of $\rho\phi$ over the control volume $V_m(t)$ can be written as

$$\frac{D}{Dt} \int_{V_m(t)} \rho \phi \, dV = \int_{V_m(t)} \rho \frac{D\phi}{Dt} dV. \tag{4.1.5}$$

Additionally, the divergence theorem states that for an arbitrary vector field $\underline{G}$, and a normal vector $\underline{n}$ to the surface $A_m(t)$ of the control volume $V_m(t)$, we have

$$\int_{V_m(t)} \underline{\nabla} \cdot \underline{G} \, dV = \int_{A_m(t)} \underline{G} \cdot \underline{n} \, dA. \tag{4.1.6}$$

Let us first apply the Reynolds transport theorem to the term on the left hand side of Equation (4.1.4). In this case, the vector velocity is written in terms of components such that $\underline{u} = u\underline{i} + v\underline{j} + w\underline{k}$. Then the term on the left hand side of Equation (4.1.4)

[1] Here, the tensor is referred as a second-order tensor and the latter can be represented in a matrix form where each component represents a force per area acting on the surface.

can be rewritten as

$$\frac{D}{Dt}\int_{V_m(t)} \rho \underline{u}\, dV = \frac{D}{Dt}\int_{V_m(t)} \rho \left(u\underline{i} + v\underline{j} + w\underline{k}\right) dV,$$

Using the distributive property of the intergral, yields

$$\frac{D}{Dt}\int_{V_m(t)} \rho \underline{u}\, dV = \left[\frac{D}{Dt}\int_{V_m(t)} \rho u\, dV\right]\underline{i} + \left[\frac{D}{Dt}\int_{V_m(t)} \rho v\, dV\right]\underline{j} + \left[\frac{D}{Dt}\int_{V_m(t)} \rho w\, dV\right]\underline{k},$$

Applying Equation (4.1.5) into the equation above by considering ϕ as the components of the vector velocity gives

$$\begin{aligned}\frac{D}{Dt}\int_{V_m(t)} \rho \underline{u}\, dV &= \int_{V_m(t)} \rho\frac{Du}{Dt}\, dV\underline{i} + \int_{V_m(t)} \rho\frac{Dv}{Dt}\, dV\underline{j} + \int_{V_m(t)} \rho\frac{Dw}{Dt}\, dV\underline{k},\\ &= \int_{V_m(t)} \rho\frac{D}{Dt}\left(u\underline{i} + v\underline{j} + w\underline{k}\right) dV\\ &= \int_{V_m(t)} \rho\frac{D\underline{u}}{Dt} dV.\end{aligned} \tag{4.1.7}$$

Now, taking the last term on the right-hand side of Equation (4.1.4) and using the divergence theorem presented in Equation (4.1.6) gives

$$\int_{A_m(t)} \underline{\underline{T}}\cdot\underline{n}\, dA = \int_{V_m(t)} \underline{\nabla}\cdot\underline{\underline{T}}\, dV. \tag{4.1.8}$$

Therefore, Equation (4.1.4) may be rewritten as

$$\int_{V_m} \left(\rho\frac{D\underline{u}}{Dt} - \rho\underline{g} - \underline{\nabla}\cdot\underline{\underline{T}}\right) dV = 0. \tag{4.1.9}$$

Equation (4.1.9) then represents the conservation of momentum for an arbitrary volume $V_m(t)$. Since the latter equation is always valid for any chosen material volume, the only way that such condition is satisfied, is that the integrand has to be zero for every point in the material volume. Therefore, we have the following equation:

$$\rho\frac{D\underline{u}}{Dt} - \rho\underline{g} - \underline{\nabla}\cdot\underline{\underline{T}} = 0. \tag{4.1.10}$$

In Equation (4.1.10) each of the terms has particular interpretations: $\rho\frac{D\underline{u}}{Dt}$ is called the inertial force per unit of volume, $\rho\underline{g}$ is called the body force per unit of volume, and $\underline{\nabla}\cdot\underline{\underline{T}}$ is called the surface force per unit of volume. The latter is used as the first equation for describing simple momentum transport in any continuum and it is called the *stress equations of motion* or *the Cauchy momentum equation* (Whitaker, 1968). The latter, has been subject to several investigations in order to treat different types of fluid motion.

4.2 Navier-Stokes equation

In this section, we are deriving the Navier-Stokes equation from Equation (4.1.10) where the fluid density varies with respect to the pressure, temperature, and salinity. According to Bachelor (1967), the Navier-Stokes equation is an equation of motion involving a viscous fluid. A fluid is said to be viscous when it exhibits internal friction. By definition, internal friction is an internal force due to the relative motion between two surfaces of the fluid that are moving at different velocities, in other words, it is defined as the internal resistance of the fluid to flow.

As we are only interested in studying the motion of Newtonian fluid. In general, there are two types of surface forces acting upon the surface of the Newtonian fluid element. There is the normal force which is due to the pressure and there are the shear forces whose directions are parallel to the surface on which they act (Whitaker, 1968). Because the stress tensor contains these two types of forces, then it can be expressed as

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}, \tag{4.2.1}$$

where p is the pressure, $\underline{\underline{I}}$ is the unit tensor, and $\underline{\underline{\tau}}$ is the viscous stress tensor.

Returning to Equation (4.1.10) and substituting the stress tensor in this equation to the expression of the stress tensor from Equation (4.2.1), it follows that the stress equations of motion can be rewritten as

$$\rho\frac{D\underline{u}}{Dt} - \rho\underline{g} + \underline{\nabla}p - \underline{\nabla}\cdot\underline{\underline{\tau}} = 0. \tag{4.2.2}$$

In Equation (4.2.2), the expression of $\underline{\underline{\tau}}$ is must be known but we will not go further into details to derive it. From Whitaker (1968), the expression of it, is given by

$$\underline{\underline{\tau}} = 2\mu\underline{\underline{D}} + \left[\left(\mathcal{K} - \frac{2}{3}\mu\right)\underline{\nabla}\cdot\underline{u}\right]\underline{\underline{I}}, \tag{4.2.3}$$

where $\underline{\underline{D}}$ is the rate of strain tensor, $\mathcal{K}$ is the bulk viscosity, and μ is the viscosity. In addition, $\underline{\underline{D}}$ is given by

$$\underline{\underline{D}} = \underline{\nabla}\left(\underline{u}\right) + \left(\underline{\nabla}\,\underline{u}\right)^{\mathrm{tr}}. \tag{4.2.4}$$

It is a well used approximation to often take the viscosity μ and $\mathcal{K}$ to be uniform over the fluid (Bachelor, 1967). As a consequence of that assumption, the divergence of Equation (4.2.3) can be expressed as

$$\underline{\nabla}\cdot\underline{\underline{\tau}} = \mu\nabla^2\underline{u} + \left(\mathcal{K} + \frac{1}{3}\mu\right)\underline{\nabla}\left(\underline{\nabla}\cdot\underline{u}\right). \tag{4.2.5}$$

Hence, Equation (4.2.2) is explicitly expressed as the momentum equation for a variable density flow

$$\rho\frac{D\underline{u}}{Dt} - \rho\underline{g} + \underline{\nabla}p - \mu\nabla^2\underline{u} - \left(\mathcal{K} + \frac{1}{3}\mu\right)\underline{\nabla}\left(\underline{\nabla}\cdot\underline{u}\right) = 0. \tag{4.2.6}$$

Since $\rho = \rho(T, S)$, we assume that the conditions of incompressibility that we have derived in the previous chapter will also be satisfied here, $\dfrac{D\rho}{Dt} = 0$, implies $\underline{\nabla}\cdot\underline{u} = 0$. As a result of that, the Equation (4.2.6) can be reduced to

$$\rho\frac{D\underline{u}}{Dt} - \rho\underline{g} + \underline{\nabla}p - \mu\nabla^2\underline{u} = 0. \tag{4.2.7}$$

The detailed dynamics of any incompressible flow problems are accurately described by Equation (4.2.7) which is the Navier-Stokes equation for incompressible flow (Bachelor, 1967). In our case, since we deal with flow problems which are related to the ocean and are subject of ocean dynamics, we need to take into consideration the effects of the Earth's rotation. However, for sake of simplicity such effects will not be considered here and in the following chapters.

Moreover, since solving Equation (4.2.7) is challenging because its complexity and its non-linearity then some simplifications would be very helpful and needed. There are several features of approximation in order to simplify Equation (4.2.7) according to the aspect of the problem. One of the commonly used approximations in the ocean dynamics is the so-called *Boussineq approximation* that we are going to discuss hereafter.

4.3 Navier-Stokes equations and the Boussinesq approximation

4.3.1 Definition

The Boussinesq approximation is based on the properties of fluid density. Following the definition given by Spiegel and Veronis (1959), this approximation consists of ignoring the variations of the fluid density except where it is multiplied by the acceleration of gravity in the vertical component of the momentum equation. Alternatively, this definition was supported by Goluskin (2015) who stated that the density variations are considered negligible so that they can be ignored everywhere except in the buoyancy force. In addition, Gray and Giorgini (1976) stated that as a result of the Boussinesq approximation, all other fluid properties are assumed constant and the viscous dissipation is assumed negligible.

Therefore, if the conditions for the Boussinesq approximation are satisfied, the governing equations which consists of the continuity equation from Equation (3.1.7), the momentum equation from Equation (4.2.7), and the conservation of salt and energy from Equations (3.1.16) and (3.1.35) can be simplified as follows:

- According to the definition of the Boussinesq approximation, the fluid density is considered constant in the continuity equation, then the latter is reduced to $\underline{\nabla} \cdot \underline{u} = 0$ (Spiegel and Veronis, 1959).
- In addition, in the momentum equation, the fluid density varies only when it is multiplied with the gravitational acceleration, resulting:

$$\frac{Du}{Dt} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} - \nu\nabla^2 u = 0,$$

$$\frac{Dv}{Dt} + \frac{1}{\rho_0}\frac{\partial p}{\partial y} - \nu\nabla^2 v = 0,$$

$$\frac{Dw}{Dt} + \frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\rho}{\rho_0}g - \nu\nabla^2 w = 0, \tag{4.3.1}$$

  where ν is the constant kinematic viscosity, ρ is variable density and ρ_0 the constant density (Samelson, 2014).

- Furthermore, using the fact that all the physical coefficients are considered constant, it allows us to write the energy equation from Equation (3.1.35) into its simple form as follows:

$$\frac{DT}{Dt} = \frac{k_T}{\rho_0 c_{p_0}}\nabla^2 T, \tag{4.3.2}$$

  where ρ_0 is a constant density, c_{p_0} is a constant specific heat capacity and k_T is a constant heat conductivity (Gray and Giorgini, 1976). In the same way, the conservation of salt in Equation (3.1.16) becomes

$$\frac{DS}{Dt} = k_S\nabla^2 S, \tag{4.3.3}$$

  where k_S is a constant salt diffusion.

4.3.2 Validation of the Boussinesq approximation from literature

Several studies have been undertaken from the literature in order to find sufficient conditions for the validation of the Boussinesq approximation. We shall not go into further details for any of these cases but only a simple description of some of them will be given in the following paragraph.

One of the most cited works has been presented by Spiegel and Veronis (1959), which is about finding the conditions under which the Boussinesq approximation is applicable for thermal convection in a compressible fluid. The work is about expressing the state variables as the sum of its mean value in absence of motion and its fluctuation resulting from the motion. By introducing a parameter ϵ from the ratio of the variation in the absence of motion of the fluid density to its mean value, they approximated the continuity equation, the momentum equation, and the energy equation. This approximation has been referred as the physical justification of the Boussinesq approximation but not complete in terms of mathematical perspective (Rajagopal et al., 2009).

To continue, Gray and Giorgini (1976) introduced a systematic method to approximate the governing equations for natural convection flows and derived from this the condition for which the Boussinesq approximation is valid. In this method, they assumed that all the physical properties vary linearly with temperature and pressure, and may be approximated by linearised Taylor expansions. They used scaling analysis to write in dimensionless form the governing equations. And resulting from this scaling, a characteristic non-dimensional parameter is introduced and by requiring the latter parameter to be small, the governing equations are reduced to the Boussinesq equations. In this work, the required conditions to approximate the value of the characteristic dimensionless parameter is referred to as the validation of the Boussinesq approximation. They stated that this approximation is mathematically straightforward.

Furthermore, LeBlond and Mysak (1981) used a similar method as Spiegel and Veronis (1959) in order to determine the condition for which the Boussinesq approximation is valid in ocean dynamics. They expressed the pressure and the fluid density as the sum of their equilibrium state and their perturbation. Using the fact that in the ocean and in terms of magnitude, the ratio of the density perturbation to fluid density in the equilibrium state is small (on order of 10^{-3}). They approximated the momentum equation by replacing the fluid density by some constant density except in the buoyancy term. This approximation is referred as the Bounssinesq approximation.

4.4 Hydrostatic approximation

In addition to the Boussinesq approximation, the hydrostatic approximation is also widely used in ocean dynamics. Characteristically, the hydrostatic approximation is a process of replacing the expression of the vertical component of the momentum equation by a simple equation where the gravity is balanced by the pressure gradient force. According to Pedlosky (2013), the hydrostatic approximation is said to be satisfied if the vertical scale is small compared to the horizontal scale. We have seen in the previous chapter the notion of scale where the vertical scale was denoted by H and the horizontal scale by L. As a consequence of this approximation, in terms of magnitude, the force which is induced by the vertical acceleration of the fluid motion $\rho_0 \dfrac{Dw}{Dt}$ is much smaller compared to the counterbalanced force due the pressure gradient and the gravitational force. Furthermore, accordingly to Samelson (2014), it is noticed that the frictional forces acting in the vertical are always very small compared to the gravitation force.

Therefore, it follows from applying the hydrostatic approximation that the vertical momentum equation from Equation (4.3.1) becomes

$$\frac{\partial p}{\partial z} + \rho g \; = \; 0, \tag{4.4.1}$$

where Equation (4.4.1) is called the hydrostatic balance (Pedlosky, 2013).

Equation (4.4.1) can be interpreted as a direct relationship between the vertical pressure gradient and the vertical distribution of density. This implies, the vertical pressure gradient is a function of the fluid density. Then, Equation (4.4.1) can be integrated to determine the pressure field in the ocean with respect to the fluid density which is obtained from the temperature and the salinity. As a result of the hydrostatic approximation, Equation (4.3.1) can be rewritten as follows:

$$\begin{aligned}
&\frac{Du}{Dt} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} - \nu\nabla^2 u = 0, \\
&\frac{Dv}{Dt} + \frac{1}{\rho_0}\frac{\partial p}{\partial y} - \nu\nabla^2 v = 0, \\
&\frac{\partial p}{\partial z} + \rho g = 0.
\end{aligned} \tag{4.4.2}$$

Furthermore, the flows that are characterized by the hydrostatic approximation may be referred to as the shallow water flows, then such approximation is also called the *shallow water approximation*. Hence, the shallow water flows are considered nearly horizontal, but not totally a two-dimensional flow since they show a three-dimensional structure due to momentum transfer in the boundary layers. In addition, the density stratification caused by the variation in temperature and salinity induces also some changes in the third direction (Vreugdenhil, 2013). In the next section, the governing equations for shallow water flows will be derived.

4.5 Shallow water equations

The shallow water equations are obtained by taking the depth-average of the reduced form of the continuity equation and the horizontal components of the momentum equation from (4.4.2). The depth-average is a very advantageous technique because due to the complexity of solving the full three-dimensional governing equations, it allows reduction of the dimension of the equations. Then we are going to perform the depth-averaging of the temperature and the salt conservation equations as well. In addition, the reduced equations from the depth-averaging process give the same information as the full original equations and at much lower cost in terms of numerical computation (Vreugdenhil, 2013).

For sake of simplicity, we will only focused on the derivation of the one-dimensional shallow water equations where the $y-$axis is not considered. In principle, the one dimensional shallow water equations will be obtained by doing vertical integration from the bottom to the surface of the domain of the governing equations with no $y-$coordinate. Let us restate all the equations as follows:

a) Following the results from the previous chapter, the continuity equation can be reduced in its incompressible form as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{4.5.1}$$

b) Momentum equation under the Boussinesq and hydrostatic assumption:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial z^2}, \\ &\frac{\partial p}{\partial z} = -\rho g. \end{aligned} \tag{4.5.2}$$

c) From the previous chapter, it has been stated that the diffusion of salt and temperature are negligible. The equation of conservation of temperature and salt can then be rewritten in their advective form:

$$\frac{\partial \Phi}{\partial t} + \frac{\partial (u\Phi)}{\partial x} + \frac{\partial (w\Phi)}{\partial z} = 0, \tag{4.5.3}$$

where Φ may be either the temperature, or the salinity.

4.5.1 Kinematic and dynamic boundary conditions

In order to derive the shallow water equations, we need the kinematic and the dynamic boundary conditions which are completely different from the boundary conditions that are needed when we solve the partial differential equations. Before discussing these boundary conditions, let us illustrate in Figure 4.2 a domain where the flow can be occured:

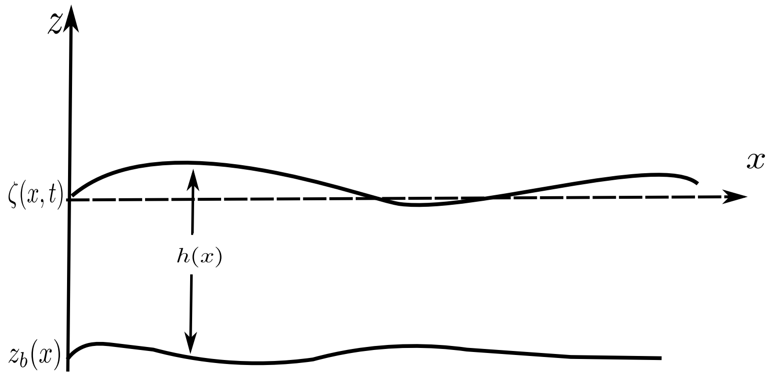

Figure 4.2: Scheme of the one-dimensional shallow water flow as an open channel flow

It can be seen from Figure 4.2, the bottom coordinate of the domain is defined as $z = z_b(x)$, and the free surface is defined as $z = \zeta(x,t)$. Then, $h = \zeta - z_b$ defines the total height of the domain. And let $\underline{u} = (u, w)$ be the vector velocity of the fluid particles.

According to Vreugdenhil (2013), the kinematic boundary condition states that the fluid particles will never cross the boundary. At the bottom of the fluid domain, this definition implies that the scalar product between the velocity of the flow and the normal vector to the bottom must be zero. We have

$$\underline{u} \cdot \underline{n} = 0 \quad \Rightarrow \quad u(z_b)\frac{\partial z_b}{\partial x} = w(z_b). \tag{4.5.4}$$

At the free-surface, the fluid particles always remain part of the surface even if the surface itself may be moving. This implies that the material derivative of particles at the free-surface has to be zero, then we have

$$u(\zeta)\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t} = w(\zeta). \tag{4.5.5}$$

Moreover, Vreugdenhil (2013) stated that the dynamic boundary conditions concern the forces acting at the boundaries. At the bottom of the domain, we have the "no-slip" boundary condition which says that the *viscous* fluid sticks or is fixed to the solid bottom. This definition has been supported by Bachelor (1967), who stated that the dynamic boundary conditions consist of continuity in the tangential velocities, and the normal stresses (without taking into consideration the surface tension) across the interface separating the fluid to another medium. This implies the following expression at the bottom of the domain: This definition implies that if we take the depth-average of the previous expression, we have

$$u = 0, \qquad v = 0, \tag{4.5.6}$$

where u and v are the horizontal components of the vector velocity fields. Furthermore, at the free surface, the stresses in the fluid are assumed to be the same as those in the air just-above with the fact that surface tension is not taken into account. Then, for the case of pressure at the open free surface, we have

$$p = p_a \tag{4.5.7}$$

where p_a is the atmospheric pressure which can be taken as zero. In our case, the dynamic boundary condition, which concerns the stresses and the viscous fluid will not be required.

4.5.2 Different assumptions

Before starting the depth-average of the governing equations, different assumptions and rules are needed, and are presented as follows:

- The fluid density is assumed to be constant in the vertical direction across the domain but it may vary in the horizontal as well as in time. This means $\rho(t,x) = \overline{\rho}$, where the latter is the depth-average of the fluid density.

- From Moran et al. (2010), we have the Leibniz's rule which states that:

  Let $A(t)$ and $B(t)$ be differentiable for $a \leq t \leq b$, and let c and d be finite constants such that $c \leq A(t) \leq d$ and $c \leq B(t) \leq d$ for each t in $a \leq t \leq b$. If $f(x,t)$ and $\dfrac{\partial f(x,t)}{\partial t}$ are continuous in the rectangle $a \leq t \leq b$, $c \leq x \leq d$, then the Leibniz rule (4.5.8) holds for each t in $a \leq t \leq b$.

$$\frac{\partial}{\partial t}\int_{A(t)}^{B(t)} f(x,t)\,dx = \int_{A(t)}^{B(t)} \frac{\partial}{\partial t} f(x,t)\,dx + B'(t)f(B(t),t) - A'(t)f(A(t),t). \tag{4.5.8}$$

- The flow velocity can be expressed as the sum of the depth-average mean value denoted by $\overline{u}$ and a deviation from its mean value denoted by $\widetilde{u}$ such that:

$$u = \overline{u} + \widetilde{u}. \tag{4.5.9}$$

This definition implies that if we take the depth-average of the previous expression, we have

$$\int_{z_b}^{\zeta} \widetilde{u}\, dz = 0.$$

- The depth-averaged velocity is defined as

$$\overline{u} = \frac{1}{h}\int_{z_b}^{\zeta} u\, dz, \tag{4.5.10}$$

where $h = \zeta - z_b$. Since $u = \overline{u} + \widetilde{u}$, we can deduce that

$$\int_{z_b}^{\zeta} u^2\, dz = \int_{z_b}^{\zeta} \left(\overline{u} + \widetilde{u}\right)^2\, dz = h\overline{u}^2 + \int_{z_b}^{\zeta} \widetilde{u}^2\, dz. \tag{4.5.11}$$

- Using the thermodynamic properties of seawater, it is noticed that the kinematic viscosity $\dfrac{\mu}{\rho_0} = \nu$ is very small (an order of 10^{-6}, without taking into account the turbulence (Mamayev, 2010)), it can be then neglected. As a consequence of that, the momentum equation can be reduced as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{4.5.12}$$

$$\frac{\partial p}{\partial z} = -\rho g. \tag{4.5.13}$$

Let us now do the depth-averaging of the governing equations, starting form the continuation equation, then the momentum equation, and finally the conservation of salt and temperature.

4.5.3 Depth-average of the continuity equation

The equation of mass conservation is given by

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{4.5.14}$$

Taking the depth-average yields

$$\underbrace{\int_{z_b}^{\zeta} \frac{\partial u}{\partial x} dz}_{I_1} + \underbrace{\int_{z_b}^{\zeta} \frac{\partial w}{\partial z} dz}_{I_2} = 0. \tag{4.5.15}$$

The term I_1 is reduced as follows using Leibniz's rule (4.5.8):

$$\begin{aligned} I_1 &= \frac{\partial}{\partial x}\int_{z_b}^{\zeta} u dz - \frac{\partial \zeta}{\partial x}\, u|_{z=\zeta} + \frac{\partial z_b}{\partial x}\, u|_{z=z_b} \\ &= \frac{\partial(\overline{u}h)}{\partial x} - u|_{z=\zeta}\frac{\partial \zeta}{\partial x} + u|_{z=z_b}\frac{\partial z_b}{\partial x}. \end{aligned} \tag{4.5.16}$$

For I_2, we have

$$I_2 = w(\zeta) - w(z_b) \tag{4.5.17}$$

Therefore,

$$I_1 + I_2 = \frac{\partial(\overline{u}h)}{\partial x} - u|_{z=\zeta}\frac{\partial\zeta}{\partial x} + u|_{z=z_b}\frac{\partial z_b}{\partial x} + w(\zeta) - w(z_b). \tag{4.5.18}$$

Applying the boundary conditions (4.5.4) and (4.5.5) into the above equation gives

$$\begin{aligned} I_1 + I_2 &= \frac{\partial(\overline{u}h)}{\partial x} - u|_{z=\zeta}\frac{\partial\zeta}{\partial x} + u|_{z=z_b}\frac{\partial z_b}{\partial x} + \frac{\partial\zeta}{\partial t} + u|_{z=\zeta}\frac{\partial\zeta}{\partial x} - u|_{z=z_b}\frac{\partial z_b}{\partial x} \\ &= \frac{\partial\zeta}{\partial t} + \frac{\partial(\overline{u}h)}{\partial x}. \end{aligned}$$

Since $h(t,x) = \zeta(t,x) - z_b(x)$ it follows that

$$I_1 + I_2 = \frac{\partial h}{\partial t} + \frac{\partial(\overline{u}h)}{\partial x}. \tag{4.5.19}$$

The latter is the depth-average of the continuity equation in one dimension.

4.5.4 Depth-average of the momentum equation with hydrostatic assumption

Taking the integral over the depth of Equation (4.5.12), gives

$$\underbrace{\int_{z_b}^{\zeta}\frac{\partial u}{\partial t}dz}_{J_1} + \underbrace{\int_{z_b}^{\zeta}\frac{\partial(u^2)}{\partial x}dz}_{J_2} + \underbrace{\int_{z_b}^{\zeta}\frac{\partial(uw)}{\partial z}dz}_{J_3} = -\frac{1}{\rho_0}\underbrace{\int_{z_b}^{\zeta}\frac{\partial p}{\partial x}dz}_{J_4}. \tag{4.5.20}$$

Evaluating J_1 yields

$$\begin{aligned} J_1 &= \frac{\partial}{\partial t}\int_{z_b}^{\zeta} u\,dz - \frac{\partial\zeta}{\partial t}\,u|_{z=\zeta} + \underbrace{\frac{\partial z_b}{\partial t}\,u|_{z=z_b}}_{=0} \\ &= \frac{\partial}{\partial t}\int_{z_b}^{\zeta} u\,dz - u|_{z=\zeta}\frac{\partial\zeta}{\partial t} \\ &= \frac{\partial(\overline{u}\,h)}{\partial t} - u|_{z=\zeta}\frac{\partial\zeta}{\partial t}. \end{aligned} \tag{4.5.21}$$

For J_2, we have

$$\begin{aligned} J_2 &= \frac{\partial}{\partial x}\int_{z_b}^{\zeta}(u^2)dz - \frac{\partial \zeta}{\partial x}\,(u^2)\Big|_{z=\zeta} + \frac{\partial z_b}{\partial x}\,(u^2)\Big|_{z=z_b} \\ &= \frac{\partial}{\partial x}\int_{z_b}^{\zeta} u^2 dz - \frac{\partial \zeta}{\partial x}\,u^2\Big|_{z=\zeta} + \frac{\partial \zeta}{\partial x}\,u^2\Big|_{z=z_b}. \end{aligned} \tag{4.5.22}$$

Aplying Equation (4.5.11) into the last equation above yields

$$\begin{aligned} J_2 &= \frac{\partial}{\partial x}\left(h\overline{u}^2 + \int_{z_b}^{\zeta}\widetilde{u}^2 dz\right) - \frac{\partial \zeta}{\partial x}\,u^2\Big|_{z=\zeta} + \frac{\partial \zeta}{\partial x}\,u^2\Big|_{z=z_b}. \\ &= \frac{\partial\,\overline{u}^2\,h}{\partial x} - \frac{\partial \zeta}{\partial x}\,u^2\Big|_{z=\zeta} + \frac{\partial \zeta}{\partial x}\,u^2\Big|_{z=z_b} + \frac{\partial}{\partial x}\left(\int_{z_b}^{\zeta}\widetilde{u}^2 dz\right). \end{aligned} \tag{4.5.23}$$

Computing J_3 gives the following expression:

$$J_3 = (uw)|_{z=\zeta} - (uw)|_{z=z_b}. \tag{4.5.24}$$

Now, combining J_1, J_2, and J_3 we have

$$\begin{aligned} J_1 + J_2 + J_3 = {} & \frac{\partial(\,\overline{u}\,h)}{\partial t} - u|_{z=\zeta}\,\frac{\partial \zeta}{\partial t} + \frac{\partial\,\overline{u}^2\,h}{\partial x} - \frac{\partial \zeta}{\partial x}\,u^2\Big|_{z=\zeta} + \\ & \frac{\partial z_b}{\partial x}\,u^2\Big|_{z=z_b} + \frac{\partial}{\partial x}\left(\int_{z_b}^{\zeta}\widetilde{u}^2 dz\right) + \Big((uw)|_{z=\zeta} - (uw)|_{z=z_b}\Big) \\ = {} & \frac{\partial(\,\overline{u}\,h)}{\partial t} + \frac{\partial\,\overline{u}^2\,h}{\partial x} - u|_{z=\zeta}\left(\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x}\,u|_{z=\zeta} - w|_{z=\zeta}\right) + \\ & u|_{z=z_b}\left(\frac{\partial z_b}{\partial x}\,u|_{z=z_b} - w|_{z=z_b}\right) + \frac{\partial}{\partial x}\left(\int_{z_b}^{\zeta}\widetilde{u}^2 dz\right) \\ = {} & \frac{\partial(\,\overline{u}\,h)}{\partial t} + \frac{\partial\,\overline{u}^2\,h}{\partial x} + \frac{\partial}{\partial x}\left(\int_{z_b}^{\zeta}\widetilde{u}^2 dz\right). \end{aligned} \tag{4.5.25}$$

Computing the term on the right hand side of Equation (4.5.20) gives

$$J_4 = \frac{1}{\rho_0}\left[-\frac{\partial}{\partial x}\int_{z_b}^{\zeta} p dz + \frac{\partial \zeta}{\partial x}\,p|_{z=\zeta} - \frac{\partial z_b}{\partial x}\,p|_{z=z_b}\right] \tag{4.5.26}$$

We need to evaluate the expression of the pressure in equation (4.5.26). In order to derive this expression, let us use Equation (4.5.13) which is expressed as

$$\frac{\partial p}{\partial z} + \rho g = 0. \tag{4.5.27}$$

Then, by integrating over the depth from the top surface ζ to an intermediate surface coordinate z, we have

$$\int_z^{\zeta} \frac{\partial p}{\partial z} dz = -\int_z^{\zeta} \rho g \, dz.$$

Since ρ does not depend on z, from our assumption,

$$\int_z^{\zeta} \frac{\partial p}{\partial z} dz = -\rho \int_z^{\zeta} g dz$$

$$p|_{\zeta} - p|_{z} = -\rho g(\zeta - z). \tag{4.5.28}$$

At the free surface, the pressure is defined as the atmospheric pressure, and at the intermediate depth $p|_z = p$, this implies

$$p = p_{\text{atm}} + \rho g(\zeta - z). \tag{4.5.29}$$

Following Vreugdenhil (2013), the absolute pressure level is not important then it can be taken zero. Substituting Equation (4.5.29) into Equation (4.5.26) yields

$$J_4 = \frac{1}{\rho_0}\left[-\frac{\partial}{\partial x} g \int_{z_b}^{\zeta} \rho(\zeta - z) dz \, + \frac{\partial \zeta}{\partial x} \rho g \, (\zeta - z)|_{z=\zeta} - \frac{\partial z_b}{\partial x} \rho g \, (\zeta - z)|_{z=z_b} \right]$$

As ρ does not depend on z, then $\rho = \overline{\rho}$ and it follows that:

$$\begin{aligned} J_4 &= \frac{1}{\rho_0}\left[-\frac{\partial}{\partial x} \overline{\rho} g \int_{z_b}^{\zeta} (\zeta - z) dz \, + \frac{\partial \zeta}{\partial x} \overline{\rho} g \, (\zeta - z)|_{z=\zeta} - \frac{\partial z_b}{\partial x} \overline{\rho} g \, (\zeta - z)|_{z=z_b} \right] \\ &= \frac{1}{\rho_0}\left[-\frac{\partial}{\partial x}\left\{ \overline{\rho} g \left[-\frac{(\zeta - z)^2}{2} \right]_{z_b}^{\zeta} \right\} - \frac{\partial z_b}{\partial x} \overline{\rho} g (\zeta - z_b) \right]. \\ &= \frac{1}{\rho_0}\left[-\frac{\partial}{\partial x}\left\{ \overline{\rho} g \left[\frac{(\zeta - z_b)^2}{2} \right] \right\} - \frac{\partial z_b}{\partial x} \overline{\rho} g (\zeta - z_b) \right]. \end{aligned} \tag{4.5.30}$$

It is known that $h = \zeta - z_b$ then it follows that:

$$J_4 = \frac{1}{\rho_0}\left[-\frac{\partial}{\partial x}\left(\overline{\rho} g \left[\frac{h^2}{2} \right] \right) - \frac{\partial z_b}{\partial x} \overline{\rho} g h \right]. \tag{4.5.31}$$

Therefore the depth-average of the momentum equation can be written as follows:

$$\frac{\partial(\overline{u}\,h)}{\partial t} + \frac{\partial}{\partial x}\left(\overline{u}^2 h + \frac{1}{2}\frac{\overline{\rho}}{\rho_0}\,g\,h^2\right) = \frac{1}{\rho_0}\left[-\frac{\partial z_b}{\partial x}\overline{\rho}\,gh + -\frac{\partial}{\partial x}\left(\overline{\rho}\int_{z_b}^{\zeta}\widetilde{u}^2 dz\right)\right],$$

and with the assumption that the viscous stresses are neglected, we have

$$\frac{\partial(\overline{u}\,h)}{\partial t} + \frac{\partial}{\partial x}\left(\overline{u}^2 h + \frac{1}{2}\frac{\overline{\rho}}{\rho_0}\,g\,h^2\right) = -\frac{\partial z_b}{\partial x}\left(\frac{\overline{\rho}}{\rho_0}\right) gh. \tag{4.5.32}$$

The latter equation represents the depth-average of the momentum equation in one-dimensional form.

4.5.5 Depth-average of the conservation equation of the state variables

Let Φ be a state variable such as temperature T or salinity S. Applying the depth-average to the advective form of the scalar conservation equation (4.5.3) yields

$$\underbrace{\int_{z_b}^{\zeta}\frac{\partial\Phi}{\partial t}dz}_{K_1} + \underbrace{\int_{z_b}^{\zeta}\frac{\partial(\Phi u)}{\partial x}dz}_{K_2} + \underbrace{\int_{z_b}^{\zeta}\frac{\partial(\Phi w)}{\partial z}\,dz}_{K_3} = 0. \tag{4.5.33}$$

Let us first evaluate K_1 by applying the Leibniz's rule. It yields

$$K_1 = \frac{\partial}{\partial t}\int_{z_b}^{\zeta}\Phi dz - \frac{\partial\zeta}{\partial t}\,\Phi|_{z=\zeta} + \frac{\partial z_b}{\partial t}\,\Phi|_{z=z_b}.$$

Since z_b does not change in time, if follows that $\dfrac{\partial z_b}{\partial t} = 0$, and then

$$K_1 = \frac{\partial(\overline{\Phi}\,h)}{\partial t} - \frac{\partial\zeta}{\partial t}\,\Phi|_{z=\zeta} \tag{4.5.34}$$

Similarly, K_2 becomes

$$\begin{aligned} K_2 &= \frac{\partial}{\partial x}\int_{z_b}^{\zeta}(\Phi u)dz - \frac{\partial\zeta}{\partial x}\,(\Phi u)|_{z=\zeta} + \frac{\partial z_b}{\partial x}\,(\Phi u)|_{z=z_b} \\ &= \frac{\partial(\overline{\Phi}\,\overline{u}h)}{\partial x} - \frac{\partial\zeta}{\partial x}\,(\Phi u)|_{z=\zeta} + \frac{\partial z_b}{\partial x}\,(\Phi u)|_{z=z_b}. \end{aligned} \tag{4.5.35}$$

It is noticed that in a similar way to the fluid density, the temperature and the salinity do not vary in the vertical direction, it then follows that:

$$\Phi|_{z=\zeta} = \Phi|_{z=z_b} = \overline{\Phi}, \tag{4.5.36}$$

where $\overline{\Phi}$ is the depth-average mean value of either the temperature, or the salinity. As a consequence of that, we can rewrite K_2 as follows

$$K_2 = \frac{\partial(\overline{\Phi}\,\overline{u}h)}{\partial x} - \overline{\Phi}\left(\frac{\partial\zeta}{\partial x}\,u|_{z=\zeta} + \frac{\partial z_b}{\partial x}\,u|_{z=z_b}\right) \tag{4.5.37}$$

Finally, evaluating K_3 gives us the following expression

$$\begin{aligned} K_3 &= (\Phi w)|_{z=\zeta} - (\Phi w)|_{z=z_b} \\ &= \overline{\Phi}\left(w|_{z=\zeta} - w|_{z=z_b}\right). \end{aligned} \tag{4.5.38}$$

Consequently, by summing all expressions for K_1, K_2, K_3, it gives

$$\begin{aligned} K_1 + K_2 + K_3 = \frac{\partial(\overline{\Phi}\,h)}{\partial t} + \frac{\partial(\overline{\Phi}\,\overline{u}h)}{\partial x} - \overline{\Phi}\left(\frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial x}\,u|_{z=\zeta} - w|_{z=\zeta}\right) + \\ \overline{\Phi}\left(\frac{\partial z_b}{\partial x}\,u|_{z=z_b} - w|_{z=z_b}\right). \end{aligned} \tag{4.5.39}$$

Using the boundary conditions (4.5.4) and (4.5.5), the previous equation above becomes:

$$K_1 + K_2 + K_3 = \frac{\partial(\overline{\Phi}\,h)}{\partial t} + \frac{\partial(\overline{\Phi}\,\overline{u}h)}{\partial x} \tag{4.5.40}$$

where Φ may be either the temperature, or the salinity. The depth-average of the continuity equation, momentum equation, and the scalar conservation equation are summarized as follows:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial(\overline{u}h)}{\partial x} = 0, \\[2ex] \dfrac{\partial(\overline{u}\,h)}{\partial t} + \dfrac{\partial}{\partial x}\left(\overline{u}^2 h + \dfrac{1}{2}\dfrac{\overline{\rho}}{\rho_0}\,g\,h^2\right) = -\dfrac{\partial z_b}{\partial x}\left(\dfrac{\overline{\rho}}{\rho_0}\right) gh, \\[2ex] \dfrac{\partial(\overline{\Phi}\,h)}{\partial t} + \dfrac{\partial(\overline{\Phi}\,\overline{u}h)}{\partial x} = 0. \end{cases} \tag{4.5.41}$$

Furthermore, the combination of the depth-average of the continuity equation and the momentum equation forms the one-dimensional shallow water equations.

4.5.6 Multi-layer shallow water models

In the case where there are some stratifications over the vertical direction of the domain, it is convenient to subdivide the domain into different layers and perform the integration of the governing equations over the height of each layer (see Figure

4.3). For instance: freshwater layer flowing above a seawater layer, or an upper layer of the sea which is heated by the sun and its density becomes different from the rest of the body of water. Here, we shall only present a model which is based on two layers of fluids since it is simple and it illustrates the principle for several layers well enough.

Furthermore, the process of computation is completely the same as we have seen in the one layer except for a small amount of manipulation in the gradient of the pressure. Performing separately the depth-average of the governing equations for each layer yields the results which are presented below:

- For layer 1, we have

$$\begin{cases} \dfrac{\partial h_1}{\partial t} + \dfrac{\partial(\,\overline{u_1} h_1)}{\partial x} = 0, \\[2ex] \dfrac{\partial(\overline{u_1}\, h_1)}{\partial t} + \dfrac{\partial}{\partial x}\left(\overline{u_1}^2 h_1 + \dfrac{1}{2}\dfrac{\overline{\rho_1}}{\rho_0}\, g\, h_1^2\right) = -\dfrac{\partial z_b}{\partial x}\left(\dfrac{\overline{\rho_1}}{\rho_0}\right) g h_1 - \dfrac{\partial h_2}{\partial x}\dfrac{\overline{\rho_1}}{\rho_0} g h_1, \\[2ex] \dfrac{\partial(\,\overline{\Phi_1}\, h_1)}{\partial t} + \dfrac{\partial(\,\overline{\Phi_1}\, u_1 h_1)}{\partial x} = 0, \end{cases} \tag{4.5.42}$$

where all the variables with subscript 1 define the depth-average over layer 1.

- For layer 2, we have

$$\begin{cases} \dfrac{\partial h_2}{\partial t} + \dfrac{\partial(\,\overline{u_2} h_2)}{\partial x} = 0, \\[2ex] \dfrac{\partial(\overline{u_2}\, h_2)}{\partial t} + \dfrac{\partial}{\partial x}\left(\overline{u_2}^2 h_2 + \dfrac{1}{2}\dfrac{\overline{\rho_2}}{\rho_0}\, g\, h_2^2\right) = -\dfrac{\partial z_b}{\partial x}\left(\dfrac{\overline{\rho_2}}{\rho_0}\right) g h_2 - \dfrac{g h_2}{\rho_0}\dfrac{\partial\,(\overline{\rho_1} h_1)}{\partial x}, \\[2ex] \dfrac{\partial(\,\overline{\Phi}_2\, h_2)}{\partial t} + \dfrac{\partial(\,\overline{\Phi}_2\, \overline{u_2} h_2)}{\partial x} = 0, \end{cases} \tag{4.5.43}$$

where all the variables with subscript 2 define the depth-average over layer 2.

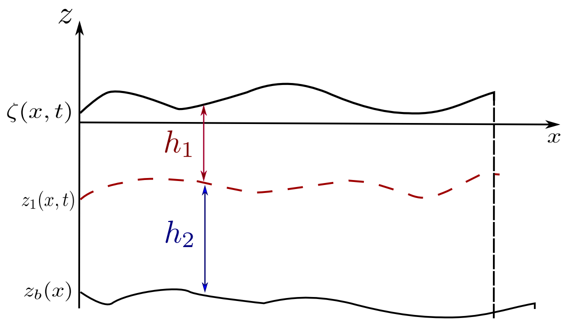

Figure 4.3: Two layers of shallow water flows

4.6 Summary

From this chapter, we have reviewed the stress equation using the momentum principle. By using the latter equation and by expanding the stress tensor, we derived the Naver-Stokes equation which satisfies the incompressible condition that we have derived in the previous chapter. Furthermore, we have discussed and have applied the hydrostatic and the Boussinesq approximations to the Navier-Stokes equations. We have seen that these approximations hold according to the physical and the geometrical property of the ocean where the flow occurs. And finally, due to the presence of the stratification in the ocean, we have applied the technique of depth-averaging to the set of governing equation in order to deduce the shallow water equations.

Chapter 5

Modified finite volume method

The purpose of this chapter is to numerically solve the one-dimensional shallow water equations that were derived in Chapter 4, and the finite volume method (FVM) is used. In order to reach this goal, a brief introduction about the FVM is first given in Section 5.1. In the following Section 5.2, the modified FVM of characteristics is described in detail by starting from the discretization of the governing equations to the construction of the numerical flux at each the interface of the control volume. Moreover, some notions of boundary conditions are presented in Section 5.3. And finally, all the steps leading to the solution of the governing equations and some test cases are presented in the last section of this chapter.

5.1 Introduction

In the FVM, the differential equations are integrated over a control volume, and the resulting equations are represented in a discretized form. In this chapter, we will only focus on the one-dimensional FVM for hyperbolic problems since the one-dimensional shallow water equations are formed by a set of hyperbolic partial differential equations. In particular, a modified FVM of characteristics formulated by Benkhaldoun and Seaïd (2010) is used. One of the features that distinguishes the FVM from others discretization methods is the construction of the numerical flux at each surface of the control volume. Several methods can be found in the literature to construct this numerical flux. For instance, a commonly used approach is to derive the numerical flux from the solution of the local one-dimensional Riemann problem in the direction normal to the cell surface of the control volume (LeVeque, 2002). Such a method is referred to as the approximate Riemann method and is defined as a fundamental tool of the FVM for the hyperbolic partial differential equations. Moreover, some specific approaches can also be used to approximate the numerical flux, for instance, the Lax-Friedrichs or the two-step Lax-Wendroff methods. The latter is second-order accurate, compared to the Lax-Friedrichs method, which is only first order. In the FVM used here, the method of characteristics will be applied to evaluate the numerical flux at each interface of the control volume. In this case, the governing equations are rewritten in non-conservative form and will be integrated along a characteristic curve from which the numerical flux can be deduced. Furthermore, according to Benkhaldoun and Seaïd (2010), this type of modified FVM of characteristics belongs to the class of *predictor-corrector* schemes where the corrector scheme is the discretized form of governing equations while the predictor scheme is the discretized form of the governing equations in non-conservative form. The advantage of using this method of characteristics is the fact that the approximate Riemann solver method will not be needed anymore. In addition, no boundary conditions will be considered when solving the governing equations in non-conservative form. It is also noticed that the non-conservative form does not mean that the variables are not conserved anymore, it is just a form of rewriting the governing equations. In non-conservative form, the material derivative appears in the governing equations. Let us first review all of the equations which have been derived in the previous chapter.

5.1.1 Governing equations

First of all, we have the conservation of mass and momentum in the one-dimensional shallow water equations:

$$\begin{aligned} &\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0, \\ &\frac{\partial (u\,h)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{1}{2}\frac{\rho}{\rho_0}\,g\,h^2\right) = -\frac{\partial z_b}{\partial x}\frac{\rho}{\rho_0}\,gh. \end{aligned} \tag{5.1.1}$$

Notice that, in the previous chapter, we have used over-bars to denote the depth-average, for simplicity of notation. We will denote here the depth-average without bars on top. Then in Equation (5.1.1), ρ is the depth-average of the fluid density, u is the depth-average of the fluid density, h is the height of the water, z_b is the coordinate of the bottom, and ρ_0 is the reference constant density. Since the fluid density ρ is not considered constant, it might depend here on the temperature T and the salinity S. Therefore, in addition to Equation (5.1.1), we need to take into consideration the equations of the conservation of temperature and salinity. According to the result from Chapter 2, it can be assumed that the heat and the salt do not diffuse, then their conservation gives the following equations:

$$\frac{\partial(T\,h)}{\partial t} + \frac{\partial(\,Thu)}{\partial x} = 0,$$

$$\frac{\partial(S\,h)}{\partial t} + \frac{\partial(\,Shu)}{\partial x} = 0, \tag{5.1.2}$$

where T is the temperature and S is the salinity. For a simple representation, Equations (5.1.1) to (5.1.2) can be rewritten in vector form as:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \\ Th \\ Sh \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ hu^2 + \frac{1}{2}g\frac{\rho}{\rho_0}h^2 \\ Thu \\ Shu \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\frac{\rho}{\rho_0}\frac{\partial z_b}{\partial x} \\ 0 \\ 0 \end{pmatrix}. \tag{5.1.3}$$

Note that Equation (5.1.3) is a coupled system. Let $\mathbf{U}$ be a vector such that $\mathbf{U} = (u_1,\, u_2,\, u_3, u_4) = (h,\, hu,\, Th,\, Sh)$, then Equation (5.1.3) can be rewritten as

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} u_2 \\ \frac{u_2^2}{u_1} + \frac{1}{2}g\frac{\rho}{\rho_0}u_1^2 \\ \frac{u_3 u_2}{u_1} \\ \frac{u_4 u_2}{u_1} \end{pmatrix} = \begin{pmatrix} 0 \\ -g\,u_1\frac{\rho}{\rho_0}\frac{\partial z_b}{\partial x} \\ 0 \\ 0 \end{pmatrix}, \tag{5.1.4}$$

which can be expressed into a compact form as follows:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U},\rho)}{\partial x} = \mathbf{Q}(\mathbf{U},\rho,z_b), \tag{5.1.5}$$

where $\mathbf{U}$ is the vector form of the conserved variables, $\mathbf{F}$ is the vector flux functions, and $\mathbf{Q}$ is the vector source term.

It is also noticed that Equation (5.1.5) can also be expressed as a function of a two-dimensional matrix form, which will be a very useful expression later in this chapter. Since ρ is not constant, Equation (5.1.5) needs to be rearranged as described below:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U},\rho)}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial x} = -\frac{\partial \mathbf{F}(\mathbf{U},\rho)}{\partial \rho}\frac{\partial \rho}{\partial x} + \mathbf{Q}(\mathbf{U},\rho,z_b). \tag{5.1.6}$$

From the equation above, let $\underline{\underline{A}}$ be a square matrix such that $\underline{\underline{A}} = \dfrac{\partial \mathbf{F}(\mathbf{U},\rho)}{\partial \mathbf{U}}$ which is defined as a Jacobian matrix and expressed as follows:

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \dfrac{g\,\rho\,u_1}{\rho_0} - \dfrac{{u_2}^2}{{u_1}^2} & \dfrac{2u_2}{u_1} & 0 & 0 \\ -\dfrac{u_2 u_3}{{u_1}^2} & \dfrac{u_3}{u_1} & \dfrac{u_2}{u_1} & 0 \\ -\dfrac{u_2 u_4}{{u_1}^2} & \dfrac{u_4}{u_1} & 0 & \dfrac{u_2}{u_1} \end{pmatrix}. \tag{5.1.7}$$

Equation (5.1.6) can be re-expressed as below:

$$\frac{\partial \mathbf{U}}{\partial t} + \underline{\underline{A}}(\mathbf{U})\,\frac{\partial \mathbf{U}}{\partial x} = -\,\frac{\partial \mathbf{F}(\mathbf{U},\rho)}{\partial \rho}\,\frac{\partial \rho}{\partial x} + \mathbf{Q}(\mathbf{U},\rho,z_b). \tag{5.1.8}$$

And computing the eigenvalues of $\underline{\underline{A}}$ will give the following results:

$$\lambda_1 = h, \qquad \lambda_2 = h, \qquad \lambda_3 = -\sqrt{\frac{g\rho h}{\rho_0}} + u, \qquad \lambda_4 = \sqrt{\frac{g\rho h}{\rho_0}} + u. \tag{5.1.9}$$

These eigenvalues above are needed later in the FVM in order to satisfy the criterion of stability. Then the main interest of representing the shallow water equations in matrix form is to derive these eigenvalues.

Moreover, as we have mentioned earlier, a non-conservative form of the governing equations will be used. This can be obtained by using the definition of the material derivative and can be written as follows:

$$\begin{cases} \dfrac{Dh}{Dt} + h\dfrac{\partial u}{\partial x} = 0, \\[2ex] \dfrac{Du}{Dt} + g\dfrac{\rho}{\rho_0}\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}h + z_b\right) = -\dfrac{1}{2}\dfrac{g}{\rho_0}\dfrac{\partial(\rho h)}{\partial x}, \\[2ex] \dfrac{DT}{Dt} = 0, \\[2ex] \dfrac{DS}{Dt} = 0, \end{cases} \tag{5.1.10}$$

where $\dfrac{D}{Dt}$ denotes the material derivative which is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}. \tag{5.1.11}$$

Furthermore, as the fluid density ρ depends on the temperature and the salinity, a relationship between them is needed in order to calculate one from the other. Several relationships can be found in the literature which has been referred as the equations of state in Chapter 1. In our case, we will use one of these equations of state to update the value of the density by using the values of temperature and salinity obtained from solving Equation (5.1.2). Let us now introduce the basic notion of the FVM starting from the integration of our governing equations over the control volumes.

5.2 Finite volume method of characteristics

The goal of this section is to derive all the discretized governing equations using the FVM.

5.2.1 Discretization of the governing equations

Let us first discretize the physical space domain into several control volumes. Here, for the one-dimensional domain, the control volume is illustrated as an interval $\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ with a uniform size $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and centered at the node x_i. We can then subdivide the space domain into $N-1$ control volumes and two half-control volumes with $x_i = i\Delta x$ for $i = 0, \cdots, N$. These intervals are illustrated in Figure 5.1.

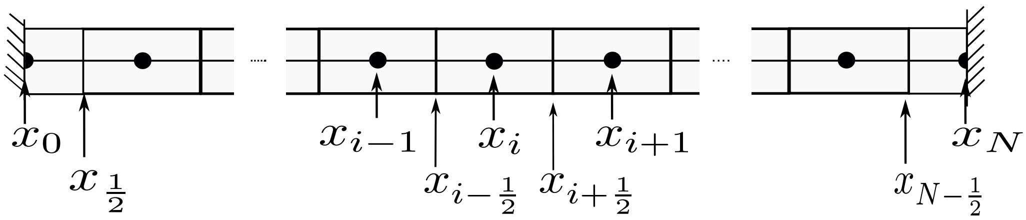

Figure 5.1: Scheme of the control volumes

Furthermore, the temporal domain is subdivided into subintervals $[t_n, t_n + \Delta t]$ with stepsize Δt where $t_n = n\Delta t$. As mentioned earlier, the numerical equivalent of the governing equations consists of using the predictor-corrector schemes. We first start by deriving the corrector scheme which can be derived by integrating Equation (5.1.5) with respect to space and time over the time-space control domain

$[t_n, t_n + \Delta t] \times \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$. This integration is performed as follows:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \frac{\partial \mathbf{U}}{\partial t}\, dt\, dx + \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial \mathbf{F}(\mathbf{U}, \rho)}{\partial x}\, dx\, dt = \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{Q}(\mathbf{U}, \rho, z_b)\, dx\, dt. \tag{5.2.1}$$

Note that $\dfrac{\partial \mathbf{U}}{\partial t}$ is firstly integrated to t and $\dfrac{\partial \mathbf{F}}{\partial x}$ is secondly integrated to space x. Then simplifying the terms on the left-hand side of Equation (5.2.1) where the integrated contains derivatives to t and x, gives

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(t_{n+1}, x) dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(t_n, x) dx + \int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{U}(t, x_{i+\frac{1}{2}}), \rho(t, x_{i+\frac{1}{2}}))\, dt - \int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{U}(t, x_{i-\frac{1}{2}}), \rho(t, x_{i-\frac{1}{2}}))\, dt = \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{Q}(\mathbf{U}, \rho, z_b)\, dx dt. \tag{5.2.2}$$

We shall use the *mean-value theorem* to approximate some of the integrals above. Let us briefly remind ourselves of this theorem by considering an arbitrary real function $\phi(x)$ defined over an interval $[a, b]$. The theorem states that if $\phi(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is alpha $\alpha \in [a, b]$ such that

$$\int_a^b \phi(x) dx = \phi(\alpha)(b - a). \tag{5.2.3}$$

By considering that all the state variables are defined as continuous functions in the control domain, applying the mean-value theorem in the two first terms on the left-hand side of Equation (5.2.2) yields

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(t_{n+1}, x) dx = \mathbf{U}(t_{n+1}, \eta) \Delta x \quad \text{and} \quad \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(t_n, x) dx = \mathbf{U}(t_n, \eta) \Delta x, \tag{5.2.4}$$

where $\eta \in \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$. In this method we replace η with x_i and using the notation which denotes $\boldsymbol{U}_i^n$ as an approximate value of $\mathbf{U}(t^n, x_i)$ gives the following expressions:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(t_{n+1}, x) dx = \boldsymbol{U}_i^{n+1} \Delta x \quad \text{and} \quad \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(t_n, x) dx = \boldsymbol{U}_i^n \Delta x. \tag{5.2.5}$$

LeVeque (2002) stated that it is not possible to have an exact evaluation of the time integrals of the physical flux $\mathbf{F}(\mathbf{U}(t, x_{i\pm\frac{1}{2}}), \rho(t, x_{i\pm\frac{1}{2}}))$ since $\mathbf{U}(t, x_{i\pm\frac{1}{2}}, t)$ and

$\rho(t, x_{i\pm\frac{1}{2}})$ change with respect to time along each interface of the control volume. Then we shall use the notation $\boldsymbol{\mathcal{F}}^n_{i\pm\frac{1}{2}}$ to indicate the numerical flux which is the approximation to the time average of the vector flux along $x_{i\pm\frac{1}{2}}$. Its expression is given by

$$\boldsymbol{\mathcal{F}}^n_{i\pm\frac{1}{2}} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{U}(t, x_{i\pm\frac{1}{2}}), \rho(t, x_{i\pm\frac{1}{2}}))\, dt. \tag{5.2.6}$$

Using also the mean-value theorem, the integral of the source term on the right-hand side of Equation (5.2.2) can be rewritten as

$$\int_{t_n}^{t_{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{Q}(\mathbf{U}, \rho, z_b)\, dt dx = \boldsymbol{\mathcal{Q}}^n_i \Delta x \Delta t, \tag{5.2.7}$$

where $\boldsymbol{\mathcal{Q}}^n_i$ is the approximate value of the source term averaged per control domain. Following the idea from Benkhaldoun and Seaïd (2010), the expression of $\boldsymbol{\mathcal{Q}}^n_i$ can represented as follows:

$$\boldsymbol{\mathcal{Q}}^n_i = \begin{pmatrix} 0 \\ \dfrac{g}{2\rho_0}\left(\widehat{\rho h}\right)^n_i \dfrac{(z_b)_{i+1} - (z_b)_{i-1}}{\Delta x} \\ 0 \end{pmatrix}, \tag{5.2.8}$$

where the expression $\left(\widehat{\rho h}\right)^n_i$ is obtained from the fact that the discretization of the governing equations satisfies the $C-$property or well-balanced property. A numerical scheme satisfies the $C-$property if it preserves the steady state of the flow, which is described as below:

$$h + z_b = \text{Constant}, \qquad \rho = \text{Constant}, \quad \text{and} \qquad u = 0. \tag{5.2.9}$$

Without going into details, the expression of $(\widehat{\rho h})^n_i$ is deduced from using the property above and given by

$$(\widehat{\rho h})^n_i = \frac{1}{4}\left((\rho h)^n_{i+1} + 2(\rho h)^n_i + (\rho h)^n_{i-1}\right). \tag{5.2.10}$$

Therefore, Equation (5.2.8) can be rewritten as below:

$$\boldsymbol{\mathcal{Q}}^n_i = \begin{pmatrix} 0 \\ \dfrac{g}{8\rho_0}\left[(\rho h)^n_{i+1} + 2(\rho h)^n_i + (\rho h)^n_{i-1}\right] \dfrac{\left((z_b)_{i+1} - (z_b)_{i-1}\right)}{\Delta x} \\ 0 \end{pmatrix}. \tag{5.2.11}$$

As a consequence of these approximations, the discretized form of the governing equations can be written as the following expression:

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{\mathcal{F}}^n_{i+\frac{1}{2}} - \boldsymbol{\mathcal{F}}^n_{i-\frac{1}{2}}\right) + \Delta t \boldsymbol{\mathcal{Q}}_i^n, \tag{5.2.12}$$

and can be expanded as below:

$$\begin{cases} h_i^{n+1} = h_i^n - \dfrac{\Delta t}{\Delta x}\left(h^n_{i+\frac{1}{2}} u^n_{i+\frac{1}{2}} - h^n_{i-\frac{1}{2}} u^n_{i-\frac{1}{2}}\right), \\ r_i^{n+1} = r_i^n - \dfrac{\Delta t}{\Delta x}\left[\left(h^n_{i+\frac{1}{2}}\left(u^n_{i+\frac{1}{2}}\right)^2 + \dfrac{1}{2}\dfrac{g}{\rho_0}\rho^n_{i+\frac{1}{2}}\left(h^n_{i+\frac{1}{2}}\right)^2\right) - \left(h^n_{i-\frac{1}{2}}\left(u^n_{i-\frac{1}{2}}\right)^2 + \dfrac{1}{2}\dfrac{g}{\rho_0}\rho^n_{i-\frac{1}{2}}\left(h^n_{i-\frac{1}{2}}\right)^2\right)\right] - \\ \qquad \dfrac{1}{8}\dfrac{g}{\rho_0}\dfrac{\Delta t}{\Delta x}\left[(\rho h)^n_{i+1} + 2(\rho h)^n_i + (\rho h)^n_{i-1}\right]\left((z_b)_{i+1} - (z_b)_{i-1}\right), \\ \mathcal{T}_i^{n+1} = \mathcal{T}_i^n - \dfrac{\Delta t}{\Delta x}\left((\mathcal{T}^n_{i+\frac{1}{2}} u^n_{i+\frac{1}{2}}) - (\mathcal{T}^n_{i-\frac{1}{2}} u^n_{i-\frac{1}{2}})\right), \\ \mathcal{S}_i^{n+1} = \mathcal{S}_i^n - \dfrac{\Delta t}{\Delta x}\left((\mathcal{S}^n_{i+\frac{1}{2}} u^n_{i+\frac{1}{2}}) - (\mathcal{S}^n_{i-\frac{1}{2}} u^n_{i-\frac{1}{2}})\right), \end{cases} \tag{5.2.13}$$

where $r = hu$, $\mathcal{T} = Th$, and $\mathcal{S} = Sh$.
Let us now discuss the method of characteristics in order to evaluate the numerical flux at each interface of the control volume.

5.2.2 Evaluation of $\boldsymbol{\mathcal{F}}^n_{i\pm\frac{1}{2}}$ using the method of characteristics

In this subsection, we are going to evaluate $\boldsymbol{\mathcal{F}}^n_{i\pm\frac{1}{2}}$ by applying the method of characteristics to Equation (5.1.10). Let us first review the basic idea of how to solve a first order partial differential equation using the method of characteristics.

5.2.2.1 Method of characteristics

Consider the simple first order linear partial differential equation

$$a(t,x)\frac{\partial u}{\partial t} + b(t,x)\frac{\partial u}{\partial x} = c(t,x). \tag{5.2.14}$$

Suppose that $u(t,x)$ is a general solution of Equation (5.2.14) and $G \equiv G(t,x,u(t,x))$ designates the solution surface. Let $\underline{v} = [a(t,x), b(t,x), c(t,x)]$ and $\underline{n} = \left[\dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}, -1\right]$ be two vectors deduced from Equation (5.2.14). As $u(t,x)$ is the solution which satisfies Equation (5.2.14) then at each point on the surface G we have

$$\underline{v} \cdot \underline{n} = 0. \tag{5.2.15}$$

We know from calculus that the vector normal to the surface G at the point A $= (t, x, u(t,x))$ is given by $\underline{n}$ and then, if the vector $\underline{v}$ is perpendicular to $\underline{n}$ at each point A, $\underline{v}$ lies in the tangent plane to G.
Consequently, to solve Equation (5.2.14) with the method of characteristics, is to find G such that at each point A on G, $\underline{v}$ lies in the tangent plane to G. Moreover, it would be preferable to look for a curve $C = C(t, x, u(t,x))$ which lies on G instead of constructing G itself. Then C is characterized in such a way that at each point on the curve, $\underline{v}$ is tangent to C. In particular, the curve can be parametrized by τ and will satisfy the following ordinary differential equations

$$\begin{cases} \dfrac{dt}{d\tau} = a(t(\tau), x(\tau)) \\ \\ \dfrac{dx}{d\tau} = b(t(\tau), x(\tau)) \\ \\ \dfrac{du}{d\tau} = c(t(\tau), x(\tau)) \end{cases} \tag{5.2.16}$$

The curve C is known as an integral curve for the vector field $\underline{v}$. For a partial differential equation, the integral curves are known as the characteristic curves and the union of these curves gives the solution surface of the partial differential equation.
Example: In order to illustrate this method, let us consider a simple transport equation as

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \tag{5.2.17}$$

where a is a constant velocity and u denotes an arbitrary function. As stated in Equation (5.2.16), we are looking for a solution of Equation (5.2.17) which satisfies

$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \\ \dfrac{dx}{d\tau} = a \\ \\ \dfrac{du}{d\tau} = 0. \end{cases} \tag{5.2.18}$$

Integrating the set of ordinary differential equations above with respect to τ yields

$$t = \tau + c_1 \tag{5.2.19}$$

$$x = a\tau + c_2 \tag{5.2.20}$$

$$u = c_3 \tag{5.2.21}$$

By eliminating τ from Equation (5.2.19) and Equation (5.2.20), we will have a characteristic curve as which is defined as line in $\mathbb{R}^3$ such that $x - at = x_0, u = u_0$ where x_0 and u_0 are constants. This implies that the solution of Equation (5.2.17) is the integral surface formed from a union of these characteristic curves. From Equation (5.2.21), it can be seen that $u(x,t)$ is constant along the lines $x - at = x_0$. In other words, $u(x,t) = f(x-at)$ for any arbitrary given continuous function f. It is noticed that the latter function can be obtained by imposing an extra condition to the surface which states that the surface contains a curve $C_0 = \{(0, x, f(x))\}$ such that the characteristic curves start from this latter curve.

5.2.2.2 Application of the method of characteristics to the governing equations

In order to evaluate the value of the numerical flux at each cell interface of the control volume presented in Equation (5.2.12), the method of characteristics will be applied to Equation (5.1.10). The latter can be rewritten as:

$$\begin{cases} \dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x} = -h\dfrac{\partial u}{\partial x}, \\[2ex] \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = -g\dfrac{\rho}{\rho_0}\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}h + z_b\right) - \dfrac{1}{2}\dfrac{g}{\rho_0}\dfrac{\partial(\rho h)}{\partial x}, \\[2ex] \dfrac{\partial T}{\partial t} + u\dfrac{\partial T}{\partial x} = 0, \\[2ex] \dfrac{\partial S}{\partial t} + u\dfrac{\partial S}{\partial x} = 0. \end{cases} \tag{5.2.22}$$

We would then like to construct a curve C' in $(x,t)-$plane defined by the vector field $(1, u)$ such that the set of equations above can be integrated along C'. Let $x = X(\tau)$ be the parametric representation of C' such that it satisfies the following set of equations:

$$\begin{cases} \dfrac{dX}{d\tau} = u(\tau, X(\tau)), \\[2ex] \dfrac{dt}{d\tau} = 1. \end{cases} \tag{5.2.23}$$

Notice from the equations above that the characteristic curve which correspond to each equation in Equation (5.2.22) are defined by the same horizontal flow velocity u. In order to derive the characteristic curve above, we need to integrate Equation (5.2.23) with respect to the parameter τ over a specific time interval. The integration is performed numerically but not in the same way as we have seen in the previous example. In order to do that, let us imagine that the fluid particles are moving along a particular path and pass through a regular grid-point. We refer here a regular-grid point as the point at each interface of the control volume.

According to Benkhaldoun and Seaïd (2010), the idea to find the characteristic curve is to impose the regular grid-point at a new time level and to back-track the fluid particle trajectory into the previous time level (see Figure 5.2). In other words, the fluid particle is set in such a way that it arrives exactly at the regular grid-point at the new time level from its departure at the characteristic root. In this case, the characteristic root is the value of $X(\tau)$ at the departure time τ and the regular grid-point is defined as $x_{i+\frac{1}{2}}$. In doing so, Equation (5.2.23) is integrated with respect to the parameter τ from the departure time t^n to the time where the fluid particles arrive at the regular grid-point $x_{i+\frac{1}{2}}$. Furthermore, it has been proved by Benkhaldoun and Seaïd (2010) that the fluid particle takes a half time step $\dfrac{\Delta t}{2}$ to arrive at $x_{i+\frac{1}{2}}$ when it starts to move from the departure position $X_{i+\frac{1}{2}}(t^n)$. Then the characteristic curve associated with Equation (5.2.22) for each fluid particles at $x_{i+\frac{1}{2}}$ satisfies the following initial value problem:

$$\frac{dX_{i+\frac{1}{2}(\tau)}}{d\tau} = u\left(\tau, X_{i+\frac{1}{2}}(\tau)\right), \qquad \text{and} \qquad X_{i+\frac{1}{2}}(t_n + \frac{\Delta t}{2}) = x_{i+\frac{1}{2}}. \tag{5.2.24}$$

Figure 5.2 illustrates the characteristic curve of a particular fluid particle.

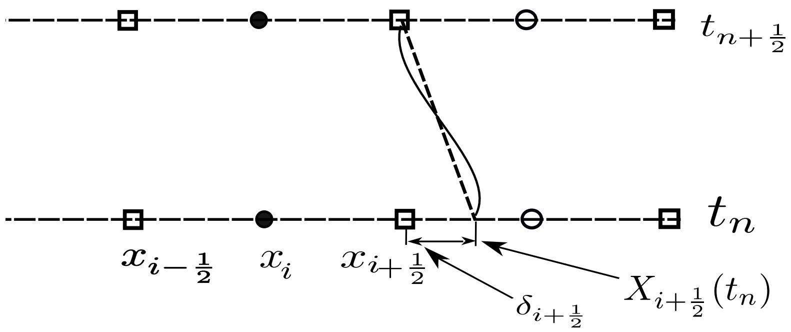

Figure 5.2: A scheme showing the trajectory of a fluid particle which is used in the calculation of the departure points.

It can be seen from Figure 5.2 that the fluid particle starts to move from the position $X_{i+\frac{1}{2}}$ at time t_n and arrives at the cell interface of the control volume after a half-time step. In this case, the exact trajectory is represented by a solid line and the approximate trajectory is represented with a dashed-line. Integrating Equation (5.2.24) over $\tau = \left[t_n, t_n + \dfrac{1}{2}\Delta t\right]$ yields

$$X_{i+\frac{1}{2}}(t_n) = x_{i+\frac{1}{2}} - \int_{t_n}^{t_n+\frac{\Delta t}{2}} u(\tau, X_{i+\frac{1}{2}}(\tau))d\tau. \tag{5.2.25}$$

Let $\delta_{i+\frac{1}{2}}$ be the distance between the regular grid-point and the characteristic foot $X_{i+\frac{1}{2}}(t_n)$ at the departure time such that

$$\delta_{i+\frac{1}{2}}(t_n) \;=\; X_{i+\frac{1}{2}} - x_{i+\frac{1}{2}}. \tag{5.2.26}$$

Furthermore,

$$\delta_{i+\frac{1}{2}}(t_n) \;=\; \int_{t_n}^{t_n+\frac{\Delta t}{2}} u(\tau, X_{i+\frac{1}{2}}(\tau))d\tau. \tag{5.2.27}$$

In order to compute the displacement $\delta_{i+\frac{1}{2}}$ in Equation (5.2.27), we first need to evaluate $u(t,x)$ at a point on the interval $[x_i, x_{i+1}]$. However, in the case where $u(t,x)$ is not a given function, we shall linearly interpolate u on this interval. Once we have the value of u, a simple numerical integration can be used to evaluate $\delta_{i+\frac{1}{2}}$, for instance, Euler's method and followed by a successive iteration procedure. Notice that after each iteration, an interpolation occurs. We have then the expression $\delta_{i+\frac{1}{2}}$ which is described below:

$$\delta_{i+\frac{1}{2}} = \frac{\Delta t}{2} u\left(t_n, x_{i+\frac{1}{2}}\right), \tag{5.2.28}$$

where $\delta_{i+\frac{1}{2}}$ is evaluated using the following iteration:

$$\begin{aligned} \delta^{(0)}_{i+\frac{1}{2}} &= \frac{\Delta t}{2} u\left(t_n, x_{i+\frac{1}{2}}\right) \\ &\;\vdots \qquad\qquad \vdots \\ \delta^{(k)}_{i+\frac{1}{2}} &= \frac{\Delta t}{2} u\left(t_n, x_{i+\frac{1}{2}} - \delta^{(k-1)}_{i+\frac{1}{2}}\right). \end{aligned} \tag{5.2.29}$$

The iteration in Equation (5.2.29) is completed when the following is satisfied,

$$\frac{\|\delta^{(k)} - \delta^{(k-1)}\|}{\|\delta^{(k-1)}\|} < \epsilon, \tag{5.2.30}$$

where the norm $\|\cdot\|$ is an infinity norm and ϵ is a given tolerance to ensure convergence of the iteration.

Once $\delta_{i+\frac{1}{2}}$ is obtained, the characteristic root $X_{i+\frac{1}{2}}(t_n)$ can be deduced using Equation (5.2.26). Then, the solutions h, u, T, S at this point can be calculated from their known values at the grid points using a linear interpolating polynomial.

Let us now integrate Equation (5.2.22) along the characteristic curve by starting to rewrite it in terms of the characteristic root $X_{i+\frac{1}{2}}(\tau)$ at the departure time τ as

$$\begin{cases} \dfrac{Dh}{D\tau}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) = -\left[h\dfrac{\partial u}{\partial x}\right]\left(\tau, X_{i+\frac{1}{2}}(\tau)\right), \\ \dfrac{Du}{D\tau}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) = -g\left[\dfrac{\rho}{\rho_0}\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}h + z_b\right)\right]\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) - \dfrac{1}{2}\dfrac{g}{\rho_0}\dfrac{\partial(\rho h)}{\partial x}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right), \\ \dfrac{DT}{D\tau}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) = 0, \\ \dfrac{DS}{D\tau}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) = 0. \end{cases} \tag{5.2.31}$$

Integrating Equation (5.2.31) above with respect to the parameter τ over the half-time interval yields

$$\begin{cases} \displaystyle\int_{t_n}^{t_n+\frac{\Delta t}{2}} \frac{Dh}{D\tau}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) d\tau = -\int_{t_n}^{t_n+\frac{\Delta t}{2}} \left[h\frac{\partial u}{\partial x}\right]\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) d\tau, \\ \displaystyle\int_{t_n}^{t_n+\frac{\Delta t}{2}} \frac{Dv}{D\tau}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) d\tau = -\int_{t_n}^{t_n+\frac{\Delta t}{2}} \left[g\frac{\rho}{\rho_0}\frac{\partial}{\partial x}\right]\left(\frac{1}{2}h + z_b\right)\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) d\tau - \\ \qquad\qquad\qquad\qquad \displaystyle\int_{t_n}^{t_n+\frac{\Delta t}{2}} \frac{1}{2}\frac{g}{\rho_0}\frac{\partial(\rho h)}{\partial x}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) d\tau, \\ \displaystyle\int_{t_n}^{t_n+\frac{\Delta t}{2}} \frac{DT}{D\tau}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) d\tau = 0, \\ \displaystyle\int_{t_n}^{t_n+\frac{\Delta t}{2}} \frac{DS}{D\tau}\left(\tau, X_{i+\frac{1}{2}}(\tau)\right) d\tau = 0. \end{cases} \tag{5.2.32}$$

Evaluating the integrals above give the following expression:

$$\begin{cases} h\left(t_n + \frac{\Delta t}{2},\, X_{i+\frac{1}{2}}(t_n + \frac{\Delta t}{2})\right) - h\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right) = -\frac{\Delta t}{2} h\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right) \overline{\overline{\frac{\partial u}{\partial x}}}\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right), \\ u\left(t_n + \frac{\Delta t}{2}, X_{i+\frac{1}{2}}(t_n + \frac{\Delta t}{2})\right) - u\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right) = -\frac{\Delta t}{2}\frac{g}{\rho_0}\rho\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right) \times \\ \overline{\overline{\frac{\partial}{\partial x}\left(\frac{1}{2}h + z_b\right)}}\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right) - \frac{\Delta t}{4}\frac{g}{\rho_0}\overline{\overline{\frac{\partial(\rho h)}{\partial x}}}\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right), \\ T\left(t_n + \frac{\Delta t}{2}, X_{i+\frac{1}{2}}(t_n + \frac{\Delta t}{2})\right) - T\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right) = 0, \\ S\left(t_n + \frac{\Delta t}{2}, X_{i+\frac{1}{2}}(t_n + \frac{\Delta t}{2})\right) - S\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right) = 0, \end{cases} \tag{5.2.33}$$

where the terms with double overbars indicate the average in time of the gradient in space. In this case, let $\overline{\overline{\frac{\partial u}{\partial x}}}\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right)$ be the average in time of the gradient of the flow velocity along the characteristic so that its value is given by

$$\overline{\overline{\frac{\partial u}{\partial x}}}\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right) = \frac{u(t_n, x_{i+1}) - u(t_n, x_i)}{\Delta x}. \tag{5.2.34}$$

Doing the same for $\overline{\overline{\frac{\partial}{\partial x}\left(\frac{1}{2}h + z_b\right)}}$ and $\overline{\overline{\frac{\partial(\rho h)}{\partial x}}}$ yields

$$\overline{\overline{\frac{\partial}{\partial x}\left(\frac{1}{2}h + z_b\right)}}\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right) = \frac{1}{\Delta x}\left[\left(\frac{1}{2}h(t_n, x_{i+1}) + z_b(x_{i+1})\right) - \left(\frac{1}{2}h(t_n, x_i) + z_b(x_i)\right)\right], \tag{5.2.35}$$

and

$$\overline{\overline{\frac{\partial(\rho h)}{\partial x}}}\left(t_n, X_{i+\frac{1}{2}}(t_n)\right) = \frac{1}{\Delta x}\left[\rho(t_n, x_{i+1})h(t_n, x_{i+1}) - \rho(t_n, x_i)h(t_n, x_i)\right]. \tag{5.2.36}$$

Substituting the expressions in Equations (5.2.34), (5.2.35) and (5.2.36) into (5.2.33) give the following result:

$$
\begin{cases}
h\left(t_n+\frac{\Delta t}{2},\, X_{i+\frac{1}{2}}(t_n+\frac{\Delta t}{2})\right)-h\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right)=-\frac{\Delta t}{2\Delta x}h\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right)u(t_n,x_{i+1})+\\
\qquad\qquad\frac{\Delta t}{2\Delta x}h\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right)u(t_n,x_i)\\
u\left(t_n+\frac{\Delta t}{2}, X_{i+\frac{1}{2}}(t_n+\frac{\Delta t}{2})\right)-u\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right)=-\frac{g}{\rho_0}\rho\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right)\times\\
\frac{\Delta t}{2\Delta x}\left[\left(\frac{1}{2}h(t_n,x_{i+1})+z_b(x_{i+1})\right)-\left(\frac{1}{2}h(x_i,t_n)+z_b(x_i)\right)\right]-\frac{\Delta t}{4}\frac{g}{\rho_0}\frac{1}{\Delta x}\rho(t_n,x_{i+1})h(t_n,x_{i+1})+\\
\qquad\qquad\frac{\Delta t}{4}\frac{g}{\rho_0}\frac{1}{\Delta x}\rho(t_n,x_i)h(t_n,x_i),\\
T\left(t_n+\frac{\Delta t}{2}, X_{i+\frac{1}{2}}(t_n+\frac{\Delta t}{2})\right)-T\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right)=0,\\
S\left(t_n+\frac{\Delta t}{2}, X_{i+\frac{1}{2}}(t_n+\frac{\Delta t}{2})\right)-S\left(t_n,\, X_{i+\frac{1}{2}}(t_n)\right)=0.
\end{cases}
\tag{5.2.37}
$$

In order to construct the numerical flux $\boldsymbol{\mathcal{F}}^n_{i+\frac{1}{2}}$ in Equation (5.2.13), let ω be a function which represents any of the solutions h, u, ρ, T and S such that

$$
\omega(t_n, X_{i+\frac{1}{2}}(t_n)) = \mathcal{P}_\omega(t_n, X_{i+\frac{1}{2}}(t_n)) = \tilde{\omega}^n_i, \tag{5.2.38}
$$

where P_ω represents an interpolating polynomial related to ω. In addition, let us use the following notation described below:

$$
\omega^n_{i+\frac{1}{2}} = \omega\left(t_n+\frac{\Delta t}{2},\, X_{i+\frac{1}{2}}(t_n+\frac{\Delta t}{2})\right) \qquad \text{and} \qquad \omega^n_i = \omega(t_n,\, x_{i+1}). \tag{5.2.39}
$$

Using the notations that we have defined above, Equation (5.2.37) becomes

$$
\begin{cases}
h^n_{i+\frac{1}{2}} = \tilde{h}^n_{x_{i+\frac{1}{2}}} - \frac{1}{2}\frac{\Delta t}{\Delta x}\widetilde{h}^n_{i+\frac{1}{2}}(u^n_{i+1}-u^n_i),\\
u^n_{i+\frac{1}{2}} = \widetilde{u}^n_{i+\frac{1}{2}} - \frac{g}{2}\frac{\Delta t}{\Delta x}\frac{\widetilde{\rho}^n_{i+\frac{1}{2}}}{\rho_0}\left(\frac{1}{2}h^n_{i+1}+(z_b)_{i+1}-\frac{1}{2}h^n_i-(z_b)_i\right)-\frac{1}{2}\frac{\Delta t}{\Delta x}\frac{g}{\rho_0}\left(\frac{1}{2}\rho^n_{i+1}h^n_{i+1}-\frac{1}{2}\rho^n_i h^n_i\right),\\
T^n_{i+\frac{1}{2}} = \widetilde{T}^n_{i+\frac{1}{2}},\\
S^n_{i+\frac{1}{2}} = \widetilde{S}^n_{i+\frac{1}{2}}.
\end{cases}
\tag{5.2.40}
$$

The set of equations above represent the predictor scheme that will be used to evaluate the numerical flux at each interface of the control volume.

5.3 Boundary conditions

The discretized form of the governing equations which has been derived in Equation (5.2.13) is valid only for all the internal points contained in a full control volume. However, these equations are not valid at the boundaries of the domain. In the case of a one-dimensional domain, a different formulation is needed for the nodes at the two boundaries. There are several types of boundary conditions encountered when we solve partial differential equations. The best-known types are classified under three canonical types: Dirichlet, Neumann, and periodic boundary conditions. We shall briefly discuss these types of boundary conditions by considering the following general second order differential equation:

$$\frac{d^2q}{dx^2}(x) = f(x, q(x), q'(x)) \tag{5.3.1}$$

to be solved on $x \in [a, b]$. In this case, the boundary conditions at a and b are required to pose the problem as a boundary value problem.

- **Dirichlet boundary conditions**: The first boundary condition that we have is the *Dirichlet boundary conditions*. In this case, the value of q is given on the boundaries and can be directly used since the nodes are situated on the boundaries, then we have

$$\begin{aligned} q(a) &= A \\ q(b) &= B, \end{aligned} \tag{5.3.2}$$

  where A, and B are the given values of q at the boundaries of the domain. In the case where $A = B = 0$, we have the perfectly absorbing boundary condition (LeVeque, 2002).

- **Periodic boundary conditions**: A particular case of the Dirichlet's boundary conditions is the periodic boundary condition where the numerical domain is topologically connected in a given direction. This condition is given by

$$q(a) = q(b). \tag{5.3.3}$$

- **Neumann's boundary conditions**: The second type of boundary conditions was formulated by a German mathematician Neumann. The latter are applied when the values of q are not directly given at the boundaries but their derivatives are specified. Then, we have

$$\frac{\partial q}{\partial x}(a) = f(a) \qquad \frac{\partial q}{\partial x}(b) = g(b), \tag{5.3.4}$$

where f and g are given arbitrary functions. The Neumann boundary conditions are often referred to as flux boundary conditions. In the case where f and g are null, the Neumann boundary conditions are referred to be the no-flux boundary conditions.

5.3.1 Additional equations on the boundaries

In the case where the values of the solutions are not directly given at the boundaries, the boundary conditions introduce additional equations at these boundaries. In our case, these additional equations come from the integration of Equation (5.1.5) with respect to time and space over the half-control volume $[x_0, x_{\frac{1}{2}}]$ and the full time interval $[t_n, t_{n+1}]$ on the left-hand side as well as on the right-hand side with half-control volume $[x_{N-\frac{1}{2}}, x_N]$. Figure 5.3 illustrates the diagram where the two half-control volumes are presented.

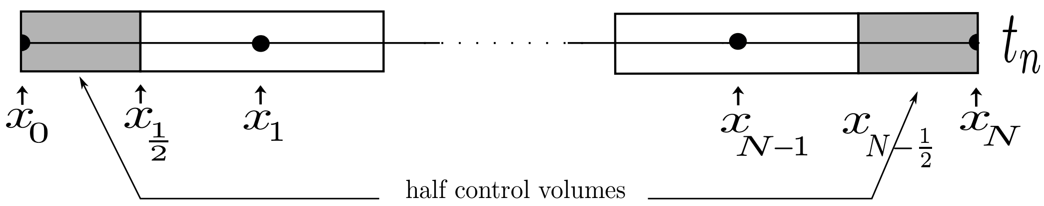

Figure 5.3: Illustration of the half control volumes

Starting from the first half-control volume, we perform the integration as shown below:

$$\int_{x_0}^{x_{\frac{1}{2}}} \int_{t_n}^{t_{n+1}} \frac{\partial \mathbf{U}}{\partial t}\, dt\, dx + \int_{t_n}^{t_{n+1}} \int_{x_0}^{x_{\frac{1}{2}}} \frac{\partial \mathbf{F}(\mathbf{U},\rho)}{\partial x}\, dx\, dt = \int_{t_n}^{t_{n+1}} \int_{x_0}^{x_{\frac{1}{2}}} \mathbf{Q}(\mathbf{U},\rho,z_b)\, dx\, dt. \tag{5.3.5}$$

Evaluating the integrals on the left-hand side of Equation (5.3.5) above yields

$$\begin{aligned}&\int_{x_0}^{x_{\frac{1}{2}}} \mathbf{U}(t_{n+1},x)dx - \int_{x_0}^{x_{\frac{1}{2}}} \mathbf{U}(t_n,x)dx + \int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{U}(t,x_{\frac{1}{2}}),\rho(t,x_{\frac{1}{2}}))\, dt - \\ &\int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{U}(t,x_0),\rho(t,x_0))\, dt = \int_{t_n}^{t_{n+1}} \int_{x_0}^{x_{\frac{1}{2}}} \mathbf{Q}(\mathbf{U},\rho,z_b)\, dxdt.\end{aligned} \tag{5.3.6}$$

Using the mean-value theorem like as we have seen before, let $\boldsymbol{U}_0^n$ approximate the average value over the first half-control volume at time t_n such that

$$\boldsymbol{U}_0^n = \frac{2}{\Delta x}\int_{x_0}^{x_{\frac{1}{2}}} \mathbf{U}(t_n, x)dx. \tag{5.3.7}$$

Let $\boldsymbol{\mathcal{F}}_{\frac{1}{2}}^n$ be the numeral flux such that

$$\boldsymbol{\mathcal{F}}_{\frac{1}{2}}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{U}(t, x_{\frac{1}{2}}), \rho(t, x_{\frac{1}{2}}))\, dt, \tag{5.3.8}$$

and

$$\boldsymbol{\mathcal{F}}_0^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{U}(t, x_0), \rho(t, x_0))\, dt. \tag{5.3.9}$$

Finally, let $\boldsymbol{\mathcal{Q}}_0^n$ be the discretized source term. As a consequence of these approximations, we have the following compact form of the discretized equation over the first half-control volume as

$$\boldsymbol{U}_0^{n+1} = \boldsymbol{U}_0^n - 2\frac{\Delta t}{\Delta x}\left(\boldsymbol{\mathcal{F}}_{\frac{1}{2}}^n - \boldsymbol{\mathcal{F}}_0^n\right) + \Delta t \boldsymbol{\mathcal{Q}}_0^n. \tag{5.3.10}$$

Doing the same integration over the second half-control volume at the right boundary of the domain gives

$$\boldsymbol{U}_N^{n+1} = \boldsymbol{U}_N^n - 2\frac{\Delta t}{\Delta x}\left(\boldsymbol{\mathcal{F}}_N^n - \boldsymbol{\mathcal{F}}_{N-\frac{1}{2}}^n\right) + \Delta t \boldsymbol{\mathcal{Q}}_N^n. \tag{5.3.11}$$

Only for the first half-control volume, let ρ_c be the reference constant density and ρ_0 be the density at the boundary x_0. In this case, using the same technique used before for the discretization of the source term in Equation (5.2.11), we have the following equations for the discretization of the source term at the left and right boundaries

$$\boldsymbol{Q}_0^n = \frac{1}{4\Delta x}\frac{g}{\rho_c}\left[(\rho h)_1^n + 2(\rho h)_0^n + (\rho h)_0^n\right]((z_b)_1 - (z_b)_0) \tag{5.3.12}$$

and

$$\boldsymbol{Q}_N^n = \frac{1}{4\Delta x}\frac{g}{\rho_0}\left[(\rho h)_N^n + 2(\rho h)_N^n + (\rho h)_{N-\frac{1}{2}}^n\right]((z_b)_N - (z_b)_{N-1}). \tag{5.3.13}$$

Consequently, we have the following additional equations at the first half-control volume

$$\begin{cases} h_0^{n+1} = h_0^n - 2\dfrac{\Delta t}{\Delta x}\left(h_{\frac{1}{2}}^n u_{\frac{1}{2}}^n - h_0^n u_0^n\right), \\[2ex] r_0^{n+1} = r_0^n - 2\dfrac{\Delta t}{\Delta x}\left[\left(h_{\frac{1}{2}}^n\left(u_{\frac{1}{2}}^n\right)^2 + \dfrac{1}{2}\dfrac{g}{\rho_c}\rho_{\frac{1}{2}}^n\left(h_{\frac{1}{2}}^n\right)^2\right) - \left(h_0^n(u_0^n)^2 + \dfrac{1}{2}\dfrac{g}{\rho_c}\rho_0^n(h_0^n)^2\right)\right] - \\ \qquad\qquad \dfrac{1}{4}\dfrac{\Delta t}{\Delta x}\dfrac{g}{\rho_c}\left[(\rho h)_1^n + 2(\rho h)_0^n + (\rho h)_0^n\right]((z_b)_1 - (z_b)_0), \\[2ex] \mathcal{T}_0^{n+1} = \mathcal{T}_0^n - 2\dfrac{\Delta t}{\Delta x}\left((\mathcal{T}_{\frac{1}{2}}^n u_{\frac{1}{2}}^n) - (\mathcal{T}_0^n u_0^n)\right), \\[2ex] \mathcal{S}_0^{n+1} = \mathcal{S}_0^n - 2\dfrac{\Delta t}{\Delta x}\left((\mathcal{S}_{\frac{1}{2}}^n u_{\frac{1}{2}}^n) - (\mathcal{S}_0^n u_0^n)\right), \end{cases} \tag{5.3.14}$$

where $r = hu$, $\mathcal{T} = Th$ and $\mathcal{S} = Sh$. By doing the same expansion for the second half-control volume and by setting ρ_0 as the constant reference fluid density,again, it yields

$$\begin{cases} h_N^{n+1} = h_N^n - 2\dfrac{\Delta t}{\Delta x}\left(h_N^n u_N^n - h_{N-\frac{1}{2}}^n u_{N-\frac{1}{2}}^n\right), \\[2ex] r_N^{n+1} = r_N^n - 2\dfrac{\Delta t}{\Delta x}\left[\left(h_N^n(u_N^n)^2 + \dfrac{1}{2}\dfrac{g}{\rho_0}\rho_N^n(h_N^n)^2\right) - \left(h_{N-\frac{1}{2}}^n\left(u_{N-\frac{1}{2}}^n\right)^2 + \dfrac{1}{2}\dfrac{g}{\rho_0}\rho_{N-\frac{1}{2}}^n\left(h_{N-\frac{1}{2}}^n\right)^2\right)\right] - \\ \qquad\qquad \dfrac{1}{4}\dfrac{\Delta t}{\Delta x}\dfrac{g}{\rho_0}\left[(\rho h)_N^n + 2(\rho h)_N^n + (\rho h)_{N-\frac{1}{2}}^n\right]((z_b)_N - (z_b)_{N-1}), \\[2ex] \mathcal{T}_N^{n+1} = \mathcal{T}_N^n - 2\dfrac{\Delta t}{\Delta x}\left((\mathcal{T}_N^n u_N^n) - (\mathcal{T}_{N-\frac{1}{2}}^n u_{N-\frac{1}{2}}^n)\right), \\[2ex] \mathcal{S}_N^{n+1} = \mathcal{S}_N^{n+1} - 2\dfrac{\Delta t}{\Delta x}\left((\mathcal{S}_N^n u_N^n) - (\mathcal{S}_{N-\frac{1}{2}}^n u_{N-\frac{1}{2}}^n)\right). \end{cases} \tag{5.3.15}$$

5.4 Solutions of the shallow water equations

Following Benkhaldoun and Seaïd (2010), we use the fixed Courant-Friedrichs-Lewy (CFL) condition denoted by C_r in the numerical method such that $C_r = 0.8$. According to LeVeque (2002), it is noticed that the CFL condition is the only necessary condition for stability (and hence convergence) of the scheme. The time step is then varied according to the stability condition which is in this case described below:

$$\Delta t = 0.8 \frac{\Delta x}{\max_k(\lambda_k^n)} \tag{5.4.1}$$

where λ_k^n represents the eigenvalues of the matrix in Equation (5.1.7). Recall that the shallow water equations (5.1.5) are represented in matrix form.
In summary, by giving h_i^n, u_i^n, T_i^n and S_i^n, we will compute $h_i^{n+1}, u_i^{n+1}, T_i^{n+1}$ and S_i^{n+1}, using the FVM of characteristics. The steps that we are going to follow in order to achieve this goal can be summarized as:

- **Step 1.** Compute the departure point (i.e. the characteristic root) $X_{i+\frac{1}{2}}(t_n)$ using the method of characteristics and adopting an iterative procedure.

- **Step 2.** Evaluate the numerical solutions of the differential Equation (5.1.5) at $X_{i+\frac{1}{2}}(t_n)$ using the one-dimensional interpolating polynomial, such that

$$\tilde{h}_{i+\frac{1}{2}}^n = P_h(X_{i+\frac{1}{2}}(t_n)), \qquad \tilde{u}_{i+\frac{1}{2}}^n = P_u(t_n, X_{i+\frac{1}{2}}(t_n)),$$

$$\tilde{S}_{i+\frac{1}{2}}^n = P_S(t_n, X_{i+\frac{1}{2}}(t_n)), \qquad \tilde{T}_{i+\frac{1}{2}}^n = P_T(t_n, X_{i+\frac{1}{2}}(t_n)).$$

- **Step 3.** Compute the intermediate states $h_{i+\frac{1}{2}}^n, u_{i+\frac{1}{2}}^n, T_{i+\frac{1}{2}}^n$, and $S_{i+\frac{1}{2}}^n$ from Equation (5.2.40).

- **Step 4.** Update the value of the fluid density $\rho_{i+\frac{1}{2}}^n$ using Eckart's formula or the UNESCO formulation. For instance, using Eckart's formulation we have

$$\rho_{i+\frac{1}{2}}^n = \frac{1000\, P_{0}{}_{i+\frac{1}{2}}^n}{1779.5 + 11.25 T_{i+\frac{1}{2}}^n - 0.0745 {T_{i+\frac{1}{2}}^n}^2 - (3.80 + 0.017 T_{i+\frac{1}{2}}^n) S_{i+\frac{1}{2}}^n + 0.6980 P_{0}{}_{i+\frac{1}{2}}^n}, \tag{5.4.2}$$

where $P_0 = 5890 + 38 T_{i+\frac{1}{2}}^n - 0.3745 {T_{i+\frac{1}{2}}^n}^2 + 3 S_{i+\frac{1}{2}}^n$.

- **Step 5.** Use the corrector stage from Equation (5.2.13) to find the values of h_i^{n+1}, r_i^{n+1}, $\mathcal{T}_i^{n+1}$ and $\mathcal{S}_i^{n+1}$.

Furthermore, the FVM has been implemented in order to solve the one-dimensional shallow water equations and PYTHON codes were developed to do the numerical method.

5.5 Test cases

5.5.1 Example 1

In order to illustrate the modified FVM of characteristics that we have explained previously, let us first solve the following simplified one-dimensional shallow water equations where the fluid density is constant:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ hu^2 + \frac{1}{2}h^2 \end{pmatrix} = \begin{pmatrix} 0 \\ h\frac{\partial z_b}{\partial x} \end{pmatrix}. \tag{5.5.1}$$

Let us consider a free-standing water column or a water hump in a rectangular channel of length 100 m with flat bottom topography $z_b(x) = 0$. The initial conditions for this case are given by

$$\begin{cases} h(0,x) = 1 + 0.5e^{-0.015(50-x)^2} & x \in [0, 100\,\mathrm{m}], \\ hu(0,x) \;=\; 0\,\mathrm{m}^2 \cdot \mathrm{s}^{-1}, & \end{cases} \tag{5.5.2}$$

where h is the water height and hu is the discharge. It is excepted that after the initial time $t = 0\,\mathrm{s}$, the water column will collapse and a diverging water wave will spread out toward the left and the right of the domain. And after a certain amount of time, the water height becomes uniform and the velocity goes to zero. Without considering the boundary conditions and by using a space discretization $\Delta x = 1\,\mathrm{m}$, the obtained results are shown in Figures 5.4 and 5.5 at time $t = 1\,\mathrm{s}$ to time $t = 40\,\mathrm{s}$:

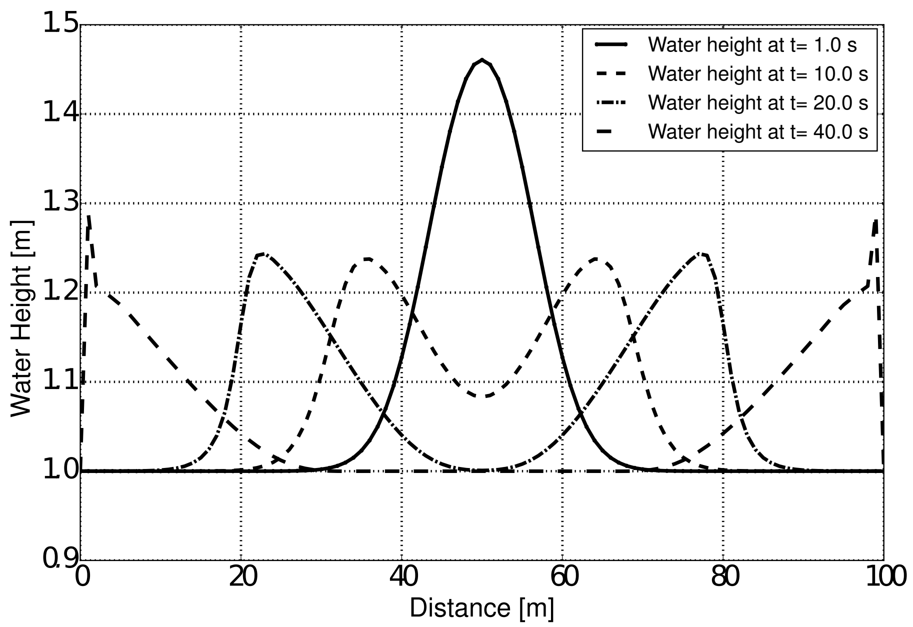

Figure 5.4: Water height profile

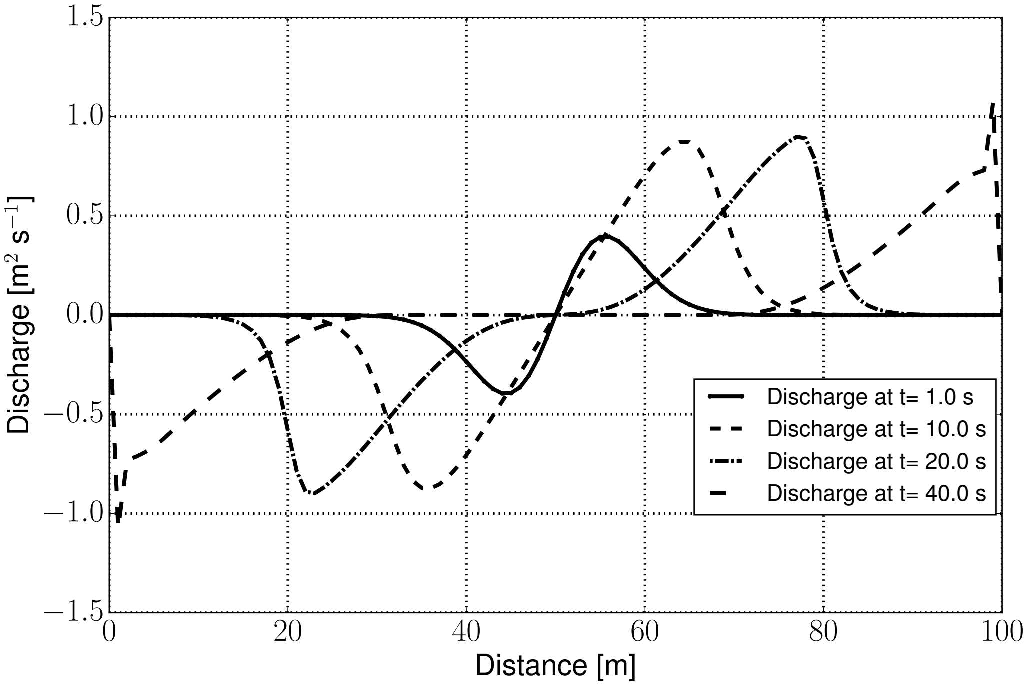

Figure 5.5: Discharge profile

The results in Figures 5.4 and 5.5 show that the modified FVM of characteristics gives the results as expected. For instance at time $t = 40\,\mathrm{s}$, it can be seen in Figure 5.4 that the graph of the solution for the water height represented by the dashed-line lies down to the level of its initial condition $h = 1\,\mathrm{m}$. In addition, the diverging water wave that spreads out to the left and the right-hand side of the domain is justified with the graph of the solution for the discharge. In this case, two waves can be seen first on the left-hand side and second on the right- hand side in Figure 5.5. These waves are reversed (upper wave and down wave) which indicate that the water waves move in opposite directions.

In order to understand the effects of the boundary conditions on the results as time evolves, we need to apply these boundary conditions at the two boundaries of the domain. As we have seen from the definition of the boundary conditions, if the values of the solutions are directly given at the boundaries then we use these values for all times $t > 0$.

Boundary case 1:

For instance, let us first consider the case where the water height is always maintained fixed at the two boundaries $x = 0\,\mathrm{m}$ and $x = 100\,\mathrm{m}$ and the velocity is set to be zero at these positions. The Dirichlet boundary conditions are then used for the water height and the discharge. Such boundary conditions are described in the following table.

Table 5.1: Boundary case 1

	$x = 0\,\mathrm{m}$	$x = 100\,\mathrm{m}$
h	$1\,\mathrm{m}$	$1\,\mathrm{m}$
hu	$0\,\mathrm{m}^2 \cdot \mathrm{s}^{-1}$	$0\,\mathrm{m}^2 \cdot \mathrm{s}^{-1}$

The results are displayed in Figures 5.6 and 5.7 at different times starting from $t = 45\,\mathrm{s}$ to $t = 80\,\mathrm{s}$:

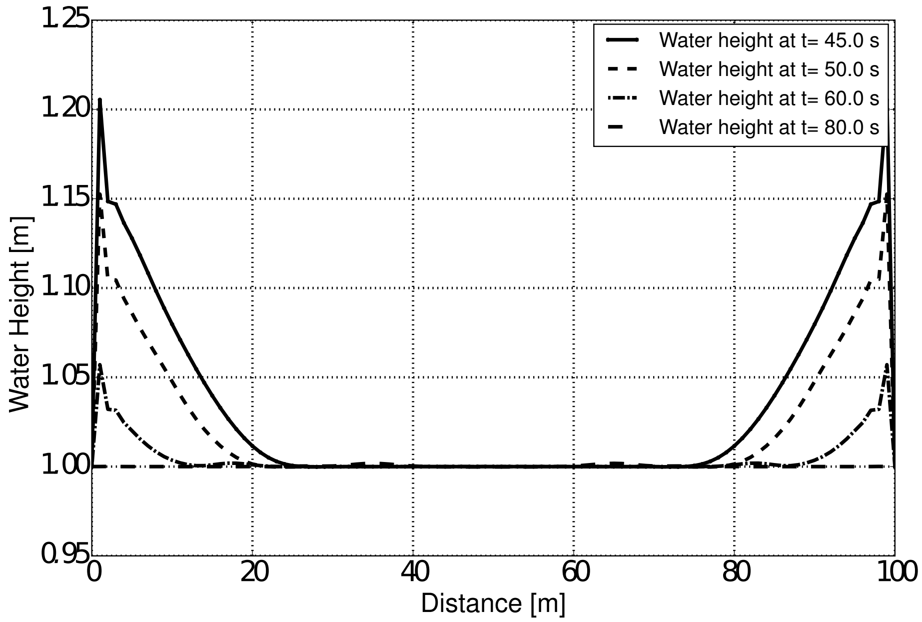

Figure 5.6: Water height profile

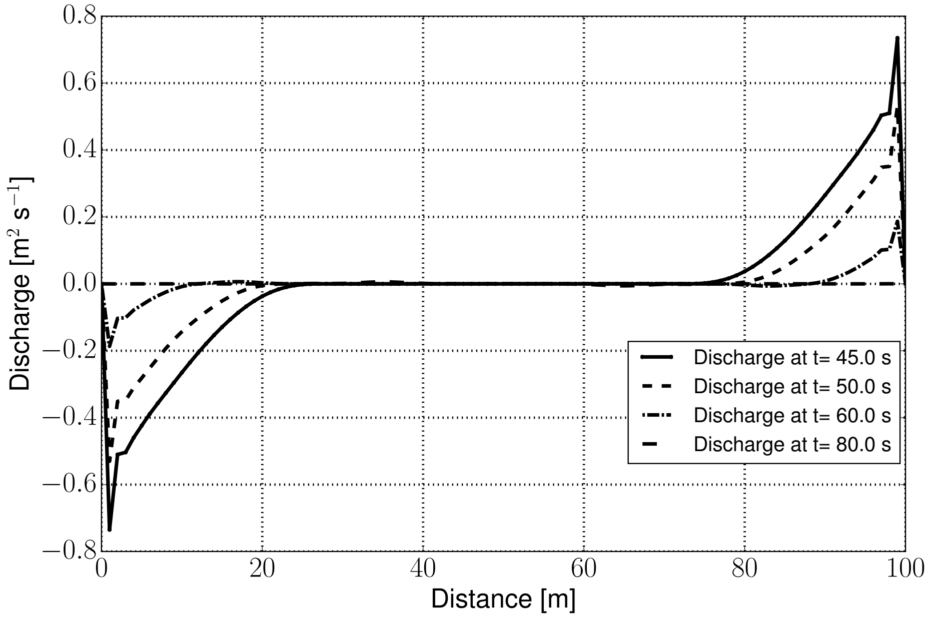

Figure 5.7: Discharge profile

From Figures 5.6 and 5.7, we can first see that water waves continue to move in their respective directions and decrease as time goes from $t = 45\,\mathrm{s}$ to $t = 60\,\mathrm{s}$. At time $t = 80\,\mathrm{s}$, we can then see that the graph of the solutions for the water height and discharge become the same as their initial graphs in the initial condition, respectively. Such results can be interpreted as that the modified FVM gives qualitatively correct results, and the chosen boundary condition is suitable for this first test case.

However, it can also be seen from Figures 5.6 and 5.7 that some instabilities occur and appear in the graph of the solution. These instabilities can be seen as pointed peaks on the graphs. For instance, the peak is very clear for time $t = 45\,\mathrm{s}$. In this example we are not going to discuss the instability of the solution. However, in order to see if changing the boundary conditions will affect these instabilities, let us consider the following new boundary conditions. **Boundary case 2**

An alternative case is to let the water height and the discharge free on the right-hand side of the domain. This implies that the Neumann boundary conditions are applied at $x = 100\,\mathrm{m}$ while Dirichlet's boundary conditions are still used at $x = 0\,\mathrm{m}$. Such boundary conditions are presented as follows:

Table 5.2: Boundary case 2

	$x = 0\,\mathrm{m}$	$x = 100\,\mathrm{m}$
h	$1\,\mathrm{m}$	$\dfrac{\partial h}{\partial x} = 0$
hu	$0\,\mathrm{m}^2 \cdot \mathrm{s}^{-1}$	$\dfrac{\partial hu}{\partial x} = 0\,\mathrm{m} \cdot \mathrm{s}^{-1}$

By using the same value of the space discretization $\Delta x = 1\,\mathrm{m}$, the results are displayed in Figures 5.8 and 5.9:

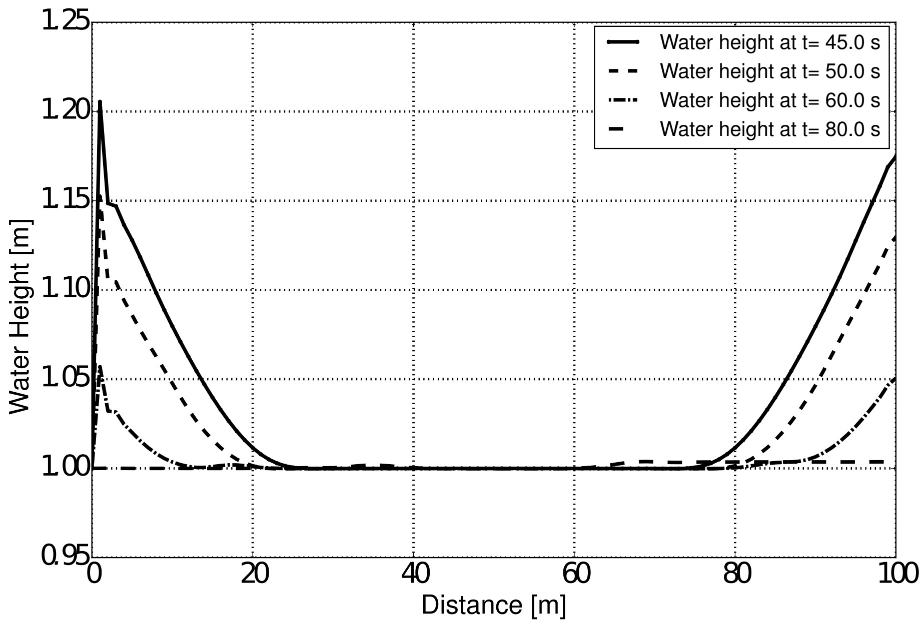

Figure 5.8: Water height profile

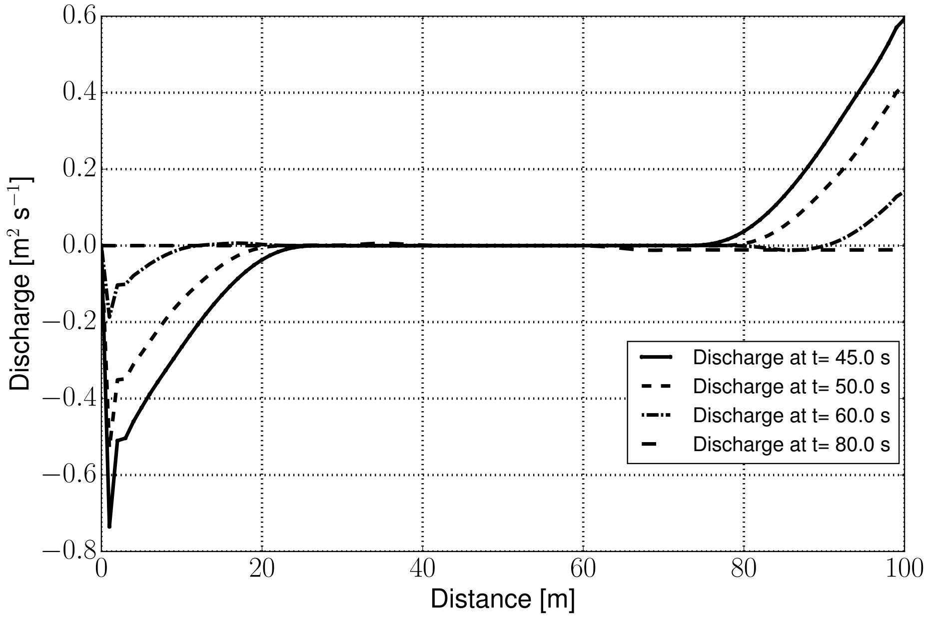

Figure 5.9: Discharge profile

It is shown from Figures 5.8 and 5.9 that on the left-hand side of the domain the peak does not appear in both graphs of the solution anymore. That is because we changed the boundary conditions at that position to the Neumann boundary condition. In terms of interpretation, applying the latter boundary conditions avoids reflection of the wave at the boundary.

From the results shown in Figures 5.8 and 5.9, we can deduce that using different boundary conditions at the two boundaries helps to avoid the stability.

5.5.2 Example 2

Let us still consider the case where the fluid density is constant. In this particular case, instead of having a flow from a water hump, we will investigate the flow over a hump in a rectangular channel of length 100 m. We impose a non uniform bottom z_b which is defined as follows:

$$z_b(x) = 0.5e^{-0.015(50-x)^2} \qquad x \in [0, 100\,\mathrm{m}]\,. \tag{5.5.3}$$

In this example, we consider the following initial conditions: The water height is initially fixed, $h = 2\,\mathrm{m}$ and no initial velocity is given, $u = 0\,\mathrm{m}\cdot\mathrm{s}$. These initial conditions are shown in Figure 5.10.

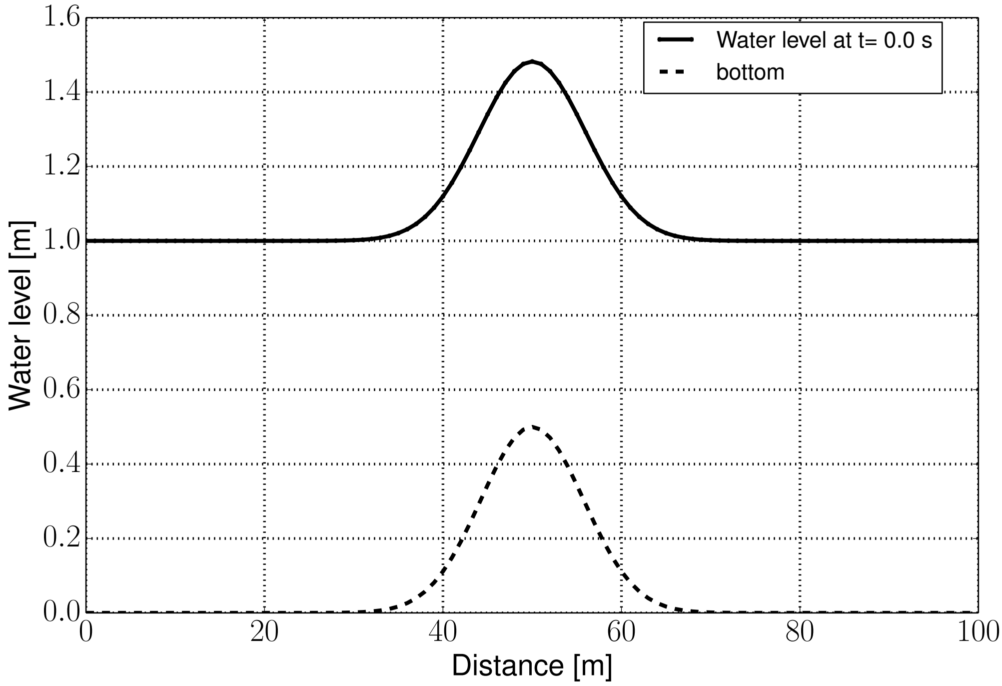

Figure 5.10: Flow with non-flat bottom

By using the boundary conditions presented in Table 5.3 with the same size discretization $\Delta x = 1\,\mathrm{m}$, we display the obtained results in Figure 5.11 at time several times $t = 1\,\mathrm{s}$, $t = 10\,\mathrm{s}$, $t = 50\,\mathrm{s}$ and $t = 150\,\mathrm{s}$.

Table 5.3: Boundary conditions for the flow over hump

	$x = 0\,\mathrm{m}$	$x = 100\,\mathrm{m}$
h	$1\,\mathrm{m}$	$1\,\mathrm{m}$
hu	$0\,\mathrm{m}^2 \cdot \mathrm{s}^{-1}$	$0\,\mathrm{m}^2 \cdot \mathrm{s}^{-1}$

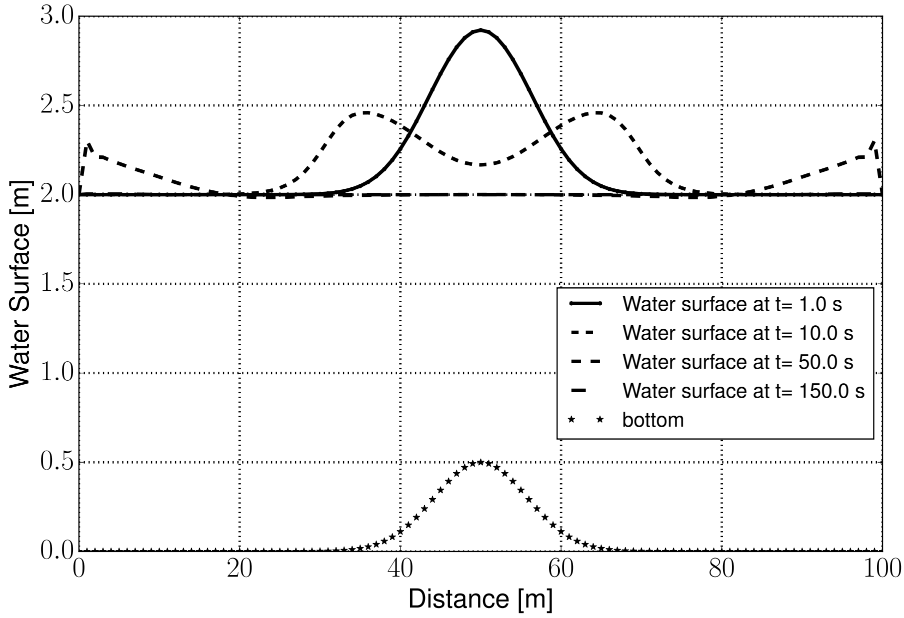

Figure 5.11: The free-surface profile at time $t = 1\,\mathrm{s}$ to $t = 150\,\mathrm{s}$

According to Benkhaldoun and Seaïd (2010) this type of example is widely used in literature in order to check the correctness of the schemes if they preserve the steady state solution or not. From Subsection 5.2.1, remember that a numerical scheme satisfies the steady state of the flow if the following conditions are satisfy:

$$h + z_b = \text{Constant}, \qquad \text{and} \qquad u = 0. \tag{5.5.4}$$

Some examples of this flow over a hump with different types of boundary conditions have been performed by Benkhaldoun and Seaïd (2010), and it has been shown that the modified FVM of characteristic preserves the steady state solution. The aim is then here to reproduce one of the results which have been presented by Benkhaldoun and Seaïd (2010) and to review that the FVM of characteristics satisfies the steady state solution. For the case of flow over an unstructured bottom, it will be expected from the results that the water free-surface remains constant and the water velocity should be zero after an amount of time. In Figure 5.11, it is shown that the graph of the solutions tends to preserve the steady-state solution. At time $t = 150\,\mathrm{s}$, we have a uniform level in the free-surface of water.

5.5.3 Example 3

Let us now take into account the fact that the governing equations with variations in density due to the changes in the temperature T and the salinity S. In this case, let us restate the governing equations as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \\ Th \\ Sh \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ hu^2 + \frac{1}{2}g\frac{\rho}{\rho_0}h^2 \\ Thu \\ Shu \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\frac{\rho}{\rho_0}\frac{\partial z_b}{\partial x} \\ 0 \\ 0 \end{pmatrix}, \tag{5.5.5}$$

where the fluid density is updated using the equation of state defined by Eckart as

$$\rho = \frac{1000P_0}{1779.5 + 11.25T - 0.0745T^2 - (3.80 + 0.01T)S + 0.6980P_0} \tag{5.5.6}$$

where $P_0 = 5890 + 38T - 0.3745T^2 + 3S$, T, and S are the temperature and the salinity respectively. In order to perform the FVM of characteristics in solving Equation (5.5.5) above, let us consider the following initial conditions:
The flow is initially taken at rest, $u = 0\,\mathrm{m}\cdot\mathrm{s}^{-1}$ with fixed water height $h = 1\,\mathrm{m}$ and a uniform temperature and salinity which is set to be zero. This implies that the fluid density is also uniform across the one-dimensional domain such that $\rho = 1\,\mathrm{kg}\cdot\mathrm{m}^3$. In addition, two boundary conditions are presented and the results from solving Equation (5.5.5) are displayed at different times according to each boundary condition. It is noticed that the results from this last example are expected to show evidence that the temperature and salt are transported by advection across the domain when time is evolving.

Boundary case 1:
The first boundary condition consists of a fixed water height, $h = 1\,\mathrm{m}$ and velocity is null, $u = 0\,\mathrm{m}\cdot\mathrm{s}^{-1}$ on the left-hand side of the domain and both free on the right-hand side. In the same way, the temperature and the salinity are maintained at $T = 5\,°\mathrm{C}$ and $S = 10\,\mathrm{ppt}$ on the left-hand side of the domain and are free on the right-hand side. Such boundary conditions are summarized in Table 5.4

Table 5.4: Boundary conditions for the flow with variations in density

	$x = 0\,\mathrm{m}$	$x = 100\,\mathrm{m}$
h	$1\,\mathrm{m}$	$\frac{\partial h}{\partial x} = 0$
hu	$0\,\mathrm{m}^2\cdot\mathrm{s}^{-1}$	$\frac{\partial hu}{\partial x} = 0\,\mathrm{m}\cdot\mathrm{s}^{-1}$
hT	$5\,\mathrm{m}\cdot{}^\circ\,\mathrm{C}$	$\frac{\partial hT}{\partial x} = 0\,°\mathrm{C}$
hS	$10\,\mathrm{m}\cdot\mathrm{ppt}$	$\frac{\partial hS}{\partial x} = 0\,\mathrm{ppt}$

Using the acceleration of the gravity $g = 9.8\,\mathrm{m}\cdot\mathrm{s}^{-2}$ with the space discretization $\Delta x = 1\,\mathrm{m}$, the results are shown in Figure 5.12 for the water height at different times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$. With the same different times, the results are displayed in Figures 5.13, 5.14 and 5.15 for the temperature, salinity and density, respectively.

a) Water height

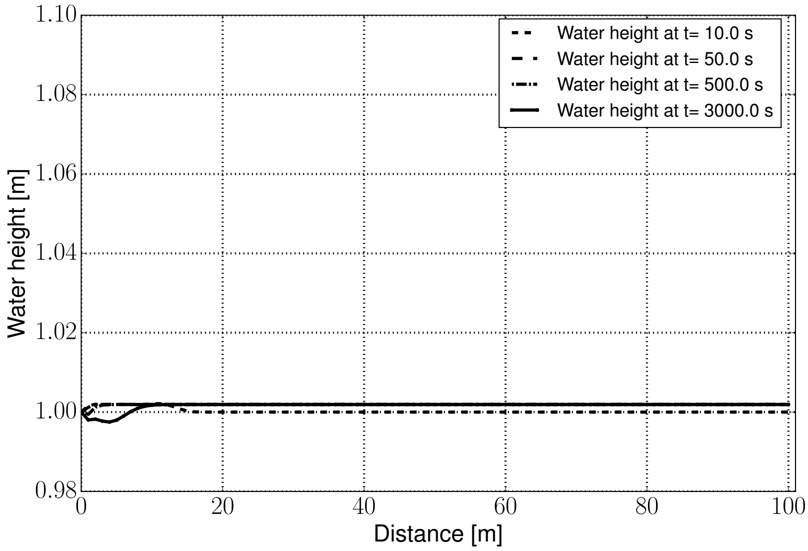

Figure 5.12: The water height profile at times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$

b) Temperature

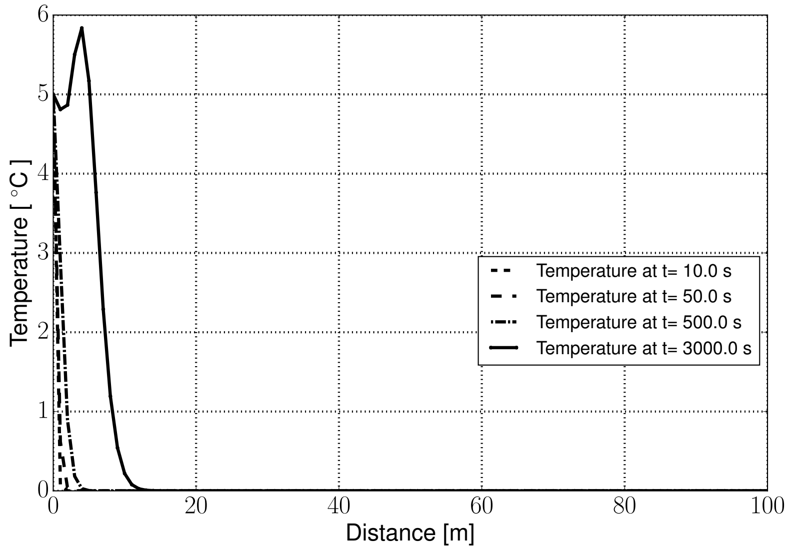

Figure 5.13: Temperature profile at times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$

c) Salinity

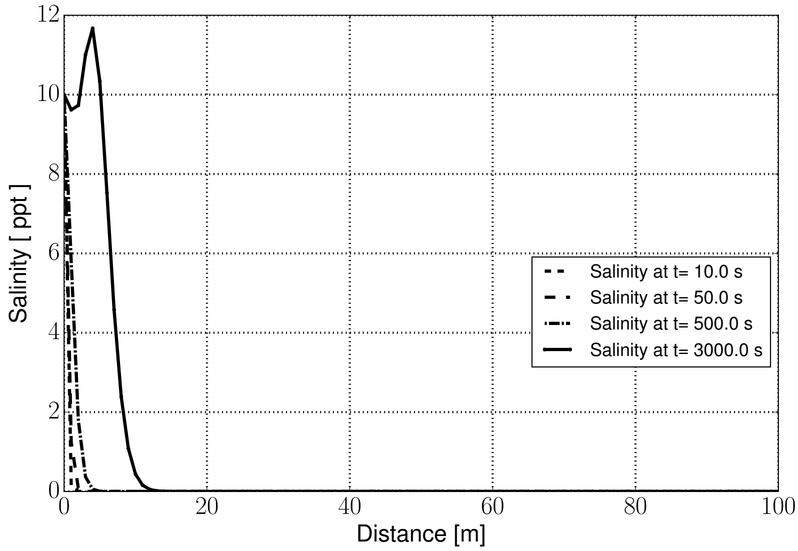

Figure 5.14: Salinity profile at times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$

d) Density

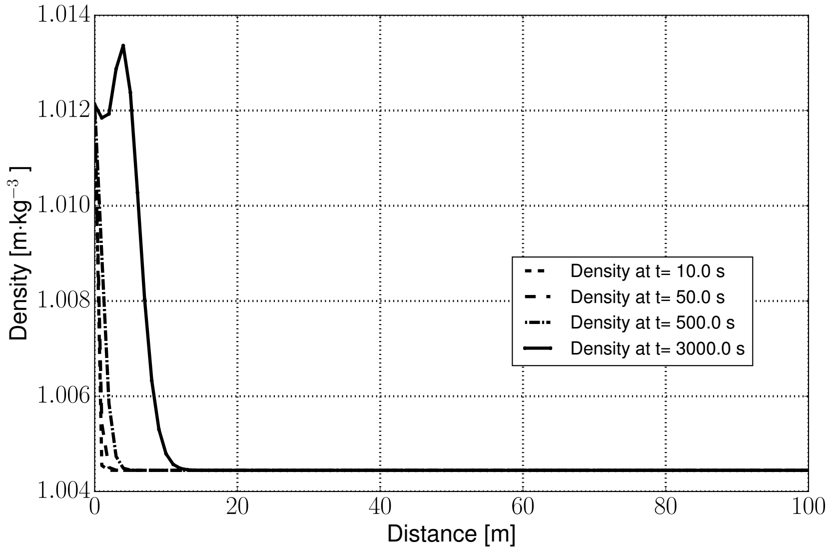

Figure 5.15: Density profile at times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$

It can be seen from the results presented above in Figure 5.12, 5.13, 5.14, and 5.15 above that there are no considerable changes in the solutions as the time evolves from $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$. This is so because there is no potential driving forces that may produce big changes in the variables. However, the small amount of changes that we can notice, for instance, in the water height, in Figure 5.12 and in the temperature, in Figure 5.13, are due to the difference of the temperature and the salinity at the upstream boundary of the domain. Since at this boundary, we keep the temperature and the salinity for all times $t > 0$ at a certain value so that the interface of the two liquids with different temperature and salinity drives a flow of higher density liquid towards the right, pushing the lower density liquid ahead.

Boundary case 2:

This case is very similar to the previous boundary case presented above except that the value of the velocity on the left-hand side of the domain is $u = 0.5\,\mathrm{m}\cdot\mathrm{s}^{-1}$. The new boundary conditions are then rewritten as follows:

Table 5.5: Boundary conditions for the flow with variations in density with discharge

	$x = 0\,\mathrm{m}$	$x = 100\,\mathrm{m}$
h	$1\,\mathrm{m}$	$\dfrac{\partial h}{\partial x} = 0$
hu	$0.5\,\mathrm{m}^2\cdot\mathrm{s}^{-1}$	$\dfrac{\partial hu}{\partial x} = 0\,\mathrm{m}\cdot\mathrm{s}^{-1}$
hT	$5\,\mathrm{m}\cdot{}^\circ\mathrm{C}$	$\dfrac{\partial hT}{\partial x} = 0\,{}^\circ\mathrm{C}$
hS	$10\,\mathrm{m}\cdot\mathrm{ppt}$	$\dfrac{\partial hS}{\partial x} = 0\,\mathrm{ppt}$

Again, using the same space discretization $\Delta x = 1\,\mathrm{m}$, the results are shown in Figures 5.16, 5.17, 5.18 and 5.19 at times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$:

a) Water height

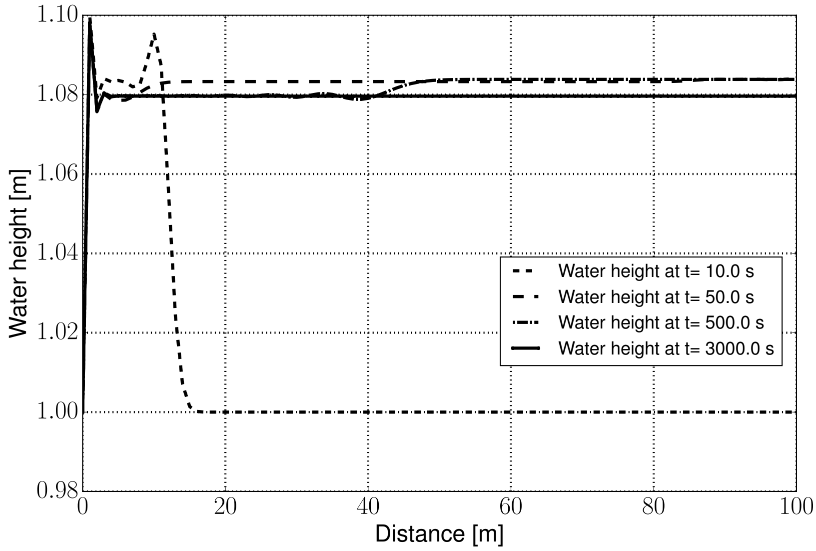

Figure 5.16: The water height profile at times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$

b) Temperature

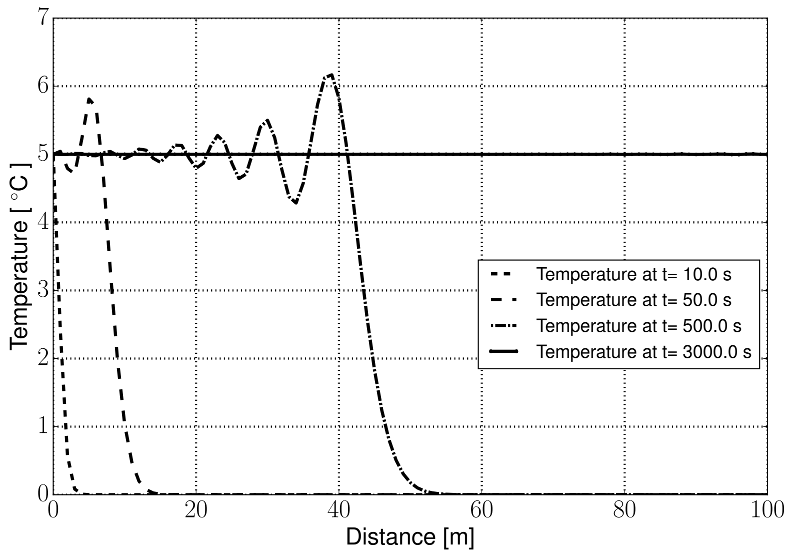

Figure 5.17: Temperature profile at times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$

c) Salinity

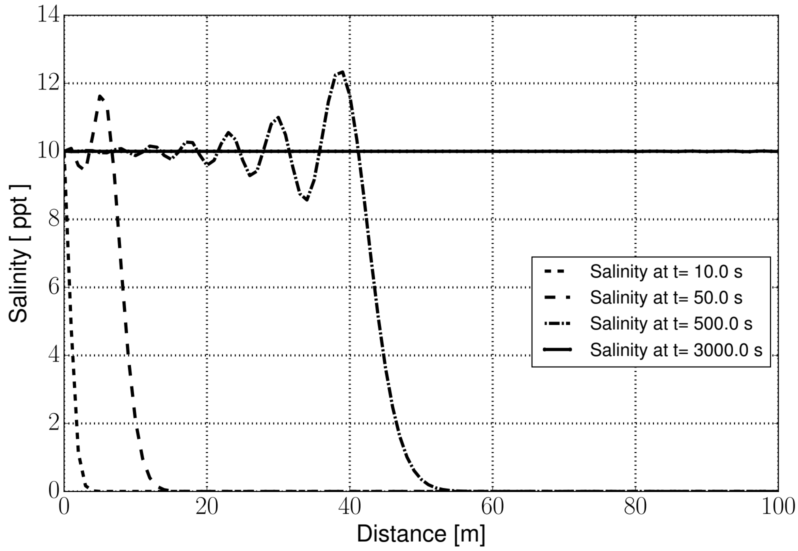

Figure 5.18: Salinity profile at times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$

d) Density

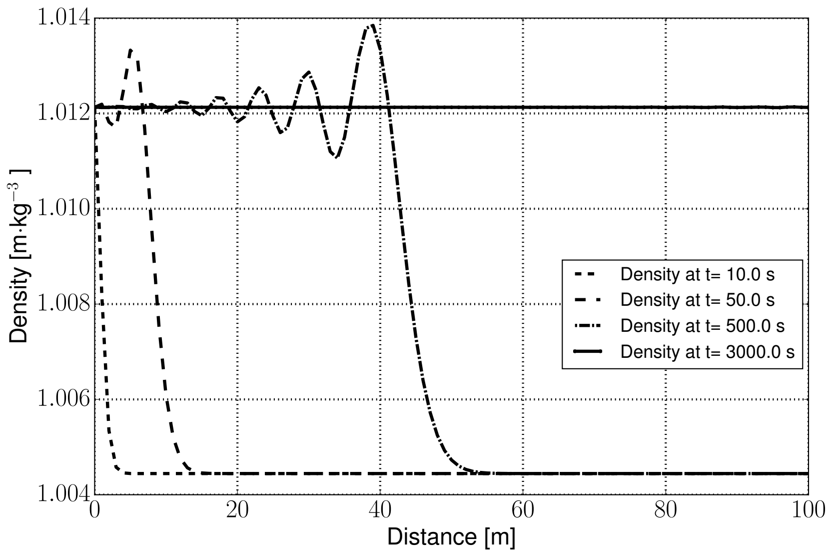

Figure 5.19: Density profile at times $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$

It is clear from the results which are shown from Figures 5.16 to 5.19 that some considerable changes occur as the time evolves from $t = 10\,\mathrm{s}$ to $t = 3000\,\mathrm{s}$. For instance, the water height increases for a some amount of level compares to the previous result shown in Figure 5.12. Such changes are due to the discharge maintained at the upstream boundary. It can be seen from the profile of the fluid density in

Figure 5.19 that the fluid with higher density pushes the fluid with lower density forward from the left to right of the domain. In the same way, the temperature and the salt are transported with the moving water which can be seen from Figures 5.18 and 5.19. Consequently, these results clearly show that the state variables are transported due to the advection.

5.6 Summary

In summary, we have reviewed the modified FVM of characteristics which has been suggested by Benkhaldoun and Seaïd (2010) in order to solve the one-dimensional shallow water equations where the fluid density is a function of the temperature and the salinity. In doing so, we have first generated the discretized form of Equation (5.1.5) by integrating it across each interface of the control volume of the domain. And then we have described in detail the construction of the numerical flux at each interface of these control volume using the method of characteristics where the latter itself has been reviewed.

In addition, the modified FVM of characteristics has been tested for some test cases. It has been concluded by the results of these test cases that the FVM of characteristics performs well and confirms some of the predicted results of one of the test examples. Furthermore, the flow over a non-uniform bottom which has been presented from Benkhaldoun and Seaïd (2010) was retested among the test cases, and the results are the same.

Finally, it has been confirmed from the results in the case where there is variations in density that the temperature and the salt are transported due to advection.

Chapter 6

Summary and Conclusions

The main objective of this study was to investigate incompressible flow where the fluid density varies with respect to the pressure, the temperature, and the salinity. During this investigation, all the objectives were accomplished including the derivation of the conditions for which the material derivative of $\rho = \rho(p, T, S)$ is negligible compared to $\nabla \cdot \underline{u}$. In addition, the governing equations which satisfy the condition of incompressibility were derived. And finally, the one-dimensional shallow water equations were solved using the modified FVM of characteristics.

6.1 Summary

In Chapter 3, after applying the structured technique of scaling analysis to the modified form of the continuity equation (3.4.24), it has been found that the main condition to qualify a flow to be incompressible is that the Mach number (M) is very small. The latter condition has been justified by Panton (2013) who says that $\mathrm{M}^2 \longrightarrow 0$ is the requirement for incompressible flow. Then for incompressible flow, the flow velocity is very small compared to the local speed of sound. In addition, it is stated from Chapter 2, Section 2.4.4 that the square of the speed of sound is inversely proportional to the changes in the fluid density with respect to the changes in the pressure, $\dfrac{1}{c^2} = \left(\dfrac{\partial \rho}{\partial p}\right)_{\eta,S}$. If the $\mathrm{M} \longrightarrow 0$ this implies that $\left(\dfrac{\partial \rho}{\partial p}\right)_{\eta,S} \approx 0$. Consequently, for incompressible flow, the changes in the pressure do not affect the fluid density. However, having a very small Mach number is not sufficient to qualify the flow to be incompressible in the case where the fluid density depends on other state variables than the pressure. For instance, in this case, the fluid density depends on the temperature and the salinity, and their material derivative has to be negligible in order to reduce the continuity equation to $\nabla \cdot \underline{u} = 0$. We found that the only condition for which such material derivative can be neglected is that the value of the Peclet number of heat and salt have to be very high. A high value of the Peclet number can be interpreted as an absence of diffusion of heat and salt when the flow occurs. This latter condition has been justified by LeBlond and Mysak (1981). To conclude, a flow is considered incompressible once its velocity is very small compared to the local velocity of sound. However, this condition is not sufficient in the case where the fluid density depends also on the temperature and the salinity, an extra condition is needed which says that the flow occurs without diffusion of heat and salt.

Chapter 4 was devoted to the derivation of the Naver-Stokes equation which satisfies the condition of incompressibility. Based on the physical properties of the ocean and the geometrical aspects of the ocean domain in which the flow occurs, the Boussinesq approximation and the hydrostatic assumption were discussed and were applied to the Navier-Stokes equation. Furthermore, due to the fact that the ocean is highly stratified, the depth-averaging of the governing equation was performed in order to deduce the shallow water equations. It is noticed that, in Chapter 4 that the investigation of the incompressible flow was directed to the momentum equation.

In Chapter 5, the modified FVM of characteristics by Benkhaldoun and Seaïd (2010) was reviewed in detail, and was performed in order to solve the one-dimensional shallow water equations in fulfilment of the last objective of this study. In doing so, the obtained numerical scheme was implemented using Python. Different test cases were presented in order to evaluate if the method works well and gives the expected results. It can be seen from the results presented in the first test case, Section 5.5.1, where the fluid density is constant, that the numerical method performs well. The same performance was seen in the case where the fluid density is not constant anymore. In addition, the results presented in Example 5.5.3 shows the evidence that the salt and heat are transported in absence of diffusion, i.e. by advection.

6.2 Conclusions

This study covers a wide range of fluid mechanics, thermodynamics, and some overview of the numerical method. From this study, we have gained knowledge about the fundamental concepts of fluid mechanics where we described in detail the kinematics of fluid, different types of flows, as well as some thermodynamics concepts. We have understood the basic idea of incompressible flow where we were familiarized with the use of the scaling analysis as the fundamental tool for deriving the conditions of incompressibility. Moreover, we have gained familiarity in deriving different conservative laws, for instance, conservation of mass, momentum and energy. And finally, we have learned skills in implementing the numerical method using PYTHON well as the interpretation of the numerical solutions.

As was mentioned above, no error, stability, or converge analyses have been given throughout this study. Possible future work could include such analyses within our investigation.

List of References

Bachelor, G.
1967. *An Introduction to Fluid Dynamics*. University Press.

Benkhaldoun, F. and M. Seaïd
2010. A simple finite volume method for the shallow water equations. *J. Comput. Appl. Math.*, 234(1):58–72.

Benkhaldoun, F. and M. Seaïd
2010. A simple finite volume method for the shallow water equations. *Journal of Computational and Applied Mathematics*, 234(1):58 – 72.

Bird, R., W. Stewart, and E. Lightfoot
2007. *Transport Phenomena*, Wiley International edition. Wiley.

Brodkey, R. and H. Hershey
2003. *Transport Phenomena: A Unified Approach*, Chemical engineering series. Brodkey Pub.

Chippada S., C.N. Dawson, M. M. M. W.
1998. A godunov-type finite volume method for the system of shallow water equations. *Computer Methods in Applied Mechanics and Engineering*, 151(1):105 – 129. Containing papers presented at the Symposium on Advances in Computational Mechanics.

Cohen, I., P. Kundu, and H. Hu
2004. *Fluid Mechanics*. Elsevier Science.

Dijkstra, H.
2008. *Dynamical Oceanography*. Springer Berlin Heidelberg.

Goluskin, D.
2015. *Internally Heated Convection and Rayleigh-Bénard Convection*, SpringerBriefs in Applied Sciences and Technology. Springer International Publishing.

Gray, D. and A. Giorgini
1976. The validity of the Boussinesq approximation for liquids and gases. *International Journal of Heat and Mass Transfer*, 19(5):545 – 551.

Krantz, W.
2007. *Scaling Analysis in Modeling Transport and Reaction Processes: A Systematic Approach to Model Building and the Art of Approximation*. Wiley.

LeBlond, P. and L. Mysak
1981. *Waves in the Ocean*, Elsevier Oceanography Series. Elsevier Science.

Lerner, R. and G. Trigg
2005. *Encyclopedia of Physics, 2 Volumes*. Wiley.

LeVeque, R.
2002. *Finite Volume Methods for Hyperbolic Problems*, Cambridge Texts in Applied Mathematics. Cambridge University Press.

Mamayev, O.
2010. *Temperature-Salinity Analysis of World Ocean Waters*, Elsevier Oceanography Series. Elsevier Science.

Moran, M., H. Shapiro, D. Boettner, and M. Bailey
2010. *Fundamentals of Engineering Thermodynamics*. Wiley.

Müller, P.
2006. *The Equations of Oceanic Motions*. Cambridge University Press.

Olbers, D., J. Willebrand, and C. Eden
2012. *Ocean Dynamics*, SpringerLink : Bücher. Springer Berlin Heidelberg.

Panton, R.
2013. *Incompressible Flow*. Wiley.

Pedlosky, J.
2013. *Geophysical Fluid Dynamics*, Springer Study Edition. Springer Berlin Heidelberg.

Rajagopal, K., G. Saccomandi, and L. Vergori
2009. On the Oberbeck-Boussinesq approximation for fluids with pressure dependent viscosities. *Nonlinear Analysis: Real World Applications*, 10(2):1139 – 1150.

Roe, P.
1981. Approximate Riemann solvers, parameter vectors, and difference schemes. *Journal of Computational Physics*, 43(2):357 – 372.

Samelson, R.
2014. *The Theory of Large-Scale Ocean Circulation*. Cambridge University Press.

Schmidt, F., R. Henderson, and C. Wolgemuth
1984. *Introduction to thermal sciences: thermodynamics, fluid dynamics, heat transfer*. Wiley.

Spiegel, A. E. and G. Veronis
1959. On the boussinesq approximation for a compressible flow.

Vallis, G.
2017. *Atmospheric and Oceanic Fluid Dynamics*. Cambridge University Press.

Van Dyke, M.
1982. *An album of fluid motion*, An Album of Fluid Motion. Parabolic Press.

Vreugdenhil, C.
2013. *Numerical Methods for Shallow-Water Flow*, Water Science and Technology Library. Springer Netherlands.

Whitaker, S.
1968. *Introduction to Fluid Mechanics*, Prentice-Hall international series in the physical and chemical engineering sciences. Krieger.

Yulong Xing, C.-W. S.
2005. High order finite difference WENO schemes with the exact conservation property for the shallow water equations. *Journal of Computational Physics*, 208(1):206 – 227.