A CATEGORICAL APPROACH TO LATTICE-LIKE STRUCTURES

by

Michael Anton Hoefnagel

Dissertation presented for the degree of Doctor of Philosophy in the Faculty of Science at Stellenbosch University

Supervisor: Prof. Z. Janelidze

Co-supervisor: Dr. J. Gray

December 2018

Declaration

By submitting this dissertation electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date: December 2018

Abstract

A CATEGORICAL APPROACH TO LATTICE-LIKE STRUCTURES

M.A.Hoefnagel

Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.

Dissertation: PhD

December 2018

This thesis is a first step in a categorical approach to lattice-like structures. Its central notion, that of a majority category, relates to the category of lattices, in a similar way as Mal'tsev categories relate to the category of groups. This notion provides a context in which to establish categorical counterparts of various lattice-theoretic results. Surprisingly, many categories of a geometric nature naturally possess the dual property; namely, they are comajority categories. We show that several characterizations of varieties admitting a majority term, extend to characterizations of regular majority categories. These characterizations then show how majority categories relate to other well known notions in the literature, such as arithmetical and protoarithmetical categories. The most interesting results, from the point of view of the author, are those that concern decomposition and factorization. For example, every subobject of a finite product of objects in a regular majority category is uniquely determined by its two-fold projections – which can be seen as a certain subobject decomposition property. One of the main points of the thesis proves that in a regular majority category, every product of directly-irreducible objects is unique.

Uittreksel

’N kategoriese benadering tot rooster soos strukture’

(“A categorical approach to
lattice-like structures”)

M.A.Hoefnagel

Department van Wiskundige Wetenskappe,
Universiteit van Stellenbosch,
Privaatsak X1, Matieland 7602, Suid Afrika.

Proefskrif: PhD

Desember 2018

Hierdie proefskrif is ’n eerste stap na ’n kategoriese benadering tot roostersoos strukture. Die sentrale begrip daarvan, dié van ’n meerderheidskategorie, het betrekking op die kategorie van roosters, op soortgelyke wyse soos Mal’tsev-kategorieë betrekking het op die kategorie van groepe. Hierdie idee bied ’n konteks waarin kategoriese eweknieë van verskillende roosterteoretiese resultate gevestig kan word. Baie kategorieë van ’n meetkundige aard het die dubbele eienskap; naamlik, hulle is (co)meerderheids kategorieë. Ons wys dat verskeie karakters van variëteite wat ’n meerderheidstermyn toelaat, uitbrei na karakterisering van gereelde meerderheidskategorieë. Hierdie karakterisering toon dan aan hoe meerderheidskategorieë verband hou met ander bekende begrippe in die literatuur, soos Arithmetical en protoarithmetical kategorieë. Die mees interessante resultate, uit die oogpunt van die skrywer, is dié wat ontbinding en faktorisering betref. Ons wys dat direkte produkte erken ’n sekere unieke faktorisering stelling soortgelyk aan die universele algebraïese teendeel.

Acknowledgements

First and foremost, I would like to thank Zurab Janelidze and James Gray for their role as supervisors over the last few years. I consider it a great honour and privilege to have been a student of mathematicians of your calibre. Your encouragement and enthusiasm toward this topic was absolutely vital during the early stages of its development, and has had a profound influence on the content of this thesis. I would like to extend this last thanks to all who expressed interest in my work during the *5th workshop in categorical methods in non-abelian algebra* at Louvain-la-Neuve in 2017, with special thanks to George Janelidze, Marino Gran, Sandra Mantovani and Giuseppe Metere for their valuable suggestions at various times. Many thanks are due to Alberto Facchini at the University of Padua, who supervised me during my stay at Padua in 2017, for which I am very grateful. These interactions eventually lead me to one of the main results presented in this thesis (Theorem 4.32). Finally, I would like to thank Marco Garuti, who made my visit to Padua possible in the first place.

I would also like to acknowledge the non-academic influence on this thesis. Many thanks are due to my family: Sherry, Peter and Kevin Hoefnagel, all of whom made significant financial contributions to my mathematical studies at some point, and gave very warm encouragement to pursue my PhD studies in mathematics. It is impossible to fully express my gratitude to Warren and Sharon Scott, who gave me very significant financial support during my first year of university studies. This gave me the freedom to explore my academic interests fully, and for this, I am eternally grateful. Finally, I would like to give thanks to Michelle Uys, who gave very helpful editing suggestions throughout the development of this text.

Dedications

To the loving memory of my two grandmothers, Dawn and Guus, may you rest in peace...

Contents

Introduction

Perhaps one of the most fundamental properties of categories of 'group-like' structures, is that they are all *Mal'tsev* [CLP91]. For varieties, being Mal'tsev amounts to the existence of a ternary term $p(x, y, z)$, satisfying the equations:

$$p(x, x, y) = y = p(y, x, x).$$

In groups, for example, a Mal'tsev term is given by $p(x, y, z) = xy^{-1}z$. Mal'tsev categories provide a suitable framework for extending various results about Mal'tsev varieties to categories, and Mal'tsev varieties provide a suitable context in which to establish certain general properties of group-like structures. Therefore, we may consider Mal'tsev categories as a categorical approach to group-like structures, in so far as Mal'tsev varieties are a universal-algebraic approach to group-like structures. Examples include the categories of groups, rings, R-modules, Heyting algebras, and lesser known structures such as quasi-groups and loops. An example of a variety which is not Mal'tsev is the variety **Lat** of lattices, and in some sense, the variety of lattices represents an opposite extreme than that which is represented by groups.

The main objective of this thesis is to initiate a categorical investigation of notions motivated from the theory of congruence distributive varieties in universal algebra. This is not the first step, as M.C. Pedicchio's paper on *arithmetical* categories (see [Ped96]) could be considered as the 'first step', followed by Bourn's *protoarithmetical* categories (see [Bou01] and [Bou05]), but ours is a first step in a new direction. We begin our investigation with the category of lattices, which is the primordial example of a congruence distributive variety. The central notion of this thesis, that of a *majority category* [Hoe18b] (which is the same as a *Pixley* category in the sense of [Jan04]), derives itself from the observation that many algebraic results which hold for lattices, generalize to any variety admitting a *majority term*, i.e., a ternary term $m(x, y, z)$ satisfying the equations:

$$m(x, x, y) = x,$$
$$m(x, y, x) = x,$$
$$m(y, x, x) = x.$$

For example, in the variety of lattices, every sublattice S of a product of lattices $L_1 \times L_2 \times \cdots \times L_n$ is uniquely determined by its 'two-fold projections', i.e., its images in $L_i \times L_j$ for $i, j \in \{1, 2, ..., n\}$. This is *Bergman's double projection theorem* for lattices (see [BP75]). From the universal-algebraic point of view, a variety satisfies Bergman's theorem for algebras, if and only if it admits a majority term. Therefore, we begin this 'categorical approach to lattice-like structures' by studying the categorical notion associated with a variety admitting a majority term, namely, majority categories.

The first two chapters of this thesis introduce the necessary categorical background for the rest of the text, and the notion of a majority category is precisely defined for the first time in Chapter 2. Their relation to existing categorical notions, such as *protoarithmetical* and *arithmetical* categories in the sense of [Bou01] and [Ped96] respectively, are considered in that chapter. It will turn out that every finitely complete Mal'tsev majority category is necessarily protoarithmetical; however, the converse does not hold in general. In the context of regular categories, we provide a counterexample showing that not every protoarithmetical category is a majority category, and for a Barr exact category $\mathbb{C}$ [BGO71], we will show that $\mathbb{C}$ is (proto)arithmetical if and only if it is both Mal'tsev and a majority category. This result generalizes a famous theorem of A. F. Pixley for varieties (see [Pix63]).

Chapter 3 begins with a categorical exploration of Bergman's double projection theorem mentioned above, and we will see that a regular category is a majority category if and only if it satisfies the categorical version of Bergman's theorem. This will allow us to show that many categories of a 'geometric' nature ($\mathbf{Top}, \mathbf{Ord}, \mathbf{Met}_\infty$, the dual of a topos, etc.) possess the dual *comajority* property. One may consider an infinite version of Bergman's theorem, which states that the a subobject S of a product $\prod_{i \in I} A_i$ is uniquely determined by its images in $A_i \times A_j$ for $i, j \in I$, and where I is not necessarily finite. If I is allowed to be countable, then there are no non-trivial finitary varieties which satisfy this countable version. However, there are varieties with operations of countable arity which do (such as the variety of lattices equipped with countable meet and join operations). The most extreme version of this property is shared by categories of a non-varietal nature, such as the category **CLat** of complete lattices, and the dual categories $\mathbf{Ord}^{\mathrm{op}}, \mathbf{Rel}_2^{\mathrm{op}}, \mathbf{Grph}^{\mathrm{op}}$ amongst others. Interestingly, the dual of the category of topological spaces $\mathbf{Top}^{\mathrm{op}}$ only satisfies the finite version. The chapter concludes with the main characterization theorem of this thesis, which presents a characterization of regular majority categories. This result extends the corresponding universal-algebraic results for varieties admitting a majority term, based mainly on the work of A. F. Pixley (see [BP75], [Pix63] and [Pix79]).

Chapter 4 presents two unique factorization results; one for a certain class of majority categories, and the other for so called *zero-majority* categories. From the point of view of the author, these are among the most interesting results of the thesis. The first unique factorization result is based on the corresponding universal-algebraic result (see Chapter 5 in [MMT87]), and is essentially an application of the results of Chapter 1.4 and Chapter 3. The second unique factorization result is proved using different techniques for pointed zero-majority categories with binary coproducts, in a context weaker than regular categories. It shows that there can be a categorical foundation to various decomposition results, that does not require the category to have an algebraic nature (regularity, exactness, ect). This theorem applies to for example, the category of *topological lattices*, which is not regular.

The last chapter of the thesis proves that under mild conditions, the only categories $\mathbb{C}$ such that $\mathbb{C}$ and $\mathbb{C}^{\mathrm{op}}$ are majority, are the preorders. The thesis ends with a brief discussion of other possible future directions, and discusses the most straightforward generalizations of the notion of a majority category towards general congruence distributive varieties. However, a fully-fledged categorical theory of congruence distributive varieties is far from being complete.

The results of Chapters 3, 4, 6 are essentially those that have already been written in [Hoe18b] and [Hoe18a], and in some cases the text is a small contextual modification of the original text. The results of Chapter 5 are new, although contain some small portion of the content of [Hoe18a].

Throughout this text we assume that the reader is familiar with some of the fundamental concepts of universal algebra such as *term*, *identity*, *variety*, *congruence*, *homomorphism*, *free algebra*, ect. Such concepts are contained in any standard introduction to the subject such as [Ber12] or [MMT87]. We also assume that the reader is familiar with the basic concepts of category theory such as *category, functor, natural transformation, limit, colimit, monomorphism, epimorphism*, as presented in [Mac98] or [Bor94a].

Convention

Throughout the remainder of this text we will always be dealing with categories with finite products. Therefore, by 'a category $\mathbb{C}$', we mean 'a category $\mathbb{C}$ with finite products'.

Chapter 1

Preliminaries

1.1 Basic categorical notions

Definition 1.1. A morphism $f : X \to Y$ in a category $\mathbb{C}$ is called a *strong epimorphism*, if for any commutative diagram of solid arrows

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[d, "m"] \\ Y \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

where m is a monomorphism, the dotted arrow exists making the diagram commute.

We then have some basic properties of strong epimorphisms: for any two morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ in a category $\mathbb{C}$:

(i) If g and f are strong epimorphisms, then so is $g \circ f$.

(ii) If $g \circ f$ is a strong epimorphism, then so is g.

Definition 1.2. A morphism $f : X \to Y$ in a category $\mathbb{C}$ is called a *regular epimorphism*, if there exist two morphisms $k_1, k_2 : K \to X$ such that the diagram

$$\begin{tikzcd} K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

is a coequalizer in $\mathbb{C}$.

Remarks 1.3. Note that any regular epimorphism is a strong epimorphism, and that if $\mathbb{C}$ is finitely complete, then any strong epimorphism is an epimorphism.

Definition 1.4. If $f : X \to Y$ is a morphism in a category $\mathbb{C}$, and $f = me$ where $e : X \to I$ is a strong epimorphism and $m : I \to Y$ a monomorphism, then the factorization $f = me$ is called an *image factorization* of f. If every morphism in $\mathbb{C}$ has an image factorization, then $\mathbb{C}$ is said to have image factorizations.

Remarks 1.5. Note that a category $\mathbb{C}$ has image-factorizations if and only if it admits an (E, M)-factorization system in the sense of [FK72], where E is the class of all strong epimorphisms and M the class of all monomorphisms.

Subobjects

If $m : M \to X$ and $n : N \to X$ are monomorphisms in a category $\mathbb{C}$, then we write $m \leqslant n$ if m factors through n, i.e., if there exists $\phi : M \to N$ such that $n\phi = m$. This defines a preorder $\mathscr{M}(X)$ on the class of all monomorphisms in $\mathbb{C}$ with codomain X. The posetal reflection of $\mathscr{M}(X)$ is called the *poset of subobjects* of X, and is denoted by $\mathrm{Sub}(X)$. Explicitly, a *subobject* $S \in \mathrm{Sub}(X)$ is an equivalence class of monomorphisms with codomain X, where two monomorphisms $n, m \in \mathscr{M}(X)$ are equivalent if and only if $n \leqslant m$ and $m \leqslant n$. If $s : S_0 \to X$ is a member of S, then we will say that S is the *subobject represented* by s in what follows.

Definition 1.6. A category $\mathbb{C}$ is said to be *well-powered* if the class of subobjects $\mathrm{Sub}(X)$ on any object X forms a set.

In any category $\mathbb{C}$ the pullback of a monomorphism along any morphisms is again a monomorphism, which is to say that if the diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "n"'] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

is a pullback diagram in $\mathbb{C}$, and m is a monomorphism, then so is n. Given that $\mathbb{C}$ has pullbacks of monomorphisms along monomorphisms and $A, B \in \mathrm{Sub}(X)$ are any subobjects represented by $a : A_0 \to X$ and $b : B_0 \to X$

respectively, then we write $A \cap B$ for the subobject of X represented by the diagonal monomorphism in any pullback

$$\begin{tikzcd} A_0 \cap B_0 \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr] & A_0 \arrow[d, "a"] \\ B_0 \arrow[r, "b"'] & X \end{tikzcd}$$

which is a monomorphism, as it is a composite of monomorphisms.

Remarks 1.7. If $f = me$ and $f = m'e'$ are two image factorizations (Definition 1.4) of a morphism $f : X \to Y$ in a category $\mathbb{C}$, then m and m' represent the same subobject of Y, which is denoted by $f(X)$. Given a subobject A of X represented by $a : A_0 \to X$ we will write $f(A)$ for the subobject represented by the mono part of an image factorization of fa. Also, we will often refer to $f(A)$ as the image of A under f.

Definition 1.8. Given a subobject $A \in \mathrm{Sub}(X)$, represented by $a : A_0 \to X$, then for any morphism x with codomain X we write $x \in_S A$ if x factors through a, and x has domain S.

$$\begin{tikzcd} & X \\ S \arrow[ur, "x"] \arrow[r, dotted] & A_0 \arrow[u, "a"'] \end{tikzcd}$$

Remarks 1.9. If x factors through one representative of A, then it factors through all representatives of A.

1.2 Internal relations in categories

A relation between sets is defined as a subset of a cartesian product, in a category we can define relations as subobjects of a cartesian product.

Definition 1.10. Given objects $A_1, A_2, ..., A_n$ in a category $\mathbb{C}$, an n-ary (internal) relation R is simply a subobject of $A_1 \times A_2 \times \cdots \times A_n$.

Remarks 1.11. If $\mathbb{C}$ did not have products, then we could still define an n-ary relation R as above as a jointly monomorphic family $(r_i : R_0 \to A_i)_{i=1,...,n}$. But for the purposes of this text, we restrict our attention to the definition above.

Given a binary relation R on an object A represented by a monomorphism $(r_1, r_2) : R_0 \to A \times A$, we can describe what it means for R to be reflexive/transitive/symmetric using the notation defined in Definition 1.8.

(i) R is *reflexive* if for any $x : S \to A$ we have $(x, x) \in_S R$. Equivalently, R is reflexive if in the diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta_A"] \\ R_0 \arrow[ur, dotted] \arrow[r, "(r_1,r_2)"'] & A \times A \end{tikzcd}$$

the dotted arrow exists making the diagram commute.

(ii) R is *symmetric* if for any $x, y : S \to A$ we have $(x, y) \in_S R$ implies $(y, x) \in_S R$, which is to say there exists a morphism $\phi : R_0 \to R_0$ such that $r_1\phi = r_2$ and $r_2\phi = r_1$.

(iii) R is transitive if for any $x, y, z : S \to A$ we have $(x, y) \in_S R$ and $(y, z) \in_S R$ implies $(x, z) \in_S R$. Equivalently, R is *transitive* if when we take any pullback

$$\begin{tikzcd} R_0 \times_A R_0 \arrow[r, "p_2"] \arrow[d, "p_1"'] & R_0 \arrow[d, "r_2"] \\ R_0 \arrow[r, "r_1"'] & A \end{tikzcd}$$

there exists a morphism $m : R_0 \times_A R_0 \to R_0$ such that

$$r_1 \circ m = r_1 \circ p_1 \quad \text{and} \quad r_2 \circ m = r_2 \circ p_2.$$

Definition 1.12. A relation R on an object A is an *equivalence relation*, if it is reflexive, transitive and symmetric.

Proposition 1.13. *If $\mathbb{C}$ is a category with pullbacks, $f : X \to Y$ and $m : A \to Y$ any morphisms with m mono, then f factors through m if and only if in the pullback:*

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "m"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

p_1 is an isomorphism.

Proof. Suppose that f factors through m via a morphism $h : X \to A$, then the pair of morphisms $(1_X, h)$ induces a morphism $k : X \to P$ such that $p_1 \circ k = 1_X$, so that p_1 is a split epimorphism. Since pullbacks of monomorphisms are monomorphisms, we have that p_1 is also a monomorphism, and therefore it is an isomorphism. On the other hand if p_1 is an isomorphism, then f factors through m via $h = p_2 p_1^{-1}$. □

Corollary 1.14. *It follows from the above proposition that if $\mathbb{C}$ is a finitely complete category and $F : \mathbb{C} \to \mathbb{D}$ is a functor which preserves finite limits, then for any equivalence relation E represented by $e : E_0 \to X \times X$, we have that $F(e) : F(E_0) \to F(X \times X)$ is a monomorphism, which when composed with the natural isomorphism $F(X \times X) \simeq F(X) \times F(X)$, represents an equivalence relation on $F(X)$.*

Proof. The morphism e is mono if and only if the kernel pair $k_1, k_2 : K \to E_0$ of e has $k_1 = k_2$. Since F preserves kernel pairs, it follows that $F(e)$ is a mono. Then it is easily seen that F preserves all the conditions on e, so that $F(e)$ represents an equivalence relation. □

Example 1.15. Given a monomorphism $E_0 \xrightarrow{e} X \times X$ in the category **Set** of sets, e represents an internal equivalence relation in **Set** if and only if $e(E)$ is an ordinary set-theoretic equivalence relation in X.

Example 1.16. Similar to the previous example, given a monomorphism $E \xrightarrow{e} X \times X$ in a variety $\mathbb{V}$ of algebras, e represents an internal equivalence relation in $\mathbb{V}$ if and only if $e(E)$ is a congruence on X in the universal algebraic sense.

Definition 1.17. For any morphism $f : X \to Y$, the *kernel equivalence relation* $\mathrm{Eq}(f)$ is the subobject of $X \times X$ represented by the kernel pair of f, i.e. , it is represented by $(k_1, k_2) : K \to X \times X$ where the following diagram is a pullback

$$\begin{tikzcd} K \arrow[r, "k_2"] \arrow[d, "k_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Definition 1.18. An equivalence relation E on an object X is said to be *effective*, if there exists $f : X \to Y$ such that $E = \mathrm{Eq}(f)$. Such equivalence relations are also called *congruences*.

Example 1.19. In any variety $\mathbb{V}$ of algebras, every equivalence relation is effective.

1.3 Internal categories

The fundamental internal notions, such as internal equivalence relation, internal preorder, internal monoid and internal group, can all be seen as aspects of one internal notion: *internal category*.

Definition 1.20. An internal category C in a category $\mathbb{C}$ is a diagram

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & C_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_2"'] & C_0 \arrow[l, "s"'] \end{tikzcd}$$

where the square

$$\begin{tikzcd} C_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_2"'] & C_0 \end{tikzcd}$$

is a pullback, and we have the following relations

(i) $d_1 \circ s = 1_{C_0} = d_2 \circ s$

(ii) $m \circ (1_{C_1}, s \circ d_2) = 1_{C_1} = m \circ (s \circ d_1, 1_{C_1})$

(iii) $d_1 \circ p_1 = d_1 \circ m$ and $d_2 \circ p_2 = d_2 \circ m$

(iv) $m \circ (p_1 q_1, m q_2) = m \circ (m q_1, p_2 q_2)$ where

$$\begin{tikzcd} C_3 \arrow[r, "q_2"] \arrow[d, "q_1"'] & C_2 \arrow[d, "p_1"] \\ C_2 \arrow[r, "p_2"'] & C_1 \end{tikzcd}$$

is a pullback, and $(p_1 q_1, m q_2), (m q_1, p_2 q_2) : C_3 \to C_2$ are the morphisms induced by the pullback (C_2, p_1, p_2).

The object C_0 is called the 'object of objects', C_1 is called the "object of arrows", $d_1, d_2 : C_1 \to C_0$ the "domain" and "codomain" morphisms respectively. The morphism m is the "composition" of C where C_2 represents the object of "pairs of composable arrows". The identities $(i) - (iv)$ then encode the familiar category axioms, where (iv) encodes the fact the composition is associative.

Example 1.21. An internal category C in **Set** essentially amounts to the ordinary notion of a small category (i.e. a category where the classes of objects and arrows form a set).

Definition 1.22. An internal category C as in Definition 1.20 is called an *internal groupoid* if there exists an "inverse" morphism $\sigma : C_1 \to C_1$ satisfying the following:

$$d_1 \circ \sigma = d_2, \quad d_2 \circ \sigma = d_1,$$

and

$$m \circ (1_{C_1}, \sigma) = s \circ d_1, \quad m \circ (\sigma, 1_{C_1}) = s \circ d_2.$$

Example 1.23. Given a topological space X, the *fundamental groupoid* $\pi_1(X)$ of X is the groupoid formed by taking the set of objects to be the underlying set of X, and the set of all homotopy equivalence classes of paths in X (a path in X is a continuous map $[0,1] \to X$) to be the set of morphisms of $\pi_1(X)$. Two paths $f, g : [0,1] \to X$ are said to be *homotopy equivalent* (written $f \simeq g$) if there exists a continuous map $H : [0,1] \times [0,1] \to X$ such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ and $H(x,0) = H(y,0)$ and $H(x,1) = H(y,1)$ for any $x, y \in [0,1]$. Then the domain of a class $[f]$ is given by $f(0)$ and the codomain by $f(1)$, for any representative f of a morphism $[f] \in \pi_1(X)$. If $[f], [g]$ are composable morphisms in $\pi(X)$, then their composite is given by $[g] \circ [f] = [h]$ where h is the path

$$h(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}.$$

Given a "base point" $x_0 \in X$, the ordinary fundamental group $\pi_1(X, x_0)$ is nothing but the automorphism group $\mathrm{Aut}(x_0)$ (which is the same as $hom(x_0, x_0)$) in $\pi_1(X)$.

Considering an extreme in Definition 1.20, if $C_0 \simeq 1$ is a terminal object in $\mathbb{C}$, then $C_2 \simeq C_1 \times C_1$ and $m : C_1 \times C_1 \to C_1$ becomes an internal monoid multiplication, where the unit $s : C_0 \to C_1$ satisfies the required conditions. Therefore, we get the following definitions:

Definition 1.24. An internal category is called an *internal monoid* when C_0 is a terminal object.

In a similar way, we can also recover the ordinary notion of an internal group.

Definition 1.25. An internal groupoid is called an *internal group* when C_0 is a terminal object.

Example 1.26. Given a monomorphism $(r_1, r_2) : R \to C_0 \times C_0$, then (r_1, r_2) represents a reflexive transitive relation (a preorder) if and only if there exist $s : X \to C_0$ and $m : R \times_{C_0} R \to R$ making the diagram

$$\begin{tikzcd} R \times_X R \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & C_0 \arrow[l, "s" description] \end{tikzcd}$$

an internal category. Conversely, if the internal category C in Definition 1.20 has d_1, d_2 jointly monomorphic, then the morphism (d_1, d_2) represents an internal preorder on C_0.

Example 1.27. Similar to the example above, every monomorphism $E \to X \times X$ which represents an equivalence relation gives rise to an internal groupoid, and every internal groupoid where the domain and codomain morphisms are jointly monomorphic gives rise to an internal equivalence relation.

Example 1.28. An internal group G in the category **Top** of topological spaces is the same as a topological group. Similarly, an internal monoid in the category **Top** is given by an ordinary topological monoid.

Example 1.29. A crossed module consists of a pair of groups G and H, an action of G on H, and a homomorphism $\sigma : H \to G$ which respects the action. If we denote the action of an element $g \in G$ on an element $h \in H$, by $g \star h$, then (G, H, σ) being a crossed module amounts to the following identities:

$$1 \star h = h, \quad g \star (g' \star h) = gg' \star h, \quad g \star (hh') = (g \star h)(g \star h'),$$

and

$$\sigma(g \star h) = g(\sigma(h))g^{-1},$$

as well as the *Peiffer identity:*

$$\sigma(h) \star h' = hh'h^{-1}.$$

Internal categories in **Grp** amount to crossed modules as above, for a proof we refer the reader to [Mac98].

A well known result of B. Jónsson in universal algebra, presents a Mal'tsev-type characterization of congruence distributive varieties, which is given below.

Theorem 1 [Jon67]. *For a variety of algebras $\mathbb{V}$, the following are equivalent.*

1. *$\mathbb{V}$ is congruence distributive.*

2. *There exist ternary terms $t_0, t_1, ..., t_n$, where $n \geqslant 2$, such that the equations*

$$\begin{aligned} t_0(x,y,z) = x, \quad t_i(x,y,x) = x, \quad t_n(x,y,z) &= z \\ t_i(x,x,z) &= t_{i+1}(x,x,z), && (i \text{ even}) \\ t_i(x,z,z) &= t_{i+1}(x,z,z), && (i \text{ odd}) \end{aligned}$$

hold in $\mathbb{V}$.

In [JP97], the authors remark that every internal groupoid in a congruence distributive variety is an equivalence relation, but actually more is true:

Proposition 1.30. *Every internal category in a congruence distributive variety is a preorder.*

Suppose that C is an internal category as in Definition 1.20 in a congruence distributive variety. Then in what follows we will write $f : X \to Y$ for arrows f of C (i.e. elements of C_1) where $d_0(f) = X$ and $d_1(f) = Y$. Also, if X is an object of C (i.e. and element of C_0) then we will write 1_X for the identity $s(X)$ of X. Finally, if $f : X \to Y$ and $g : Y \to Z$ are any two composable morphisms, then we shall write $g \circ f = m(g, f)$ for their composition in C.

Proof. Suppose that $f, g : X \to Y$ are any two parallel arrows in C, and let $t_0, t_1, ..., t_n$ be the Jónsson terms as in the proposition above. Then for any

$i = 1, 2, ..., n$ we have

$$\begin{aligned} t_i(f,g,g) &= t_i(m(f,1_X), m(1_Y,g), m(g,1_X)) \\ &= m(t_i((f,1_X),(1_Y,g),(g,1_X))) \\ &= m(t_i(f,1_Y,g), t_i(1_X,g,1_X)) \\ &= m(t_i(f,1_Y,g), t_i(1_X,f,1_X)) \\ &= m(t_i((f,1_X),(1_Y,f),(g,1_X))) \\ &= t_i(m(f,1_X), m(1_Y,f), m(g,1_X)) \\ &= t_i(f,f,g) \end{aligned}$$

And since $t_1, t_2, ..., t_n$ are Jónsson terms we have:

$$\begin{aligned} f = t_0(f,f,g) = t_1(f,f,g) = t_1(f,g,g) = \\ t_2(f,g,g) = t_2(f,f,g) = t_3(f,f,g) = \cdots \\ \cdots = t_n(f,g,g) = g \end{aligned}$$

And therefore $f = g$, so that there are no parallel arrows in C, and hence we have that C is a transitive relation. □

In particular, if $\mathbb{V}$ is a variety of algebras which admits a majority term, then $\mathbb{V}$ is congruence distributive by Jónsson's theorem, so that every internal category in $\mathbb{V}$ is a preorder. Since majority categories (see Definition 2.4) are seen as the categorical counterparts of varieties admitting a majority term, and the fact that internal categories in congruence distributive varieties are preorder, the question of whether internal categories in majority categories are preorders is natural. It will be shown later that internal groupoids in majority categories are equivalence relations, but it remains open whether or not internal categories in majority categories are preorders. However, we conjecture that in every finitely complete majority category, every internal category is a preorder.

1.4 Regular categories

Most of the examples of majority categories that are known, are at the same time regular categories [BGO71], and so the most striking aspects of the theory of majority categories are seen within the regular context.

We have seen that (binary) relations in categories may be defined simply as subobjects of a product of two objects. It is then straightforward to define the corresponding notions of reflexive, symmetric, transitive, difunctional relations, and establish basic properties of them such as, for example, the fact that every reflexive difunctional relation is an equivalence relation. One of the most fundamental operations on relations, is that we can compose them: given two relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ between sets, their *composite* $R \circ S$ is defined as

$$R \circ S = \{(x,z) \mid \exists y \in Y((x,y) \in R \wedge (y,z) \in S)\}.$$

Remarks 1.31. The above notion for $R \circ S$ is not the standard notion — which agrees with function composition.

Let $r_1 : R \to X$ and $r_2 : R \to Y$ be the canonical projections $(x,y) \mapsto x$ and $(x,y) \mapsto y$ respectively, and similarly let $s_1 : S \to Y$ and $s_2 : S \to Z$ be the canonical projections. In order to define a categorical counterpart of the above set-theoretic construction, consider the following diagram:

$$\begin{tikzcd}
 & & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
X & & Y & & Z
\end{tikzcd}$$

where (P, p_1, p_2) is the pullback of r_2 along s_1, and r_1, r_2, s_1, s_2 are the canonical projections. Set theoretically, P is given by

$$P = \{(x,y,z) \mid (x,y) \in R \wedge (y,z) \in S\}.$$

Clearly the image of the map $P \to X \times Z$ defined by $(x,y,z) \longmapsto (x,z)$ is precisely $R \circ S$, and therefore the composite of two relations $R \leqslant X \times Y$ and $S \leqslant Y \times Z$ could be constructed in any category $\mathbb{C}$ with finite-limits and image factorizations. However, this composition need not be associative – and the usual composite of set-theoretic relations is. The answer to the question of when this relation composition is associative is: if and only if $\mathbb{C}$ is a *regular* category.

Definition 1.32 ([BGO71])**.** A category $\mathbb{C}$ is said to be regular if

(i) $\mathbb{C}$ has finite limits, and coequalizers of kernel pairs.

(ii) For any pullback square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "g"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

if f is a regular epimorphism, then so is g.

The category $\mathbb{C}$ is said to be *weakly* regular, if it satisfies (i) and a weakening of (2): if f is a regular epimorphism, then g is an epimorphism.

Remarks 1.33. Every regular category is weakly regular.

The theorem below is a standard theorem of regular categories, and the proof below is essentially the same as the proof that can be found in [Bor94b].

Theorem 1.34. *In every weakly regular category, every morphism factors as a regular epimorphism followed by a monomorphism.*

Proof. Suppose that $X \xrightarrow{f} Y$ is any morphism, and consider the diagram

$$\begin{tikzcd} K \arrow[r, shift left, "k_2"] \arrow[r, shift right, "k_1"'] \arrow[d, dotted, "h"'] & X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ R \arrow[r, shift left, "r_2"] \arrow[r, shift right, "r_1"'] & Q \arrow[ur, "g"] & \end{tikzcd}$$

where K is the kernel pair of f, q the coequalizer of (k_1, k_2), R is the kernel pair of g, where q is the coequalizer of (k_1, k_2) and g is the unique morphism making the triangle commute. We will show that h is an epimorphism, which would then imply that $r_1 = r_2$ so that g is a monomorphism.

Consider the diagram below:

$$\begin{tikzcd}
K \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, "h"] \arrow[rr, bend left, "k_2"] \arrow[dd, bend right, "k_1"'] & \bullet \arrow[r] \arrow[d, two heads] & X \arrow[d, two heads, "q"'] \\
\bullet \arrow[r, two heads] \arrow[d] & R \arrow[r, "r_2"] \arrow[d, "r_1"'] & Q \arrow[d, "g"] \\
X \arrow[r, two heads, "q"'] & Q \arrow[r, "g"'] & Y
\end{tikzcd}$$

Each of the interior squares are pullbacks, which implies that every morphism in the upper left-hand square can be realized as a pullback of q along some morphism. Therefore, they are all epimorphisms, so that

$$r_1h = k_1q = k_2q = r_2h \implies r_1 = r_2.$$

□

Corollary 1.35. *In a weakly-regular category every strong epimorphism is a regular epimorphism.*

Proposition 1.36. *If $\mathbb{C}$ is any category with finite limits and coequalizers of kernel pairs, and $F : \mathbb{C} \to \mathbb{D}$ any functor which preserves pullbacks and regular epimorphisms, and also reflects epimorphisms, then if $\mathbb{D}$ is weakly regular, so is $\mathbb{C}$*

Proof. All we need to show is that the pullback of a regular epimorphism in $\mathbb{C}$ is an epimorphism. Therefore, suppose that the diagram

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y \arrow[r, "b"'] & Z
\end{tikzcd}$$

is a pullback, where f is a regular epimorphism. By the assumptions on F, it follows that the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(a)"] \arrow[d, "F(g)"'] & F(X) \arrow[d, "F(f)"] \\
F(Y) \arrow[r, "F(b)"'] & F(Z)
\end{tikzcd}$$

is a pullback in $\mathbb{D}$. Since $F(f)$ is a regular epimorphism and $\mathbb{D}$ is weakly regular (see Definition 1.32), it follows that $F(g)$ is an epimorphism, which implies that g is an epimorphism, since F reflects epimorphisms. □

Recall that the category $\mathbf{Met}_\infty$ of *extended* metric spaces has as its objects (X, d_X) metric spaces, where the metric d_X takes values in the extended real line $\mathbb{R} \cup \{\infty\}$. Morphisms in $\mathbf{Met}_\infty$ are *subcontractions*, i.e., maps $f : X \to Y$ satisfying:

$$d_Y(f(x), f(y)) \leqslant d_X(x, y).$$

The category $\mathbf{Met}_\infty$ has all limits and colimits (see [Wei17]), and admits a forgetful functor $\mathbf{Met}_\infty \to \mathbf{Set}$

Example 1.37. The forgetful functors $\mathbf{Top} \to \mathbf{Set}$, $\mathbf{Ord} \to \mathbf{Set}$, $\mathbf{Met}_\infty \to \mathbf{Set}$ all preserve finite-limits and regular epimorphisms, and reflect epimorphisms. Thus, since $\mathbf{Set}$ is regular, it follows that $\mathbf{Top}, \mathbf{Ord}, \mathbf{Met}_\infty$ are weakly regular categories.

Example 1.38. A topological lattice L is a lattice equipped with a topology on the underlying set of L for which the meet and join operations $\wedge, \vee : L \times L \to L$ are continuous (see [Str68]). A morphism of topological lattices is a lattice homomorphism which is continuous with respect to the underlying topologies. The category $\mathbf{Lat}(\mathbf{Top})$ of all topological lattices, admits a forgetful functor $U : \mathbf{Lat}(\mathbf{Top}) \to \mathbf{Lat}$ which preserves regular epimorphisms and finite limits, and also reflects epimorphisms. Since $\mathbf{Lat}$ is regular, it follows that $\mathbf{Lat}(\mathbf{Top})$ is weakly-regular.

Theorem 1.39. *Let $\mathbb{C}$ be a regular category. Given the reasonably commutative diagram*

$$\begin{tikzcd} K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] \arrow[d] & X \arrow[r, "f"] \arrow[d, "q"] & A \\ T \arrow[r, shift left, "t_1"] \arrow[r, shift right, "t_2"'] & Y \arrow[ur, "g"'] & \end{tikzcd}$$

where q is a regular epi, (t_1, t_2) jointly monic, if (K, k_1, k_2) is the kernel pair of f, then (T, t_1, t_2) is the kernel pair of g.

Proof. Let (E, e_1, e_2) be the kernel pair of g, then it suffices to show that (e_1, e_2) factors through (t_1, t_2). Consider the diagram below, where p is the pull-back of q making the square reasonably commute.

$$\begin{tikzcd}
P \arrow[d, "p"'] \arrow[dr, dotted, "h"'] \arrow[drr, shift left, "p_2"] \arrow[drr, shift right, "p_1"'] & & & \\
E \arrow[drr, shift left] \arrow[drr, shift right] & K \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X \arrow[r, "f"] \arrow[d, "q"] & A \\
 & T \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[ur, "g"'] &
\end{tikzcd}$$

Then the morphism h exists in the diagram above (since K is the kernel pair of f). This gives the following commutative diagram:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "p"'] & K \arrow[r] & T \arrow[d, "(t_1,t_2)"] \\
E \arrow[urr, dotted] \arrow[rr, "(e_1,e_2)"'] & & Y \times Y
\end{tikzcd}$$

The dotted arrow exists, since p is a regular epimorphism and (t_1, t_2) is a monomorphism. $\square$

Definition 1.40. A diagram

$$\begin{tikzcd}
E_0 \arrow[r, shift left, "e_1"] \arrow[r, shift right, "e_2"'] & X \arrow[r, "q"'] & X/E
\end{tikzcd}$$

in a category is said to be *exact*, if (e_1, e_2) is the kernel pair of q and q is the coequalizer of E_0. A functor $F : \mathbb{C} \to \mathbb{D}$ is said to be exact, if it sends exact sequences in $\mathbb{C}$ to exact sequences in $\mathbb{D}$.

Corollary 1.41. *Let $(e_1, e_2) : E \to X^2$ and $(f_1, f_2) : F \to X^2$ represent two effective equivalence relations on X, where $E \leqslant F$, and consider the following diagram:*

$$\begin{tikzcd}
E \arrow[r, shift left, "e_1"] \arrow[r, shift right, "e_2"'] \arrow[d, two heads] & X \arrow[d, two heads, "q_E"'] \arrow[dr, two heads, "q_F"] & \\
F/E \arrow[r, shift left, "l_1"] \arrow[r, shift right, "l_2"'] & X/E \arrow[r, dotted, two heads] & X/F
\end{tikzcd}$$

q_F and q_E are the quotients of E and F respectively, and the two parallel arrows in the bottom row are obtained from taking the mono part of the regular image of F under q_E. Then (l_1, l_2) is the kernel pair of the dotted arrow.

Proof. This is a straightforward application of Theorem 1.39. $\square$

Proposition 1.42. *If $f : X \to Y$ and $g : A \to B$ are regular epimorphisms in a regular category $\mathbb{C}$, then the product $f \times g : X \times A \to Y \times B$ is a regular epimorphism.*

Proof. Consider the pullback squares

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d, two heads, "p_1"'] & X \arrow[d, two heads, "f"] \\ Y \times B \arrow[r, "\pi_1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} P_2 \arrow[r] \arrow[d, two heads, "p_2"'] & A \arrow[d, two heads, "g"] \\ Y \times B \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

And also the pullback square obtained from the above pullbacks

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, two heads, "p"] & P_1 \arrow[d, "p_1"] \\ P_2 \arrow[r, "p_2"'] & Y \times B \end{tikzcd}$$

Then there exists a morphism $q : P \to X \times A$ such that $(f \times g)q = p$, which implies that $f \times g$ is a regular epimorphism since p is a regular epimorphism. $\square$

Relations in regular categories

Let $\mathbb{C}$ be a regular category, and R and S relations represented by $(r_1, r_2) : R_0 \rightarrowtail X \times Y$ and $(s_1, s_2) : S_0 \to Y \times Z$ respectively. Suppose that (P, p_1, p_2) is the pullback of s_1 along r_2:

$$\begin{tikzcd} & & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\ & R_0 \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & S_0 \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ X & & Y & & Z \end{tikzcd}$$

The composite $R \circ S$ is the relation represented by the monomorphism $r \circ s : R_0 \circ S_0 \rightarrowtail X \times Z$, which is obtained by taking the regular epi, mono factorization of $(r_1 p_1, s_2 p_2) : P \to X \times Z$ as in the diagram:

$$\begin{tikzcd} P \arrow[r, two heads, "e"] \arrow[rr, bend right, "(r_1p_1, s_2p_2)"'] & R_0 \circ S_0 \arrow[r, "r \circ s"] & X \times Z \end{tikzcd}$$

We have the following lemma for this relation composition.

Proposition 1.43. *If $(x,z) : A \to X \times Z$ is any morphism, then $(x,z) \in_A R \circ S$ if and only if there exists a regular epimorphism $\alpha : Q \to A$ and a $y : Q \to Y$ such that $(x\alpha, y) \in_Q R$ and $(y, z\alpha) \in_Q S$.*

Proof. If (x,z) factors though $R_0 \circ S_0$, then the dotted arrow exists making the diagram

$$\begin{tikzcd}
Q \arrow[r, two heads, "\alpha"] \arrow[d, "q"'] & A \arrow[d, dotted, "h"] \arrow[dr, "(x{,}z)"] & \\
P \arrow[r, two heads, "e"] \arrow[rr, bend right, "(r_1p_1{,}s_2p_2)"'] & R_0 \circ S_0 \arrow[r, "r \circ s"] & X \times Z
\end{tikzcd}$$

commute. Then we can pull h back along e, to produce α and q in the diagram above. Then setting $y = qr_2p_1$, we have that α and y satisfy the required conditions.

For the "only if" part, suppose that $(x\alpha, y) \in_Q R$ and $(y, z\alpha) \in_Q S$, then it is easy to see that $(x\alpha, z\alpha) \in_Q R \circ S$ which gives the diagram below

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, two heads, "\alpha"'] & R_0 \circ S_0 \arrow[d, "r \circ s"] \\
A \arrow[r, "(x{,}z)"'] \arrow[ur, dotted] & X \times Z
\end{tikzcd}$$

where the dotted arrow exists, since α is a regular epimorphism, and $r \circ s$ is a monomorphism. $\square$

Proposition 1.44. *Let $\mathbb{C}$ be a regular category, and let F_1, F_2, K be any effective equivalence relations on any object X in $\mathbb{C}$ such that $F_1 \cap F_2 = K$ and $F_1 \circ F_2 = 1$. Then the canonical morphism*

$$X/K \to X/F_1 \times X/F_2,$$

is an isomorphism, where $q : X \to X/K$, $q_1 : X \to X/F_1$, and $q_2 : X \to X/F_2$ are the respective coequalizer morphisms.

Proof. Let $q_1 : X \to X/F_1$ and $q_2 : X \to X/F_2$ be the quotients of F_1 and F_2 respectively. Consider the product diagram:

$$X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X.$$

Then we have that $(\pi_1, \pi_2) \in_{X\times X} F_1 \circ F_2$, so that there exists a regular epimorphism $\alpha : Q \to X \times X$ and a morphism $f : Q \to X$ such that $(\pi_1\alpha, f) \in_Q F_1$ and $(f, \pi_2\alpha) \in_Q F_2$. This implies that the following diagram is commutative:

$$\begin{tikzcd}
Q \arrow[r, "f"] \arrow[d, "\alpha"'] & X \arrow[d, "{(q_1,q_2)}"] \\
X \times X \arrow[r, "q_1 \times q_2"'] & X/F_1 \times X/F_2
\end{tikzcd}$$

Note that $q_1 \times q_2$ is a regular epimorphism by Proposition 1.42. This implies that $(q_1 \times q_2)\alpha$ is a regular epimorphism by Corollary 1.35. Since

$$(q_1 \times q_2)\alpha = (q_1, q_2)f,$$

we have that (q_1, q_2) is a regular epimorphism since $(q_1, q_2)f$ is regular. Then, since the kernel of $(q_1 \times q_2)$ is given by $K = F_1 \cap F_2$, it follows that the canonical morphism

$$X/K \to X/F_1 \times X/F_2,$$

is an isomorphism. $\square$

1.5 Duals of geometric categories are regular

One of the surprising aspects of the theory of majority categories, is that there are many examples of 'geometric' categories whose duals turn out to be regular majority categories.

The category $\mathbf{Rel}_n$ has as its objects pairs (U_X, R_X) where U_X is a set, and R_X a n-ary relation on U_X. A morphism $f : X \to Y$ in $\mathbf{Rel}_n$ is a map $f : U_X \to U_Y$ which satisfies

$$(x_1, x_2, ..., x_n) \in R_X \implies (f(x_1), f(x_2), ..., f(x_n)) \in R_Y,$$

for any $x_1, x_2, ..., x_n \in U_X$. A morphism $m : A \to X$ is a regular monomorphism if and only if it is a mono and satisfies

$$(f(x_1), f(x_2), ..., f(x_n)) \in R_X \implies (x_1, x_2, ..., x_n) \in R_A,$$

for any $x_1, x_2, ..., x_n \in U_A$. The limit/colimit of a diagram D in $\mathbf{Rel}_n$ has as its underlying set, the set-theoretic limit/colimit of the underlying diagram in $\mathbf{Set}$, equipped with the largest/smallest relation making the canonical projections/inclusions morphisms in $\mathbf{Rel}_n$.

Proposition 1.45. $\mathbf{Rel}_n^{\mathrm{op}}$ *is a well-powered (co)complete regular category.*

Proof. Consider the following pushout diagram in $\mathbf{Rel}^{\mathrm{op}}$:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & X \arrow[d, "b'"] \\ Y \arrow[r, "a'"'] & Y +_A X \end{tikzcd}$$

where a is a regular monomorphism. Then a' is a monomorphism since pushouts of monomorphisms along any morphism in **Set** are monomorphisms. Now, suppose that $y_1, y_2, ..., y_n \in Y$ are any elements. Then the construction of the pushout gives only two possibilities that would yield

$$(a'(y_1), a'(y_2), ..., a'(y_n)) \in R_{Y+_A X}.$$

One of these possibilities is that $(y_1, y_2, ..., y_n) \in R_Y$. The other possibility is that there exist $x_1, x_2, ..., x_n \in A \subseteq X$ such that $(a(x_1), a(x_2), ..., a(x_n)) \in R_X$. This would imply that $(x_1, x_2, ..., x_n) \in R_A$ since a is assumed to be a regular monomorphism, and since a' is a monomorphism (injective), we must have that $b(x_i) = y_i$ for $i = 1, 2, ..., n$ since the diagram above commutes. Since b is a morphism in $\mathbf{Rel}_n$, it immediately follows that $(y_1, ..., y_n) \in R_Y$. Thus, in all of the two cases considered, we have the implication:

$$(a'(y_1), a'(y_2), ..., a'(y_n)) \in R_{Y+_A X} \implies (y_1, y_2, ..., y_n) \in R_Y,$$

so that a' is a regular monomorphism. $\square$

Recall that the category $\mathbf{Met}_\infty$ of *extended metric spaces* consists of metric spaces (X, d_X) where the metric d_X could take ∞ as a value. A morphism $f : X \to Y$ in $\mathbf{Met}_\infty$ is a set theoretic map satisfying $d_Y(f(x), f(y)) \leqslant d_X(x, y)$, such maps are usually called *subcontractions*. It was shown in [Wei17] that $\mathbf{Met}_\infty^{\mathrm{op}}$ is a regular category, which we state below without a proof.

Proposition 1.46. $\mathbf{Met}_\infty^{\mathrm{op}}$ *is a well-powered (co)complete regular category.*

As was shown in [BP95], $\mathbf{Top}^{\mathrm{op}}$ is a quasi-variety. Thus, it immediately follows that $\mathbf{Top}^{\mathrm{op}}$ is a regular category, however, we will give a direct proof below.

Proposition 1.47. $\mathbf{Top}^{\mathrm{op}}$ *is a well-powered (co)complete regular category.*

Recall that an injective continuous map $f : X \to Y$ between topological spaces is an embedding if and only if for every open set $U \subseteq X$, there exists an open set $V \subseteq Y$ such that $U = f^{-1}(V)$.

Proof. Consider the pushout diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & X \arrow[d, "b'"] \\ Y \arrow[r, "a'"'] & Y +_A X \end{tikzcd}$$

where a is a regular monomorphism. Suppose that $V \subset Y$ is any open set. Since a is an embedding of spaces, there exists $U \subseteq X$ such that $b^{-1}(V) = a^{-1}(U)$. The set $V + U$ is open in $Y + X$, and the image $[V + U]$ of $V + U$ in $Y +_A X$ under the canonical quotient map, is an open set in $Y +_A X$. Then, set-theoretically we have that $a'^{-1}([V + U]) = V$, so that a' is a regular monomorphism. $\square$

Chapter 2

The notion of a majority category

2.1 Definition of a majority category

Let $\mathbb{V}$ be a Mal'tsev variety, with p the corresponding Mal'tsev term. If $R \subseteq X \times Y$ is any subalgebra of a product of algebras X and Y, then R is *difunctional*, i.e., it satisfies:

$$(x,v) \in R \quad \text{and} \quad (u,v) \in R \quad \text{and} \quad (u,y) \in R \implies (x,y) \in R. \qquad (*)$$

Indeed, applying p to the elements on the left, we get

$$p((x,v),(u,v),(u,y)) = (p(x,u,u), p(v,v,y)) = (x,y),$$

which implies that $(x,y) \in R$. Moreover, this property of internal relations characterizes Mal'tsev varieties among all varieties over a given signature:

Theorem 2.1. *The following are equivalent for a variety $\mathbb{V}$ of algebras.*

1. *$\mathbb{V}$ is a Mal'tsev variety*
2. *Every homomorphic binary relation in $\mathbb{V}$ is difunctional.*

The condition $(*)$ above is a condition on internal relations in a category, and can therefore be reformulated for an abstract category. This leads to the following definition which can be found in [CPP91]:

Definition 2.2. A finitely complete category $\mathbb{C}$ is Mal'tsev when every internal relation R in $\mathbb{C}$ is difunctional, i.e., satisfies

$$(x,v) \in_S R \quad \text{and} \quad (u,v) \in_S R \quad \text{and} \quad (u,y) \in_S R \implies (x,y) \in_S R. \quad (*)$$

Remarks 2.3. Note that the original definition of a Mal'tsev category is one which is Barr exact, and where the composite of two equivalence relations on the same object is again an equivalence relation (see [CLP91]).

The notion above provides a context in which to establish categorical counterparts of theorems for Mal'tsev varieties. For example, in a Mal'tsev category it is easy to see that every reflexive internal relation is an internal equivalence relation. Clearly the Mal'tsev term equations determine a matrix:

$$N = \left(\begin{array}{ccc|c} x & u & u & x \\ v & v & y & y \end{array}\right),$$

which captures the equations defining the Mal'tsev term. Just as the Mal'tsev term determines an elementary matrix of terms, so do the majority term equations, and the resulting matrix is given by:

$$M = \left(\begin{array}{ccc|c} x & x & x' & x \\ y & y' & y & y \\ z' & z & z & z \end{array}\right).$$

This leads to the following definition:

Definition 2.4 ([Hoe18b])**.** A ternary relation R between objects X, Y, Z in $\mathbb{C}$ is said to be *majority-selecting* if its satisfies

$$(x,y,z') \in_S R \quad \text{and} \quad (x,y',z) \in_S R \quad \text{and} \quad (x',y,z) \in_S R \implies (x,y,z) \in_S R.$$

Then $\mathbb{C}$ is said to be a *majority category* if every internal relation in $\mathbb{C}$ is majority-selecting. In other words, every internal relation R in $\mathbb{C}$ is strictly M-closed in the sense of [Jan06].

There is nothing particularly special about the Mal'tsev term nor the majority term, other than the fact that the equations for them take the form of a matrix M as given above. Therefore, the above technique generalizes to varieties which admit an n-ary term p satsifying some "elementary equations", which take the form of an elementary matrix as above. This has been fully elaborated in [Jan06], where the author establishes general properties of categories defined by such a matrix condition.

2.2 Algebraic examples of majority categories

Example 2.5. A variety of algebras $\mathbb{V}$ is a majority category if and only if it admits a majority term, i.e., a ternary term $m(x, y, z)$ satisfying the equations:

$$m(x, x, y) = x,$$
$$m(x, y, x) = x,$$
$$m(y, x, x) = x.$$

For a proof of this statement, we refer the reader to [Jan06].

Example 2.6. The only subvarieties of monoids which are majority categories are trivial.

Proof. The free algebra $F_{\mathbb{V}}(x)$ over one element $\{x\}$ is some monoid quotient of $\mathbb{N}$ - the free monoid over one element. Therefore any ternary polynomial $p(x, y, z)$ in $F_{\mathbb{V}}(x)$ has the form

$$p(x, y, z) = ax + by + cz,$$

for some natural numbers $a, b, c \in \mathbb{N}$. Therefore,

$$x = p(x, x, x) = p(x, 0, 0) + p(0, x, 0) + p(0, 0, x) = 0,$$

if p is a majority term. This would imply that $F_{\mathbb{V}}(x)$ has one element, and therefore every algebra of $\mathbb{V}$ has at most one element. □

Example 2.7. A subvariety $\mathbb{V}$ of the variety of rings is a majority category if and only if $\mathbb{V}$ satisfies the equation $x^n = x$ for some $n \geqslant 2$. In particular, the category **BoRg** of Boolean rings is a majority category.

Proof. It was shown in [MW70] that for any variety of rings admitting a majority term, there exists $n \in \mathbb{N}$ with $n \geqslant 2$ such that $x^n = x$. Now if $\mathbb{V}$ is a variety of rings where $\mathbb{V} \models x^n = x$ for some $n \geqslant 2$, then the polynomial

$$p(x, y, z) = x - (x - y)(x - z)^{n-1},$$

is a majority term for $\mathbb{V}$. □

Example 2.8. The category **NReg** of von Neumann regular rings (see [Neu36]) is the class of all rings R such that for any $a \in R$ there exists $x \in R$ such that $a = axa$. The category **NReg** is a majority category.

Proof. Suppose that A, B, C are rings and that R is a subring of $A \times B \times C$ which is a von Neumann regular ring. Let $\bar{a} = (a, b, c'), \bar{b} = (a, b', c), \bar{c} = (a', b, c)$ be any elements of R. Then since R is von Neumann regular, there exists $x = (x_1, x_2, x_3) \in R$ such that

$$(\bar{a} - \bar{b})x(\bar{a} - \bar{b}) = (\bar{a} - \bar{b}).$$

Then it is easy to see that

$$\bar{a} - (\bar{a} - \bar{b})x(\bar{a} - \bar{c}) = (a, b, c),$$

so that $(a, b, c) \in R$ □

Example 2.9. The category **HLat** of Heyting semi-lattices, also known as implicative semi-lattices (see [Nem65]) is a majority category.

Proof. This is a consequence of Pixley's theorem (see [Pix63]), since **HLat** has both distributive and permutable congruences (see [Nem65]). □

Example 2.10. The above arguments can all be repeated for internal-structures of the previous kind, so that the category $\mathbf{NReg}(\mathbf{Top})$ of topological von Neumann regular rings, $\mathbf{Lat}(\mathbf{Top})$ of topological lattices, $\mathbf{HLat}(\mathbf{Top})$ of topological Heyting semi-lattices, are all majority categories.

2.3 Relation to antilinear and protoarithmetical categories

The notion of an *arithmetical* category was first introduced by M. C. Pedicchio in [Ped96], as a Barr exact Mal'tsev category (see Definition 2.2) with coequalizers, whose lattice of equivalence relations on each object is distributive. It was proved there that in an arithmetical category, every internal groupoid is an equivalence relation. Moreover, this property characterizes arithmetical categories among Barr exact Mal'tsev categories with coequalizers. In [Bou01], the author introduces the notion of a *protoarithmetical* category, which is the same as a finitely complete Mal'tsev category in which every internal groupoid is an equivalence relation. In the Barr exact context, protoarithmetical categories are characterized as congruence distributive Mal'tsev categories (Mal'tsev categories whose lattice of equivalence relations on each object is a distributive lattice). Thus, in [Bou01], an arithmetical category is simply a Barr exact Mal'tsev category which is congruence distributive (dropping coequalizers from the original definition), which is what we will mean by arithmetical category. This section shows that in the Barr exact context, arithmetical categories are precisely Mal'tsev majority categories.

Remarks 2.11. Pedicchio's orginial proof of the fact that among all Barr exact Mal'tsev categories with coequalizers, those that are arithmetical are precisely those in which every internal groupoid is an equivalence relation, required a certain commutator defined for Barr exact Mal'tsev categories with coequalizers (see [Ped95]). By working with *connectors* (see [BG02]) between equivalence relations instead, Bourn was able to obtain several characterizations of Mal'tsev categories in which every internal groupoid is an equivalence relation in the left exact context.

Definition 2.12. A protoarithmetical category is a finitely complete Mal'tsev category in which every internal groupoid is an equivalence relation.

Remarks 2.13. The orginal definition of a protoarithmetical category, which is equivalent to Definition 2.12, is that of a finitely complete category $\mathbb{C}$ where the category of points $\mathrm{Pt}_I(\mathbb{C})$ above any object I is unital [Bou96], and such that every internal group in $\mathrm{Pt}_I(\mathbb{C})$ is trivial (see [Bou01]).

One of the main results of [Bou01] is the following theorem, which links the original notion of an arithmetical category to the notion of a protoarithmetical category.

Theorem 2.14 ([Bou01]). *A Barr exact category $\mathbb{C}$ is protoarithmetical if and only if it is Mal'tsev and congruence distributive (i.e. it is arithmetical).*

The next theorem has been proved in [Hoe18b] with no limit assumptions whatsoever. It can actually be proved for categories more general than majority categories, and will be revisited in Chapter 6.

Theorem 2.15. *Every internal groupoid in a majority category $\mathbb{C}$ is an equivalence relation.*

Proof. Suppose that the diagram

$$\begin{tikzcd} G_2 \arrow[r, "m"] & G_1 \arrow[loop above, "\sigma"] \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] & G_0 \arrow[l, "s"'] \end{tikzcd}$$

is an internal groupoid (see Definition 1.22) in a majority category $\mathbb{C}$, then we show that d_0 and d_1 are jointly monomorphic. Let $p_1 : G_2 \to G_1$ and $p_2 : G_2 \to G_1$ be the canonical pullback projections. Then let R be the relation represented by the monomorphism (p_1, p_2, m). Suppose that $f, g : S \to G_1$ are morphisms with $d_1 f = d_1 g$ and $d_0 f = d_0 g$, then $(f, \sigma f, sd_1 f) \in_S R$ and $(g, \sigma g, sd_1 g) \in_S R$ and $(f, \sigma g, m(f, \sigma g)) \in_S R$ which implies that $(f, \sigma g, sd_1 g) \in_S R$ by Definition 2.4. This implies that $m(f, \sigma g) = sd_1 g$, which implies that $f = g$. $\square$

Corollary 2.16. *Every finitely complete Mal'tsev majority category is protoarithmetical.*

Definition 2.17 ([Bou02])**.** Let $\mathbb{C}$ be a pointed category with binary products, and let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbb{C}$. A morphism $\phi : X \times Y \to Z$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[dr, "f"'] & X \times Y \arrow[d, "\phi"] & Y \arrow[l, "\iota_Y"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commute, is called a cooperator for f and g. If $g = 1_Z$ in the diagram above, then f is said to be central when such a ϕ exists.

Definition 2.18 ([Bou02])**.** A unital category $\mathbb{C}$ is said to be antilinear if the only central morphisms are the null morphisms.

Proposition 2.19. *Let $\mathbb{C}$ be a pointed finitely complete majority category, and let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbb{C}$. If f and g admit a cooperator, then the square*

$$\begin{tikzcd} \ker(f) \times \ker(g) \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a pullback, where p_1 and p_2 are the canonical product projections composed with the canonical inclusions.

Corollary 2.20. *If $\mathbb{C}$ is a pointed finitely complete majority category, then $f : X \to Y$ is central if and only if $f = 0$.*

Proof. By Definition 2.17, f being central, it cooperates with the identity on Y, so that by Proposition 2.19 the pullback of 1_Y along f is given by $\ker(f) \times \ker(1_Y) \simeq \ker(f)$. This implies that the identity on 1_X is the kernel of f, so that $f = 0$. □

In particular, this gives that every unital majority category is antilinear in the sense of Definition 2.18, as the next corollary shows.

Corollary 2.21. *A unital majority category is necessarily antilinear in the sense of Definition 2.18.*

Proof of Proposition 2.19. Suppose that ϕ is a cooperator between f and g, then it suffices to show that for any commutative square

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

we have $g\beta = 0 = f\alpha$. Consider the ternary relation R represented by $r : R_0 \to X \times Y \times Z$ - which is defined by the equalizer:

$$R_0 \xrightarrow{r} X \times Y \times Z \underset{\pi_3}{\overset{\phi(\pi_1,\pi_2)}{\rightrightarrows}} Z.$$

Then since we have $\phi(\alpha, 0) = f\alpha$ and $\phi(0, \beta) = g\beta$, by the universal property of the equalizer it follows that $(\alpha, 0, f\alpha) : A \to X \times Y \times Z$ and $(0, \beta, g\beta) : A \to X \times Y \times Z$ and $(0, 0, 0) : A \to X \times Y \times Z$ all have:

$$(\alpha, 0, f\alpha) \in_A R \quad \text{and} \quad (0, \beta, g\beta) \in_A R \quad \text{and} \quad (0, 0, 0) \in_A R.$$

Since $f\alpha = g\beta$, we have that $(0, 0, f\alpha) \in_A R$, which implies that $f\alpha = 0 = g\beta$. □

Mal'tsev + Majority = Arithmetical

In any regular Mal'tsev category $\mathbb{C}$, the join of two equivalence relations C_1 and C_2 on an object X is given by their composite $C_1 \circ C_2$ (see [CLP91]). Therefore, a regular Mal'tsev category $\mathbb{C}$ is congruence distributive if and only if for any three equivalence relations α, β, γ on X we have

$$\alpha \cap (\beta \circ \gamma) = (\alpha \cap \beta) \circ (\alpha \cap \gamma).$$

In Theorem 3.27 we will see that regular majority categories are characterized as those regular categories for which the above equation holds for the lattice of equivalence relations on any object. Then the following theorem is a straightforward corollary of Theorem 3.27:

Theorem 2.22. *If $\mathbb{C}$ is a regular Mal'tsev category such that the lattice of equivalence relations on each object is a distributive lattice, then $\mathbb{C}$ is a majority category.*

Corollary 2.23. *For a Barr exact category $\mathbb{C}$ the following are equivalent:*

(1) *$\mathbb{C}$ is arithmetical (i.e. Mal'tsev and congruence distributive);*

(2) *$\mathbb{C}$ is Mal'tsev and a majority category.*

Proof. (1) $\Rightarrow$ (2) is immediate by Theorem 2.22. For (2) $\Rightarrow$ (1) suppose that $\mathbb{C}$ is a Mal'tsev majority category, then by Corollary 2.16 we have that $\mathbb{C}$ is protoarithmetical, and thus $\mathbb{C}$ is arithmetical by Theorem 2.14. $\square$

The above corollary motivates the question of whether protoarithmetical categories are, in general, the same as Mal'tsev majority categories. Or if there are naturally weaker conditions (than Barr exactness) under which "Mal'tsev + majority = arithmetical". In what follows, we will show that even regular protoarithmetical categories need not be majority categories.

Majority objects

The notion of a *majority object* below is the exact analogue of the notion of a *Mal'tsev object* in the sense of [Wei17]. The general results of majority objects, which are given below, derive them self from the corresponding general results of Mal'tsev objects.

Definition 2.24 (Majority object)**.** An object S in a category $\mathbb{C}$ is said to be a majority object if for any ternary relation R in $\mathbb{C}$, we have

$$(x,y,z') \in_S R \quad \text{and} \quad (x,y',z) \in_S R \quad \text{and} \quad (x',y,z) \in_S R \implies (x,y,z) \in_S R.$$

The full subcategory of majority objects in a category is denoted by $\mathrm{Maj}(\mathbb{C})$.

The following proposition is the analogue of Proposition 2.3 in [Wei17].

Proposition 2.25. *Let $\mathbb{C}$ be a category with binary products, binary coproducts and image factorizations, then the following are equivalent for $\mathbb{C}$.*

(i) *$\mathbb{C}$ is a majority category;*

(ii) *For any object S in $\mathbb{C}$, there exists a morphism $f : S \to R$ making the diagram*

$$\begin{tikzcd}
3S \arrow[dr, "e"] \arrow[dd, "M=\begin{pmatrix} \iota_1 & \iota_1 & \iota_2 \\ \iota_1 & \iota_2 & \iota_1 \\ \iota_2 & \iota_1 & \iota_1 \end{pmatrix}"'] & & \\
& R \arrow[dl, tail, "r"] & \\
(2S)^3 & & S \arrow[ul, dotted, "f"'] \arrow[ll, "(\iota_1,\iota_1,\iota_1)"]
\end{tikzcd}$$

commute, where $M = re$ is an image factorization.

Proof. Composing e with each of the canonical inclusions $S \to 3S$, and applying the fact that S is a majority object, we have (i) implies (ii). We show (ii) implies (i): let $\mathbb{C}$ be a category with image factorizations and binary products and binary coproducts. Let A, B, C be any objects in $\mathbb{C}$ and $r' : R' \rightarrowtail A \times B \times C$ any monomorphism. Suppose that $a, a' \in \hom(S, A)$, $b, b' \in \hom(S, B), c, c' \in \hom(S, C)$ and $f_1, f_2, f_3 \in \hom(S, R')$ are such that

$$\begin{tikzcd}
& R' \arrow[d, "r'"] \\
S \arrow[ur, "f_3"] \arrow[r, "{(a,b,c')}"'] & A \times B \times C
\end{tikzcd}
\qquad
\begin{tikzcd}
& R' \arrow[d, "r'"] \\
S \arrow[ur, "f_2"] \arrow[r, "{(a,b',c)}"'] & A \times B \times C
\end{tikzcd}
\qquad
\begin{tikzcd}
& R' \arrow[d, "r'"] \\
S \arrow[ur, "f_1"] \arrow[r, "{(a',b,c)}"'] & A \times B \times C
\end{tikzcd}$$

commute. This implies that the dotted arrow f exists, making the diagram

$$
\begin{tikzcd}
3S \arrow[d, "e"'] \arrow[dr, dotted, "f"] \arrow[dd, bend right=60, "M"'] \arrow[ddrr, bend left, "\begin{pmatrix} a & b & c' \\ a & b' & c \\ a' & b & c \end{pmatrix}"] & & \\
R \arrow[d, "r"'] & R' \arrow[dr, "r'"] & \\
(2S)^3 \arrow[rr, "\begin{pmatrix} a \\ a' \end{pmatrix} \times \begin{pmatrix} b \\ b' \end{pmatrix} \times \begin{pmatrix} c \\ c' \end{pmatrix}"'] & & A \times B \times C
\end{tikzcd}
$$

commute. By assumption, we have that $(\iota_1, \iota_1, \iota_1) : S \to (2S)^3$ factors through R (α in the diagram below), and also by the fact that $M = re$ is an image-factorization, there exists $\beta : R \to R'$ making the diagram

$$
\begin{tikzcd}
 & 3S \arrow[d, "e"'] \arrow[dr, "f"] \arrow[ddrr, bend left, "\begin{pmatrix} a & b & c' \\ a & b' & c \\ a' & b & c \end{pmatrix}"] & & \\
 & R \arrow[r, dotted, "\beta"'] \arrow[d, "r"'] & R' \arrow[dr, "r'"] & \\
S \arrow[ur, dotted, "\alpha"] \arrow[r, "{(\iota_1,\iota_1,\iota_1)}"'] & (2S)^3 \arrow[rr, "\begin{pmatrix} a \\ a' \end{pmatrix} \times \begin{pmatrix} b \\ b' \end{pmatrix} \times \begin{pmatrix} c \\ c' \end{pmatrix}"'] & & A \times B \times C
\end{tikzcd}
$$

commute. Then $r'(\beta\alpha)$ is a factorization of (a, b, c) through R'. Thus, S is a majority object. $\square$

The theorem below is the analogue of Proposition 2.1 in [Wei17].

Proposition 2.26. *Given any category $\mathbb{C}$, the full subcategory $\mathrm{Maj}(\mathbb{C})$ of majority objects in $\mathbb{C}$ is closed under colimits and regular quotients in $\mathbb{C}$.*

Proof. Suppose that $D : I \to \mathbb{C}$ is any functor where I is a small category, and for any $i \in I_0$ we have $D(i)$ a majority object. Suppose that C is a colimit object of the diagram D, and suppose that R is any internal relation in $\mathbb{C}$ between objects X, Y and Z with morphisms $x, x' : C \to X$, $y, y' : C \to Y$ and $z, z' : C \to X$, such that:

$$(x, y, z') \in_C R \quad \text{and} \quad (x, y', z) \in_C R \quad \text{and} \quad (x', y, z) \in_C R.$$

Then for each object $i \in I_0$, we compose with the canonical inclusions $\iota_i : D(i) \to C$ to get

$$(x\iota_i, y\iota_i, z'\iota_i) \in_{D(i)} R \quad \text{and} \quad (x\iota_i, y'\iota_i, z\iota_i) \in_{D(i)} R \quad \text{and} \quad (x'\iota_i, y\iota_i, z\iota_i) \in_{D(i)} R.$$

And since $D(i)$ is a majority object, it follows that$(x\iota_i, y\iota_i, z\iota_i) \in_{D(i)} R$. Thus, since C is a colimit, the relevant factorization exists, so that $(x, y, z) \in_C R$. If $S \to Q$ is a regular epimorphism, then by using the diagonal fill in property (see Definition 1.1) of regular epimorphisms, it is easy to see that if S is a majority object, then so is Q. □

The corresponding proposition for Mal'tsev objects is given by Corollary 2.1 in [Wei17], the proof is essentially the same as the one found there.

Proposition 2.27. *Let $\mathbb{C}$ be a well-powered regular category admitting coproducts, then* $\mathrm{Maj}(\mathbb{C})$ *is a coreflective subcategory of* $\mathbb{C}$.

Proof. If $\mathbb{D}$ is a full subcategory of $\mathbb{C}$ which is closed under regular quotients and coproducts, then $\mathbb{D}$ is coreflective. If X is any object in $\mathbb{C}$, then let M be a set of subobjects of X which lie in $\mathbb{D}$ and let $\sqcup M$ be the coproduct of their domains. Then the coreflection of X in $\mathbb{D}$ is given by the mono part of the regular epi-mono factorization of the canonical morphism $\sqcup M \to X$. Therefore, by Proposition 2.26, $\mathrm{Maj}(\mathbb{C})$ is a coreflective subcategory of $\mathbb{C}$. □

Again, we have an analogue of Corollary 2.5 in [Wei17].

Proposition 2.28. *Consider the following conditions on a regular category $\mathbb{C}$ with binary coproducts.*

(i) Every morphism in $\mathrm{Maj}(\mathbb{C})$ *which is a regular epimorphism in $\mathbb{C}$, is also a regular epimorphism in* $\mathrm{Maj}(\mathbb{C})$.

(ii) Every jointly monomorphic triple of morphisms $r_1 : R \to X$ and $r_2 : R \to Y$ and $r_3 : R \to Z$ in $\mathrm{Maj}(\mathbb{C})$ *is also jointly monomorphic in $\mathbb{C}$.*

(iii) $\mathrm{Maj}(\mathbb{C})$ *is the largest full subcategory of $\mathbb{C}$ which is a majority category and closed under binary coproducts and regular quotients in $\mathbb{C}$.*

Then $(i) \implies (ii) \implies (iii)$.

Proof. Suppose that r_1, r_2, r_3 are the jointly monomorphic tripple in $\mathrm{Mal}(\mathbb{C})$ as in (ii), and consider regular epi-mono factorization the morphism $r = (r_1, r_2, r_3) : R \to X \times Y \times Z$ in $\mathbb{C}$:

$$\begin{tikzcd} R \arrow[r, two heads, "e"'] \arrow[rr, bend left, "r"] & S \arrow[r, "m"'] & X \times Y \times Z \end{tikzcd}$$

Then since $\mathrm{Maj}(\mathbb{C})$ is closed under regular quotients (Proposition 2.26), the morphisms $e, \pi_1 m, \pi_2 m, \pi_3 m$ are all morphisms in $\mathrm{Maj}(\mathbb{C})$. By (i), e is a regular epimorphism in $\mathrm{Maj}(\mathbb{C})$. But since we have

$$(\pi_1 m)e = r_1, \quad (\pi_2 m)e = r_2, \quad (\pi_3 m)e = r_3,$$

which is to say the family $\pi_1 me, \pi_2 me, \pi_3 me$ is jointly monomorphic, it follows that e is a monomorphism, and therefore an isomorphism. For $(ii) \implies (iii)$ suppose that $\mathbb{D} \subseteq \mathbb{C}$ is a full-subcategory, which is majority and closed under binary coproducts and regular quotients in $\mathbb{C}$. For any object S in $\mathbb{D}$, it follows that the diagram in (ii) of Proposition 2.25, both R and $3S$ an $2S$ are objects of $\mathbb{D}$. The morphisms $r_1, r_2, r_3 : R \to 2S$ where $r = (r_1, r_2, r_3) : R \to (2S)^3$ in the diagram of (ii) in Proposition 2.25, are jointly monomorphic in $\mathbb{C}$, and therefore they are jointly monomorphic in $\mathbb{D}$. This implies that the internal relation defined by r_1, r_2, r_3 being majority selecting gives the existence of a morphism $f : S \to R$ which gives the required factorization so that S is a majority object, i.e., that $\mathbb{D} \subseteq \mathrm{Maj}(\mathbb{C})$. Finally, it is easy to see that if $\mathbb{C}$ satisfies (ii) that $\mathrm{Maj}(\mathbb{C})$ is a majority category (since every ternary relation in $\mathrm{Maj}(\mathbb{C})$ is a ternary relation in $\mathbb{C}$ under the assumption of (ii)). □

Using the results of majority objects above, we are able to construct a regular protoarithmetical (Definition 2.12) category which is not a majority category.

A counterexample

Recall that the category of ternary relations $\mathbf{Rel}_3$ has as its objects pairs $X = (U_X, R_X)$ where U_X is a set and R_X is a ternary relation on U_X. A morphism $f : X \to Y$ in $\mathbf{Rel}_3$ is a function $f : U_X \to U_Y$ for which $(x, y, z) \in R_X \implies (f(x), f(y), f(z)) \in R_Y$. The limit/colimit of a diagram D in $\mathbf{Rel}_3$ has as its underlying set U_L the set-theoretic limit/colimit of the underlying diagram in **Set**, equipped with the largest/smallest relation making the canonical projections/inclusions homomorphisms. A morphism $m : A \to X$ in $\mathbf{Rel}_3$ is a regular monomorphism if and only if m is relation-reflecting, which is to say m satisfies

$$(m(x), m(y), m(z)) \in R_X \implies (x, y, z) \in R_A,$$

for any $x, y, z \in U_A$. We have seen in Proposition 1.45 that $\mathbf{Rel}_3^{\mathrm{op}}$ is a regular category. We state below as a lemma:

Lemma 2.29. *The category $\mathbf{Rel}_3^{\mathrm{op}}$ is a complete and cocomplete regular category.*

Remarks 2.30. For any morphism $f : X \to Y$ in $\mathbf{Rel}_3$ denote $f(X)$ for the subrelation of Y restricted to the set-theoretic image of f. Then the coimage factorization of f is given by $f = me$ where $e : X \to f(X)$ is the canonical projection, and $m : f(X) \to Y$ is the canonical inclusion.

As mentioned above, the notion of a majority object in a category $\mathbb{C}$ derives itself from the notion of a Mal'tsev object, which is defined below.

Definition 2.31 ([Wei17])**.** Let S be an object in a category $\mathbb{C}$, then S is a Mal'tsev object in $\mathbb{C}$ if for any binary relation $r : R \to X \times Y$, the induced relation on sets

$$\hom(S, R) \rightarrowtail \hom(S, X) \times \hom(S, Y),$$

is difunctional.

Remarks 2.32 ([Wei17]). A topological space S is a Mal'tsev object in $\mathbf{Top}^{\mathrm{op}}$ if and only if the map $f : R \to S$ defined by $f(x, x, y) = y = f(y, x, x)$ is continuous, where R is the subspace of S^3 generated by

$$\{(x, x, y), (y, x, x) \mid x, y \in S\}.$$

This happens if and only if the space S is an R_1-space, which is to say S satisfies the separation axiom: for any $x, y \in S$ if there exists an open U such that $x \in U$ and $y \notin U$, then there exists V and W open, such that $x \in V$ and $y \in W$, and $V \cap W = \varnothing$. Furthermore, a metric space S is a Mal'tsev object in $\mathbf{Met}^{\mathrm{op}}$ if and only if it is an ultra-metric space.

In what follows we will be concerned with Mal'tsev objects in $\mathbf{Rel}_3^{\mathrm{op}}$.

Lemma 2.33. *Let S be any object in $\mathbf{Rel}_3$, and let $M = (U_M, R_M)$ be the subrelation of $S \times S \times S$ defined by*

$$U_M = \{(x,x,y) \mid x,y \in U_S\} \cup \{(y,x,x) \mid x,y \in U_S\},$$

and where R_M is the restriction of R_{S^3} to U_M. Then S is a Mal'tsev object in $\mathbf{Rel}_3^{\mathrm{op}}$ if and only if the map $f : U_M \to U_S$ defined by

$$f(x,x,y) = y = f(y,x,x),$$

preserves the relation structure (is a morphism in $\mathbf{Rel}_3$).

Sketch. By Proposition 2.3 in [Wei17], an object S in $\mathbf{Rel}_3^{\mathrm{op}}$ is a Mal'tsev object if and only if there exists $f : M \to S$ making the diagram

$$\begin{tikzcd}
S^3 & & \\
& M \arrow[ul, "m"'] \arrow[dr, dotted, "f"] & \\
2S^2 \arrow[uu, "\begin{pmatrix}\pi_2 & \pi_2 & \pi_1 \\ \pi_1 & \pi_2 & \pi_2\end{pmatrix}"] \arrow[ur, "e"] \arrow[rr, "\begin{pmatrix}\pi_1 \\ \pi_1\end{pmatrix}"'] & & S
\end{tikzcd}$$

in $\mathbf{Rel}_3$ commute, where me is an image-factorization of the vertical morphism. Now by Remark 2.30, M can be taken to be the set-theoretic image of the vertical morphism, together with the restriction of R_{S^3}. Then

$$U_M = \{(x,x,y) \mid x,y \in U_S\} \cup \{(y,x,x) \mid x,y \in U_S\},$$

and if f exists it must be defined by

$$f(x,x,y) = y = f(y,x,x).$$

$\square$

As mentioned earlier, the full subcategory of majority objects $\mathrm{Maj}(\mathbb{C})$ is the analogue of the full subcategory of Mal'tsev objects in a category $\mathbb{C}$, which is denoted by $\mathrm{Mal}(\mathbb{C})$, and has the following properties similar to those properties of $\mathrm{Maj}(\mathbb{C})$ (see [Wei17]):

(i) $\mathrm{Mal}(\mathbb{C})$ is closed under colimits and regular quotients in $\mathbb{C}$. So that in particular if $\mathbb{C}$ is cocomplete, then so is $\mathrm{Mal}(\mathbb{C})$.

(ii) If $\mathbb{C}$ is a regular well-powered category admitting coproducts, then $\mathrm{Mal}(\mathbb{C})$ is a coreflective subcategory of $\mathbb{C}$.

(iii) If $\mathbb{C}$ is a regular category with binary coproducts, such that every morphism in $\mathrm{Mal}(\mathbb{C})$, which is a regular epimorphism in $\mathbb{C}$ is a regular epimorphism in $\mathrm{Mal}(\mathbb{C})$, then $\mathrm{Mal}(\mathbb{C})$ is the largest full subcategory of $\mathbb{C}$ which is Mal'tsev, and closed under binary coproducts and regular quotients in $\mathbb{C}$.

These properties are the analogues of the properties we have already seen for majority objects. By Lemma 2.29 and (ii) above, $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is a coreflective subcategory of $\mathbf{Rel}_3^{\mathrm{op}}$. Explicitly, this coreflection $r : \mathbf{Rel}_3^{\mathrm{op}} \to \mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ acts on objects as follows: if X is an object of $\mathbf{Rel}_3^{\mathrm{op}}$, then define $U_{r(X)} = U_X$, and define $R_{r(X)}$ as the smallest ternary relation R on U_X such that $R_X \subseteq R$ and (U_X, R) is a Mal'tsev object in $\mathbf{Rel}_3^{\mathrm{op}}$. Then, it can be checked that $r(X)$ is indeed a Mal'tsev object in $\mathbf{Rel}_3^{\mathrm{op}}$. If $f : X \to Y$ is a morphism in $\mathbf{Rel}_3^{\mathrm{op}}$ then we define $r(f) = f$. To summarize, we have the following lemma:

Lemma 2.34. *The functor $r : \mathbf{Rel}_3^{\mathrm{op}} \to \mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is right adjoint to the inclusion functor $\iota : \mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}}) \to \mathbf{Rel}_3^{\mathrm{op}}$, and for any object X in $\mathbf{Rel}_3^{\mathrm{op}}$ we have $U_{r(X)} = U_X$.*

The above lemma implies that $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ has limits, and that the limit of any diagram D in $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ has the same underlying set as the corresponding limit of D in $\mathbf{Rel}_3^{\mathrm{op}}$—which itself has the same underlying set as the corresponding limit in $\mathbf{Set}^{\mathrm{op}}$. This is to say that the forgetful functor $U : \mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}}) \to \mathbf{Set}^{\mathrm{op}}$ preserves limits. Since every discrete relation (X is discrete if $R_X = U_X \times U_X \times U_X$) is an object of $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$, it will follow that a morphism in $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is a monomorphism if and only if it is a monomorphism in $\mathbf{Rel}_3^{\mathrm{op}}$. This implies that the forgetful functor $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}}) \to \mathbf{Set}^{\mathrm{op}}$ reflects monos. Thus, we have the following lemma:

Lemma 2.35. *The forgetful functor $U : \mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}}) \to \mathbf{Set}^{\mathrm{op}}$ preserves limits and reflects monos.*

Proposition 2.36. *The category $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is a complete and cocomplete regular protoarithmetical category.*

Proof. Again, since $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ contains all discrete relations, it will follow that every morphism in $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ which is a regular epimorphism in $\mathbf{Rel}_3^{\mathrm{op}}$ is also a regular epimorphism in $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$. Moreover, since $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is coreflective, it follows that a morphism in $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is a regular epi if and only if it is a regular epi in $\mathbf{Rel}_3^{\mathrm{op}}$. Therefore, since $\mathbf{Rel}_3^{\mathrm{op}}$ is regular, so is $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$. Also, by (iii) above, it follows that $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is a Mal'tsev category, and by (i) it is cocomplete. By Lemma 2.34, $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ inherits its completeness from $\mathbf{Rel}_3^{\mathrm{op}}$. Next, we show that any internal groupoid in $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is an equivalence relation. Suppose that G is an internal groupoid in $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$, where G_1 is the object of arrows and $d_0, d_1 : G_1 \to G_0$ the domain and codomain morphisms respectively. By Lemma 2.35, the forgetful functor $U : \mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}}) \to \mathbf{Set}^{\mathrm{op}}$ preserves limits, so that UG is an internal groupoid in $\mathbf{Set}^{\mathrm{op}}$ — which is a majority category. Thus, $U(d_0, d_1)$ is a monomorphism by Theorem 2.15, and thus, (d_0, d_1) is a monomorphism since U reflects monos. □

As an easy application of Proposition 2.25 to $\mathbf{Rel}_3^{\mathrm{op}}$, we have the following lemma:

Lemma 2.37. *A ternary relation S is a majority object in $\mathbf{Rel}_3^{\mathrm{op}}$ if and only if the map $f : U_N \to U_S$ defined by $f(x,x,y) = f(x,y,x) = f(y,x,x) = x$ is a morphism in $\mathbf{Rel}_3$ where*

$$U_N = \{(x,x,y) \mid x,y \in U_S\} \cup \{(x,y,x) \mid x,y \in U_S\} \cup \{(y,x,x) \mid x,y \in U_S\},$$

and R_N is the restriction of R_{S^3} to U_N.

Proposition 2.38. $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ *is not a majority category.*

Proof. Since $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is closed under binary products and regular quotients in $\mathbf{Rel}_3^{\mathrm{op}}$, if $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ were a majority category, then we would have $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}}) \subseteq \mathrm{Maj}(\mathbf{Rel}_3^{\mathrm{op}})$ by Proposition 2.28. Thus to show that $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is not a majority category, it suffices to produce a Mal'tsev object S which is not a majority object. Consider the ternary relation S where $U_S = \{0,1\}$ and

$$R_S = \{(1,1,0),(0,1,1),(0,0,0)\}.$$

Then it is routine to verify that S satisfies the conditions of Lemma 2.33, and is thus an object of $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$. If the f in the statement of Lemma 2.37

above were a morphism in $\mathbf{Rel}_3$, then we would have

$$((1,0,0),(1,1,0),(0,1,0)) \in R_{S^3} \implies (f(1,0,0), f(1,1,0), f(0,1,0)) \in R_S.$$

But this would immediately imply that $(0,1,0) \in R_S$ so that $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ can not be a majority category. □

It was shown in [Bou01], that regular protoarithmetical categories are characterized by a *weak* form of congruence distributivity:

Theorem 2.39 ([Bou01])**.** *A regular Mal'tsev category* $\mathbb{C}$ *is protoarithmetical if and only if for any three equivalence relations* R, S, T *on any object* X

$$R \cap S = 0 \quad \textit{and} \quad R \cap T = 0 \implies R \cap (S \vee T) = 0.$$

Clearly any Mal'tsev category whose lattice of equivalence relations is distributive satisfies the above property. The categories $\mathbf{NReg}(\mathbf{Top})$ and $\mathbf{BoRg}(\mathbf{Top})$ are both examples of regular protoarithmetical categories, as they satisfy the weak congruence distributivity mentioned above. Now, in [Bou05] the author remarks that *'it is far less clear, at this point, if they are fully congruence distributive or not'*, but theorem 3.27 clarifies the situation, since both $\mathbf{NReg}(\mathbf{Top})$ and $\mathbf{BoRg}(\mathbf{Top})$ are regular Mal'tsev majority categories, and hence they are fully congruence distributive. The general question of whether or not weak congruence distributivity is the same as full congruence distributivity for regular categories (or even cocomplete pre-exact categories), is fully answered by the counterexample $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ above. Since for a regular Mal'tsev category $\mathbb{C}$, being fully congruence distributive is equivalent to being a majority category (see Theorem 3.27).

Chapter 3

Characterizations of majority categories

The main theorem of this chapter is the characterization Theorem 3.27, which extends the universal algebraic results of [BP75] and [Pix63] to regular categories.

3.1 Subobject decompositions

If S and T are any sublattices of a finite product of lattices $A_1 \times A_2 \times \cdots \times A_n$, such that $\pi_{i,j}(S) = \pi_{i,j}(T)$ where $A_1 \times \cdots \times A_n \xrightarrow{\pi_{i,j}} A_i \times A_j$ are the canonical two-fold projections, then $S = T$. This is the so called *Bergman's double projection theorem* which is mentioned in [BP75]. Clearly the statement of Bergman's theorem admits a categorical reformulation, when there is a suitable notion of image factorization. Recall a category $\mathbb{C}$ is said to have *image factorizations* if every morphism $f : X \to Y$ in $\mathbb{C}$ factors as $f = me$ where m is a monomorphism and e a strong epimorphism (see Definition 1.4). The factorization $f = me$ is then called an image factorization, which is unique up to unique isomorphism.

Recall that given a morphism $f : X \to Y$ and a subobject N represented by $n : N_0 \to X$, then the subobject of Y represented by the mono part of the image factorization of nf is called the image of N under f, and is denoted by $f(N)$.

Definition 3.1. Let $\mathbb{C}$ be a category with image factorizations and let I be a set, and let $J \subseteq I$ be any subset. Suppose that $(A_i)_{i\in I}$ is a family of objects in $\mathbb{C}$, such that both products $\prod\limits_{i\in I} A_i$ and $\prod\limits_{j\in J} A_j$ exist. Then for any subobject S of $\prod\limits_{i\in I} A_i$, the image of S under the canonical map

$$\prod_{i\in I} A_i \xrightarrow{\pi_J} \prod_{j\in J} A_j,$$

is called the J-image of S in $\prod\limits_{j\in J} A_j$ and is denoted by S_J.

With the notion J-image above, we are in a position to categorically investigate Bergman's double projection theorem.

Definition 3.2. Let $\mathbb{C}$ be a category with image factorizations, and let I be a set and $\mathscr{J} = (I_j)_{j\in J}$ a family of subsets of I. Then the product $\prod\limits_{i\in I} A_i$ is said to have $\mathscr{J}$-fold subobject decompositions if it satisfies the following property: for any two subobjects S, T of $\prod\limits_{i\in I} A_i$, if $S_{I_j} = T_{I_j}$ for any $j \in J$, then $S = T$. In other words, we say that every subobject of $\prod\limits_{i\in I} A_i$ is uniquely determined by its $\mathscr{J}$-fold images.

Proposition 3.3. *Let $\mathbb{C}$ be a complete category with image-factorizations, let I be a set and $\mathscr{J} = (I_j)_{j\in J}$ a family of subsets of I. The following are equivalent for a family $(A_i)_{i\in I}$ of objects in $\mathbb{C}$.*

(i) $\prod\limits_{i\in I} A_i$ *has $\mathscr{J}$-fold subobject decompositions.*

(ii) *For any monomorphism $s : S \to \prod\limits_{i\in I} A_i$, the diagram*

$$\begin{tikzcd} S \arrow[rr, "(e_{I_j})_{j\in I}"] \arrow[d, "s"'] & & \prod\limits_{j\in J} S_{I_j} \arrow[d, "\prod\limits_{j\in J} s_{I_j}"] \\ \prod\limits_{i\in I} A_i \arrow[rr, "(\pi_{I_j})_{j\in J}"'] & & \prod\limits_{j\in J}\Big(\prod\limits_{k\in I_j} A_k\Big) \end{tikzcd}$$

is a pullback, where

$$S \xrightarrow{e_{I_j}} S_{I_j} \xrightarrow{s_{I_j}} \prod_{k\in I_j} A_k,$$

is an image factorization of $\pi_{I_j} s$.

Proof. For (i) $\implies$ (ii): let $s : S \to \prod_{i\in I} A_i$ be any monomorphism, and consider the diagram below where the square is a pullback:

$$\begin{tikzcd}
S \arrow[dr, dotted, "\phi"] \arrow[drr, bend left, "(e_{I_j})_{j\in I}"] \arrow[ddr, bend right, "s"'] & & \\
& P \arrow[r, "\alpha"] \arrow[d, "p"] & \prod_{j\in J} S_{I_j} \arrow[d, "\prod_{j\in J} s_{I_j}=\beta"] \\
& \prod_{i\in I} A_i \arrow[r, "(\pi_{I_j})_{j\in J}"'] & \prod_{j\in J}\left(\prod_{k\in I_j} A_k\right)
\end{tikzcd}$$

By construction, the outer rectangle commutes, so that the dotted arrow ϕ exists. We claim that the subobject represented by p has the same two fold images as the subobject represented by s. Let $j \in J$ be any element then since $(\pi_j\alpha)\phi = e_{I_j}$ is a strong epimorphism, it follows that $\pi_j\alpha$ is a strong epimorphism. Then the factorization $s_{I_j}(\pi_j\alpha) = \pi_{I_j}p$ is an image factorization, therefore the I_j-image of the subobject represented by p in $\prod_{i\in I} A_i$ is S_{I_j}. Therefore, the subobjects represented by s and p are the same, so that ϕ is an isomorphism, which implies that the outer rectangle is a pullback. Finally, (ii) $\implies$ (i) follows from the universal property of pullback. $\square$

Definition 3.4. For any set I and any natural number $k \in \mathbb{N}$, we shall say that $\prod_{i\in I} A_i$ has k-fold subobject decompositions of size I if it has $\mathscr{J}$-fold decompositions, where $\mathscr{J}$ is the set of all subsets of I of size k. If I is a countable set, then we say that $\prod_{i\in I} A_i$ has countable k-fold subobject decompositions. If $\mathbb{C}$ is a category with image-factorizations which has products indexed by I, and every such product has k-fold subobject decompositions, then we shall say that $\mathbb{C}$ has k-fold subobject decompositions of size I. If $\mathbb{C}$ has k-fold subject decompositions of any size, then we shall simply say that $\mathbb{C}$ has k-fold subobject decompositions. We will also say that $\mathbb{C}$ has countable k-fold subobject decompositions, if every countable product in $\mathbb{C}$ has countable k-fold subobject decompositions.

Remarks 3.5. If $\mathbb{C}$ has k-fold subobject decompositions, it has countable k-fold subobject decompositions. And if $\mathbb{C}$ has countable k-fold subobject decompositions, then $\mathbb{C}$ has finite k-fold subobject decompositions.

Remarks 3.6. It has been shown in [BP75] that a finitary variety $\mathbb{V}$ has finite k-fold subobject decompositions if and only if $\mathbb{V}$ admits a k-ary near-unanimity term, i.e., a k-ary term p satisfying:

$$\begin{aligned} p(x, x, ..., x, y) &= x, \\ p(x, x, ..., y, x) &= x, \\ &\vdots \\ p(y, x, ..., x, x) &= x. \end{aligned}$$

So that in particular, any variety $\mathbb{V}$ with finite two-fold subobject decompositions admits a majority term.

The link between majority categories and subobject decompositions can be seen from the following theorem.

Theorem 3.7. *Let $\mathbb{C}$ be a category with finite limits and image factorizations. If $\mathbb{C}$ has finite 2-fold subobject decompositions, then $\mathbb{C}$ is a majority category.*

Proof. Suppose that R is any subobject of $A \times B \times C$ represented by (r_1, r_2, r_3) : $R_0 \to A \times B \times C$. Let $r_{1,2} : R_{1,2} \to A \times B$ and $r_{1,3} : R_{1,3} \to A \times C$ and $r_{2,3} : R_{2,3} \to B \times C$ be the monomorphisms formed from taking the mono part of the image-factorization (r_1, r_2, r_3) composed with the canonical projections $A \times B \times C \xrightarrow{(\pi_1,\pi_2)} A \times B$ and $A \times B \times C \xrightarrow{(\pi_1,\pi_3)} A \times C$ and $A \times B \times C \xrightarrow{(\pi_2,\pi_3)} B \times C$, respectively. Consider the pullback square below:

$$\begin{tikzcd} P_0 \arrow[r] \arrow[d, "p=(p_1,p_2,p_3)"'] & R_{1,2} \times R_{1,3} \times R_{2,3} \arrow[d, "r_{1,2}\times r_{1,3}\times r_{2,3}"] \\ A \times B \times C \arrow[r, "((\pi_1,\pi_2),(\pi_1,\pi_3),(\pi_2,\pi_3))"'] & (A \times B) \times (A \times C) \times (B \times C) \end{tikzcd}$$

It is easily seen that P is majority selecting in the sense of Definition 2.4, and therefore by Proposition 3.3 we have $P = R$ so that R is majority selecting. $\square$

Given any relation R on a product $X \times Y$, we can consider the image of R under the canonical projections $(X \times Y)^2 \to X^2$ and $(X \times Y)^2 \to Y^2$ which give two relations R_1 on X and R_2 on Y, respectively. Conversely, given R_1

and R_2 represented by $r_1 : R_0 \to X \times X$ and $r_2 : R_0' \to Y \times Y$ respectively, then the composite morphism

$$R_0 \times R_0' \xrightarrow{r_1 \times r_2} (X \times X) \times (Y \times Y) \xrightarrow{\phi} (X \times Y)^2,$$

is a mono (where ϕ is the canonical 'transpose' isomorphism), which represents a relation $R_1 \times_T R_2$ on $(X \times Y)$. Note, that we always have $R \leqslant R_1 \times_T R_2$.

Definition 3.8. A category $\mathbb{C}$ with image factorizations is said to have *directly decomposable reflexive relations*, if for any reflexive relation R on a product $X \times Y$ in $\mathbb{C}$, we have $R_1 \times_T R_2 = R$.

Example 3.9. The category **Ring** of unitary rings has directly decomposable reflexive relations, and the category **Grp** does not. To see this, suppose that $R \leqslant (X \times Y)^2$ is a reflexive internal relation of unitary rings. Now suppose that $((x, y), (a, b)) \in R_1 \times_T R_2$, then by definition, there exists $x', a', y', b' \in R$ such that

$$((x, y'), (a, b')) \in R \quad \text{and} \quad ((x', y), (a', b)) \in R.$$

Since R is reflexive we have that

$$\begin{aligned}
&((x, y'), (a, b')) \cdot ((1, 0), (1, 0)) = ((x, 0), (a, 0)) \in R \quad \text{and} \\
&((x', y), (a', b)) \cdot ((0, 1), (0, 1)) = ((0, y), (0, b)) \in R \implies \\
&((x, 0), (a, 0)) + ((0, y), (0, b)) = ((x, y), (a, b)) \in R.
\end{aligned}$$

Therefore **Ring** has directly decomposable reflexive relations. To see that **Grp** does not, consider the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. If R is the reflexive relation generated by the element $((1, 0), (0, 1))$, then it is easy to see that $((1, 1), (0, 1)) \in R_1 \times_T R_2$, but $((1, 1), (0, 1)) \notin R$, so that **Grp** does not have directly decomposable reflexive relations.

Proposition 3.10. *Let $\mathbb{C}$ be any category with image-factorizations, and two-fold subobject decompositions. Then $\mathbb{C}$ has directly decomposable reflexive relations.*

Proof. For any reflexive relation R on a product $X \times Y$, its easy to see that both R and $R_1 \times_T R_2$ have the same two fold subobject decompositions, when viewed as subobjects of $X \times Y \times X \times Y$. $\square$

In Theorem 3.27, we will see that any regular majority category has finite 2-fold subobject decompositions. This is then the categorical analogue of the lattice-theoretic double-projection theorem of Bergman [BP75]. This motivates the investigation of general (possibly infinite) subobject decompositions. As we will see in the next section, there are no finitary varieties which have countable finite-subobject decompositions, however, there are infinitary varieties which do.

3.2 Infinite subobject decompositions

Proposition 3.11. *The only finitary varieties of algebras* $\mathbb{V}$ *which have countable* 2*-fold subobject decompositions are trivial, i.e., each algebra in* $\mathbb{V}$ *has at most* 1 *element.*

Proof. Suppose that $\mathbb{V}$ is a finitary variety which has countable 2-fold subobject decompositions. Consider the set theoretic maps:

$$f_n : \mathbb{N} \to \{1, 2, \dots, n\}, \quad x \longmapsto \begin{cases} x & x \leqslant n \\ n & x > n \end{cases}.$$

Let $F = F_{\mathbb{V}}(\mathbb{N})$ and $F_n = F_{\mathbb{V}}(\{1, 2, \dots, n\})$, then each f_n induces a homomorphism $\overline{f_n} : F \to F_n$ via the free algebra in $\mathbb{V}$. Now let f be the induced homomorphism into the product of the $F_i^{\prime}s$

$$\begin{tikzcd} F \arrow[r, dotted, "f"] \arrow[dr, "\overline{f_n}"'] & \prod\limits_{n\in\mathbb{N}} F_n \arrow[d, "\pi_n"] \\ & F_n \end{tikzcd}$$

Then f is a monomorphism. Let $F_{i,j}$ be the two-fold image of F in $F_i \times F_j$, and $f_{i,j} : F_{i,j} \to F_i \times F_j$ the canonical inclusion. Let $g_n : F_1 \to F_n$ be the homomorphism sending 1 to n, and let $g = \prod\limits_{n\in\mathbb{N}} g_n$. Now, for any $i, j \in \mathbb{N}$ we have that $(i, j) \in F_{i,j}$ since if $i \leqslant j$ then $f_i(j) = i$ and $f_j(j) = j$. Consider the homomorphism $g_{i,j} : F_1 \to F_{i,j}$ sending 1 to (i, j), this gives the following commutative diagram:

$$\begin{tikzcd}
F_1 \arrow[r, "((g_{i,j})_{i,j\in\mathbb{N}})"] \arrow[d, "g"'] & \prod_{i,j\in\mathbb{N}} F_{i,j} \arrow[d, "\prod_{i,j\in\mathbb{N}} f_{i,j}"] \\
\prod_{n\in\mathbb{N}} F_n \arrow[r, "((\pi_i,\pi_j))_{i,j\in\mathbb{N}}"'] & \prod_{i,j\in\mathbb{N}} F_i \times F_j
\end{tikzcd}$$

Thus by Proposition 3.3, the square

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "f"'] & \prod_{i,j\in\mathbb{N}} F_{i,j} \arrow[d, "\prod_{i,j\in\mathbb{N}} f_{i,j}"] \\
\prod_{n\in\mathbb{N}} F_n \arrow[r, "((\pi_i,\pi_j))_{i,j\in\mathbb{N}}"'] & \prod_{i,j\in\mathbb{N}} F_i \times F_j
\end{tikzcd}$$

is a pullback, where the top arrow is the canonical morphism into the product. Therefore, there exists a morphism $F_1 \to F$, making the relevant triangle commute. This amounts to the existence of an element $x \in F$ such that $\overline{f_n}(x) = n$ for any $n \in \mathbb{N}$. Since x is an element of F it follows that $x = t(a_1, a_2, ..., a_k)$ where t is a k-ary term, and $a_1, a_2, ..., a_k \in \mathbb{N}$. Now, let $m = \max\{a_1, a_2, ..., a_k\}$, then it follows that

$$m = \overline{f_m}(t(a_1, a_2, ..., a_k) = t(f_m(a_1), f_m(a_2), ..., f_m(a_k)) = t(a_1, a_2, ..., a_k),$$

but also we have

$$m+1 = \overline{f_{m+1}}(t(a_1, a_2, ..., a_k) = t(f_{m+1}(a_1), f_{m+1}(a_2), ..., f_{m+1}(a_k)) = t(a_1, a_2, ..., a_k).$$

So that in $F_{m+1} \models m = m+1$. This implies that every algebra in $\mathbb{V}$ has at most one element. $\square$

In the above proof it is crucial that $\mathbb{V}$ be finitary, as the finiteness of t allows us to select the maximum of $a_1, a_2, ..., a_k$. As the proposition below shows, there are varieties which have operations of countable arity which have countable 2-fold subobject decompositions. In what follows, we will see that there can be infinitary varieties with 2-fold subobject decompositions of infinite size.

Recall that if I is an arbitrary set, then an I-complete lattice L is one in which any family $(x_i)_{i\in I}$ has a meet and join. And a homomorphism

$f : L \to M$ of I-complete lattices is a function which preserves joins and meets of families indexed by I. In what follows we shall denote the category of I-complete lattices by $\mathbf{Lat}_I$.

Proposition 3.12. *The category $\mathbf{Lat}_I$ of I-complete lattices has arbitrary 2-fold subobject decompositions of size I.*

Proof. Suppose that $S \subseteq \prod_{i \in I} L_i = L$ is any I-complete sublattice of a product of I-complete lattices, and suppose that $\pi_k : \prod_{i \in I} L_i \to L_k$ are the canonical product projections. Then it suffices to show that S has the following property (*): for any $x \in L$, if for any $i, j \in I$ there exists $s \in S$ such that $\pi_i(x) = \pi_i(s)$ and $\pi_j(x) = \pi_j(s)$, then $x \in S$. To that end, suppose that $x \in L$ satisfies (*). Now let $s_{i,j} \in S$ be elements of S with $\pi_i(s_{i,j}) = x_i$ and $\pi_j(s_{i,j}) = x_j$. Define the elements s_j of S as $s_j = \bigwedge_{i \in I} s_{i,j}$. Then for any $i, j \in I$ we have $\pi_i(s_j) \leqslant \pi_i(s_i) = x_i$, since

$$s_j = \bigwedge_{i \in I} s_{i,j} \leqslant s_{i,j} \implies \pi_i(s_j) \leqslant \pi_i(s_{i,j}) = x_i = \pi_i(s_i).$$

This implies that

$$x = \bigvee_{i \in I} s_i,$$

so that $x \in S$. $\square$

Proposition 3.13. *If I, J are infinite sets and $|I| < |J|$, then $\mathbf{Lat}_I$ does not have 2-fold subobject decompositions of size J.*

Proof. Consider the subset S of $\prod_{j \in J} \mathbf{2}$ consisting of all elements $s \in \prod_{j \in J} \mathbf{2}$ such that

$$|\{j \in J \mid \pi_j(s) = 1\}| \leqslant |I|.$$

Now suppose that $(s_i)_{i \in I}$ is a collection of elements of S and let $s = \bigvee_{i \in I} s_i$. Then it is easy to see that

$$\{j \in J \mid \pi_j(s) = 1\} = \bigcup_{i \in I} \{j \in J \mid \pi_j(s_i) = 1\}.$$

But then since I is infinite, it follows that $|I \times I| = |I|$. Therefore we have:

$$|\{j \in J \mid \pi_j(s) = 1\}| = |\bigcup_{i \in I} \{j \in J \mid \pi_j(s_i) = 1\}| \leqslant |\bigsqcup_{i \in I} I| = |I \times I| = |I|,$$

so that $s \in S$. Thus, S is a sub I-complete lattice of $\prod\limits_{j\in J} \mathbf{2}$. Moreover, S has the same 2-fold projections as $\prod\limits_{j\in J} \mathbf{2}$, but is not equal to $\prod\limits_{j\in J} \mathbf{2}$. As, for example, the top element of $\prod\limits_{j\in J} \mathbf{2}$ is not contained in S. $\square$

3.3 Geometric examples of majority categories

Surprisingly, the dual subobject decomposition property is one which is shared by many categories of a geometric nature, and thus many categories of a geometric nature will turn out to be majority categories. For example, in what follows we will show that $\mathbf{Set}^{\mathrm{op}}, \mathbf{Top}^{\mathrm{op}}, \mathbf{Rel}^{\mathrm{op}}, (G - \mathbf{Set})^{\mathrm{op}}$ are majority categories by showing that they satisfy the conditions of Theorem 3.7. Therefore, in each of these categories we will show that if $r : \coprod\limits_{i\in I} X_i \to R$ and $t : \coprod\limits_{i\in I} X_i \to T$ are any two epimorphisms with the same two-fold coimages, then $r \simeq s$ as epimorphisms. This amounts to showing that if for any $i, j \in I$ we have the following commutative diagram of solid arrows

$$\begin{tikzcd}
 & R_{i,j} \arrow[rr, "r_{i,j}"] \arrow[dd, "\phi_{i,j}"] & & R \arrow[dd, dotted, "\phi"] \\
X_i \sqcup X_j \arrow[ur, "\alpha_{i,j}"] \arrow[rr] \arrow[dr, "\beta_{i,j}"'] & & \coprod\limits_{i\in I} X_i \arrow[ur, "r"] \arrow[dr, "t"] & \\
 & T_{i,j} \arrow[rr, "t_{i,j}"'] & & T
\end{tikzcd}$$

Figure 31

where $\phi_{i,j}$ are isomorphisms, $r_{i,j}\alpha_{i,j}$ and $t_{i,j}\beta_{i,j}$ are the canonical coimage factorizations, then the dotted arrow ϕ exists, is an isomorphism, and makes the diagram commute. We will refer to this diagram in what follows, and since we will always be dealing with categories whose objects have underlying sets and functions, we will assume without loss of generality that the $X_i's$ above have disjoint underlying sets.

Proposition 3.14. *The category* $\mathbf{Set}^{\mathrm{op}}$ *has general 2-fold subobject decompositions.*

Proof. We define the ϕ as follows: if $x \in R$, then select an element $y \in r^{-1}(x)$ and set $\phi(x) = t(y)$. To see that this is well-defined, suppose that $y, y' \in r^{-1}(x)$ then there exists $i, j \in I$ such that $y \in X_i$ and $y' \in X_j$. Now, $\alpha_{i,j}(y) = \alpha_{i,j}(y')$ since $r_{i,j}$ is a monomorphism, and therefore $\phi_{i,j}\alpha_{i,j}(y) = \beta_{i,j}(y) = \beta_{i,j}(y') = \phi_{i,j}\alpha_{i,j}(y')$ so that $t_{i,j}(y) = t_{i,j}(y')$ which implies $t(y) = t(y')$. $\square$

Proposition 3.15. *The dual category* $\mathbf{Ord}^{\mathrm{op}}$ *of ordered sets has general 2-fold subobject decompositions.*

Proof. It suffices to show that the set-theoretic ϕ in Example 3.14 is relation preserving. Suppose that $a \leqslant b$ in R. Then there exists $x \in X_i$ and $y \in X_j$ such that $r(x) = a$ and $r(y) = b$, hence $a, b \in R_{i,j}$. Since $r_{i,j}$ is a regular monomorphism, we have $r_{i,j}(a) \leqslant r_{i,j}(b) \implies a \leqslant b$ in $R_{i,j}$. It follows that $\phi(a) \leqslant \phi(b)$ since $\phi_{i,j}(a) \leqslant \phi_{i,j}(b)$. $\square$

Proposition 3.16. *If* $\mathbb{C}$ *is complete and has 2-fold subobject decompositions, then so does* $\mathbb{C}^{\mathbb{D}}$ *for any category* $\mathbb{D}$. *Similarly, if* $\mathbb{C}^{\mathrm{op}}$ *has two-fold subobject decompositions, then so does* $(\mathbb{C}^{\mathbb{D}})^{\mathrm{op}}$. *Thus, for example, the category* $(G - \mathbf{Set})^{\mathrm{op}}$ *of G-sets has two-fold subobject decompositions.*

Proof. Since $\mathbb{C}^{\mathbb{D}}$ inherits (co)limits and (co)image factorizations from $\mathbb{C}$ point-wise, if $\mathbb{C}$ satisfies the conditions of Proposition 3.3 (ii), then so will $\mathbb{C}^{\mathbb{D}}$. $\square$

Perhaps, it is natural to expect that the dual category $\mathbf{Top}^{\mathrm{op}}$ of topological spaces and continuous maps has two-fold subobject decompositions, given the geometric nature of the previous examples. However, as the next example shows, $\mathbf{Top}^{\mathrm{op}}$ does not even have countable 2-fold subobject decompositions, however, we will see that $\mathbf{Top}^{\mathrm{op}}$ has finite two-fold subobject decompositions.

Counterexample 1. Consider $\mathbb{Q}$ together with the subspace topology induced by $\mathbb{R}$. Define the continuous maps $f_a : \mathbb{Q} \to \mathbb{Q}^2$ by $f_a(x) = (a, x)$. The induced continuous map f in the diagram

$$\begin{tikzcd}
\coprod_{a \in \mathbb{Q}} \mathbb{Q} \arrow[r, dotted, "f"] & \mathbb{Q}^2 \\
\mathbb{Q} \arrow[u, "\iota_a"] \arrow[ur, "f_a"'] &
\end{tikzcd}$$

is an epimorphism. Moreover, f has the same two-fold co-images as the identity on $\coprod_{a\in\mathbb{Q}} \mathbb{Q}$, so that if $\mathbf{Top}^{\mathrm{op}}$ had two-fold coimages, then we would have $\coprod_{a\in\mathbb{Q}} \mathbb{Q} \simeq \mathbb{Q}\times\mathbb{Q}$ — which is a contradiction.

Example 3.17. $\mathbf{Top}^{\mathrm{op}}$ has finite two-fold subobject decompositions.

Proof. We will show that ϕ in the above Figure 31 is a homeomorphism, provided that I is finite. We show that ϕ preserves open sets: let $U \subseteq R$ be any open set in R, then for any i, j we have that $R_{i,j} \cap U$ is open in $R_{i,j}$, which implies that $\phi_{i,j}(U \cap R_{i,j})$ is open in $T_{i,j}$ since each $\phi_{i,j}$ is a homeomorphism. Therefore, there exists an open set $V_{i,j} \subseteq T$ such that $\phi_{i,j}(R_{i,j} \cap U) = T_{i,j} \cap V_{i,j}$, and hence we have:

$$T_{i,j} \cap \phi(U) = \phi(R_{i,j} \cap U) = \phi_{i,j}(R_{i,j} \cap U) = T_{i,j} \cap V_{i,j}.$$

Let $V_i = \bigcap_{j\in I} V_{i,j}$ and $T_i = T_{i,i}$. Then we have

$$\bigcap_{j\in J}(T_{i,j} \cap \phi(U)) = \bigcap_{j\in J}(T_{i,j} \cap V_{i,j}) \implies T_i \cap \phi(U) = T_i \cap V_i,$$

then we will show that $\bigcup_{i\in I} V_i = \phi(U)$. For the direction $\phi(U) \subseteq \bigcup_{i\in I} V_i$: let $x \in \phi(U)$ then there exists $j \in I$ such that $x \in T_j$, so that

$$x \in \phi(U) \cap T_j \implies x \in T_j \cap V_j \implies x \in \bigcup_{i\in I} V_i.$$

For the reverse inclusion $\bigcup_{i\in I} V_i \subseteq \phi(U)$: suppose that $x \in V_i$ for some $i \in I$. Then there exists $j \in I$ such that $x \in T_{i,j}$ and therefore,

$$x \in T_{i,j} \cap V_i \implies x \in T_{i,j} \cap V_{i,j} \implies x \in T_{i,j} \cap \phi(U) \implies x \in \phi(U).$$

The reverse argument can then be applied to ϕ^{-1}, so that ϕ is a homeomorphism. □

3.4 The Pairwise Chinese Remainder Theorem in a category

The so called *Pairwise Chinese Remainder Theorem* for varieties of universal algebras (see [BP75]) presents itself as a Mal'tsev condition equivalent to the existence of a majority term. This particular theorem can be suitably reformulated for regular categories, which is the subject of this section.

Let $\mathbb{C}$ be a regular category and X an object of $\mathbb{C}$. If θ is an equivalence relation on X and $a, b : S \to X$ morphisms in $\mathbb{C}$, then we will write $a \equiv b \mod \theta$ if $(a, b) \in_S \theta$ in what follows. Given an object X of a category $\mathbb{C}$, morphisms $a_1, a_2, ..., a_m : S \to X$ and equivalence relations $\theta_1, \theta_2, ..., \theta_m$, we will be concerned with solving the system of congruence equations:

$$\begin{aligned} x \equiv& a_1 \mod \theta_1, \\ x \equiv& a_2 \mod \theta_2, \\ &\vdots \\ x \equiv& a_m \mod \theta_m. \end{aligned} \tag{$*$}$$

Definition 3.18. An approximate solution to the system above consists of a morphism $a : Q \to X$ (the approximate solution), together with a regular epimorphism $\alpha : Q \to S$ (the approximation of a), such that for any $i \in \{1, 2, ..., m\}$ we have

$$a \equiv a_i\alpha \mod \theta_i.$$

If such an $a : Q \to X$ and $\alpha : Q \to S$ exist, then the above system is said to be approximately solvable. The above system $(*)$ is said to be approximately pairwise solvable, if for any $i, j \in \{1, 2, ..., m\}$ the system

$$\begin{aligned} x \equiv& a_i \mod \theta_i, \\ x \equiv& a_j \mod \theta_j \end{aligned}$$

is approximately solvable.

Remarks 3.19. The above notion is similar to the notion of an approximate operation in the sense of [BJ08], in how it compares with the ordinary notion of solution of a system of equations.

Definition 3.20 (PCRT)**.** Let X be an object of a regular category $\mathbb{C}$, then X is said to satisfy the Pairwise Chinese Remainder Theorem, if for any morphisms $a_1, a_2, ..., a_m : S \to X$, and any effective equivalence relations

$\theta_1, \theta_2, ..., \theta_m$, if the system

$$
\begin{aligned}
x \equiv & a_1 \mod \theta_1, \\
x \equiv & a_2 \mod \theta_2, \\
& \vdots \\
x \equiv & a_m \mod \theta_m
\end{aligned}
$$

is approximately pairwise solvable, then it is approximately solvable. If every object of $\mathbb{C}$ satisfies the PCRT, then we say that $\mathbb{C}$ satisfies the PCRT, or that the PCRT holds in $\mathbb{C}$.

Lemma 3.21. *If $\alpha'_i : Q_i \to A$ and $\beta'_i : Q_i \to B$ are regular epimorphisms making the diagram*

$$
\begin{tikzcd}
Q_i \arrow[r, "\beta'_i"] \arrow[d, "\alpha'_i"'] & B \arrow[d, "b_i"] \\
A \arrow[r, "a_i"'] & C_i
\end{tikzcd}
$$

commute, then there exist regular epimorphisms $\alpha : Q \to A$ and $\beta_i : Q \to B$ making the diagram

$$
\begin{tikzcd}
Q \arrow[r, "\beta_i"] \arrow[d, "\alpha"'] & B \arrow[d, "b_i"] \\
A \arrow[r, "a_i"'] & C_i
\end{tikzcd}
$$

commute for any $i \in \{1, 2, ..., n\}$

Proof. Simply consider the limit of the diagram:

$$Q_i \xrightarrow{\alpha'_i} A$$

where i ranges from 1 to n. This produces a family of regular epimorphisms $p_i : Q \to Q_i$ making the diagram

$$
\begin{tikzcd}
Q \arrow[d, "p_i"'] \arrow[dr, "\alpha"] & \\
Q_i \arrow[r, "\alpha'_i"'] & A
\end{tikzcd}
$$

commute, where α is any composite $\alpha'_i p_i$ where $i \in \{1, 2, ..., n\}$. Then defining $\beta_i = \beta'_i p_i$, it follows that α and β_i satisfy the required properties. $\square$

The next theorem is the categorical generalization of the Pairwise Chinese Remainder Theorem for varieties of algebras (see [BP75]).

Lemma 3.22. *Let $\mathbb{C}$ be a regular category, then (i) $\implies$ (ii) where:*

(i) The PCRT holds in $\mathbb{C}$.

(ii) $\mathbb{C}$ has 2-fold subobject decompositions.

Proof. Suppose that $C_1, C_2, ..., C_r$ are any objects in $\mathbb{C}$, and let A, B be any subobjects of $C = C_1 \times C_2 \times \cdots \times C_r$, with representatives $a : A_0 \to C$ and $b : B_0 \to C$, and which have the same 2-fold images. Let $\pi_i(a) = a_i$ and $\pi_i(b) = b_i$, then we will show that $A \leqslant B$. Consider the regular-epi mono factorizations of the morphisms (a_i, a_j) and (b_i, b_j) below:

$$A_0 \xrightarrow{\alpha'_{i,j}} R \xrightarrow{(r_i,r_j)} C_i \times C_j,$$
$$B_0 \xrightarrow{\beta'_{i,j}} T \xrightarrow{(t_i,t_j)} C_i \times C_j.$$

Since A and B have the same two-fold images, there exists an isomorphism $\phi : R \to T$ such that $(t_i, t_j)\phi = (r_i, r_j)$. Now, we can pullback $\phi\alpha'_{i,j}$ along $\beta'_{i,j}$, and get two regular epimorphisms $\alpha''_{i,j} : Q_{i,j} \to A_0$ and $\beta''_{i,j} : Q_{i,j} \to B_0$ making the diagram

$$\begin{tikzcd} Q_{i,j} \arrow[r, two heads, "\beta''_{i,j}"] \arrow[d, two heads, "\alpha''_{i,j}"'] & B_0 \arrow[d, "(b_i,b_j)"] \\ A_0 \arrow[r, "(a_i,a_j)"'] & C_i \times C_j \end{tikzcd}$$

commute. Then by Lemma 3.21, there exist regular epimorphisms $\alpha : Q \to A_0$ and $\beta_{i,j} : Q \to B_0$ such that the diagram

$$\begin{tikzcd} Q \arrow[r, two heads, "\beta_{i,j}"] \arrow[d, two heads, "\alpha"'] & B_0 \arrow[d, "(b_i,b_j)"] \\ A_0 \arrow[r, "(a_i,a_j)"'] & C_i \times C_j \end{tikzcd}$$

commutes for any $i, j \in \{1, 2, ..., r\}$. Now define $\beta_i = \beta_{i,i}$, and let θ_i be the kernel equivalence relation on B defined by b_i. Then we have that

$$\beta_{i,j} \equiv \beta_i \mod \theta_i \quad \text{and} \quad \beta_{i,j} \equiv \beta_j \mod \theta_j,$$

so that the system

$$x \equiv \beta_i \mod \theta_i \qquad \text{(for } i = 1, 2, ..., r.\text{)}$$

is pairwise approximately solvable (the approximation in each case is the identity on Q). Therefore, by (i) there exists a regular epimorphism $\alpha' : Q' \to Q$ and a morphism $\beta : Q' \to B_0$ such that

$$\beta \equiv \beta_i \alpha' \mod \theta_i \qquad \text{(for } i = 1, 2, ..., r.\text{)}$$

This implies that

$$b_i \beta_i \alpha' = b_i \beta \implies a_i \alpha\alpha' = \pi_i b\beta \implies \pi_i(a\alpha\alpha') = \pi_i(b\beta),$$

for any $i \in \{1, 2, ..., n\}$, and therefore, $a\alpha\alpha' = b\beta$. Therefore the diagram of solid arrows

$$\begin{tikzcd} Q' \arrow[r, "\beta"] \arrow[d, two heads, "\alpha\alpha'"'] & B_0 \arrow[d, "b"] \\ A_0 \arrow[r, "a"'] \arrow[ur, dotted] & C \end{tikzcd}$$

commutes, and the dotted arrow exists since $\alpha\alpha'$ is a regular epimorphism and b is a monomorphism. □

Lemma 3.23. *Let $\mathbb{C}$ be a majority category with finite products, and $R \leqslant A_1 \times A_2 \times \cdots \times A_n$ any n-ary relation with $n \geqslant 3$. If*

$$(x, a_2, a_3, ..., a_n) \in_S R \quad \textit{and} \quad (a_1, y, a_2, a_3, ..., a_n) \in_S R \quad \textit{and} \quad (a_1, a_2, z, ..., a_n) \in_S R,$$

then

$$(a_1, a_2, a_3, ..., a_n) \in_S R.$$

Proof. Follows trivially from the fact that R is a ternary relation between A_1, A_2 and $A_3 \times \cdots \times A_n$, which must be majority-selecting. □

Lemma 3.24. *If $\mathbb{C}$ is a regular majority category, then the Pairwise Chinese Remainder Theorem holds for $\mathbb{C}$.*

Consider the system of congruences from Definition 3.20:

$$\begin{aligned} x \equiv & a_1 \mod \theta_1, \\ x \equiv & a_2 \mod \theta_2, \\ & \vdots \\ x \equiv & a_m \mod \theta_m. \end{aligned} \qquad (*)$$

In the proof below, we will show that in any regular majority category $\mathbb{C}$, if any system of congruences of length m is approximately solvable as soon as it is pairwise approximately solvable, then any system of length $m + 1$ is approximately solvable as soon as it is pairwise approximately solvable. The result will then follow by induction, since in any regular category $\mathbb{C}$, any system of length 2 is approximately solvable if and only if it is pairwise approximately solvable.

Proof. Suppose that $m > 2$ is any natural number, and suppose that any system of congruences in $\mathbb{C}$ of length m is approximately solvable as soon as it is pairwise approximately solvable. Let X be any object in $\mathbb{C}$, $a_1, a_2, ..., a_{m+1} : S \to X$ any morphisms, and $\theta_1, \theta_2, ..., \theta_{m+1}$ any effective equivalence relations of the morphisms $f_1 : X \to X_1, f_2 : X \to X_2, ..., f_{m+1} : X \to X_{m+1}$ respectively. Suppose that the system

$$\begin{aligned} x \equiv & a_1 \quad \mod \theta_1, \\ x \equiv & a_2 \quad \mod \theta_2, \\ & \vdots \\ x \equiv & a_{m+1} \quad \mod \theta_{m+1}, \end{aligned} \tag{$*$}$$

is pairwise approximately solvable. By assumption, the three systems obtained from removing the first, second and third rows from $(*)$ are approximately pairwise solvable and therefore they are approximately solvable. Let $\alpha_1 : Q_1 \to S$ together with $x'_1 : Q_1 \to X$, $\alpha_2 : Q_2 \to S$ together with $x'_2 : Q_2 \to X$ and $\alpha_3 : Q_3 \to S$ together with $x'_3 : Q_3 \to X$ be the approximate solutions of $(*)$ after removing the first, second and third rows respectively. Consider the limit of the diagram:

$$Q_i \xrightarrow{\alpha_i} S \qquad (i = 1, 2, 3)$$

which gives an object Q together with three regular epimorphisms p_1, p_2, p_3 making the diagram

$$\begin{tikzcd} Q \arrow[d, "p_i"'] \arrow[dr, "\alpha"] & \\ Q_i \arrow[r, "\alpha_i"'] & S \end{tikzcd}$$

commute, where α is any composite $\alpha_i p_i$. Define $x_i = x'_i p_i$ for $i = 1, 2, 3$, then we have that α together with x_1, α together with x_2, and α together with x_3, are approximate solutions of the $(*)$ after removing the first, second and third row respectively. Now, let $e : X \to R_0$ and $r : R_0 \to X_1 \times X_2 \times \cdots \times X_{m+1}$ be the regular epi and mono part of the regular image factorization of $(f_1, f_2, ..., f_{m+1}) : X \to X_1 \times X_2 \times \cdots \times X_{m+1}$, and let R be the $(m+1)$-ary relation represented by r. Then we have

$$\begin{aligned}
&(f_1x_1, f_2a_2\alpha, f_3a_3\alpha, ..., f_{m+1}a_{m+1}\alpha) \in_Q R \quad \text{and,}\\
&(f_1a_1\alpha, f_2x_2, f_3a_3\alpha, ..., f_{m+1}a_{m+1}\alpha) \in_Q R \quad \text{and,}\\
&(f_1a_1\alpha, f_2a_2\alpha, f_3x_3, ..., f_{m+1}a_{m+1}\alpha) \in_Q R,
\end{aligned}$$

which by Lemma 3.23, implies that $(f_1a_1\alpha, f_2a_2\alpha, ..., f_{m+1}a_{m+1}\alpha) \in_Q R$. Therefore, there exists $g : Q \to R_0$ making the square

$$\begin{tikzcd}
Q \arrow[r, two heads, "\alpha"] \arrow[d, "g"'] & S \arrow[dl, dotted, "h"'] \arrow[d, "{(f_1a_1, f_2a_2, ..., f_{m+1}a_{m+1})}"] \\
R_0 \arrow[r, "r"'] & X_1 \times X_2 \times \cdots \times X_{m+1}
\end{tikzcd}$$

commute. The morphism h exists because α is a regular epimorphism. Finally, by pulling back h along e, we get the commutative diagram:

$$\begin{tikzcd}
Q' \arrow[r, two heads, "\alpha'"] \arrow[d, "a'"'] & S \arrow[d, "h"] \arrow[dr, "{(f_1a_1, f_2a_2, ..., f_{m+1}a_{m+1})}"] & \\
X \arrow[r, two heads, "e"'] & R_0 \arrow[r, "r"'] & X_1 \times X_2 \times \cdots \times X_{m+1}
\end{tikzcd}$$

where a' is an approximate solution of the system $(*)$ with approximation α'. □

3.5 The characterization theorem

It is well known in universal algebra that varieties admitting a majority term admit several characterizations, as can be seen from the work of A .F. Pixley in [Pix63], [BP75] and [Pix79]. The lemmas preceding this section were put in place, in order to establish the categorical counterparts of these universal algebraic results. The main theorem of this section is Theorem 3.27.

Lemma 3.25. *Let $\mathbb{C}$ be a regular majority category. Then for any three reflexive relations A, B, C on any object X in $\mathbb{C}$ we have:*

$$(A \circ B) \cap (A \circ C) \leqslant A \circ (B \cap C).$$

Proof. Let $a = (a_1, a_2) : A_0 \to X \times X$, $b = (b_1, b_2) : B_0 \to X \times X$ and $c = (c_1, c_2) : C_0 \to X \times X$ represent the three reflexive relations A, B, C respectively. Consider the quaternary relation $R \leqslant X^4$ represented by r, which is formed from the following pullback:

$$\begin{tikzcd} R_0 \arrow[r, "p_2"] \arrow[d, "r"'] & A_0 \times B_0 \times C_0 \arrow[d, "a \times b \times c"] \\ X^4 \arrow[r, "(\pi_2,\pi_1),(\pi_1,\pi_3),(\pi_1,\pi_4)"'] & (X \times X) \times (X \times X) \times (X \times X). \end{tikzcd}$$

Set theoretically, R is the relation defined by:

$$R = \{(x, y, z, w) \in X^4 \mid (y, x) \in A \wedge (x, z) \in B \wedge (x, w) \in C\}.$$

Consider the image factorization er' of $(\pi_2, \pi_3, \pi_4)r$ in the diagram below

$$\begin{tikzcd} R_0 \arrow[r, "r"] \arrow[d, "e"'] & X^4 \arrow[d, "(\pi_2,\pi_3,\pi_4)"] \\ R'_0 \arrow[r, "r'"'] & X^3 \end{tikzcd}$$

Now, let $(x, y) : S \to X \times X$ be such that $(x, y) \in_S (A \circ B) \cap (A \circ C)$ then there exist regular epis $\alpha_1 : Q_1 \to S$ and $\alpha_2 : Q_2 \to S$ as well as morphisms $z_1 : Q_1 \to X$ and $z_2 : Q_2 \to X$ such that $(x\alpha_1, z_1) \in_{Q_1} A$ and $(z_1, y\alpha_1) \in_{Q_1} B$, together with $(x\alpha_2, z_2) \in_{Q_2} A$ and $(z_2, y\alpha_2) \in_{Q_2} C$. We may assume that $\alpha_1 = \alpha = \alpha_2$, since if not, we could pullback α_1 along α_2. Then note that we have that

$$(z_1, x\alpha, y\alpha, z_1) \in_Q R \quad \text{and} \quad (y\alpha, y\alpha, y\alpha, y\alpha) \in_Q R \quad \text{and} \quad (z_2, x\alpha, z_2, y\alpha) \in_Q R,$$

which implies that

$$(x\alpha, y\alpha, z_1) \in_Q R' \quad \text{and} \quad (y\alpha, y\alpha, y\alpha) \in_Q R' \quad \text{and} \quad (x\alpha, z_2, y\alpha) \in_Q R',$$

and since R' is majority selecting, it follows that $(x\alpha, y\alpha, y\alpha) \in_Q R'$. Thus, there exists $\phi : Q \to R'_0$ such that $r'\phi = (x, y, y)\alpha$. Now, take the pullback of e along ϕ, to obtain the diagram below:

$$\begin{tikzcd} Q' \arrow[r, "z"] \arrow[d, "\alpha'"'] & R_0 \arrow[d, "e"] \arrow[dr, "r"] & \\ Q \arrow[r, "\phi"] \arrow[d, "\alpha"'] & R'_0 \arrow[d, "r'"] & X^4 \arrow[dl, "(\pi_2,\pi_3,\pi_4)"] \\ S \arrow[r, "(x,y,y)"'] & X^3 & \end{tikzcd}$$

Then, if we let $p = \pi_1 rz$, it follows that $rz = (p, x\alpha\alpha', y\alpha\alpha', y\alpha\alpha')$. Now by construction of R, it follows that $(x\alpha\alpha', p) \in_{Q'} A$ and $(p, y\alpha\alpha') \in_{Q'} B \cap C$, so that

$$(x\alpha\alpha', y\alpha\alpha') \in_{Q'} A \circ (B \cap C) \implies (x, y) \in_S A \circ (B \cap C).$$

□

Lemma 3.26. *Let $\mathbb{C}$ be a regular category such that for any three effective equivalence relations α, β, γ on an object X, we have*

$$\alpha \cap (\beta \circ \gamma) = (\alpha \cap \beta) \circ (\alpha \cap \gamma),$$

then $\mathbb{C}$ is a majority category.

Proof. Consider the ternary relation R represented by the monomorphism $(r_1, r_2, r_3) : R_0 \to X \times Y \times Z$, we will show that it is majority selecting in the sense of Definition 2.4. Let

$$x, x' : S \to X, \quad y, y' : S \to Y, \quad z, z' : S \to Z,$$

and $a, b, c : S \to R_0$ be any morphisms in $\mathbb{C}$ such that the diagrams:

$$\begin{tikzcd} & R_0 \arrow[d] \\ S \arrow[ur, "a"] \arrow[r, "{(x,y,z')}"'] & X \times Y \times Z \end{tikzcd} \qquad \begin{tikzcd} & R_0 \arrow[d] \\ S \arrow[ur, "b"] \arrow[r, "{(x,y',z)}"'] & X \times Y \times Z \end{tikzcd} \qquad \begin{tikzcd} & R_0 \arrow[d] \\ S \arrow[ur, "c"] \arrow[r, "{(x',y,z)}"'] & X \times Y \times Z \end{tikzcd}$$

commute. Consider the kernel congruences α, β, γ on R formed from taking the kernel pairs of r_1, r_2, r_3 respectively. Then $(a, c) \in_S \beta \cap (\alpha \circ \gamma) \implies (a, c) \in_S (\beta \cap \alpha) \circ (\beta \cap \gamma)$, so that there exists a regular epimorphism $e : Q \to S$ and a morphism $b : Q \to R$ such that $(ae, b) \in_Q (\beta \cap \alpha)$ and $(b, ce) \in_Q (\beta \cap \gamma)$. This implies that $xe = r_1 b$ and $ye = r_2 b$ and $ze = r_3 b$, and therefore we have the commutative diagram:

$$\begin{tikzcd} Q \arrow[r, "b"] \arrow[d, "e"'] & R \arrow[d, "{(r_A,r_B,r_C)}"] \\ S \arrow[ur, dotted, "f"] \arrow[r, "{(x,y,z)}"'] & X \times Y \times Z \end{tikzcd}$$

where f exists, since e is a regular epimorphism. □

We are now ready to prove the main theorem of this chapter.

Theorem 3.27. *The following are equivalent for a regular category $\mathbb{C}$:*

(i) *The Pairwise Chinese Remainder Theorem holds for $\mathbb{C}$.*

(ii) *$\mathbb{C}$ has finite 2-fold subobject decompositions.*

(iii) *$\mathbb{C}$ is a majority category.*

(iv) *For any three reflexive relations A, B, C on an object X in $\mathbb{C}$ we have*

$$(A \circ B) \cap (A \circ C) \leqslant A \circ (B \cap C).$$

(v) *For any three reflexive relations A, B, C on an object X in $\mathbb{C}$ we have*

$$A \cap (B \circ C) \leqslant (A \cap B) \circ (A \cap C).$$

(vi) *For any equivalence relations α, β, γ on an object X in $\mathbb{C}$ we have*

$$\alpha \cap (\beta \circ \gamma) = (\alpha \cap \beta) \circ (\alpha \cap \gamma).$$

(vii) *For any effective equivalence relations α, β, γ on an object X in $\mathbb{C}$ we have*

$$\alpha \cap (\beta \circ \gamma) = (\alpha \cap \beta) \circ (\alpha \cap \gamma).$$

We have the chain of implications $(i) \implies (ii) \implies (iii) \implies (iv)$ by Lemma 3.22, Theorem 3.7 and Lemma 3.25, respectively. We also have $(vii) \implies (iii) \implies (i)$ by Lemma 3.26 and Lemma 3.24, respectively. Trivially, we have $(v) \implies (vi) \implies (vii)$. Thus, to prove the theorem above, it suffices to show $(iv) \implies (v)$.

Proof. Note that if $\mathbb{C}$ satisfies (iv), then for any three reflexive relations A, B, C on an object X in $\mathbb{C}$ we have

$$(B \circ A) \cap (C \circ A) \leqslant (B \cap C) \circ A.$$

This is because we may take the double opposite of the left-hand side:

$$\begin{aligned}(B \circ A) \cap (C \circ A) &= (((B \circ A) \cap (C \circ A))^{\mathrm{op}})^{\mathrm{op}} \\ &= ((B \circ A)^{\mathrm{op}} \cap (C \circ A)^{\mathrm{op}})^{\mathrm{op}} \\ &= ((A \circ B) \cap (A \circ C))^{\mathrm{op}} \leqslant (A \circ (B \cap C))^{\mathrm{op}} = (B \cap C) \circ A\end{aligned}$$

Now, for $(iv) \implies (v)$ we have the following:

$$\begin{aligned}(A \cap B) \circ (A \cap C) &\geqslant ((A \cap B) \circ A) \cap ((A \cap B) \circ C)\\ &\geqslant ((A \cap B) \circ A) \cap (A \circ C) \cap (B \circ C)\\ &\geqslant A \cap (B \circ C)\end{aligned}$$

by repeated application of (iv). □

Definition 3.28. Let X be an object in a category $\mathbb{C}$, then a factor congruence K on X is a congruence relation represented by the kernel equivalence relation of π_1 where

$$A \xleftarrow{\pi_1} X \xrightarrow{\pi_2} B$$

is a binary product diagram. The factor congruence K' obtained as the kernel equivalence of π_2 is the complement of K, and more over, $K \circ K' = 1$. The sub-poset of all factor congruences of an object X is denoted by $F(X)$.

Remarks 3.29. By Proposition 1.44, the pair of congruences K, K' are a pair of factor congruences if and only if $K \circ K' = 1$ and $K \cap K' = 0$.

The notion of a *factor permutable* variety was introduced in [Gum83], which has a straightforward generalization given below.

Definition 3.30 ([Gra04])**.** A regular category $\mathbb{C}$ is said to be factor permutable if for any factor congruence F and any congruence E on an object X we have

$$F \circ E = E \circ F.$$

Corollary 3.31. *Every regular majority category is factor permutable.*

Proof. Just note that $E \circ F$ and $F \circ E$ as subobjects of $X \times Y \times X \times Y$ have the same two-fold projections. Therefore, by Theorem 3.27 (ii), we must have

$$E \circ F = F \circ E.$$

□

Given a morphism $f : X \to Y$ in a category $\mathbb{C}$ with image factorizations, and a subobject $S \leqslant X \times X$, we will write $f(S)$ for the subobject $(f \times f)(S)$, and similarly we write $f^{-1}(S)$ for $(f \times f)^{-1}(S)$. If $\mathbb{C}$ is regular, then we have:

$$f^{-1}f(S) = K \circ S \circ K,$$

where K is the kernel equivalence relation on X associated to f. Bourn showed in [Bou05], that a regular Mal'tsev category is congruence distributive if and only if for any regular epimorphism $f : X \to Y$ and any equivalence relations $\alpha, \beta \in \mathrm{Eq}(X)$, we have $f(\alpha \cap \beta) = f(\alpha) \cap f(\beta)$ (this holds, in fact, for Goursat categories). The proof of this fact essentially reduces to the proposition below.

Proposition 3.32. *Let $\mathbb{C}$ be a regular category, then the following are equivalent.*

(i) For any regular epimorphism $f : X \to Y$, and any reflexive relations $R, S \in \mathrm{Eq}(X)$ we have $f(R \cap S) = f(R) \cap f(S)$.

(ii) For any three reflexive relations R, S, T on any object X in $\mathbb{C}$, we have

$$(T \circ R \circ T) \cap (T \circ S \circ T) = (T \circ R \cap S \circ T).$$

The proof below is essentially that which can be found in [Bou05], however we include a sketch for completeness.

Sketch. For $(i) \implies (ii)$: suppose that $(r_1, r_2) : R_0 \to X \times X$ and $(s_1, s_2) : S_0 \to X \times X$ and $(t_1, t_2) : T_0 \to X \times X$ are represent R, S, T respectively. Note that $t_2(t_1^{-1}(R)) = T \circ R \circ T$ and $t_2(t_1^{-1}(R)) = T \circ S \circ T$, so that we have:

$$\begin{aligned}
(T \circ R \circ T) \cap (T \circ S \circ T) &= t_2(t_1^{-1}(R)) \cap t_2(t_1^{-1}(S)) \\
&= t_2(t_1^{-1}(R) \cap t_1^{-1}(RS)) \\
&= t_2(t_1^{-1}(R \cap S)) \\
&= T \circ (R \cap S) \circ T
\end{aligned}$$

For $(ii) \implies (i)$: suppose that $f : X \to Y$ is any regular epimorphism, then we have that

$$\begin{aligned} f(R \cap S) &= f(f^{-1}f(R \cap S)) \\ &= f(K \circ (R \cap S) \circ K) \\ &= f((K \circ R \circ K) \cap (K \circ S \circ K)) \\ &= f(f^{-1}f(R) \cap f^{-1}f(S)) \\ &= ff^{-1}(f(R) \cap f(S)) \\ &= f(R) \cap f(S) \end{aligned}$$

□

Corollary 3.33. *For any regular epimorphism $f : X \to Y$ in a regular majority category $\mathbb{C}$, and any reflexive relations R, S on X we have*

$$f(R \cap S) = f(R) \cap f(S)$$

Proof. We show that any regular majority category $\mathbb{C}$ satisfies (ii) of Proposition 3.32: suppose that R, S, T are reflexive relations on an object X in $\mathbb{C}$. Then we always have:

$$T \circ (R \cap S) \circ T \leqslant (T \circ R \circ T) \cap (T \circ S \circ T).$$

For the reverse inequality, we have

$$\begin{aligned} T \circ (R \cap S) \circ T &\geqslant ((T \circ R) \cap (T \circ S)) \circ T \\ &\geqslant (T \circ R \circ T) \cap (T \circ S \circ T) \end{aligned}$$

□

Majority categories are Gumm categories

The *shifting property* was first introduced for varieties in [Gum83], and its categorical formulation is the following definition.

Definition 3.34 (Shifting property [BG04])**.** Let $\mathbb{C}$ be a category with finite limits, X an object of $\mathbb{C}$, and α, β, γ congruences on X and $a, b, c, d : S \to X$ any morphisms in $\mathbb{C}$. Then $\mathbb{C}$ is said to satisfy the shifting property if whenever $\alpha \cap \beta \leqslant \gamma$, then if $(a,b),(d,c) \in_S \beta$, $(a,c),(b,d) \in_S \alpha$ and $(b,d) \in \gamma$,

$$\begin{tikzcd}
b \arrow[r, dash, "\gamma"] \arrow[r, dash, bend left, "\alpha"] & d \\
a \arrow[u, dash, "\beta"] \arrow[r, dash, "\alpha"] \arrow[r, dash, dotted, bend right, "\gamma"'] & c \arrow[u, dash, "\beta"']
\end{tikzcd}$$

then $(a,c) \in_S \alpha$.

Categories satisfying the shifting property are called *Gumm* categories, after H.P. Gumm – the author of [Gum83]. A variety is a Gumm category if and only if it is congruence modular, however, for general categories the shifting property is strictly weaker than congruence modularity (see [Jan16]). Regular majority categories satisfy a stronger property, which is obtained by dropping the requirement that $(b,d) \in_S \alpha$.

Theorem 3.35 (Strong Shifting)**.** *Let $\mathbb{C}$ be a regular majority category, X an object of $\mathbb{C}$, α, β, γ congruences on X, and $a,b,c,d : S \to X$ any morphisms. Then, if $(a,b),(d,c) \in_S \beta$, $(a,c),(b,d) \in_S \alpha$ and $(b,d) \in \gamma$,*

$$\begin{tikzcd}
b \arrow[r, dash, "\gamma"] & d \\
a \arrow[u, dash, "\beta"] \arrow[r, dash, "\alpha"] \arrow[r, dash, dotted, bend right, "\gamma"'] & c \arrow[u, dash, "\beta"']
\end{tikzcd}$$

then $(a,c) \in_S \gamma$.

Proof.

$$(a,c) \in_S \alpha \cap (\beta \circ \gamma \circ \beta) = (\alpha \cap \beta) \circ (\alpha \cap \gamma) \circ (\alpha \cap \beta) \leqslant \gamma.$$

□

Theorem 3.36. *If $\mathbb{C}$ is a regular category satisfying the strong shifting property as above, then $\mathbb{C}$ is factor permutable.*

Proof. Suppose that $\mathbb{C}$ a regular category satisfying the strong shifting property, and that

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

is a product diagram in $\mathbb{C}$. Suppose that $\theta = \mathrm{Eq}(\pi_1)$ and that $\theta' = \mathrm{Eq}(\pi_2)$, and that γ is any congruence on $X \times Y$. Suppose that $a, d : S \to X$ are morphisms in $\mathbb{C}$, with $(a, d) \in_S \gamma \circ \theta$. Then, there exists a regular epimorphism $e : T \to S$ and a morphism $b : T \to X$ such that $(ae, b) \in_T \gamma$ and $(b, de) \in_T \theta$. Since $\theta \circ \theta' = 1$, there exists a regular epi $e : T' \to T$ and a morphism $c : T' \to X$ such that $(aee', c) \in_{T'} \theta'$ and $(c, dee') \in_{T'} \theta$. Therefore, since $\mathbb{C}$ satisfies the strong shifting property, it follows that $(aee', c) \in_{T'} \gamma$.

$$\begin{tikzcd}
be' \arrow[r, dash, "\gamma"] \arrow[d, dash, "\theta"'] & dee' \arrow[d, dash, "\theta"] \\
aee' \arrow[r, dash, "\theta'"] \arrow[r, dash, dotted, bend right, "\gamma"'] & c
\end{tikzcd}$$

Therefore, $(aee', dee') \in_{T'} \theta \circ \gamma$, which implies $(a, d) \in_S \theta \circ \gamma$. $\square$

This gives another way to see that every regular majority category is factor permutable in the sense of Definition 3.30.

Corollary 3.37. *Every regular majority category is factor permutable.*

Chapter 4

Unique factorization

In this chapter, we prove two uniqueness of direct-decomposition theorems. The first is essentially an application of Theorem 3.27, which has been motivated by the corresponding universal algebraic result (see Chapter 5 in [MMT87]). The second Theorem (Theorem 4.32) is one of the main results of the thesis, which presents a uniqueness of factorization result which is much more general than the first, provided that the base category is pointed and has binary coproducts. It shows that there can be a categorical foundation to various decomposition theorems, that do not require the base category to be very algebraic.

4.1 Pre-exact categories

Throughout this section, we fix a (finitely) complete regular category $\mathbb{C}$. Given a (finite) family $(E_i)_{i\in I}$ of effective equivalence relations on an object X in $\mathbb{C}$, with representatives

$$E_i' \rightrightarrows X \xrightarrow{f_i} A_i,$$

the kernel equivalence relation of the induced morphism

$$E \rightrightarrows X \xrightarrow{f} \prod_i A_i,$$

represents the meet $\bigwedge_i E_i$ in the semi-lattice $\mathrm{Ef}(X)$ of effective equivalence relations on X. The notion of a pre-exact category is a weakening of the notion of a Barr exact category [BGO71], which includes duals of geometric categories such as $\mathbf{Top}^{\mathrm{op}}$ and $\mathbf{Rel}_n^{\mathrm{op}}$.

Definition 4.1. A (finitely) complete category $\mathbb{C}$ is said to be *pre-exact* if it is regular, has coequalizers of equivalence relations, and for any (finite) family of equivalence relations represented by $(E_i')_{i\in I}$ on any object X, the canonical morphism

$$\bigwedge_{i\in I} E_i' \to \bigwedge_{i\in I} \overline{E_i'},$$

is an epimorphism, where $\overline{E_i'}$ denotes the kernel pair of the coequalizer of E_i'.

Remarks 4.2. The morphism in the definition above is always a monomorphism, and therefore a pre-exact category $\mathbb{C}$ is exact if and only if it is a regular epimorphism. Thus in particular, every exact category is pre-exact.

Some of the details in the proof below are left out, as the argument below is straightforward .

Proposition 4.3. *Let $\mathbb{C}$ and $\mathbb{D}$ be (complete) regular categories which have coequalizers of equivalence relations. Also, let $F : \mathbb{C} \to \mathbb{D}$ be a functor which preserves limits and coequalizers of equivalence relations, and also reflects epimorphisms. Then, if $\mathbb{D}$ is pre-exact, then so is $\mathbb{C}$.*

Proof. Suppose that $(E_i \xrightarrow{e_i} X^2)_{i\in I}$ is a family of monomorphisms which represent equivalence relations on X and let $X \xrightarrow{q_i} X_i$ be a coequalizer of the E_i. Then, we are required to show that in the diagram,

$$\begin{tikzcd}
\bigwedge_{i\in I} \overline{E_i} \arrow[dr, shift left] \arrow[dr, shift right] & & \\
\bigwedge_{i\in I} E_i \arrow[u, "\sigma"] \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, "(q_i)_{i\in I}"'] & \prod_{i\in I} X_i
\end{tikzcd}$$

the vertical arrow is an epimorphism. Now, we have that $F(\bigwedge_{i\in I} E_i) \simeq \bigwedge_{i\in I} F(E_i)$ since $\bigwedge_{i\in I} E_i$ is constructed as a certain limit, and F preserves limits. We also have that $F(\overline{E_i}) \simeq \overline{F(E_i)}$ since F preserves coequalizers (and also kernel pairs). Now since E_i is an equivalence relation, and F preserves limits, it follows that each $F(E_i) \to F(X \times X)$ is also an equivalence relation. Therefore, we have the following diagram in $\mathbb{D}$:

$$\begin{tikzcd}
F(\bigwedge_{i\in I}\overline{E_i}) \arrow[r, "\sim"] & \bigwedge_{i\in I}\overline{F(E_i)} \arrow[dr, shift left] \arrow[dr, shift right] & & \\
F(\bigwedge_{i\in I} E_i) \arrow[u, "F(\sigma)"] \arrow[r, "\sim"] & \bigwedge_{i\in I} F(E_i) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & F(X) \arrow[r, "(F(q_i))_{i\in I}"'] & \prod_{i\in I} F(X_i)
\end{tikzcd}$$

Finally, since $\mathbb{D}$ is a pre-exact category, the right-hand vertical morphism is an epimorphism, which implies that $F(\sigma)$ is an epimorphism, so that σ is an epimorphism. □

Proposition 4.4. *Let $\mathbb{C}$ be a pre-exact category. For any family of equivalence relations $(E_i)_{i\in I}$ on any object X represented by the family of monomorphisms $(E'_i \to X^2)_{i\in I}$ in $\mathbb{C}$, we have*

$$\overline{\bigwedge_{i\in I} E_i} = \bigwedge_{i\in I} \overline{E_i}.$$

Proof. There is a canonical triangle in the preorder of monomorphisms into $X \times X$:

$$\begin{tikzcd}
\overline{\bigwedge_{i\in I} E'_i} \arrow[r] & \bigwedge_{i\in I} \overline{E'_i} \\
\bigwedge_{i\in I} E'_i \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Since $\mathbb{C}$ is pre-exact, it follows by definition that the diagonal morphism is an epimorphism, which implies that the horizontal morphism above is an epimorphism.

$$\begin{tikzcd}
\overline{\bigwedge_{i\in I} E_i} \arrow[dd] \arrow[dr, shift left] \arrow[dr, shift right] & & \frac{X}{\bigwedge_{i\in I} \overline{E_i}} \arrow[dd, dotted, bend left, "f"] \\
& X \arrow[ur, two heads] \arrow[dr, two heads] & \\
\bigwedge_{i\in I} \overline{E_i} \arrow[ur, shift left] \arrow[ur, shift right] & & \frac{X}{\overline{\bigwedge_{i\in I} E_i}} \arrow[uu, dotted, bend left, "g"]
\end{tikzcd}$$

Since the upper parallel pair of morphisms factor through the bottom parallel pair, it follows that the dotted arrow g exists in the diagram above. The morphism f exists since the vertical solid arrow is an epimorphism. These morphisms f and g are inverse isomorphisms, and therefore

$$\overline{\bigwedge_{i \in I} E_i} = \bigwedge_{i \in I} \overline{E_i}.$$

□

Example 4.5. The categories $\mathbf{Top}^{\mathrm{op}}, \mathbf{Rel}^{\mathrm{op}}, \mathbf{Ord}^{\mathrm{op}}, \mathbf{Met}_{\infty}^{\mathrm{op}}$ all admit forgetful functors to $\mathbf{Set}^{\mathrm{op}}$. These forgetful functors preserve limits, colimits and reflect epimorphisms. Since they are regular, it follows from Proposition 4.3 that they are all pre-exact.

4.2 Direct decomposition in pre-exact majority categories

Definition 4.6. An object X in a category $\mathbb{C}$ with a terminal object is said to be *non-trivial* if the terminal map $t_X : X \to 1$ is a regular epimorphism which is not an isomorphism.

Throughout this section we will be working with a complete pre-exact (see Definition 4.1) majority category $\mathbb{C}$, where the class of non-trivial objects in $\mathbb{C}$ is closed under products.

Definition 4.7. An object X in a regular category $\mathbb{C}$ is said to be *directly irreducible* if it is non-trivial and satisfies the following property: for any isomorphism $\phi : X \to A \times B$, either $\pi_1\phi$ or $\pi_2\phi$ is an isomorphism. The object X is said to be *strongly directly irreducible* if and only if X is non-trivial, and $X \simeq A \times B$ implies $A \simeq 1$ or $B \simeq 1$.

Remarks 4.8. Every strongly directly irreducible object is directly irreducible.

Example 4.9. A topological space/ordered set/graph S is strongly irreducible in $\mathbf{Top}^{\mathrm{op}}/\mathbf{Ord}^{\mathrm{op}}/\mathbf{Grph}^{\mathrm{op}}$ if and only if S is *connected* in the usual sense.

Example 4.10. If S is a $G - \mathbf{Set}$, then S is strongly irreducible in $(G - \mathbf{Set})^{\mathrm{op}}$ if and only if S is *transitive* (has at most one orbit). To see this, suppose that S is strongly directly irreducible in $(G - \mathbf{Set})^{\mathrm{op}}$, and let $x \in S$ be any element. Note that $\mathrm{orb}(x)$ and also $S - \mathrm{orb}(x)$ are both G-sets together with the restricted action of S. We have a canonical isomorphism:

$$\mathrm{orb}(x) \sqcup (S - \mathrm{orb}(x)) \simeq S,$$

and since S is assumed to be strongly directly irreducible in $(G - \mathbf{Set})^{\mathrm{op}}$, and $\mathrm{orb}(x) \neq \emptyset$, it follows that $S - \mathrm{orb}(x) = \emptyset$, so that S is transitive. The converse statement is left to the reader.

Lemma 4.11. *Suppose that A is a directly irreducible object in $\mathbb{C}$, and let $X = A \times B$ where B is any object in $\mathbb{C}$. Then the factor congruence K represented by the kernel pair of the projection $\pi_1 : X \to A$ is maximal in $F(X)$.*

Proof. Suppose that $F \in F(X)$ has complement F' and $K \leqslant F$. Then by Corollary 3.31, it follows that K permutes with F' so that $K \circ F'$ is an equivalence relation on X. Moreover, the join $K \vee F'$ in $\mathrm{Ef}(X)$ is given by $\overline{K \circ F'}$, which is the effective closure of $K \circ F'$. Then we have

$$F \circ (K \circ F') \geqslant F \circ F' \ = 1 \implies F \circ (\overline{K \circ F'}) = 1.$$

And since $\mathbb{C}$ is a pre-exact majority category, it follows that:

$$F \cap (K \vee F') = \overline{F} \cap \overline{K \circ F'} = \overline{F \cap (K \circ F')} = \overline{F \cap K \circ F \cap F'} = K,$$

by Theorem 4.4 and Theorem 3.27. Thus by Proposition 1.44, the canonical morphism

$$X/K \xrightarrow{\sigma} X/F \times X/(K \vee F'),$$

is an isomorphism. Since $X/K \simeq A$ - which is directly irreducible, it follows that either of the morphisms $\pi_1\sigma$ or $\pi_2\sigma$ is an isomorphism. If $\pi_2\sigma$ were an isomorphism, then $K \vee F' = K$ which would imply that $F' = 0$ and therefore $F = 1$. On the other hand, if $\pi_1\sigma$ is an isomorphism, then $K = F$. Therefore, K is a maximal in $F(X)$. □

Throughout the remainder of this section, we fix a (finitely) complete pre-exact majority category $\mathbb{C}$. Let $(X_i)_{i \in I}$ be a (finite) family of directly irreducible objects in $\mathbb{C}$, and denote $X = \prod_{i \in I} X_i$. In what follows, K_i is the kernel congruence of the projection $\pi_i : X \to X_i$.

Lemma 4.12. *For any object X in $\mathbb{C}$ the poset of factor congruences $F(X)$ on X forms a complete boolean lattice.*

Proof. Suppose that F, K are any two factor congruences with complements F' and K' respectively. Then we claim that $F \cap K$ is also a factor congruence, with complement $\overline{F' \circ K'}$ - the effective closure of $F' \circ K'$. Applying both Corollary 3.31 and Theorem 3.27 we have that:

$$(F \cap K) \cap (\overline{F' \circ K'}) = \overline{(F \cap K) \cap (F' \circ K')} = \overline{F \cap K \cap F' \circ F \cap K \cap K'} = 0.$$

And also we have

$$\begin{aligned} F \cap K \circ (\overline{F' \circ K'}) &\geqslant F \cap K \circ (F' \circ K'), \\ &= (F \circ F' \circ K') \cap (K \circ F' \circ K') = 1, \end{aligned}$$

so that $F \cap K \circ (\overline{F' \circ K'}) = 1$. This shows that $F \cap K$ is a factor congruence, whose complement is given by $\overline{F' \circ K'}$. Thus, $F(X)$ is a complemented lattice, which is distributive since

$$F \cap (K \vee T) = F \cap (\overline{K \circ T}) = \overline{F \cap (K \circ T)} = \overline{(F \cap K) \circ (F \cap T)} = (F \cap K) \vee (F \cap T).$$

Thus $F(X)$ is a Boolean lattice, and it also easily seen to be complete (which is left to the reader). □

Lemma 4.13. *Any (small) complete Boolean lattice is infinitely distributive, which is to say satisfies*

$$b \vee \bigwedge_{i \in J} a_i = \bigwedge_{i \in J} (b \vee a_i),$$

where J is any set.

Proof. The inequality $b \vee \bigwedge_{i \in J} a_i \leqslant \bigwedge_{i \in J} (b \vee a_i)$ is trivial, therefore we will show the reverse. Let b' be the complement of b, then for any $i \in J$ we have:

$$a_i = a_i \vee (b \wedge b') = (a_i \vee b) \wedge (a_i \vee b') \geqslant (a_i \vee b) \wedge b'.$$

This implies

$$\bigwedge_{i \in J} a_i \geqslant \bigwedge_{i \in J} (a_i \vee b) \wedge b' = (\bigwedge_{i \in J} (a_i \vee b)) \wedge b',$$

so that

$$b \vee \bigwedge_{i \in J} a_i \geqslant b \vee (\bigwedge_{i \in J} (a_i \vee b) \wedge b') = (b \vee \bigwedge_{i \in J} (a_i \vee b)) \wedge (b \vee b') = \bigwedge_{i \in J} (a_i \vee b).$$

□

Lemma 4.14. *If $F \in F(X)$ is a maximal factor congruence, then $F = K_i$ for some unique $i \in I$.*

Proof. Suppose that $F \neq K_i$ for any $i \in I$, then by Lemma 4.11, since each K_i is a maximal factor congruence, we have that $F \vee K_i = 1$ for any $i \in I$. By Lemma 4.12, $F(X)$ is a (small) complete Boolean lattice, and is thus infinitely distributive by Lemma 4.13. Therefore we have

$$F = F \vee (\bigwedge_{i \in I} K_i) = \bigwedge_{i \in I} (F \vee K_i) = 1,$$

which is a contradiction, so that $F = K_i$ for some $i \in I$. For uniqueness, suppose that K_i and K_j are represented by K'_i and K'_j, and have $K_i = K_j$. Suppose that (T, t_1, t_2) in

$$T \overset{t_1}{\underset{t_2}{\rightrightarrows}} X_i \longrightarrow 1,$$

is a kernel pair of the terminal morphisms $X_i \to 1$. Then we have that the diagram

$$(\prod_{k \neq i} X_k) \times T \overset{id \times t_1}{\underset{id \times t_2}{\rightrightarrows}} X \xrightarrow[\pi_j]{} X_j$$

commutes, so that $((id, t_1), (id, t_2))$ factors through K_j, and hence through K_i. This implies that $t_1 = t_2$, which implies that the terminal map $X_i \to 1$ is a mono, which is a contradiction since X_i is directly irreducible, and therefore non-trivial. $\square$

Lemma 4.15. *The product projections $\pi_i : \prod_{i \in I} X_i \to X_i$ are regular epimorphisms.*

Proof. Since the class of non-trivial objects in $\mathbb{C}$ is closed under products, the terminal map $\prod_{k \neq i} X_k \to 1$ is a regular epimorphism. The projection $\pi_i : \prod_{i \in I} X_i \to X_i$ may be obtained as the pullback of $\prod_{k \neq i} X_k \to 1$ along $X_i \to 1$, and since $\mathbb{C}$ is regular it must be a regular epimorphism.

$$\begin{tikzcd}
\prod_{i \in I} X_i \arrow[r] \arrow[d, "\pi_i"'] & \prod_{k \neq i} X_k \arrow[d] \\
X_i \arrow[r] & 1.
\end{tikzcd}$$

$\square$

Suppose that $\mathbb{C}$ is a (finitely) complete pre-exact category, such that the class of non-trivial objects is stable under products. Let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ be two families of directly irreducible objects in $\mathbb{C}$, and let

$$X = \prod_{i\in I} X_i \xrightarrow{f} \prod_{j\in J} Y_j = Y,$$

be any isomorphism in $\mathbb{C}$.

Definition 4.16. If there exists a bijection $\sigma : I \to J$ and a family of isomorphisms $(f_i : X_i \to Y_{\sigma(i)})_{i\in I}$ such that the diagram

$$\begin{tikzcd} \prod_{i\in I} X_i \arrow[r, "f"] \arrow[d, "\pi_i"'] & \prod_{j\in J} Y_j \arrow[d, "\pi'_{\sigma(i)}"] \\ X_i \arrow[r, "f_i"'] & Y_{\sigma(i)} \end{tikzcd}$$

commutes, then we shall say that $\mathbb{C}$ has *strong refinements*.

Remarks 4.17. Note that in the universal algebraic setting, a variety has strong refinements if and only if it has strict refinements in the sense of [MMT87].

Theorem 4.18. *Every pre-exact majority category $\mathbb{C}$ has strong refinements.*

Proof. Suppose that K_i is the kernel equivalence relation of π_i and L_j is the kernel equivalence of π'_j. Then for any i, we have that K_i is a maximal factor congruence by Lemma 4.11, and thus the image $f(K_i)$ of K_i under f is a maximal factor congruence on Y. Thus, by Lemma 4.14 we have that $f(K_i) = L_{\sigma_i}$ for some unique $\sigma_i \in J$. Since the π_i is a regular epimorphism, it is the coequalizer of K_i, and hence there exists a morphism $f_i : X_i \to Y_{\sigma_i}$.

$$\begin{tikzcd} K_i \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & f(K_i) = L_{\sigma(i)} \arrow[d, shift left] \arrow[d, shift right] \\ \prod_{i\in I} X_i \arrow[r, "f"] \arrow[d, "\pi_i"'] & \prod_{j\in J} Y_j \arrow[d, "\pi'_{\sigma(i)}"] \\ X_i \arrow[r, dotted, "f_i"'] & Y_{\sigma(i)} \end{tikzcd}$$

By applying the whole argument above to the inverse isomorphism of f, it will follow that the map $i \mapsto \sigma_i$ is a bijection, and the morphisms f_i are isomorphisms. $\square$

Remarks 4.19. The above theorem recaptures the familiar lattice-theoretic fact that a direct product of directly irreducible lattices is unique in the sense above. Interestingly, the theorem applies to the categories $\mathbf{Top}^{\mathrm{op}}, \mathbf{Ord}^{\mathrm{op}}, \mathbf{Met}_{\infty}^{\mathrm{op}}$ and $(G - \mathbf{Set})^{\mathrm{op}}$ to reproduce the fact that a coproduct of connected topological spaces/ordered sets/metric spaces is unique up to a permutation of the connected components. Furthermore, a connected coproduct of transitive G-sets is also unique up to a permutation of its transitive components.

Corollary 4.20. *If* $\mathbb{C}$ *is any (co)complete topos, then* $\mathbb{C}^{\mathrm{op}}$ *has strong refinements.*

Remarks 4.21. Note that the category $\mathbf{Lat}(\mathbf{Top})$ of topological lattices is a majority category, but it is not regular so that we may not apply the above theorem to it. However, the main theorem of the next section gets around this problem.

Remarks 4.22. The results of this section can be done more generally, by discussing refinement properties for general (regular) categories. This is based on the paper [CJT64], and is not presented here.

4.3 Direct decompositions in zero-majority categories

The proof in the previous section was possible because the notion of a pre-exact category was sufficiently algebraic, so that we could borrow ideas from its universal algebraic counterpart. This section proves the same result for pointed categories, in a much weaker context than pre-exact categories, or majority categories, provided the base category has binary coproducts and is pointed.

Definition 4.23. Let $\mathbb{C}$ be a pointed category with finite products, then $\mathbb{C}$ said to be zero-majority if any ternary relation R in $\mathbb{C}$ satisfies

$$(x,y,z) \in_S R \quad \text{and} \quad (x,y',0) \in_S R \quad \text{and} \quad (x',y,0) \in_S R \implies (x,y,0) \in_S R.$$

Example 4.24. Any pointed majority category is necessarily zero-majority.

The below example is a consequence of the more general results of [Jan06].

Example 4.25. A pointed variety of algebras $\mathbb{V}$ is zero-majority if and only if it admits a ternary term $m(x,y,z)$ satisfying the equations:

$$
\begin{aligned}
m(x,x,z) &= x,\\
m(x,y,x) &= x,\\
m(x,0,0) &= 0.
\end{aligned}
$$

Example 4.26. The category **Imp** of implication algebras was introduced in [Abb67], as a variety of algebras with a single binary operation satisfying:

$$
\begin{aligned}
(x \cdot y) \cdot x &= x\\
(x \cdot y) \cdot y &= y \cdot (x \cdot x)\\
(x \cdot y) \cdot z &= y \cdot (x \cdot z)
\end{aligned}
$$

It can be shown that every non-empty implication algebra A admits a constant 1 which is given by $x \cdot x$ for any $x \in A$, and satisfies $x \cdot 1 = 1$ and $1 \cdot x = x$. This is why implication algebras are assumed to be non-empty, and therefore the category **Imp** is a pointed category. In what follows, we will write xy for $x \cdot y$. Set

$$m(x,y,z) = (y(zx))x,$$

then

$$
\begin{aligned}
m(x,x,z) &= (x(zx))x = x\\
m(x,y,x) &= (y(xx))x = 1x = x\\
m(x,1,1) &= (1(1x))x = (1x)x = xx = 1
\end{aligned}
$$

which implies that **Imp** is a zero-majority category by Example 4.25. This provides an example of a zero-majority category, which is not a majority category.

Remarks 4.27. Any finitary variety of algebras admitting a majority term is necessarily congruence distributive (see [Jon67]). But zero-majority finitary varieties are not even congruence modular, necessarily. To see this, suppose that $X = \{0,1,2,3\}$ and that m_X is defined by

$$
m_X(x,y,z) = \begin{cases} x & x = z\\ y & x = y\\ 0 & \text{otherwise} \end{cases}
$$

Then the subsets L, M, N of $X \times X$ defined by:

$$\begin{aligned} L &= \{(0,0),(1,1),(2,2),(3,3),(3,0),(0,3)\} \\ M &= \{(0,0),(1,0),(0,1),(1,1),(2,2),(3,3)\} \\ N &= \{(0,0),(1,0),(0,1),(1,1),(2,2),(3,3),(1,2),(2,1),(0,2),(2,0)\} \end{aligned}$$

are congruences, and moreover we have that they give us the pentagon

$$\begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ N \arrow[d, dash] & & L \arrow[ddl, dash] \\ M \arrow[dr, dash] & & \\ & 0 & \end{tikzcd}$$

in the congruence lattice of X. So that X is not congruence modular.

In what follows, we will make use of a weakened version of the *approximate operations* of Z. Janelidze and D. Bourn (see [BJ08]). It what follows, we will be working with categories that have binary coproducts. Given three morphisms $x, y, z : S \to A$ the induced morphism will be denoted by a vector with square brackets:

$$3S \xrightarrow{[x,y,z]} A.$$

Definition 4.28. Let $\mathbb{C}$ be a pointed category with binary coproducts. An approximate zero-majority co-operation on an object S, is a pair of morphisms: $m_S : D_S \to 3S$ (the approximate co-operation), and an epimorphism $\alpha_S : D_S \to S$ (the approximation of m_s), with the property that for any morphisms $x, y : S \to A$ we have

$$[x,x,y]m_s = x\alpha_S, \quad [x,y,x]m_s = x\alpha_S, \quad [x,0,0]m_s = 0,$$

where 0 is a zero-morphism from D_S to A.

Recall the definition of a weakly regular category from Definition 1.32:

Definition 4.29. We call a category $\mathbb{C}$ weakly-regular if it is finitely complete, has coequalizers of kernel pairs, and the pullback of a regular epimorphism is again an epimorphism.

We have already seen that in every weakly regular category, every morphism factors as a regular epimorphism followed by a monomorphism.

Theorem 4.30. *Let $\mathbb{C}$ be a weakly-regular zero-majority category with binary coproducts, then every object X in $\mathbb{C}$ admits an approximate zero-majority co-operation.*

Sketch. Consider the pullback diagram below:

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "\alpha"'] & 3X \arrow[d, "\begin{pmatrix} \iota_1 & \iota_1 & \iota_2 \\ \iota_1 & \iota_2 & 0 \\ \iota_2 & \iota_1 & 0 \end{pmatrix}"] \\ X \arrow[r, "(\iota_1, \iota_1, 0)"'] & (2X)^3 \end{tikzcd}$$

Let $3X \xrightarrow{e} R \xrightarrow{r} (2X)^3$ be an epi-mono factorization of the right-hand vertical morphism. Then we have that $(\iota_1, \iota_1, \iota_2)$ factors though R, and $(\iota_1, \iota_2, 0)$ factors though R, and $(\iota_2, \iota_1, 0)$ factors though R, which implies that $(\iota_1, \iota_1, 0)$ factors though R (via h). Therefore, the diagram

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "\alpha"'] & 3X \arrow[d, "e"] \\ X \arrow[r, "h"'] & R \end{tikzcd}$$

is a pullback, so that α is an epimorphism. And by construction, p is an approximate zero-majority co-operation. □

For the remainder of this section, we fix a complete weakly regular, zero-majority category $\mathbb{C}$ with binary coproducts. Let $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$ be a family of directly irreducible objects in $\mathbb{C}$, and let

$$X = \prod_{i \in I} X_i \xrightarrow{f} \prod_{j \in J} Y_j = Y$$

be any isomorphism. Consider the regular epi-mono factorization $m_{i,j} e_{i,j}$ of $\pi_j f \iota_i$ in the diagram below:

$$\begin{tikzcd} X_i \arrow[r, "\iota_i"] \arrow[dr, "e_{i,j}"'] & X \arrow[r, "f"] & Y \arrow[d, "\pi_j"] \\ & Y_{i,j} \arrow[r, "m_{i,j}"'] & Y_j \end{tikzcd}$$

In what follows, we will denote $\prod Y_{i,j} = S$ and each $y_j : S \to Y_j$ for $m_{i,j}\pi'_j$, where $\pi'_j : S \to Y_{i,j}$ are the canonical product projections. Then $y = (y_j)_{j \in I}$ is a monomorphism, as it is a product of monomorphisms. If we denote $\overline{S}$ for the subobject represented by y, and $\overline{X_i}$ for the subobject of Y represented by $f\iota_i$. Then we have $\overline{X_i} \leqslant \overline{S}$, and in what follows we will show that the reverse inequality $\overline{S} \leqslant \overline{X_i}$ also holds.

Lemma 4.31. *The morphism $f^{-1}y$ factors through ι_i.*

Proof. Suppose that $x = (x_i)_{i \in I} : S \to X$ is the morphism with $fx = y$, and fix $q \in J$. Since the pullback of a regular epimorphism is an epimorphism, there exists an epimorphism $\alpha : Q \to S$ and a morphism $x' : Q \to X$ such that $\pi_q f x' = y_q \alpha$, and x' factors through $X_i \xrightarrow{\iota_i} X$ (which implies that $x'_k = 0$ if $k \neq i$). Let $y' = fx'$ and let $y'' = (y''_j)_{j \in J} : Q \to Y$ be the morphism defined by $y''_j = 0$ for $j \neq q$, and $y''_q = y_q \alpha$, and let $x'' : Q \to X$ be the morphism with $fx'' = y''$.

Now, let $m_Q : A_Q \to Q$ be an approximate zero-majority co-operation with approximation $\alpha_Q : A_Q \to Q$ (see Definition 4.28). Then we have

$$[y'', 0, y']m_Q = y''\alpha_Q,$$

since for $j \in J$ we have:

$$\pi_j[y'', 0, y']m_Q = [y''_j, 0, y'_j]m_Q = \begin{cases} 0 = [0, 0, y'_j]m_Q & j \neq q \\ y''_q \alpha_Q = [y_q\alpha, 0, y_q\alpha]m_Q & j = q \end{cases}.$$

This implies that

$$f^{-1}[y'', 0, y']m_Q = f^{-1}y''\alpha_Q \implies [x'', 0, x']m_Q = x''\alpha_Q,$$

so that if $k \neq i$, then we have

$$\pi_k[x'', 0, x']m_Q = \pi_k x''\alpha_Q \implies [x''_k, 0, x'_k]m_Q = \pi_k x''\alpha_Q \implies 0 = \pi_k x''\alpha_Q,$$

which implies that $x''\alpha_Q$ factors through $X_i \xrightarrow{\iota_i} X$, and hence so does x'' (since α is an epimorphism, and ι_i a split mono). Now, let $x^* : S \to X$ be the morphism that factors through $X_i \xrightarrow{\iota_i} X$ with $\pi_i(x^*) = x_i$. Then we have that

$$\pi_k[x\alpha, x'', x^*\alpha]m_Q = \begin{cases} 0 = [x_k\alpha, x''_k, x^*_k\alpha]m_Q & k \neq i \\ x_k\alpha\alpha_Q = [x_k\alpha, x''_k, x^*_k\alpha]m_Q & k = i \end{cases},$$

which shows that

$$[x\alpha, x'', x^*\alpha]m_Q = x^*\alpha\alpha_Q.$$

Now consider the following argument:

$$\begin{aligned}
f[x\alpha, x'', x^*\alpha]m_Q &= fx^*\alpha\alpha_Q \implies \\
[y\alpha, y'', y^*\alpha]m_Q &= y^*\alpha\alpha_Q \implies \\
\pi_q[y\alpha, y'', y^*\alpha]m_Q &= \pi_q y^*\alpha\alpha_Q \implies \\
[y_q\alpha, y''_q, y^*_q\alpha]m_Q &= y^*_q\alpha\alpha_Q \implies \\
[y_q\alpha, y_q\alpha, y^*_q\alpha]m_Q &= y^*_q\alpha\alpha_Q \implies \\
y_q\alpha\alpha_Q &= y^*_q\alpha\alpha_Q \implies \\
y_q &= y^*_q.
\end{aligned}$$

Finally, since the q above is arbitrary, and x^* is fixed once y is fixed, it follows that $f(x^*) = y = f(x) \implies x = x^*$, which completes the proof. $\square$

Theorem 4.32. *Let $\mathbb{C}$ be a weakly regular, zero-majority category with binary coproducts, then $\mathbb{C}$ has strong refinements.*

In the proof below, we will be working with the diagram in Definition 4.16.

Proof. By Lemma 4.31, it follows that the canonical morphism

$$X_i \to \prod_{j\in J} Y_{i,j},$$

is an isomorphism. Since X_i is strongly-directly irreducible, there exists a unique $\sigma_i \in J$ such that the canonical projection

$$\prod_{j\in J} Y_{i,j} \xrightarrow{\pi_{\sigma_i}} Y_{i,\sigma_i},$$

is an isomorphism. Thus the morphism $f_i = \pi_{\sigma_i} f \iota_i$ is a monomorphism.

$$\begin{tikzcd}
\prod_{i\in I} X_i \arrow[r, "f"] \arrow[d, "\pi_i"] & \prod_{j\in J} Y_j \arrow[d, "\pi'_{\sigma(i)}"'] \\
X_i \arrow[r, "f_i"'] \arrow[u, bend left=60, "\iota_i"] & Y_{\sigma(i)} \arrow[u, bend right=60, "\iota'_{\sigma_i}"']
\end{tikzcd}$$

Consider the morphism $g = \pi_i f^{-1} \iota'_{\sigma_i}$, then we have

$$f_i g = f_i \pi_i f^{-1} \iota'_{\sigma_i} = f_i = \pi_{\sigma_i} f \iota_i \pi_i f^{-1} \iota'_{\sigma_i} = 1_{Y_{\sigma_i}},$$

which implies that f_i is a split epimorphism, and therefore an isomorphism. Since $g = \pi_i f^{-1} \iota'_{\sigma_i} = f^{-1}$, it follows that the assignment $i \longmapsto \sigma_i$ is a bijection. $\square$

The above result applies to the categories $\mathbf{Lat}_*(\mathbf{Top})$ of pointed topological lattices, or $\mathbf{Imp}(\mathbf{Top})$ of topological implication algebras, and gives us the above uniqueness of direct-decomposition results. Also, the category $\mathbf{NReg}(\mathbf{Top})$ of topological von Neumann regular rings, which is regular, admits the above the above direct deccomposition results as a special case, as it is regular and pointed.

Chapter 5

Comajority excludes Majority

In Section 3.3 we saw that many categories of a geometric nature, such as topological spaces, metric spaces, any topos, etc., form comajority categories. This raises the question of whether there are categories which are simultaneously majority and comajority. This section proves that the only finitely complete categories $\mathbb{C}$ with binary coproducts, such that $\mathbb{C}$ and $\mathbb{C}^{\mathrm{op}}$ are majority categories are the preorders having finite meets and joins. This result is similar to the result that if a category $\mathbb{C}$ is such that both $\mathbb{C}$ and $\mathbb{C}^{\mathrm{op}}$ are distributive categories, then $\mathbb{C}$ is a preorder.

In what follows, by a *majority* algebra we mean a set X equipped with a majority operation $p_X : X^3 \to X$. A homomorphism of majority algebras $f : (X, p_X) \to (Y, p_Y)$ is a function $f : X \to Y$ satisfying $p_Y(f(x), f(y), f(z)) = f(p_X(x, y, z))$. A majority algebra is said to be *commutative* if the majority operation is a homomorphism.

Lemma 5.1. *Let $\mathbb{C}$ be a finitely complete majority category and A any object in $\mathbb{C}$, then the morphisms*

$$\begin{tikzcd} A^3 \arrow[r, "{(\pi_1,\pi_1,\pi_3)}"] & A^3 & A^3 \arrow[l, "{(\pi_1,\pi_2,\pi_2)}"'] \\ & A^3 \arrow[u, "{(\pi_3,\pi_2,\pi_3)}"] & \end{tikzcd}$$

are jointly strongly epimorphic.

Proof. Suppose r is a monomorphism, such that each of the morphisms above factor through R:

$$
\begin{tikzcd}
 & R \arrow[d, "r"] \\
A^3 \arrow[ur, "m_1"] \arrow[r, "(\pi_1,\pi_1,\pi_3)"'] & A^3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \arrow[d, "r"] \\
A^3 \arrow[ur, "m_2"] \arrow[r, "(\pi_1,\pi_2,\pi_2)"'] & A^3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \arrow[d, "r"] \\
A^3 \arrow[ur, "m_3"] \arrow[r, "(\pi_3,\pi_2,\pi_3)"'] & A^3
\end{tikzcd}
$$

Then there exists $m : A^3 \to R$ making the diagram

$$
\begin{tikzcd}
 & R \arrow[d] \\
A^3 \arrow[ur, dotted, "m"] \arrow[r, "(\pi_1,\pi_2,\pi_3)"'] & A^3
\end{tikzcd}
$$

commute, so that r is a split epimorphism, and hence an isomorphism. □

Lemma 5.2. *Let $\mathbb{C}$ be a finitely complete majority category with binary coproducts. If $\mathbb{C}$ and $\mathbb{C}^{\mathrm{op}}$ are majority categories, then every* hom-*set can be equipped with a commutative majority operation.*

Proof. Let A be any object of $\mathbb{C}$, then by Lemma 5.1 the morphism

$$
3A^3 \xrightarrow{M_A = \begin{pmatrix} \pi_1 & \pi_1 & \pi_3 \\ \pi_1 & \pi_2 & \pi_2 \\ \pi_3 & \pi_2 & \pi_3 \end{pmatrix}} A^3
$$

is an epimorphism. In particular, A^3 together with M_A is a ternary corelation on A^3 (a ternary relation in $\mathbb{C}^{\mathrm{op}}$). Composing M_A with each of the projections $\pi_i : A^3 \to A$, we have the following commutative diagrams:

$$
\begin{tikzcd}
3A^3 \arrow[r, two heads, "M_A"] \arrow[dr, "\begin{pmatrix}3\\ \pi_1\\ \pi_1\end{pmatrix}\pi_3"'] & A^3 \arrow[d, "\pi_1"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
3A^3 \arrow[r, two heads, "M_A"] \arrow[dr, "\begin{pmatrix}3\\ \pi_1\\ \pi_2\end{pmatrix}\pi_2"'] & A^3 \arrow[d, "\pi_2"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
3A^3 \arrow[r, two heads, "M_A"] \arrow[dr, "\begin{pmatrix}3\\ \pi_3\\ \pi_2\end{pmatrix}\pi_3"'] & A^3 \arrow[d, "\pi_3"] \\
 & A
\end{tikzcd}
$$

Since $\mathbb{C}$ is a comajority category, there exists a morphism $p_A : A^3 \to A$ making the diagram

$$
\begin{tikzcd}
3A^3 \arrow[r, two heads, "M_A"] \arrow[dr, "\begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\end{pmatrix}"'] & A^3 \arrow[d, dotted, "p_A"] \\
 & A
\end{tikzcd}
$$

commute. Thus we have constructed an internal majority operation p_A on A, for each object A in $\mathbb{C}$. Next, to see that every morphism in $\mathbb{C}$ is a homomorphism with respect to the internal majority operation constructed above, let $f : A \to B$ be any morphism in $\mathbb{C}$, then the commutativity of the diagram below follows from the commutativity of the top and outer rectangles, and the fact that M_A is an epimorphism.

$$\begin{tikzcd}
3A^3 \arrow[r, "3f^3"] \arrow[d, two heads, "M_A"] & 3B^3 \arrow[d, two heads, "M_B"] \\
A^3 \arrow[r, "f^3"] \arrow[d, "p_A"] & B^3 \arrow[d, "p_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

The commutativity of the bottom square is precisely the statement that f is a homomorphism with respect to the internal majority operations p_A and p_B. Therefore, for any objects S and A the composite

$$\hom(S, A)^3 \simeq \hom(S, A^3) \xrightarrow{\hom(S, p_A)} \hom(S, A),$$

is a commutative majority operation. $\square$

Lemma 5.3. *Let (X, p_X) be a commutative majority algebra, then X has at most one element.*

Proof. Let $x, y \in X$ be any two elements, then

$$\begin{aligned}
x &= p_X(x, x, y) \\
&= p_X(p_X(x, x, y), p_X(x, y, x), p_X(y, y, y)) \\
&= p_X(p_{X^3}((x, x, y), (x, y, y), (y, x, y))) \\
&= p_X(p_X(x, x, y), p_X(x, y, y), p_X(y, x, y)) \\
&= p_X(x, y, y) = y
\end{aligned}$$

$\square$

As an immediate corollary of Lemma 5.2 and Lemma 5.3, we have:

Theorem 5.4. *If $\mathbb{C}$ has finite limits and binary coproducts, and $\mathbb{C}$ and $\mathbb{C}^{\mathrm{op}}$ are majority categories, then $\mathbb{C}$ is a preorder.*

Remarks 5.5. It is possible to prove the theorem above under different limit and colimit assumptions, but it is impossible without at least some limit and colimit assumptions. This is because the category consisting of just two parallel arrows is both majority and comajority.

Remarks 5.6. The proof above depends on the fact that the morphisms in the statement of Lemma 5.1 are epimorphic. In a unital category, they are jointly strongly epimorphic. Therefore, by the proof above we may also conclude that a unital category $\mathbb{C}$ with binary coproducts such that $\mathbb{C}$ is comajority is equivalent to the terminal category **1**.

5.1 Future directions

As mentioned in the introduction, majority categories provide a categorical way to analyze and relate properties of the category of lattices, at various levels of generality. If we allow our base category to be regular, we can similarly study congruence distributivity categorically:

Definition 5.7. A CD(n) category $\mathbb{C}$, where $n \geqslant 1$, is a regular category such that for any reflexive relations R, S, T on an object X in $\mathbb{C}$, we have:

$$R \cap (S \circ T) \leqslant (R \cap S) \circ_n (R \cap T)$$

where $A \circ_0 B = A$ and $(A \circ_{n+1} B) = (A \circ_n B) \circ B$ if n is even, and $(A \circ_{n+1} B) = (A \circ_n B) \circ A$ if n is odd for any relations A, B on the same object.

Then a variety is CD(n) if and only if it admits n-Jonsson terms (see 1). CD(n) categories automatically satisfy the strong shifting principle (see 3.35) and are therefore factor permutable, which implies that the lattice of factor relations on any object in a CD(n) category is a Boolean algebra. If the base category of a CD(n) category is pre-exact, then we can deduce a similar unique factorization result as for majority categories (see Chapter 4).

In universal algebra, a k-ary near unanimity term is a k-ary term $p(x_1, ..., x_k)$ satisfying the equations

$$\begin{aligned}
p(x, x, ..., x, y) &= x, \\
p(x, x, ..., y, x) &= x, \\
&\vdots \\
p(y, x, ..., x, x) &= x.
\end{aligned}$$

Clearly, the above system of equations determines a matrix condition in the sense of [Jan06] (see Section 2).

$$M_k = \left(\begin{array}{cccccc|c} x_1 & x_1 & \cdots & x_1 & y_1 & & x_1 \\ x_2 & x_2 & \cdots & y_2 & x_2 & & x_2 \\ & \vdots & & \vdots & & & \vdots \\ y_k & x_k & \cdots & x_k & x_k & & x_k \end{array}\right)$$

Definition 5.8. Let $k \geqslant 3$, then a category $\mathbb{C}$ with M_k-closed relations is called a *k-unanimous* category.

Remarks 5.9. A majority category is nothing but a 3-unanimous category.

For the proof of the theorem below, we refer the reader to [Mit78].

Theorem 5.10. *If* $\mathbb{V}$ *is a variety which is k-unanimous, then* $\mathbb{V}$ *is congruence distributive.*

It is currently unknown, whether or not regular k-unanimous categories are CD(n), as is the case for varieties.

Corollary 5.11. *If* $\mathbb{V}$ *is a Mal'tsev variety which is k-unanimity, then* $\mathbb{V}$ *admits a majority term, which is to say that* $\mathbb{V}$ *is* 3*-unanimity.*

$$\textit{(Mal'tsev)} + \textit{(k-unanimity)} \implies \textit{(Majority)}.$$

Theorem 5.12. *Let* $\mathbb{C}$ *be a k-unanimous category, then every internal groupoid in* $\mathbb{C}$ *is an equivalence relation.*

We give a set-theoretic sketch of the proof of the above theorem.

Proof. Suppose that G is an internal groupoid in a k-unanimous category, with G_1 the object of arrows. Consider the k-ary relation R on G_1 defined by

$$(f_1, f_2, ..., f_k) \in R \quad \text{iff} \quad f_1 \circ f_2 \circ \cdots \circ f_{k-1} = f_k.$$

Now, suppose that f, g are any parallel arrows in G, then we have

$$\begin{aligned}
(fg^{-1}, 1, 1, ..., fg^{-1}) &\in R, \\
(1, fg^{-1}, 1, ..., fg^{-1}) &\in R, \\
(1, 1, fg^{-1}, ..., fg^{-1}) &\in R, \\
&\vdots \\
(1, 1, 1, ..., fg^{-1}, fg^{-1}) &\in R, \\
(1, 1, 1, ..., 1, 1) &\in R \implies \\
(1, 1, 1, ..., 1, fg^{-1}) &\in R.
\end{aligned}$$

So that

$$1 = 1 \circ 1 \circ \cdots \circ 1 = fg^{-1} \implies f = g.$$

□

Thus we have the following corollary due to Theorem 2.14:

Corollary 5.13. *If $\mathbb{C}$ is a Barr exact Mal'tsev category, then $\mathbb{C}$ is majority category if and only if $\mathbb{C}$ is k-unanimous for some $k \geqslant 3$.*

This above condition shows that there are some matrix conditions which can be derived from each other, provided that the base category is Barr exact. However, it is not true that the above condition holds when the base category is regular, as for example the category $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ is a regular Mal'tsev 4-unanimous category which is not a majority category.

Varieties that admit a majority term are the simplest possible congruence distributive varieties. And thus majority categories are the simplest possible categorical counterpart of congruence distributive varieties. However, it remains open whether we can develop a satisfactory categorical algebra of general congruence distributive varieties. The next simplest approach to doing this would be to study k-unanimous categories in general, as these are frequently occurring examples of congruence distributive varieties. In this direction, we can make the following conjecture:

Conjecture 1. *The only finitely complete and cocomplete categories where $\mathbb{C}$ and $\mathbb{C}^{\mathrm{op}}$ are k-majority, are the preorders, where $k \geqslant 3$.*

Even for varieties the question above is still interesting, and remains open.

Conjecture 2. *The only varieties $\mathbb{V}$ such that $\mathbb{V}$ and $\mathbb{V}^{\mathrm{op}}$ are k-majority, are trivial, i.e. , every algebra in $\mathbb{V}$ has at most one element.*

Varieties admitting a near-unanimity term also have a characterization in terms of congruences (see [Pix79]). This characterization could conceivably be extended to regular categories, where it might be possible to show that regular k-unanimous categories are factor permutable. With this in mind, it would be interesting to know whether or not exact k-unanimous categories have strong refinements or not. This question is straightforward for varieties, but not for regular k-unanimous categories, since there are regular Mal'tsev categories which are 4-unanimous, which are not even congruence distributive, such as the category $\mathrm{Mal}(\mathbf{Rel}_3^{\mathrm{op}})$ from Chapter 2.

List of References

[Abb67] J. C. Abbott. Semi-boolean algebra. *Matematički Vesnik*, 4:177–198, 1967.

[Ber12] C. Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Graduate Texts in Mathematics. Taylor and Francis, 2012.

[BG02] D. Bourn and M. Gran. Centrality and connectors in Maltsev categories. *Algebra Universalis*, 48:309–331, 2002.

[BG04] D. Bourn and M. Gran. Categorical aspects of modularity. *Fields Institute Communications*, 43:77–100, 2004.

[BGO71] M. Barr, P.A. Grillet, and D. H. van Osdol. Exact categories and categories of sheaves. *Springer, Lecture Notes in Mathematics*, 236, 1971.

[BJ08] D. Bourn and Z. Janelidze. Approximate Mal'tsev operations. *Theory and Applications of Categories*, 21:152–171, 2008.

[Bor94a] F. Borceux. *Handbook of Categorical Algebra*, volume 1. Cambridge University Press, 1994.

[Bor94b] F. Borceux. *Handbook of Categorical Algebra*, volume 2. Cambridge University Press, 1994.

[Bou96] D. Bourn. Mal'cev categories and fibration of pointed objects. *Applied Categorical Structures*, 4:307–327, 1996.

[Bou01] D. Bourn. A categorical genealogy for the congruence distributive property. *Theory and Applications of Categories*, 8:391–407, 2001.

[Bou02] D. Bourn. Intrinsic centrality and associated classifying properties. *Journal of Algebra*, 256:126–145, 2002.

[Bou05] D. Bourn. Congruence distributivity in Goursat and Mal'cev categories. *Applied Categorical Structures*, 13:101–111, 2005.

[BP75] K. A. Baker and A. F. Pixley. Polynomial interpolation and the Chinese remainder theorem for algebraic systems. *Mathematische Zeitschrift*, 143:165–174, 1975.

[BP95] M. Barr and M. C. Pedicchio. Top^{op} is a quasi-variety. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 36:3–10, 1995.

[CJT64] C. Chang, B. Jónsson, and A. Tarski. Refinement properties for relational structures. *Fundamenta Mathematicae*, 55(3):249–281, 1964.

[CLP91] A. Carboni, J. Lambek, and M. C. Pedicchio. Diagram chasing in Mal'cev categories. *Journal of Pure and Applied Algebra*, 69:271–284, 1991.

[CPP91] A. Carboni, M. C. Pedicchio, and N. Pirovano. Internal graphs and internal groupoids in Mal'cev categories. 13:97–109, 1991.

[FK72] P. Freyd and M. Kelly. Categories of continuous functors I. *Journal of Pure and Applied Algebra*, 2:169–191, 1972.

[Gra04] M. Gran. Applications of categorical Galois theory in universal algebra. *Fields Institute Communications*, 43:243–280, 2004.

[Gum83] H. P. Gumm. *Geometrical Methods in Congruence Modular Algebras*. Memoirs of the American Mathematical Society, 1983.

[Hoe18a] M. A. Hoefnagel. Characterizations of regular majority categories (in preparation). 2018.

[Hoe18b] M. A. Hoefnagel. Majority categories. *Theory and Applications of Categories (submitted)*, 2018.

[Jan04] Z. Janelidze. Generalized difunctionality, Pixley categories, and a general Bourn localization theorem. *67th Workshop on General Algebra*, 2004.

[Jan06] Z. Janelidze. Closedness properties of internal relations I: a unified approach to Mal'tsev, unital and subtractive categories. *Theory and Applications of Categories*, 16:236–261., 2006.

[Jan16] G. Janelidze. A history of selected topics in categorical algebra I: From Galois theory to abstract commutators and internal groupoids. *Categories and General Algebraic Structures with Applications*, 8:1–54, 2016.

[Jon67] B. Jonnson. Algebras whose congruence lattices are distributive. *Mathematica Scandinavica*, 21:110–121, 1967.

[JP97] G. Janelidze and M. C. Pedicchio. Internal categories and groupoids in congruence modular varieties. *Journal of Algebra*, 193:552–570, 1997.

[Mac98] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer New York, 1998.

[Mit78] A. Mitschke. Near unanimity identities and congruence distributivity in equational classes. *Algebra Universalis*, 8:29–38, 1978.

[MMT87] R. McKenzie, G. F. McNulty, and W. Taylor. *Algebras, lattices, varieties*. Number v. 1 in Wadsworth & Brooks/Cole mathematics series. Wadsworth & Brooks/Cole Advanced Books & Software, 1987.

[MW70] G. Michler and R. Wille. Die primitiven klassen arithmetischer ringe. *Mathematische Zeitschrift*, 113:369–372, 1970.

[Nem65] W. C. Nemitz. Implicative semi-lattices. *Transactions of the American Mathematical Society*, 117:128–142, 1965.

[Neu36] J. Neumann. On regular rings. *Proceedings of the National Academy of Sciences of the United States of America*, 22:707–713, 1936.

[Ped95] M. C. Pedicchio. A categorical approach to commutator theory. *Journal of Algebra*, 177:647–657, 1995.

[Ped96] M. C. Pedicchio. Arithmetical categories and commutator theory. *Applied Categorical Structures*, 4:297–305, 1996.

[Pix63] A. F. Pixley. Distributivity and permutability of congruence relations in equational classes of algebras. *Proceedings of the American Mathematical Society*, 14:105–109, 1963.

[Pix79] A. F. Pixley. Characterizations of arithmetical varieties. *Algebra Universalis*, 9:87–98, 1979.

[Str68] D. P. Strauss. Topological lattices. *Proceedings of the London Mathematical Society*, 18(2):217–230, 1968.

[Wei17] T. Weighill. Mal'tsev objects, r_1-spaces and ultrametric spaces. *Theory and Applications of Categories*, 32:1485–1500, 2017.