

**A STUDY OF THE DEVELOPMENT OF
MATHEMATICAL KNOWLEDGE IN A GEOGEBRA-
FOCUSED LEARNING ENVIRONMENT**

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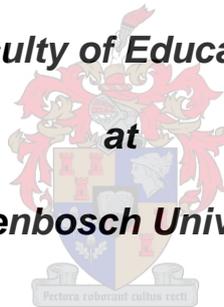
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Co-promoter: Dr J. H. Smit**

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DECLARATION

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ABSTRACT

This study was about a GeoGebra-focused learning environment in which students could develop mathematical knowledge. It was done during an intervention with a convenient sample of 48 Science and Mathematics bridging programme students at Stellenbosch University (SciMathUS).

Students in the SciMathUS year have to improve their Mathematics marks in order to qualify for admission into Science, Technology, Engineering and Mathematics (STEM) oriented programmes the following year at any tertiary institution in South Africa. The teacher-researcher felt that these students needed to be exposed to a hybridisation of the student-centred approach and teacher as facilitator of learning approach which had the potential to enhance conceptual and deeper understanding of Mathematics. GeoGebra was a pivotal teaching tool in the teacher-researcher's interactive lecturing. Students were afforded the opportunity to engage individually, or in groups, with the learning material on transformations of functions and circle geometry by making use of GeoGebra. The prepared tasks in the different learning trajectories used in this research were guided by a social constructivist view of learning, Realistic Mathematics Education (RME), the Van Hiele theory, the Theory of Instrumental Genesis (TIG) and the Technological Pedagogical and Content Knowledge (TPACK) framework for technology integration. A learning route for the students in the instruction of transformations of functions and circle geometry was established through the creation of learning trajectories intertwined with transformation geometry, straight line, triangle and quadrilateral geometry, respectively.

The methodology used was a mixed methods exploratory case study. The quantitative part of the study was a pretest-posttest experimental design. This study utilised pre- and post-intervention questionnaires, pre- and post-tests for transformations of functions and circle geometry, observations, and in-depth and focus group interviews. The results of the quantitative and qualitative data were triangulated, with a higher priority given to analysis of the qualitative data to answer the research questions.

The results from both quantitative and qualitative data showed that the pedagogical processes involved in using GeoGebra were more than just technology. They showed that real-life activities, guidance, exploration and interaction were important RME principles to be borne in mind when using GeoGebra. The results also underlined the fact that students had certain preferences on how GeoGebra could be used as a pedagogical tool.

The qualitative and quantitative results also revealed that GeoGebra afforded students an opportunity to better understand transformations of functions, circle geometry and general solutions of trigonometric equations. The qualitative content analysis of pre- and post-test for both transformations of functions and circle geometry showed that students moved to higher levels of abstraction. Students attributed this to the instrumentation processes of GeoGebra, i.e. *visual affordance*, *potentialities* and *enablements*.

Moreover, the results revealed that exploring with GeoGebra made the teaching and learning of mathematics more fun. The students utilised hand movements to articulate their ideas with a classmate, or the teacher-researcher. During *Guide-and-explain* orchestrations students explained to their group members, or teacher-researcher, how they understood the intended mathematics from the activities and in this way used informal reasoning (horizontal mathematizing) and then moved within the discussions to formal reasoning (vertical mathematizing). The results also showed that GeoGebra afforded the students an opportunity to acquire physical and logico-mathematical knowledge.

Challenges observed whilst the students were working with GeoGebra in the computer lab and based on students' responses in the interviews, were constraints posed by syntax and menu commands. A small percentage of students found the different teaching and learning approach to be challenging at times. A few others could not see the intended mathematics from activities with GeoGebra, or felt that it further confused them, or that this approach required more thinking.

Consequently it is recommended that GeoGebra should be part of a teacher's arsenal to teach mathematical concepts that lend themselves to technology integration.

SAMEVATTING

Hierdie studie handel oor 'n GeoGebra-gefokusde leeromgewing omgewing waarin studente wiskundige kennis kan ontwikkel. Dit is deur middel van 'n intervensie met 'n gerieflike steekproef van 48 Wetenskap- en Wiskunde-studente in die Universiteit van Stellenbosch se oorbruggingsprogram (SciMathUS) uitgevoer.

Studente in die SciMathUS-jaar moet hulle Wiskunde-punte verbeter om die volgende jaar toelating te verkry tot Wetenskap-, Tegnologie-, Ingenieurswese- en Wiskunde georiënteerde programme (STEM) aan enige tersiêre instelling in Suid-Afrika. Die onderwyser-navorsers is van mening dat hierdie studente aan 'n hibriede van student-gesentreerde leer en leer waarin die onderwyser as fasiliteerder optree, blootgestel moet word. Dit het die potensiaal om konseptuele en dieper begrip van wiskunde te bevorder. GeoGebra is 'n belangrike tegnologiese onderrighulpmiddel in die onderwyser-navorsers se interaktiewe lesaanbiedingbenadering. Studente is 'n geleentheid gebied om met leermateriaal oor transformasies van funksies en sirkel-meetkunde met behulp van GeoGebra individueel, of in groepe, te werk. Die voorbereide take in die onderskeie leertrajekte wat in hierdie navorsing gebruik is, is begelei deur 'n sosiaal-konstruktivistiese benadering van leer, Realistiese Wiskunde-Onderwys (RWO), die Van Hiele-teorie, die Teorie van Instrumentele Genesis (TIG) en die Tegnologiese, Pedagogiese en Inhoudskennis (TPACK) raamwerk vir die integrasie van tegnologie. Die transformasies van funksies en sirkel-meetkunde se onderrig is op 'n leerroete met behulp van 'n verskeidenheid van leertrajekte geplaas en is met transformasiemeetkunde reguitlyn-, driehoek- en veelhoekmeetkunde, verweef.

Die metodologie was 'n gemengde metode verkennende gevallestudie. Die kwantitatiewe deel van die studie was 'n voortoets-natoets eksperimentele ontwerp. Hierdie studie het voor- en na-intervensie vraelyste, voor- en na-toetse vir transformasies van funksies en sirkelmeetkunde, waarnemings, en in-diepte en fokusgroep-onderhoude gebruik. Die resultate van die kwantitatiewe en kwalitatiewe data is getrianguleer om die navorsingsvrae te beantwoord. Hoër prioriteit is aan die ontleding van die kwalitatiewe data gegee.

Die resultate van beide kwantitatiewe en kwalitatiewe data het getoon dat die pedagogiese prosesse verbonde aan die gebruik van GeoGebra meer as net tegnologies van aard was. Dit het getoon dat kontekstuele aktiwiteite, begeleiding, ontdekking en interaksie belangrike RWO-beginsels is wat in gedagte gehou moet word wanneer GeoGebra gebruik word. Die

resultate het voorts die feit benadruk dat studente sekere voorkeure gehad het oor hoe GeoGebra as 'n pedagogiese instrument gebruik moet word.

Die kwantitatiewe en kwalitatiewe resultate het vervolgens daarop gedui dat GeoGebra studente 'n geleentheid gebied het om beide transformasies van funksies, sirkelmeetkunde en algemene oplossings van trigonometriese vergelykings beter te verstaan. Die kwalitatiewe ontleding van inhoudskennis in die voor- en na-toetse vir beide transformasies van funksies en sirkelmeetkunde het bewys dat studente na hoër vlakke van abstraksie beweeg het. Studente skryf dit toe aan die instrumentasie-prosesse van GeoGebra, onder meer visuele voorstelling, potensialiteite en instaatstelling.

Uit die resultate het dit ook geblyk dat die ondersoek met GeoGebra die onderrig en leer van wiskunde meer pret maak. Studente gebruik handbewegings om hulle idees aan 'n klasmaat, of die onderwyser-navorsers, oor te dra. Tydens die begelei- en verduidelik-orkestrasie het die studente aan hulle groeplede, of die onderwyser-navorsers, verduidelik hoe hulle die wiskunde van die aktiwiteite verstaan het en hoe hulle informele beredenering (horisontale matematisering) gebruik het om tydens die gesprekke deur te beweeg tot formele redenasie (vertikale matematisering). Die resultate het getoon dat GeoGebra die studente die geleentheid gebied het om fisiese en logies-wiskundige kennis te verkry.

Studente wat met GeoGebra in die rekenaarlokaal besig was, het uitdagings ervaar met sintaksis en menu-opdragbeperkings. Studente se terugvoering het hierdie aspek bevestig. 'n Klein persentasie studente het die gebruik van verskillende onderrig- en leerbenaderings soms uitdagend gevind. 'n Paar ander kon nie die beoogde wiskunde in die aktiwiteite met GeoGebra raaksien nie en het gevoel dat dit hulle meer verward maak. Soms het hulle besef dat die benadering meer dinkwerk vereis.

Derhalwe word aanbeveel dat GeoGebra deel moet uitmaak van 'n onderwyser se arsenaal van hulpmiddels wat hom/haar spesifiek leen tot die gebruikmaking van tegnologie vir die onderrig van wiskundige konsepte.

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ACRONYMS

ANA	Annually National Assessment
ANOVA	Analysis of Variance
CAPS	Curriculum Assessment Policy Statement
CAQDAS	Computer Aided Qualitative Data Analysis Software
DoE	Department of Education of South Africa (before 2009)
DBE	Department of Basic Education (from 2009)
HLT	Hypothetical Learning Trajectory
IEA	International Association for the Evaluation of Educational Achievement
NSC	National Senior Certificate
RME	Realistic Mathematics Education
SACMEQ	Southern and Eastern Africa Consortium for Monitoring Education Quality
SciMathUS	Science and Mathematics at Stellenbosch University (bridging programme)
SPSS	Statistical Package for the Social Sciences
STEM	Science, Technology, Engineering and Mathematics
SU	Stellenbosch University
TIG	Theory of Instrumental Genesis
TIMSS	Trends in International Mathematics and Science Study
TPACK	Technological, Pedagogical and Content Knowledge
WCED	Western Cape Education Department

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CHAPTER 1

TECHNOLOGY IS PART OF THE PACKAGE: A NEED TO CHANGE OUR WAY OF TEACHING

1.1 INTRODUCTION: TECHNOLOGY AND MATHEMATICS

The initial motivation to conduct this research was driven by the researcher's curiosity to understand why algorithms in mathematics are working in a certain way, as well as his passion for teaching mathematics and using technology as a strengthening tool in teaching mathematics. During a visit in 1999 to the University of Leuven in Belgium and the University of Fontys in Netherlands, the researcher experienced using graphic calculators and Cabri dynamic mathematics software for the first time. The following year, the researcher started using a graphic calculator to teach Mathematics to Grade 8 – 12 students.

According to Kelman *et al.* (1983), computers have been a part of the teaching of Mathematics since the 1960s. Yanik and Porter (2009) state that the use of technology “has increased incredibly and gained recognition and acceptance as an instructional tool in school mathematics both at elementary level and at the secondary level” (p. 3). This increase in use also brought drastic changes to the type of technologies used in education in South Africa. Overhead projection technology has changed from slide projectors or overhead projectors (which are still in use) to data projectors, video machines and digital video disc (DVD) players. The internet also gives students and teachers online learning material. The use of technology generally has been an integral part of the teaching of Mathematics for decades even in South Africa, but not in every classroom. However, initially it has been for a minority of well-resourced elite schools. Today more teachers and students have access to computers, interactive whiteboards, and the internet, and more educational software packages have become available. Technology is here to stay and teachers need to make effective use of technology in the teaching of Mathematics.

The use of different teaching strategies and resources are synonymous with good teaching, especially in the teaching of Mathematics. “Blended learning, the mixing of different teaching methods and resources, is nothing new” (McCabe, 2007, p. 55). Students are growing up with technology and teachers have to expose them to the use of technology in the teaching and learning of Mathematics and even other subjects. This study utilised technology with an

active learning approach and the traditional approach of teaching. GeoGebra, dynamic mathematics software such as Cabri and Geometer's Sketchpad, was used as an ingredient in both the traditional and active teaching and learning approach. GeoGebra was used in a traditional approach as a teaching tool in the class by the teacher-researcher and also in an interactive approach by the students as a learning tool where they explore mathematics with it.

1.2 BACKGROUND TO THE STUDY

1.2.1 An overview of ICT in schools in South Africa

The Department of Education of South Africa (DoE, 2004) introduced Information and Communication Technologies (ICT) to improve the quality of teaching and learning across the curriculum in the education and training system. In a report of 2009 the Department of Education (DBE, 2011d) furnished the following statistics on public ordinary schools' access to computers.

Table 1.1 : Availability of computer labs in primary and high schools (DBE, 2011d))

Province	Number of schools	With Computer Centre	% With Computer Centre	Without Computer Centre	% Without Computer Centre
Eastern Cape	5715	596	10	5119	90
Free State	1643	353	21	1290	79
Gauteng	1994	1510	76	484	24
KwaZulu Natal	5835	982	17	4853	83
Limpopo	3918	428	11	3490	89
Mpumalanga	1540	254	16	1286	84
North West	1740	391	22	1349	78
Northern Cape	609	314	52	295	48
Western Cape	1466	886	60	580	40
TOTAL	24 460	5 714	23	18 746	77

According to the Department of Basic Education (DBE, 2011d), the Western Cape and Gauteng Provinces were far ahead in 2009 (see Table 1.1) compared to other provinces when it came to having computers in schools for teaching and learning. The number of well-equipped computer centres in the Western Cape and Gauteng are reflective of the success

of computer lab deployment in the provinces from the Khanya and Gauteng online projects respectively (Butcher & Associates, 2011). In 2001, the Western Cape Education Department (WCED) started to provide schools with technology and had invested around R530 million on the Khanya project by the end of 2010 (Khanya, 2011). Their goal was to ensure that every teacher in every school of the Western Cape would be empowered to use appropriate and available technology to deliver the curriculum to every student in the province (Khanya, 2011) by the start of the 2012 academic year. According to the WCED (2011), by March 2011 the 1 329th school was supplied with the abovementioned technology and 195 schools were to follow to reach their ambitious goal to provide all schools in the Western Cape with technology. The statistics tell us that by March 2011 Khanya had helped and provided 87% of the schools in the Western Cape with computer labs, interactive whiteboards, etc. to enhance the curriculum.

According to the premier of the Western Cape at the time, Helen Zille (Fredericks, 2013), every school in the province already had a computer laboratory by 2013, but the wish was “to take this much further in the next five years”. The WCED wants to give each student in the province further access to tablets so that “[they] will be able to download their subject content and work out where they are supposed to be in the curriculum at a particular point in the year” (Fredericks, 2013). Zille (Fredericks, 2013) adds that “[t]echnology has become a very important component in education. It’s hardly possible to think of a child getting a decent school education today without technology”. Educational departments are therefore recognising the importance of technology and want technology to be integrated into teaching. Some of the high schools are also provided with educational software packages for Mathematics such as Master Maths, CAMI, Evalunet, MathsPro, Geometer’s Sketchpad, etc. Many of these software packages are quite expensive and some of them require an annual licence fee. Freeware and shareware, such as GeoGebra dynamic software, can address this problem and can be downloaded from the internet free of charge.

Although some schools do have access to technology they do not use it optimally. Mdlongwa (2012) argues that the failure of schools in South Africa to use ICT as a means of enhancing teaching and learning has led to South Africa failing to close the ‘digital divide’. The digital divide is defined as the gap between those individuals who benefit from digital technology and those who do not. The use of ICT in schools to enhance learning could help overcome some of the challenges in trying to improve the efficiency and productivity of both learning

and teaching in South African schools, thereby narrowing the digital divide (Mdlongwa, 2012). The general state of ICT readiness in South Africa remains very low according to the Global Information Technology Report (Dutta, Geiger, & Lanvin, 2015). According to this report, South Africa was 102nd out of 143 countries in terms of ICT readiness. The report indicates that the potential of ICTs has not been fully unlocked in South Africa. This is not only a problem in South Africa but also in the USA. Hohenwarter and Lavicza (2010, p. 132) state that in Florida in the USA, “[t]oday many teachers and students have access to computers and although appropriate software is available both in schools and at home, it is rarely integrated substantially into everyday teaching”. Not as many teachers and students in South Africa have access to computers compared to the USA. Nkula and Krauss (2014) allude to the fact that many of the schools in South Africa that do have access to different ICT tools tend to use them in a limited manner. Instead, they focus on learning about computers or acquiring ICT hardware skills. According to Nkula and Krauss (2014), ICTs are often implemented without integration as opposed to implementation with integration, where students use ICTs to learn and where ICTs are an integral part of teaching and learning practices. The use of ICT has to be an integral part of teaching and in the case of Mathematics teaching it has to be a tool to help teachers make Mathematics fun and more attractive. It can be done by affording students the opportunity to explore Mathematics on a computer.

However, Yanik and Porter (2009) argue in their research that teachers need people who can help them design activities and show them new ways to use the computer to teach Mathematics, because it is time consuming and not all teachers are computer literate and creative enough. Certain software packages have prepared activities and although they focus on the curriculum they sometimes do not suit the needs of the teachers. Teachers can create their own activities in software such as GeoGebra and Geometer’s Sketchpad and also afford students an opportunity to create their own activities to explore. The researcher in this study was of the opinion that, by creating their own activities, teachers will gain an understanding of the activity and why it is developed that way. This study used the teacher-researcher’s instructional design activities that were given to students. Although the participants in this study were not teachers but students, the teacher-researcher in this study wished to map out some of the knowledge and skills that Mathematics teachers need for effective Mathematics teaching with technology. The researcher used the requisite knowledge and skills to explore the potential of integrating GeoGebra in Mathematics

teaching and learning with the participants in this study. The responses of the participants were the focus of this action research study.

1.2.2 Current situation in the teaching of Mathematics in South Africa

1.2.2.1 Purposes and aims of the Mathematics curriculum

The National Curriculum Statement (NCS) for Mathematics Grade 10 – 12 describes Mathematics as “the human activity practised by all cultures” (DoE, 2003). The DBE started to implement the Curriculum and Assessment Policy Statement (CAPS) for Grade 10 – 12 in 2012. The CAPS started with Grade 10 in 2012 and was implemented in Grade 12 in 2014. The DBE (2011a) replaced the NCS of 2003 with CAPS. One of the specific aims set out by the CAPS document was that mathematical modelling is an important focal point of the curriculum. Real life problems should also be incorporated into all sections whenever appropriate. Students should be provided with the opportunity to develop the ability to be methodical, to generalise, make conjectures and try to justify or prove them. Students should also develop problem-solving and cognitive skills. Teaching should not be limited to ‘how’ but should rather feature the ‘when’ and ‘why’ of problem types.

The question that can be raised is: Are we achieving these goals?

1.2.2.2 Is South Africa there yet?

South African mathematics students are underperforming compared to their counterparts at international level. Firstly the Trends in International Mathematics and Science Study (TIMSS) suggest that South Africa was one of three countries that wrote the TIMSS Grade 8 assessments with their Grade 9 students in 2011 (Mullis, Martin, Foy, & Arora, 2012). Countries where students were expected to find the TIMSS assessments too difficult for their fourth or eighth grade students were given the option to assess students at a higher grade (Mullis *et al.*, 2012). The report reflects that countries assessing their eighth grade students had an average achievement of between 338 and 397 which was still below the Low International Benchmark of 400 for eighth grade students. The report also revealed that the ninth grade students of South Africa scored an average of 352 (Mullis *et al.*, 2012).

TIMSS has the goal of helping countries to make informed decisions about how to improve their teaching and learning in Mathematics and Science. Since 1995, this cross-national assessment is conducted by the International Association for the Evaluation of Educational Achievement (IEA) in South Africa. These assessments are repeated every four years with

Grade 4 and 8 students. The Low International Benchmark (400) is the level of achievement to compare most of the weaker performing countries at Grade 4 and 8. The Grade 9 students of South Africa could not outscore Grade 8 students of other countries. South Africa's Grade 9 students in 2011 were in the 42nd place out of 45 countries (Mullis *et al.*, 2012). We are globally linked and it is important to compare ourselves with other countries.

Secondly, the World Economic Forum (Schwab, 2015) for 2015 – 2016 also conducted a survey to assess the quality of Mathematics and Science for higher education and training amongst 140 countries in the world. South Africa was ranked last in this survey. The survey of the World Economic Forum and the results from TIMSS are an indication that we are not faring well compared to other countries.

Thirdly, the National Senior Certificate (NSC) Mathematics results in South Africa are not good. Figure 1.1 shows that only 46.3% of South African students who wrote the NSC Mathematics examination achieved 30% and above, and 30.1% of these students achieved 40% and higher in NSC Mathematics in 2011.

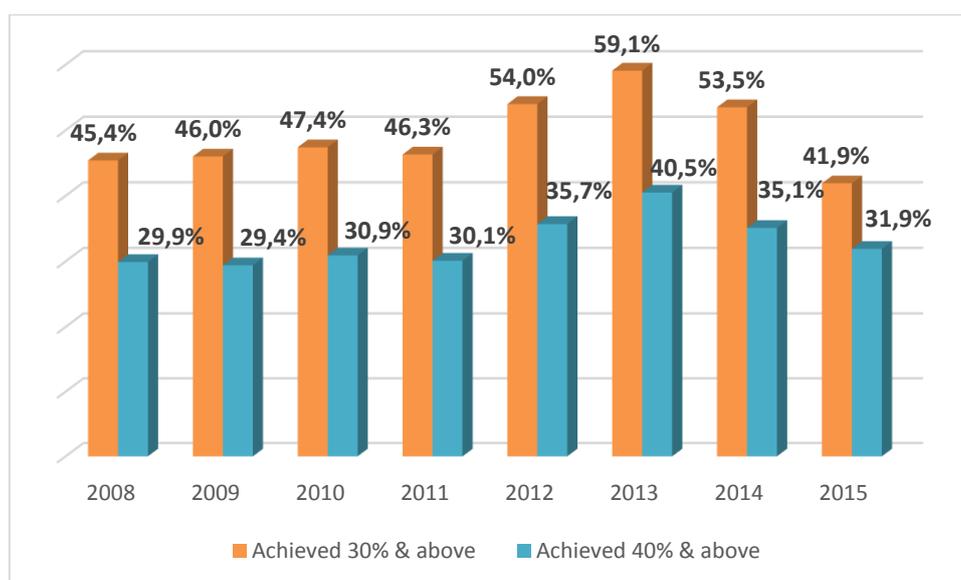


Figure 1.1: Students' performances in NSC Mathematics examinations (2008 – 2015)

Figure 1.1 also reflects that the results in 2012 improved with 54% of students achieving 30% and above, and 35.7% achieving 40% and higher. The results improved further in 2013 and 2014 with more students achieving 40% and higher. Only 41.9% of the South African students who wrote the NSC Mathematics examination achieved 30% and higher, and 31.9% of these students achieved 40% and higher in 2015, but the number of students doing pure Mathematics has been decreasing.

Figure 1.2 shows that only 13.4% of the students who wrote Mathematics achieved more than 60%, 15.6% in 2012; 13.4% in 2014 and 12% in 2015.

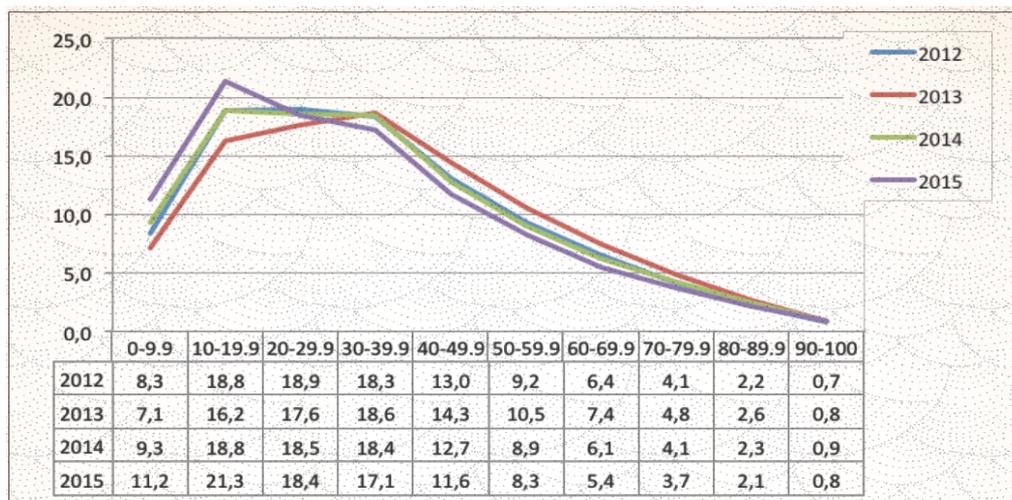


Figure 1. 2: Performance distribution curves in Mathematics (DBE, 2015)

Figure 1.2 shows in what a dismal situation students are if they wish to study in a field at a tertiary institution where a Mathematics mark of 60% is a requirement. Nationally an average of 13.6% of the students achieved above 60% for the past four years. Bernstein (2013) reports that only 12% of students in South Africa will qualify to gain access to university.

1.2.2.3 Shortcomings in the teaching of functions

DBE (2011b) suggests that teachers should note that the need for students to know the effect of the various parameters in the equation of the parabola is clearly stipulated in the Subject Assessment Guidelines document. The report also emphasises that in teaching the function concept, students should first know the basic curve and then be afforded the opportunity, via worksheets, to investigate the effects of changing the values of the various parameters. DBE (2011b) identified a few basics and foundational competencies in Paper 2. One of those was that the students understood that the vertical shift of a trigonometric function will impact the minimum value of the function. The DBE (2011b) requires that trigonometric functions and graphs should be taught in a way that leads to an understanding of the effect of the different parameters. Students must also have an understanding of transformation when doing questions on the effect of the different parameters. The report also points out that it is important that students are familiar with the characteristics such as shape, amplitude and period, as well as asymptotes where appropriate. Moreover, the report recommends that students must be afforded the opportunity to sketch the graphs by making use of the characteristics. Students can easily investigate the characteristics of functions

with GeoGebra because they can create families of functions and then investigate the characteristics from the different graphs.

Students therefore tend not to struggle with the sketching of graphs when given the equation of the function but experience problems with the sketching of graphs from the characteristics of the function, the interpretation of the graphs and transformations of functions. All three questions on functions and graphs in the Mathematics National Senior Certificate paper 1 in 2009 were interpretation of graphs and transformations of functions. In Figure 1.3 below the analysis of the Western Cape Mathematics results (Sasman, 2011) shows that the students scored an average of 18.3% in 2009 for the three questions on functions and graphs. In 2010 one question on functions and graphs required the students to sketch the exponential graphs and interpret them. They scored an average of 44.6% for that question. The other two questions in 2010 were again interpretation of graphs and the students scored an average of 33.8% in those two questions. The results show that Western Cape students therefore struggle with the higher order level questions in the question papers and this prevents students from scoring higher marks in Mathematics. Interventions that expose students to topics and questions that require higher order level thinking must therefore be researched. Part of this study focuses on functions and their transformations.

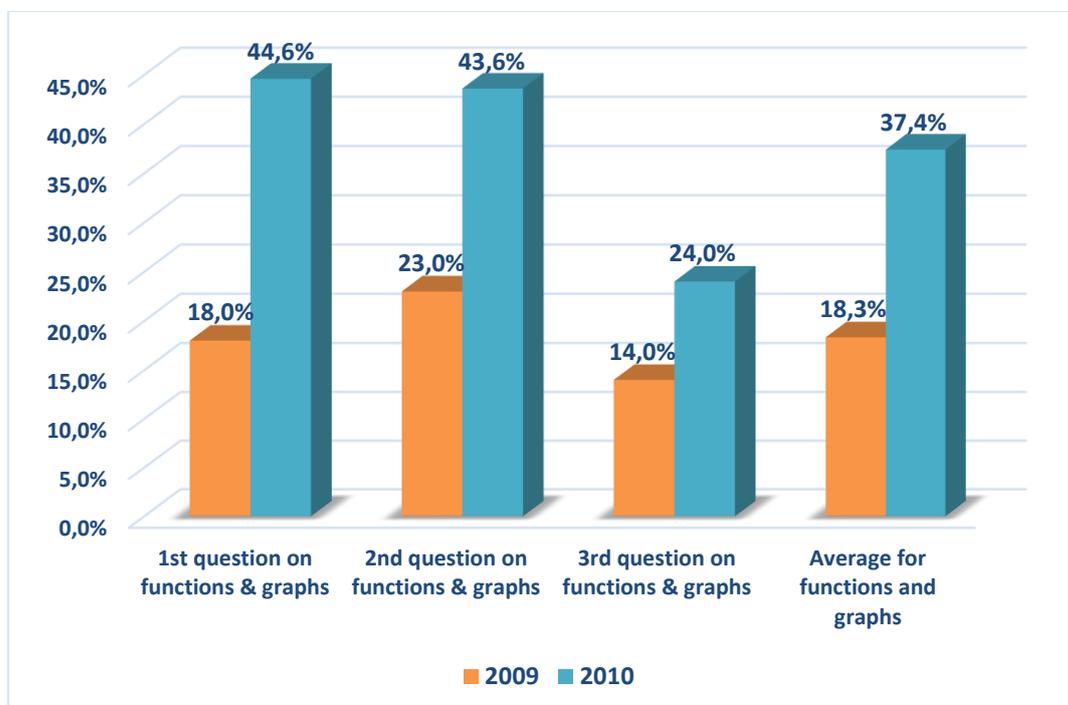


Figure 1.3: Western Cape average % obtained in each questions on functions and graphs in 2009 & 2010 NSC Mathematics (Sasman, 2011, p. 6)

Sasman (2011) also reports that many candidates in the Western Cape did not have a clear understanding of the characteristics of various families of functions and were unable to sketch graphs. She also notes that the candidates lack an understanding of the behaviour of functions. She adds that the notation embedded in functions and the transformations of functions were poorly understood in 2008 and 2009, but there was an improvement in these aspects in 2010.

The average percentage for questions on functions and graphs within all provinces from 2012 to 2013 (see Figure 1.4 below) is not high. The average for these questions is 44% from 2012 to 2013. The first question on functions and graphs in Mathematics National Senior Certificate paper 1 in 2012 entailed sketching a graph (35% of the question) and interpretation of graphs and transformations of functions (65% of the question). The second question in this paper was an illustration of the graph of the inverse of the parabola and the students had to find the equation and interpret the graphs. The third question in the paper asked the students to sketch the hyperbola from its properties. The respective national aggregate scores in these three questions were 46.3%; 34.1% and 44.6% with an average of 41.6% across the three questions.

In Mathematics National Senior Certificate paper 1 in 2013 the first question on functions and graphs required the students to sketch the parabola and they scored an average of 44.6% for that question. The other two questions in 2013 were the interpretation of exponential and hyperbolic graphs. The candidates attained an average of 31.2% and 40.1% respectively. They scored an average of 46.4% for the three questions on functions in 2013. The dynamic nature of GeoGebra lends itself to explore questions that require interpretation. Questions for example: For which x -values is $f(x) > 0$ or $g^{-1}(x) \leq 0$ can be easily explained with GeoGebra. This research focuses on how the students experienced this approach of teaching and learning.

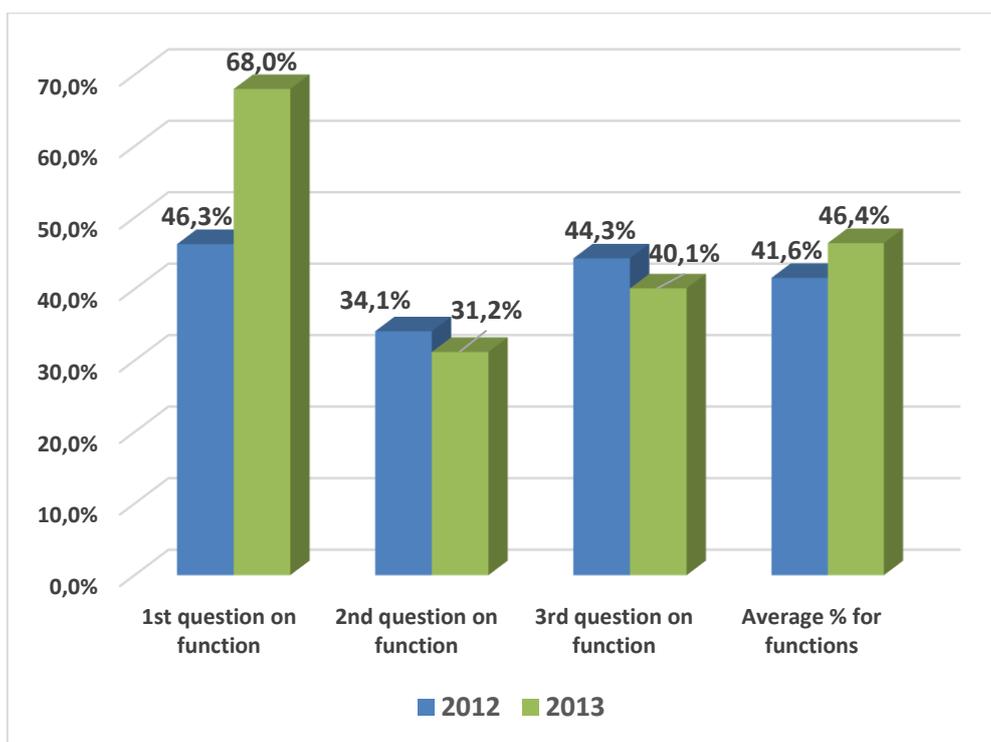


Figure 1.4: National average % obtained in each question on functions in 2012 – 2013
NSC Mathematics DBE (2012, 2013)

No analysis per question was provided in the National diagnostic report for 2011 but the report does indicate that questions on functions were not answered very well. As a basis for their argument, DBE (2011b) stated that teaching does not stop at sketching the curve, but with questions which require using the graph to deduce properties of the function when teaching functions. The average scores in functions as presented in Figures 1.3 and 1.4 affirm that Grade 12 students in South Africa are not performing well in questions on functions and graphs. When students are asked to sketch the graphs they tend to score a higher mark but struggle to score above 60% when given the graph and asked to interpret it.

The results above reflect that the students did not perform as well in the questions where they were required to interpret graphs. It is therefore imperative that research is done to find possible interventions to address the issues raised by the DBE reports. Wessels (2009, p. 320) affirms that “[t]he teaching methodology that will make a difference in the classroom falls in the broad category of problem solving. The day-to-day teaching method should be the problem-centred teaching and learning approach”. This implies that students should be exposed to teaching methodologies that enhance deeper understanding so that they are able to answer questions that contain complex procedures and are of a problem solving

nature. The problem-centred approach is a constructivist approach to teaching. Constructivist approaches support critical thinking and can help students to develop deeper understanding.

GeoGebra can also be used to explore the effect of parameters of algebraic and trigonometric functions. Students can also explore with GeoGebra and familiarise themselves with the characteristics of functions. The researcher is of the opinion that the integration of GeoGebra into the teaching of functions and graphs can enhance candidates' understanding of this problematic mathematical content. Petre (2010) emphasises that the use of the software will not be as helpful if it is not combined with teaching techniques, with student-centred teaching methods, with active learning methods, such as solving problems creatively, critical thinking, learning through discovery, through practice, and learning through experiments. The researcher uses a Hypothetical Learning Trajectory (HLT) that involves active learning with GeoGebra for the teaching of transformations of functions in this study.

1.2.2.4 Shortcomings in the teaching of circle geometry

Euclidean Geometry was also examined between 2008 and 2013 in the NSC Examination in paper 3. This paper was optional and around 4% (DBE, 2011b; 2014) of the students who took Mathematics from 2008 to 2013 wrote it. Euclidean Geometry again became compulsory in 2014 in the NSC Examination in paper 2. The teaching of Euclidean Geometry in the NSC after seven years of absence as a compulsory section in the curriculum can become a challenge for many teachers. Euclidean Geometry has to be taught in such a way to foster development from one level to the next through sequences of activities, beginning with an exploratory phase, gradually building concepts and related language, and culminating summary activities that help students integrate what they have learned into what they already know (Olkun, Sinoplu, & Deryakulu, 2005). According to Olkun *et al.*, (2005) school geometry is presented in an axiomatic fashion, assuming that students think on a formal deductive level. Geometry lessons have to be designed so that they are accessible to all students, allowing them to work at their own level of development. Designing lessons within a constructivist approach can therefore be helpful. The use of Geogebra can also enhance student learning with circle geometry in a constructivist way.

According to Malati (1999b) many high school teachers can attest to the difficulties they have teaching Euclidean Geometry and that

“[d]espite the best efforts of teachers, students continue to have difficulty with deduction and proof. It appears, firstly, that in trying to cope with this form of geometry, students and teachers turn it into something algorithmic: Many students simply memorise proofs or rules” (Malati, 1999b).

According to the Van Hiele theory (1986), a student has to be on the ordering Van Hiele level to cope meaningfully with an axiomatic system. Research in South Africa (De Villiers & Njisane, 1987) and elsewhere (Senk, 1989; Usiskin, 1982; Shaughnessy & Burger, 1985) has shown that many school students are only on the Van Hiele visual or analysis levels. Ndlovu, Wessels and De Villiers (2013b) and, Ndlovu and Mji (2012) also observed similarly low Van Hiele levels of attainment in pre-service and in-service teachers’ understanding of geometry.

The National diagnostic report of the DBE (2013) indicates that some students do not understand or know theorems and their applications. The report mentions that a common misconception is that students treat any quadrilateral as a cyclic quadrilateral. The students also made certain assumptions that were not proven. GeoGebra can be used to visualise why not any quadrilateral can be a cyclic quadrilateral and why there are conditions for theorems to be true. Students can also use GeoGebra to make conjectures and is a good platform for experimentation, which supports the development of mathematical concepts and the abilities to explain geometric properties (Chan, 2013).

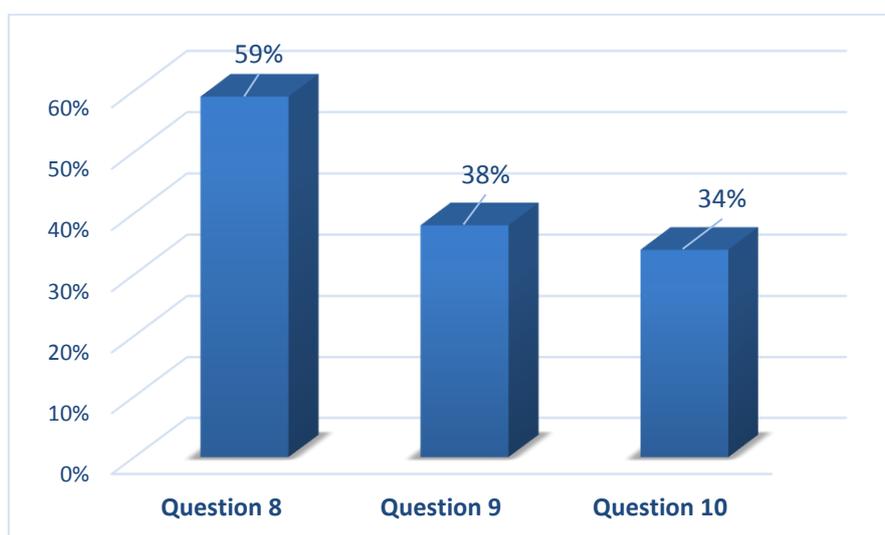


Figure 1.5: All the provinces average % obtained in each questions on Euclidean Geometry in 2014 in Mathematics DBE (2014)

The diagnostic report of the DBE (2014) also mentions that candidates of 2014 performed reasonably well in the lower order questions in Euclidean Geometry. The DBE (2014) points

out that a number of candidates lacked the necessary insight to deal with interpretative questions and complex questions in Euclidean Geometry. Figure 1.5 shows that the students scored only an average of 59% for Question eight in Mathematics National Senior Certificate paper 2 in 2014. Question eight covered circle geometry and it assessed knowledge and routine type of questions. Question nine assessed the triangle proportional theorem and applications of it and Question ten applications of the similarity theorem combined with circle geometry. Questions nine and ten were poorly answered and students scored an average of 38% and 34% respectively.

Interventions can be done with students to expose them to the different levels of Van Hiele or re-teach certain topics that students are struggling with. The Van Hiele theory of geometric thought describes the different levels of understanding through which students' progress when learning geometry (Van Hiele 1984). As students see, touch, and manipulate shapes, they begin to develop spatial reasoning skills (Howse & Howse, 2014). Education departments have revision intervention programmes and if students do not work on all the Van Hiele levels, for example on circle geometry, then the revision sessions are not going to serve any purpose. Circle geometry was new for most of the SciMathUS (Science and Mathematics at Stellenbosch University) students in 2014 and the researcher used a Hypothetical Learning Trajectory (HLT) that involved active learning with GeoGebra for teaching of circle geometry in this study.

1.3 STATEMENT OF THE PROBLEM

The researcher has been involved with FET Mathematics teaching for more than 25 years and started teaching Mathematics at SciMathUS in 2013. Through interaction with students in the SciMathUS programme, the researcher also experienced that their perceptions were that mathematics should be memorised. The students' perceptions are that they should be given notes and a few examples so that they can memorise the content. The researcher in this study experienced that students could not explain why they are doing certain algorithms in a specific way. Students' responses in explaining their methodology are: "My teacher told me to do it in this way" or "I did it like that last year".

If students enter the SciMathUS programme with 45% for Mathematics they need to improve their mark by at least 15 percentage points if they wish to study at Stellenbosch University. For students to achieve the required mark in Mathematics they need to start performing better, especially in topics such as functions, circle geometry and similar questions which

require deeper understanding. The DBE (2009) indicates that all the National Senior Certificate Mathematics examination papers include questions across four cognitive levels: Knowledge (25%); Routine procedures (30%); Complex procedures (30%) and Solving problems (15%). This implies that students should be exposed to complex procedures and problem solving questions if they are to achieve the minimum requirement of 60% and above for Mathematics.

The DBE (2011b; 2012a; 2013; 2014) reports that many students experienced difficulties with concepts in the curriculum that required deeper conceptual understanding in the two NCS Mathematics question papers. The researcher decided to focus his study on algebraic functions (23% of paper 1), trigonometric functions (6% of paper 2) and Euclidean geometry (33% of paper 2). Students must not only improve their marks in these topics to achieve 60% and more in Mathematics, but also in all the other topics in the curriculum. The mentioned topics are also identified by DBE (2011b; 2012a; 2013; 2014) as those needing additional attention. The topics identified require sketching or drawing and also visualisation. The researcher felt that GeoGebra has the characteristics of visualisation and, therefore, decided to use it in this study to investigate how the students' experienced their explanations whilst being taught with GeoGebra.

To complicate matters, the teaching of Mathematics in most schools in South Africa is often done through traditional instruction where students are positioned as passive recipients of knowledge (Van der Walt & Maree, 2007). Wessels (2009) also points to South African students' inability to solve problems in Mathematics. Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier and Human (1997) and, Wessels (2009) contend Mathematics is taught in South Africa apart from the reality and students see it as simply computation and manipulation. Barnes and Venter (2008) say that is the reason why South African students struggle far more in the TIMSS than the rest of the world when required to perform mathematics within a context.

Gravemeijer (2004) refers to teaching by telling and viewing students as passive recipients of knowledge as the transmission model of teaching. According to Freudenthal (1991, p. 48) "only a minority learn Mathematics in this way". If only the minority are learning through teaching Mathematics by telling, then this way of teaching must be changed. Lokan *et al.* (2003) argue that it is common for Mathematics teachers, especially from middle primary years onwards, to demonstrate specific procedures to their students, supplemented by

repetitious practice of similarly constructed examples, the intention of which is to develop procedural fluency. Lokan *et al.* (2003) argue that the process can become both boring and restrictive for students. Students must have a sense why Mathematics works in a specific way and they must be able to talk and share their ideas. Gervasoni (2004) argues that low-achieving students can easily lose confidence in their ability, and that they can also develop poor attitudes to learning and to school. This may result in such students not participating in the classroom and losing interest in Mathematics. Richard, Kirschner and Sweller (2012) argue that teachers can be more effective when they provide explicit guidance accompanied with practice and feedback. This study therefore utilised guidance with an active learning approach and also with the traditional approach of teaching.

Freudenthal (1991) offers an alternative in arguing that mathematics education should start and stay within reality. Barnes and Venter (2008) maintain that if students do not go through the process of horizontal mathematisation, there is a strong possibility that they can forget their memorised algorithm. Horizontal mathematisation is when students use their informal strategies to describe and solve a contextual problem and vertical mathematisation occurs when the students' informal strategies lead them to solve the problem using mathematical language or to find a suitable algorithm (Treffers, 1987; Gravemeijer, 1994). The danger is that if students experience difficulties with a problem they will not have a strategy in place to assist them in solving the problem. Barnes and Venter (2008) attest that students who achieved low marks often exhibit a lack of strategy. Students struggle to fall back to a strategy when they are faced with problems that require higher levels of understanding.

Research by Yanik and Porter (2009), Furner & Marinas (2013) and Lin (2008) show that Mathematics lessons can be made more stimulating if technology is introduced. Padayachee, Boshoff, Olivier and Harding (2011) support the contention that there is a lack of Mathematics interventions that focuses on the use of technology in the teaching and learning of Mathematics. The researcher therefore used GeoGebra as an ingredient for teaching and learning because of the lack of research done with Mathematics intervention with technology, especially with GeoGebra in South Africa.

Nieuwoudt, Nieuwoudt and Monteith (2007) put forward the argument that in “this technologically advanced era it would seem natural to question whether ‘relatively accessible and affordable technologies’ can contribute towards addressing the poor quality of teaching and learning”. Our students are born in a technological era and enjoy the rapidly

increasing availability of technology. Teachers can make use of technology to create a learning environment where students can enjoy, discover and explore Mathematics. The user can create with GeoGebra, in less time, the same number of activities as done with pencil and paper and this gives students more time to investigate and create more activities to do more investigations. Therefore, students who do not use technology such as GeoGebra are missing out on exploring and investigating multiple activities. Freudenthal (1991, p. 147) argues that “[l]earners should be allowed to find their own levels and explore the paths leading there with as much and little guidance as each particular case requires”. Discovery for them must be enjoyable and learning by reinvention can possibly motivate students.

Freudenthal (1991) emphasised that the process of reinvention should be guided. Students should be offered a learning environment in which they construct mathematical knowledge and have possibilities of coming to higher levels of comprehension. Teachers can design activities in such a way that students can use technology to help them understand mathematical concepts better. Consequently, this study utilises the guided approach with GeoGebra. It also utilises an active learning approach combined with interactive instruction. Therefore, some topics in this study were taught with interactive lecturing and others with a student-centred approach.

1.4 PURPOSE OF THE STUDY

The traditional teaching of Mathematics as a ready-made subject is still prevalent in South African schools and, therefore, the focus in this study is on the instructional design of activities with GeoGebra to expose students to the use of technology in the teaching and learning of Mathematics.

Functions and graphs, and circle geometry, are the topics in Grade 10 – 12 that were selected from the Curriculum Assessment Policy Statement (CAPS) (DBE, 2011a), as the subject matter that could be covered, using GeoGebra for the instructional activities. The focus for functions and graphs was on linear functions ($y = ax + q$); quadratic functions $y = a(x + p)^2 + q$; exponential functions ($y = ab^{x+p} + q$); hyperbolic functions $y = \frac{a}{x+p} + q$ and periodic functions: $y = a \sin(x + p) + q$; $y = a \cos(x + p) + q$ and $y = a \tan(x + p) + q$. Students used real-life applications modelling and problem solving activities and the researcher used GeoGebra to explore the concept of a function, domain

and range and the shapes of some of the functions mentioned above. Students then explored the effect of the parameters a, b, p and q on the graphs of the abovementioned functions on their own, using GeoGebra. Students also used GeoGebra applets to explore and investigate the different circle geometry theorems. Both the teaching of transformation of functions and circle geometry to the students were presented with a Hypothetical Learning Trajectory (HLT).

Diković (2009) emphasises that GeoGebra was “[de]signed specifically for educational purposes, [and] can help students grasp experimental, problem-oriented and research-oriented learning of Mathematics, both in the classroom and at home” (p. 191). This design promotes the activity and guided reinvention principles of Realistic Mathematics Education (RME). “GeoGebra promotes students structural conception of function in that it gives them an opportunity to observe the global behaviour of the graph of a function in accord with the changes in its algebraic form” (Bayazit, Aksoy, & İlhan, 2010, p. 5). Instructional activities were designed to introduce students to the topics and these activities were done in GeoGebra.

In short, the purpose of this study was to investigate the viability of using GeoGebra dynamic mathematics software in the teaching and learning of Mathematics over more than one topic.

1.5 RESEARCH QUESTION AND SUB-QUESTIONS

The following main research question will guide the inquiry:

What is the role of a GeoGebra-focused learning environment in helping students to develop mathematical knowledge?

The sub-questions for this question are:

- (i) What are the processes involved in using GeoGebra as a pedagogical tool?
- (ii) How did GeoGebra afford students an opportunity to understand mathematical concepts and move to higher levels of abstraction?
- (iii) What are students’ experiences and gestures when taught with GeoGebra?
- (iv) What are the challenges that students experience whilst working with GeoGebra?

1.6 RESEARCH METHODOLOGY

The research questions will be answered using both quantitative and qualitative data. This research will make use of combined/ mixed methods to answer the research questions. According to Gorard and Taylor (2004, p. 7) “[c]ombined methods research, and the combination of data derived through the use of different methods, has been identified by a variety of authorities as a key element in the improvement of social sciences, including education research”. Creswell (2003) also points out that mixed methods approaches employ strategies that involve collecting data simultaneously or sequentially to best understand research problems.

The data were collected through questionnaires, pre- and post-tests, interviews (both individual and focus groups) and from observation. The pilot study was done from February 2013 to September 2013 and the main study from February 2014 to September 2014. The topics that were covered in the pilot study were functions and for the main study they were functions and circle geometry. Students were using GeoGebra as a learning tool for these topics. GeoGebra was also used as a teaching tool by the teacher-researcher. He used applets to compliment his traditional approach of teaching where he was lecturing.

1.7 CONTEXT OF THE STUDY

The target group for the pilot was 54 students in the SciMathUS in 2013 and 48 students for the study in 2014. SciMathUS is a bridging programme that is offered by the Stellenbosch University Centre for Pedagogy (SUNCEP). It was founded in 2001. This bridging programme offers educationally disadvantaged (either financially, by school and/or home circumstances) students a second chance to qualify for admission into Science, Technology, Engineering and Mathematics (STEM) oriented programmes at Stellenbosch University or any other university of their choice. It is a year-long programme that offers students an opportunity to improve their Grade 12 marks.

Students can choose between Mathematics with Physical Sciences and Mathematics with Accountancy options. In addition subjects such as Computer Literacy and Academic Literacy and Language, Thinking and Studying Skills are offered. However, the students re-write the National Senior Certificate Examination in Mathematics and Physical Sciences only. The curriculum approach encourages students to become critical thinkers. The students of SciMathUS are encouraged to become independent thinkers through a holistic curriculum

that integrates practical skills such as research, essay writing, critical reading, thinking, life and computer skills (Malan, 2008; Malan, Ndlovu, & Engelbrecht, 2014). To be selected into the programme, a prospective student should have at least four subjects from the group of subjects designated for university admission; have a Grade 12 average of at least 55% (excluding Life Orientation) and have obtained a minimum of 50% in Afrikaans or English (Home Language or First Additional Language). For students to qualify for the Science stream, they should have obtained a minimum of 45% in Mathematics or a minimum of 80% in Mathematical Literacy and a minimum of 45% in Physical Sciences. For the Accountancy stream, prospective students should have obtained a minimum of 45% in Mathematics or a minimum of 80% in Mathematical Literacy. Thirty nine (81.3%) of the 48 SciMathUS students in the two groups achieved less than 60% in their previous NSC examinations for Mathematics. The population is therefore inclined towards bridging or access programmes.

The teacher-researcher was also a Mathematics facilitator for both the pilot and study group in 2013 and 2014. The researcher used GeoGebra as a teaching and learning tool and afforded students the opportunity to explore Mathematics using the software. Hence this research had a strong action research orientation.

1.8 SIGNIFICANCE OF THE STUDY

The implementation of active learning, guided reinvention with GeoGebra combined with the traditional approach teaching of Mathematics for Grade 12 in the SciMathUS programme at Stellenbosch University, is a relatively new approach of teaching to the programme. Students from 2007 – 2013 (Malan, 2008) that were part of SciMathUS were exposed to active learning during their year in the programme. Technology, especially GeoGebra, was used only by one facilitator as a teaching tool in 2012. GeoGebra was used to teach and learn Mathematics in only two groups in SciMathUS since 2013. The significance of this research is, firstly, to show students who are taking Mathematics as a subject that it can be fun with the use of GeoGebra for exploration and that Mathematics is not just about rules, computation and manipulations. Secondly, to investigate the potential that GeoGebra has to help students gain a better understanding of certain topics, such as functions, circle geometry and also higher order level questions or concepts which they usually struggle with. Additionally, there is the aim to help understand these topics better with the hope of increasing their achievement in the NSC exams in order to gain access to tertiary institutions in STEM careers. The findings of this study will help Mathematics teachers and students

understand that the teaching of Mathematics should not only be based upon explaining an example, then producing a few activities to drill, but rather that to teach to the point of understanding by using guided-reinvention and exploration with GeoGebra. Successful completion of this study may offer insight to Mathematics specialists in the Department of Basic Education, Mathematics teachers and foundation programmes at tertiary institutions how students experienced the teaching and learning Mathematics with a mixed approach of technology and a traditional approach of teaching.

1.9 DELIMITATIONS OF THE STUDY

The study draws data from only 45 students in the SciMathUS programme. There was only an experimental group and no control. Although the 45 students were in two different classes they could not divide into a control and an experimental group because they had a combined class every third day. The combined class made it impossible to teach the one group differently to the other. The teacher-researcher was also a Mathematics facilitator for both the pilot and study group in 2013 and 2014.

1.10 DEFINITION OF KEY CONCEPTS

Realistic Mathematics Education (RME): It focuses on making something real in your mind, and this gave RME its name. The problems to be presented to the students can be a real-world context but this is not always necessary. Freudenthal (1977) views RME to be connected to reality, staying close to children's experience and being relevant to society, in order to be of human value. The fantasy world of fairy tales and even the formal world of mathematics can be most suitable contexts for a problem, as long as they are real in the student's mind (Van den Heuvel-Panhuizen & Drijvers, 2014).

Dynamic geometry environments (DGEs): DGE "allows the user to drag parts of a geometry window, while measurements of the figure change dynamically in an algebra window" (Lingefjärd, 2015, p. 50). A DGE is therefore a dynamic mathematics tool for visualising and exploring geometry and algebra. The drag mode is a key element in DGEs and the most important tool provided by the software. The drag-mode allows the user to drag; e.g. the points of a given sketch and the diagram will change according to the new position of the dragged point, while all the geometrical relations used in the construction are preserved.

GeoGebra: It is a Dynamic Mathematics Software (DMS) for teaching and learning Mathematics from middle school through to college level.

Applets: Applets often have a specific conceptual focus, so they can be used selectively by instructors to support understanding of key concepts or to enhance an instructor's 'story-telling' or educational narrative. Applets are flexible, allowing use in a classroom or by students outside of class, and are usually easy for users to master without training or previous experience with their use (Morphett, Gunn, & Maillardet, 2015).

Functions: The concept of a function will be used as defined by DBE (2011a, p. 24). It is where a certain quantity (output value) uniquely depends on another quantity (input value). Work with relationships between variables using tables, graphs, words and formulae. Convert flexibly between these representations.

Hypothetical Learning Trajectories (HTL): It is the imagined route by which the teacher thinks students will learn a particular concept (Simon, 1995).

Science, Technology, Engineering and Mathematics (STEM): Career paths where Mathematics is a requirement.

Expository instruction or lecturing: The teacher is the main source and has the knowledge in expository instruction (Martin, 2003).

1.11 OVERVIEW OF CHAPTERS

Chapter 1 provided the rationale and the orientation for this research study. It also outlined its purpose and significance and presented its background. Moreover, it focused on the current situation of Mathematics teaching in South Africa and what ICT infrastructure schools have.

Chapter 2 is an extension of the literature review introduced in Chapter 1. In this chapter Constructivism; the RME approach and Van Hiele levels will be discussed and its South African version mentioned. RME and Constructivism will be the theoretical frameworks that underpin this study.

In Chapter 3 the role of ICT will be discussed. The focus will be on what can be done with GeoGebra – the affordances of the software. The role of HTL will also be discussed.

Chapter 4 will focus in greater detail on the research methodology and design. The philosophical and epistemological justifications will also be discussed in detail for the research paradigms underpinning the mixed methods approach. The research design (mixed methods approaches) and the specific methods that will be used in the study to generate the data, will also be covered in this chapter. Moreover, the learning trajectories for teaching function transformation and circle geometry will be discussed in detail.

In Chapter 5 the data from the various collection procedures, e.g. pre- and post-test, observations, questionnaires and interviews (individual and focus group) will be analysed, interpreted and discussed.

In Chapter 6 the deductions made from the analysis will be presented and the researcher will make concluding remarks and recommendations. The findings of this study will be aligned to its main aims, objectives and purpose. It will also offer suggestions regarding the areas for further research – i.e. the research gap which this study intended to fulfil.

1.12 SUMMARY

This chapter has presented an introduction to this study. It has provided the study's rationale and background. It also focused on the current situation of Mathematics teaching in South Africa and what access schools have to technology. It also presented the research questions, purpose, context, significance and the delimitations of the study. An overview of Constructivism, RME and Van Hiele levels will be given in Chapter 2.

CHAPTER 2

WHAT INGREDIENTS DO WE NEED FOR A CHANGE IN THE TEACHING AND LEARNING OF MATHEMATICS?

“Education is not filling a bucket but lighting a fire.”

– *William Butler*

2.1 INTRODUCTION

This chapter will review literature that underpins the teaching and learning of mathematics. It will therefore review theories that support the teacher-researcher to prepare tasks for the students in the teaching and learning of transformations of functions and circle geometry. The chapter will discuss the features of the constructivist approach to teaching and give an overview of Realistic Mathematics Education (RME) with guided reinvention, which is one of the six principles of RME, as the main theoretical framework of the study. The literature on RME will be used to explore the role it plays in the teaching of Mathematics. The Van Hiele theory will also be discussed to support a Hypothetical Learning Trajectory (HLT) for teaching and learning of circle geometry.

2.2 TEACHING MATHEMATICS EFFECTIVELY

Teaching mathematics effectively is not a challenge only to South Africa, it is a concern to countries throughout the world (Van der Walt & Maree, 2007). Mathematics has become the standard measure of the health of nations with respect to education and development. Almost every national and international test of achievement includes, or singles out, numeracy or mathematics (Jansen, 2012). Mathematics is the gatekeeper to Science, Technology, Engineering and Mathematics (STEM) oriented career paths. National and international reports such as the National Senior Certificates results, South Africa’s Annual National Assessment (ANA), Trends in International Mathematics and Science Study (TIMSS), Southern and Eastern Africa Consortium for Monitoring Education Quality (SACMEQ), Global Competitiveness reports, as well as additional research, show that student performance in numeracy and mathematics are a concern all over the world. These reports and research are intended to inform governments, schools, teachers, parents and

even students on what needs to be done to change the approach of teaching so that we can perform better in Mathematics as a nation.

As mentioned in Chapter 1, the TIMSS reports show that the Grade 8 and 9 South African students cannot perform on a Grade 8 level. The Global Competitiveness Reports (2013; 2014; 2015; 2016) consistently indicate that the quality of mathematics and science for higher education and training in South Africa is dismal. Ndlovu (2013b) argues that although the reports are based on perceptions rather than actual performance, it confirms the country's unsatisfactory performance in international benchmark tests. It also confirms why students in South Africa are underperforming in the National Senior Certificate.

The DBE (2011a; 2012a; 2013; 2014) also concedes that many candidates struggled with certain concepts in the curriculum that required deeper conceptual understanding. For students to have a deeper understanding of a concept, they need to understand why it is like that and why it is working in that way. Students need to be engaged in activities that enhance critical thinking because they help with the deeper understanding of concepts. Scriven and Paul (2007) define critical thinking as:

“the intellectually disciplined process of actively and skilfully conceptualizing, applying, analyzing, synthesizing, and/or evaluating information gathered from, or generated by, observation, experience, reflection, reasoning, or communication, as a guide to belief and action” (p. 1).

Snyder and Snyder (2008) are of the view that there are instructional methods that can help students to think critically and give them a better understanding of concepts. Traditional teaching methods focus predominantly on rote learning or memorising of information from textbooks or lectures (McTighe & Self, 2003; Snyder & Snyder, 2008). According to Snyder and Snyder (2008), these instructional methods do not promote conceptual understanding. McTighe and Self (2003) add that the knowledge acquired at the level of rote learning seldom promotes the student to apply this knowledge to new contexts. The students will therefore experience challenges to solve problems that are at higher cognitive levels. Research on teachers in South Africa (Webb & Webb, 2004; Morar, 2000; Stoker, 2003) shows that those who use traditional approaches mislead students into thinking that Mathematics is a subject that can be memorised. Petty (2012) also argues that if students only remembered what they were told, they would not make such mistakes and they would either remember or not. He adds that conceptual blunders show that humans make their own conceptual constructs and do not remember other people's. It is therefore important to

investigate learning theories, such as constructivism that supports critical thinking and can help students with deeper understanding. The next section will give an overview on constructivism.

2.3 A CONSTRUCTIVIST APPROACH TO TEACHING

Constructivism, as a psychological theory, stems from the escalating field of cognitive science, particularly the later work of Jean Piaget, the socio-historical work of Lev Vygotsky and academics who support his view, and the work of Jerome Bruner, Howard Gardner, and Nelson Goodman (Fosnot & Perry, 2005). Constructivism has its origin in cognitive psychology, which explains how people might acquire knowledge and learn. It, therefore, has a direct application to education. Fosnot and Perry (2005) emphasises that constructivism is not a theory of teaching. Constructivist teaching methods are therefore based on constructivist learning theory.

Constructivism, as a theoretical framework, holds that learning always builds upon prior knowledge. In a more traditional, teacher-centric classroom, where information transfer is prevalent, students are less likely to be engaged as the meaning is being interpreted on their behalf by the teacher. Any type of constructivists recognise that the students bring knowledge into the classroom (Liu & Matthews, 2005; Ndlovu, 2013a). Ndlovu (2013a) adds that the students bring with them prior knowledge and predispositions, unique to the individual student or cognising agent, which they use to “actively construct knowledge within the constraints and offerings of the learning environment” (Liu & Matthews, 2005, p. 387). Students also give meaning based on their experiences, individually or socially (Fox, 2001; Narayan, Rodriguez, Araujo, Shaqlaih, & Moss, 2013). Narayan *et al.* (2013) add that constructivism posits that knowledge is constructed by a student and not transferred to the student. The students are therefore not clean slates, as they are bringing prior knowledge and/or experiences to the learning situations. This prior knowledge or experiences will make an impact on, or be integrated in, the new knowledge that students construct (Merriam & Caffarella, 1999; Cooper, 2007; Woolfolk, 2007).

Constructivist approaches are considered critical to deeper understanding and internalising material or concepts (Narayan *et al.*, 2013). Fosnot and Perry (2005) reiterate that the constructivist approach concentrates on a holistic view of learning mathematics, and focuses on deep understanding and strategies, rather than facts and rote memorisation. Mvududu and Thiel-Burgess (2012) also state that constructivism is widely touted as an approach

where we are able to probe childrens' levels of understanding, and to show that their understanding can increase and change to higher level thinking. Constructivism describes the way that the students can make sense of the material and also how the materials can be taught effectively. Goodwin and Webb (2014) point out that constructivism changes the student's role from passive recipient of knowledge to active participant in the learning process. The students become involved in applying their existing knowledge and real-life experience, learn to make and test their conjectures, and ultimately assess their findings (Goodwin & Webb, 2014). This is exactly what is needed for the students at SciMathUS – to have a deeper and conceptual understanding so that they can perform better in those questions that pose a challenge to them. Roblyer (2006) contends that knowledge is constructed by students through experience-based activities rather than direct instruction, which is based on behaviourist and information-processing models of human learning. Constructivism according to Ndlovu (2013a) has, without doubt, been the touchstone of many postmodern Western educational reform efforts.

Cognitive constructivism, radical constructivism; and social constructivism are amongst the most prominent and espoused variants of constructivism.

2.3.1 Cognitive constructivism

Narayan *et al.* (2013) view cognitive constructivism as the root of all other types of constructivism. Cognitive constructivism is viewed as an individualistic perspective. Jean Piaget (1896-1980) views the development of human intellect through adaptation and organisation. Adaptation is a process of assimilation and accommodation, where, on the one hand, external events are assimilated into thoughts and, on the other, new and unusual mental structures are accommodated into the mental environment. As Piaget identifies knowledge with action, he considers that mental development organises these schemes in more complex and integrated ways to produce the adult mind. Piaget (1980) views knowledge as actively constructed by the student – not passively received from the environment.

Piaget (1952; 1954) differentiates between three types of knowledge and how students learn: social, physical and logico-mathematical knowledge. According to Waite-Stupiansky (1997) all three types of knowledge are essential, but they are learned in different ways. She attests that constructivist teachers adjust their teaching approaches to help students discover, construct, or memorise according to the knowledge or concepts they want students

to learn. She further points out that constructivist teachers should remember that all three types of knowledge must be integrated. Kamii and Ewing (1996) contend that traditional teachers are usually unaware of Piaget's distinction amongst the three types of knowledge. They point out that much of traditional elementary Mathematics is taught as if Mathematics is social knowledge.

Social knowledge, also known as conventional knowledge, is conventions that are created by people over a period of time (Kamii, 2014). Teaching rules, or algorithms such as "carrying" and "borrowing" when teaching adding and subtracting of numbers, are examples of social-knowledge approach (Kamii & Ewing, 1996). Social knowledge can only be gained through interaction with others (Lutz & Huitt, 2004). Just to teach that $f(x + 3)$ means that the graph moves three units to the left, is an example of social knowledge because students are just being told and not shown why the graph moves to the left. It is possible that the students can apply the algorithm, or the knowledge, without understanding. (Lutz & Huitt, 2004). A student can memorise this but it does not mean that (s)he understands the implication of the transformation on the input and output values or critical points (turning points, asymptotes, etc.) of the function. Faulkenberry and Faulkenberry (2010) emphasise that the focus on transformations of functions should be taught with the concept of the input and output values of the function. According to Waite-Stupiansky (1997) social knowledge is learned by repetition and drilling.

Physical knowledge, also called empirical knowledge, is obtained through hands-on interaction with the environment or discovered by exploring objects. Exploring concepts with GeoGebra can help students to acquire physical knowledge. Physical knowledge forms the basis for logico-mathematical thinking (Piaget, 1952; 1954).

Logico-mathematical knowledge is an abstract reasoning that is applicable beyond physical interaction with a concrete stimulus (Lutz & Huitt, 2004). Lutz and Huitt (2004) point out that physical knowledge is discovered and logico-mathematical knowledge is created through actions. They further allude that logico-mathematical knowledge can only be gained by frequent exposure and interaction with multiple objects in multiple settings in order for mental structures to be modified and created. The manipulation of objects in different patterns and contexts will allow for generalizations and abstractions to be formed (Lutz & Huitt, 2004). According to Waite-Stupiansky (1997) logico-mathematical knowledge affords opportunities to solve realistic problems, manipulating objects in thought-provoking ways, questioning

others and oneself, and articulates one's own logical reasoning without fear of failure. Teachers who pose questions that create disequilibrium in the children's minds and allow children time and opportunity to grapple with their own solutions, are teaching logico-mathematical thinking.

According to Berk (1997), Piaget believed that children develop gradually throughout the stages of life and that the experiences in one stage form the foundations for movement to the next. Every human being therefore moves from one stage to the next one, without missing one. This implies older children, and even adults, who have not passed through later stages process information in ways that are characteristic of young children at the same developmental stage (Eggen & Kauchak, 2000). Ojose (2008) supports Piaget's intimation that the child should be encouraged to self-check, estimate, reflect and reason while the teacher studies the child's work to better understand his thinking. Students in the same class are not necessarily functioning at the same cognitive level. Therefore, it is important to determine students' cognitive levels, because that can inform the instructional process.

However, Piaget's theory shows several deficiencies, especially because the different phases were set too rigidly. Firstly, critics argue that by describing tasks with confusing, abstract terms and using more than usually difficult tasks, Piaget underestimates children's abilities. Researchers have found that young children can succeed on simpler forms of tasks requiring the same skills. Eggen and Kauchak (2000) contend that middle school teachers interpreting Piaget's work may assume that their students can always think logically in the abstract, yet this is often not the case. The teaching of Euclidean geometry, (for this study circle geometry), is a good example of students being introduced to abstract theorems very rapidly. Teachers start teaching the proof of theorems and expect students to think logically. Students are also informed of what the meaning is, for example $f(x)$, $f(x + 3)$, $f(2x)$, etc. but with no understanding as to why. Secondly, Piaget's theory predicts that thinking within a particular stage would be similar across different tasks. In other words, pre-schoolchildren should perform at the pre-operational level in all cognitive tasks. Research has shown diversity in children's thinking across cognitive tasks. Thirdly, according to Piaget, efforts to teach children developmentally advanced concepts would be unsuccessful. Researchers have found that in some instances, children often learn more advanced concepts with relatively little instruction. Researchers now believe that children may be more competent than Piaget originally thought, especially in their practical knowledge. The stage dependency

of Piaget's theory has also been challenged. Van Hiele views that to move from one stage is not dependent on age, but more on the education experiences (Mason, 1998). They then proposed instructional phases as the scaffolding by which students could be assisted to move from one stage to the next rather than by mere maturation.

Despite the critiques of cognitive constructivism, the researcher in this study acknowledged the individual perspective of cognitive constructivism and how an individual acquires knowledge. He felt that students have to actively construct knowledge and this can also be done when students work independently.

2.3.2 Radical constructivism

Radical constructivism evolved from Piaget's genetic epistemology, although neo-Piagetian views may deviate from Piaget's original tenets (Derry, 1996) with followers such as Bruner, Ausubel, and Von Glasersfeld (Liu & Matthews, 2005). Von Glasersfeld (1995) describes radical constructivism as an unconventional approach to the problems of knowledge and knowing. Radical constructivism is a theory of knowing which, for reasons that had nothing to do with teaching mathematics or education, does not accept the common-sense perspective (Von Glasersfeld, 1991). According to Von Glasersfeld (1995) knowledge is in the heads of the persons and the thinking subject has no alternative, but to construct what he or she knows on the basis of his or her experiences.

Cognitive or radical constructivists consequently emphasise student-centred and discovery-oriented learning processes. In the process, social environment and social interaction work merely as a stimulus for individual cognitive conflict (Liu & Matthews, 2005). Teaching has to be concerned with understanding rather than performance, or the rote-learning, or training the mechanical performance of algorithms – because training is suitable only for animals that one does not credit with a thinking mind (Von Glasersfeld, 1994). The student's construction does not necessarily reflect knowledge of the real world. Radical constructivists view the student as an active participant that explores or discovers in the learning process. They acknowledge that social interaction can help the individual, but view the learning as an individual process where the student constructs meaning. According to Belbase (2011) group discussion, interaction, and exchange of ideas do not mean that mental construction of knowledge is social, but always individual. Social interaction facilitates such construction of knowledge by the individual mind. This study will therefore also embrace aspects of radical constructivism as part of the theoretical framework.

2.3.3 Social constructivism

The social or realist constructivist tradition is often said to derive from the work of Vygotsky. Ernest (1994) views the social constructivist theory as a theory which acknowledges that both social processes and individual sense have a central and essential part to play in the learning of mathematics. Other authors classified in this category include Kuhn, Greeno, Lave, Simon, and Brown. Varied as these theorists' ideas are, they are popularly understood to be proponents of the idea that the social environment plays a central role in learning. Students are believed to be enculturated into their learning community and appropriate knowledge, based on their existing understanding, through their interaction with the immediate learning environment. As such, students are encouraged to construct their own understandings and then to validate, through social negotiation, these new perspectives (Ertmer & Newby, 2013). Vygotsky (1978) argues that all cognitive functions begin as a product of social interactions. These participants must be involved in some form of interaction for knowledge to be constructed and they must have knowledge of prior social experience (Gergen, 1995). Gergen (1995) further points out that this is a shared understanding among individuals whose interaction is based on mutual interests that form the basis for their communication. Therefore, during the interaction between the participants' this prior knowledge is exchanged in a transaction in order to negotiate meaning. Vygotsky therefore contends that students can learn from each other. Unlike Piaget, Vygotsky believes that they can learn from those who are not necessarily of the same age. Within a social constructivist instructional framework, students are provided opportunities to interact with their peers for the purpose of discussing, generating, and sharing knowledge (Roessingh & Chambers, 2011). A student can therefore learn from someone, from a more knowledgeable adult or peer. This is precisely what makes it social. Add to this the fact that through negotiation of meanings the come to taken-to-be-shared understandings is no longer the individual's understanding per se. Vygotsky also proposed the idea of Zone of Proximal Development (ZPD), which proposes that a child can work with more capable peers or adults to achieve something that they could not achieve on their own. This is sometimes called scaffolding (Bruner, 1966; Wood, 1998). Therefore students can also learn from other students who are at different development levels.

Social constructivism thus assumes that cognitive growth first occurs at a social level and later at individual level, emphasising the role of ZPD (Vygotsky, 1978). Thus, instructors who are facilitators in social constructivism first provide support and help for students, and

this support is gradually decreased as students begin to learn independently (Amineh & Asl, 2015). Roosevelt (2008) posits that the main goal of education from the Vygotskian perspective is to keep students in their own ZPDs as often as possible by giving them interesting and culturally meaningful learning and problem solving tasks that are slightly more difficult than what they do alone. To achieve this outcome, they will need to work together either with another, a more competent peer or with a teacher or adult to finish the task.

According to Christmas, Kudzai and Josiah (2013), in mathematics teaching the teacher should avoid exposing students to tantalisers that lead students into a mental cul-de-sac. The tasks given should be challenging to such an extent that mediation by the knowledgeable teacher, or peer, is needed. After the student has attained mastery of the concept with the assistance from others, he/she should be able to do the task independently. If the student accomplishes the task individually through that process, then the student's ZPD for that particular task will have been raised. This process is then repeated at the higher level of task difficulty – that is the student's new ZPD (Christmas, Kudzai, & Josiah, 2013). Christmas *et al.* (2013) also contend that if the ZPD theory is well applied, it can incrementally lead to higher mathematical achievement.

In Vygotsky's theory, interpersonal interactions with adults, or more skilled peers, mediate the cognitive structures created by the larger culture. Mediation is the process of introducing concepts, knowledge, skills and strategies to the child (Karpov & Haywood, 1998; Vygotsky, 1981). Littlefield-Cook and Cook (2005) also emphasise that mediation is the process where adults and more skilled peers are used to introduce concepts and cognitive structures to lesser skilled children.

For the researcher of this study, social constructivism is the appropriate approach to teach students concepts. The researcher believes that interaction between the students and the teacher, as well as amongst students, is important. The researcher is also of the view that students cannot be left alone to learn and that an expert, or more capable person, must be available to help with misconceptions or stumbling blocks that may surface during students' interaction. Students can after conceptual understanding work independently. This is the point where the student can start to consolidate what (s)he understands of the concept, or topic, and then do practising or drilling.

2.3.4 Misconceptions and criticisms of constructivism

There are a few misconceptions about constructivism. Davis and Sarma (2002) point out that the most prevalent concern seems to be the fragmented and inherent character of the literature on constructivism. This lack of clarity, according to Davis and Sarma (2002), has contributed to a misinterpretation of the major principles of the theory. This inefficiency can also be the result of the many faces of constructivism (Phillips, 1995; Perkins, 1999).

Existing in many versions, constructivism has become a somewhat uncritically accepted textbook account of learning (Eggen & Kauchak, 2000; Fosnot, 2005). With the increased attention, many variations emerged and in the near future it can be viewed as a church of theoretical accounts (Liu & Matthews, 2005). Many researchers contend that constructivism has been elevated as the only human epistemology that can help with effective learning (Liu & Matthews, 2005; Ndlovu, 2013a; Phillips, 1995). These researchers view it as the only gospel or religion for human epistemology and it operates in isolation. Such an elevation has led to constructivism being viewed as an elitist theory that has been most successful with children from privileged backgrounds who are fortunate enough to have outstanding teachers, committed parents and materially rich home environments (Thirteen.org, n.d.). Above all, teachers and educational theorists in general should remember that, as with any effective model of teaching and learning, constructivism is not a remedy that can cure us of all of our educational woes. Gordon (2009) emphasises that the constructivist approach of teaching can produce remarkable results when followed correctly and cautiously. Gordon adds that if it is misapprehended, or abused, it can result in ineffective learning and students can still show poor results.

Other researchers (e.g. Biggs, 1998; Jin & Cortazzi, 1998) have noted that while constructivist teaching approaches, which include one-on-one or small group classroom interaction, do not always guarantee teaching effectiveness, didactic lecturing (mostly traditional) in large classes of 50 to 70 students in China, has not always meant the doom of teaching efforts. Ndlovu (2013a) further argues that constructivism does not imply teaching only through small groups and the students cannot learn from the teacher's lecture. Instead he states that telling, or lecturing, should not be the main classroom discourse. Ndlovu (2013a) also emphasises that teachers of mathematics (and science) should be aware that constructivism is one worldview amongst many competing for the same pedagogical space, i.e. constructivism and the latter is not the only theory emphasising

active learning. Gärdenfors and Johansson (2014) also support the notion that maybe a mixture of constructivism and more traditional methods will yield better learning results. Ndlovu (2013a) also alludes to the fact that direct teaching can be more time efficient, and can also produce excellent test results as evidenced by TIMSS and Programme for International Student Assessment (PISA) results for the East Asian countries. Fox (2001) argues that the naïve constructivist views' of classroom teaching and learning is that learning is not about remembering, or 'filling empty vessels' with facts, therefore implying that remembering is not important. A constructivist approach to teaching alone is, therefore, not the sole solution to improve students' understanding, or even for students to excel in Mathematics. The effective teaching of Mathematics is therefore more than just active learning. The researcher of this study concurs with the views of the abovementioned researchers and subsequently opted for the approach that was a combination of active learning approach and direct teaching.

The direct teaching, or transmission methods of teaching, refers to formal, expository and teacher-centred approaches. (Serbessa, 2006). According to Martin (2003) the teacher is the focal source and has the knowledge in expository instruction. Although this approach allows the teacher to present concepts and information efficiently, it can run the risk of diminishing students to passive receivers of knowledge (Baldin, 2002; Davis, 1993). Research in education shows the effectiveness of interactive lecturing techniques implemented during expository lectures encourages active learning (Crowe & Pemberton, 2000). According to Tang and Titus (2002) interactive techniques promote interaction between the teacher and the students, and also amongst the students. They add that students can learn in expository lectures but students' interest and activity have to increase. Eison (2010) points out that the use of technology can be one of the strategies that transforms traditional expository lectures into interactive lectures, and this way a teacher can create excitement and enhance learning. The teaching and learning approach in study was therefore a mixture of active learning and interactive lecturing with GeoGebra.

According to Gordon (2009) another misuse of constructivist teaching is when teachers essentially require their students to teach themselves. While the constructivist notion that students should be encouraged to create their own interpretations of the text is a sound idea, this is not the same as leaving students to their own devices and requiring them to teach themselves (Gordon, 2009). Kirschner and Sweller (2006) also point out that in some

studies, constructivism is viewed as a theory of instruction to allow students free reign to explore topics with minimal teacher input and not to reinvent the wheel. Gordon (2009) emphasises that students have to learn from the constructivist approach and not just be entertained and kept busy. Dewey (1956) also contends that:

“Nothing can be developed from nothing; nothing but the crude can be developed out of the crude – and this is what surely happens when we throw the child back upon his achieved self as a finality, and invite him to spin new truths of nature or of conduct out of that. It is certainly as futile to expect a child to evolve a universe out of his own mind as it is for a philosopher to attempt that task” (p. 18).

Students cannot therefore be actively involved in the learning process if they do not have the prerequisite content knowledge to do the activities presented to them. It is thus important to guide them, hence Freudenthal (1991) recommends guided reinvention.

We need to be careful not to confuse constructivism with student-centred teaching, or to assume that teachers who espouse this approach, have no content expertise. In a student-centred approach the teacher intervenes as little as possible. In contrast, a constructivist classroom is one in which there is a balance between teacher- and student-directed learning and teachers are required to take an active role in the learning process, including formal teaching (Gordon, 2009). Ndlovu (2013a) points out that in its avid quest to be student-centred, constructivism does not define the extent to which educators must do more listening to students before intervening. It is important to learn about students’ learning and construction of understandings about specific mathematical content (Ndlovu, 2013a).

Biesta (2014) also contends that constructivist thinking has discredited the transmission model of teaching and thus gives lecturing and so-called didactic teaching a bad name. According to Biesta (2014) it seems as if constructivism has given up on the idea that teachers have something to teach and that students have something to learn from their teachers. Fox (2001) also observes that, in its emphasis on students’ active participation, it is often seen as if constructivism dismisses the roles of passive perception, memorisation, and all the mechanical learning methods in traditional didactic lecturing too easily. Fox (2001) also posits that there is a tendency for constructivism to seem to offer learning without tears and that it brushes all manner of obvious problems and difficulties that it can present aside. Some critics consequently blame social constructivism for leading to ‘group think’ which tends to produce a ‘tyranny of the majority’, wherein a few students’ voices dominate

the group's conclusions, and "dissenting students are forced to conform to the emerging consensus" (Thirteen.org, n.d).

According to Howell (2006) students that are exposed for the first time to a constructivist approach can find this challenging, as the shift from a conventional, or traditional approach of teaching, can be disturbing to students who may hesitate at having the rules changed in a game that they have come to know so well. Students are likely to have been conditioned to sit quietly in their seats, take notes, perhaps ask a question, or discuss a point or two, and cram the knowledge into their short-term memories in order to pass the test. The SciMathUS students can also be unsettled with the different approach of teaching that they are used to because they come from different schools.

Ndlovu (2013a) also sees the lack of clarity about the meaning of construction as a criticism of constructivism. Freudenthal (1991) questions the meaning of 'construction' or 'reconstruction' as subsumed by constructivism and charges that these are words that can mean everything and their opposites. Freudenthal (1991) thus proposes 'reinvention' as a more appropriate term.

Research done by Lim (2007) in five schools in Shanghai, the top scorers in PISA (2000, 2009, 2012), shows that although teachers are doing direct teaching, they also include ICT such as Power Points and Geometer's Sketchpad. The teachers also place a great emphasis on logical reasoning, mathematical thinking and proof during teaching. From the research done by Lim (2007) it is evident from the classroom observations that high-level-thinking-skill questions such as 'why?', 'how?', 'what if?' were asked during lessons. Direct teaching with visualisation, posing high level questions, etc. can also help students to perform at a higher level. Ndlovu (2013a) also indicates that constructivism is still a developing theory that needs more research support. The researcher of this study consequently has a strong belief in the constructivist approach, but will explore it with RME.

According to Ndlovu (2013a) the teacher's role is to create optimal conditions for successful constructivist learning to take place, e.g., students should have the prior knowledge necessary to scaffold them beyond their zone of proximal development. Learning mathematics therefore is not about algorithms or 'recipes' only. The researcher in this study shares the same view in this regard, especially in the case of the SciMathUS students. Students can be lectured or taught, but in this research teaching and learning was done with

interactive lecturing and with a constructivist approach. Students can be taught through lecturing, but it has to be in a way that the content is not only transferred to them. As mentioned earlier in this section, the students have already passed Grade 12 Mathematics and were therefore exposed to topics in a specific approach. The properties of quadrilaterals, for example, were not taught in a way that students explored the properties of quadrilaterals on their own. Instead, the teacher-researcher used GeoGebra applets (see example no. 8 in Appendix R) to teach the properties of quadrilaterals in such a way that students could see that squares, rhombi and rectangles are all parallelograms. The students are actively part of the lecturing by completing the properties of different quadrilaterals on the worksheet (see Appendix K) given to them. The teacher-researcher, therefore, tried to teach the students in this manner to give them a deeper understanding of the properties of the quadrilaterals, and he did not teach it in a way that they could only memorise it. On the other hand, circle geometry was taught in a way that students explore the theorems themselves.

Although constructivism has provided Mathematics teachers with useful ways to understand learning and students, the task of reconstructing mathematics pedagogy on the basis of a constructivist view of learning, is a considerable challenge, one that the mathematics education community has only begun to tackle. It provides a useful framework for thinking about mathematics learning in classrooms and therefore can contribute in important ways to the effort to reform classroom mathematics teaching. It does not tell us how to teach mathematics; that is, it does not stipulate a particular mode (Simon, 1995). De Villiers (2007) points out that students cannot translate abstract or 'pure' mathematics into practical situations when mathematics is mostly taught in a decontextualised way. For the researcher in this study, the teaching of mathematics needs more than a student-centred approach. The introduction to new topics or concepts is, for the researcher, the most important part of teaching. Using real or experientially real context is a very sound point of departure for students to connect with mathematics and experience mathematics in a way that is not just about symbolic manipulation. Kline (1977) points out that there must be some motivation why this topic is being studied. The approach that the researcher felt answers his concerns in this study was Realistic Mathematics Education.

2.4 REALISTIC MATHEMATICS EDUCATION (RME)

Realistic Mathematics Education (RME) is a teaching and learning theory in mathematics education that was first introduced and developed by the Freudenthal Institute in the Netherlands in the 1970s. Realistic Mathematics Education has its roots in Hans Freudenthal's interpretation of mathematics as a human activity (Freudenthal, 1973). Freudenthal argued against structuralism saying that mathematics should be thought of as a human activity of 'mathematising'. South African Mathematics curriculum documents define mathematics similarly. The (DBE, 2011a, p. 10) describes mathematics as a "human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects themselves". This confirms that RME has had a strong influence in the design of South Africa's Mathematics curricula (in fact, since the RNCS times).

It should not be seen as a discipline of structures to be transmitted, discovered, or even constructed, but as schematising, structuring, and modelling the world mathematically. From as early as the 1960s, Freudenthal had a constructivist view of education and began to work with Dutch mathematics educators and researchers to develop what he called, Realistic Mathematics Education. Volumes of research and curricula were developed after extensive research (Fosnot, 2005).

De Lange (1996) points out that RME has been adopted by a large number of countries all over the world such as England, Germany, Denmark, Spain, Portugal, South Africa (as a problem-centred approach), Brazil, USA, Japan, Thailand, South Korea and Malaysia. A characteristic of RME is that rich, 'realistic' situations are awarded a prominent position in the learning process. These situations serve as a source for initiating the development of mathematical concepts, tools and procedures, and provide context in which students can, at a later stage, apply their mathematical knowledge, which gradually becomes more formal, more general, and less context specific (Van den Heuvel-Panhuizen & Drijvers, 2014).

In South Africa there are few programmes based on RME. The Realistic Mathematics Education in South Africa (REMESA) project of the Freudenthal Institute at the University of the Western Cape in the 1990s was aimed at developing and researching the impact of innovative mathematics learning and teaching materials based on the premise that 'reality is the basis of and the domain of application of mathematics' (Niehaus, 1996; Niehaus, 1997; Hadi, 2002). The materials developed by REMESA were intended to form a useful resource

from which teachers, textbook authors and others can develop school mathematics programmes relevant to the South African situation. Several other researchers (Barnes, 2005; Julie, 1993; Julie, 2004; Ndlovu, 2014) investigated the possibility of adapting and applying the RME approach in South Africa. Ndlovu (2014) investigated the perceptions and performance of Mathematics teachers in a teacher professional learning programme (TPL) based on RME principles. RME was researched in professional development for teachers, focusing on secondary school teachers (Ndlovu, 2014) and on low attaining learners in Mathematics (Barnes, 2005), as well as on implementing RME in the school curriculum (Julie, 2004). Research done by Murray, Olivier and Human (1998) re-inforce the view that weaker students in Mathematics are able to construct mathematical concepts if they are allowed to explore Mathematics on their own. In other words, mathematically weaker students have the capacity to learn mathematics with conceptual understanding if given the right opportunity to explore their own thinking in a supportive environment of inquiry-based learning, which GeoGebra can potentially afford them.

The Malati Project was an initiative in the Western Cape Province of South Africa. Malati was a co-operative project of mathematics educators at the Universities of the Western Cape, Stellenbosch and Cape Town. The project, which ended in 1999, had a fundamental pedagogical principle that the need for each of these aspects should be motivated by the context. The rationale of the project material was that if students choose to abandon Mathematics at the end of General Education Certificate level (at the end of Grade 9) they would do so having acquired basic mathematical knowledge, and having experienced some authentic mathematical activity in a range of contexts. The choice of content was motivated by providing access to different mathematical processes e.g.: modelling; designing algorithms; making definitions, etc. (Malati, 1999a). The project focused on problem solving and was not always driven by realistic contexts.

Another initiative that was closely linked to socio-constructivist theory was the Problem-Centred Learning (PCL) approach, developed in South Africa in the mid-1980s by researchers at Stellenbosch University. The PCL approach is based on a socio-constructivist theory of the nature of knowledge and learning (Olivier, Murray, & Human, 1992; 1993; 1998). Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier & Wearne (1997) point out that in a problem-centred approach, instruction begins with reality-based problems, dilemmas and open-ended questions. The learners acquire knowledge from the

solution of problems. They engage in a variety of problem situations and, in the process, learn mathematical content. The students also use mathematical knowledge to solve real life problems.

Ground breaking work, especially by the Freudenthal Institute in the Netherlands over several decades, and all the abovementioned initiatives and research done by researchers in South Africa, using such a modelling approach, has shown not only its feasibility, but also substantial gains in children's ability to relate Mathematics to the real world meaningfully (as well as increased conceptual understanding generally). De Villiers (2007, p. 4) argues that "[u]sing modelling as a teaching strategy by starting with real world (or theoretical) applications has much higher motivational value, as it immediately places the usefulness, or value of the content, in the foreground." Armanto (2002) and Fauzan (2002) argue that research done in numerous countries (including developing countries such as Indonesia) over the past decades has shown that RME theory is a promising direction to improve and enhance students' understanding in Mathematics. Dickson and Hough (2012) also support this view by pointing out the effectiveness of RME in the Netherlands, USA and UK. Ndlovu (2013a) also alludes to this stating that learning of mathematics should involve contexts that students can identify with – to enhance relational understanding. The researcher of this study is also of the view that it is important to make students excited about mathematics, in such way that they are motivated to perform better in Mathematics. For the research in this study, it was viewed to be crucial to start a new lesson with context or in a way that makes sense to the students at that point in time.

Researchers such as Gravemeijer (1994), De Lange (1996) and Kanselaar (2002) view RME as one example of learning environments based on socio-constructivism. Ndlovu (2013b) also refers to the RME approach as a neo-constructivist approach which stresses that mathematics learning should, in Freudenthal's (1977) view, be connected to reality, stay close to children's experience and be relevant to society, in order to be of human value. De Lange (1996) also points out two main similarities between RME and socio-constructivist mathematics education. First, both the socio-constructivist and realistic mathematics education are developed independently of constructivism. The second and main difference between RME and constructivism is that RME is only applied to mathematics education while constructivism is used in many subjects (De Lange, 1996). Gravemeijer (1994) points out that the difference between socio-constructivist approach and realistic approach is that

the former does not offer heuristics for developing instructional activities for students. In other words, in socio-constructivist approach, the teacher does not use heuristics, but rather a method of solving problems by learning from past experience and investigating practical ways of finding a solution. In RME, it is known as guided reinvention.

Freudenthal (1991) also addressed constructivism. Although he criticises the constructivist epistemology from an observer's point of view, it can be argued that the way he sees mathematics from an actor's point of view is compatible with this epistemology. From the perspective of an active mathematician, he characterises mathematics as a form of (well developed) common sense – a notion that is strongly tied to his idea of 'expanding reality'. Moreover, his educational goal is to make sure that the students experience 'objective mathematical knowledge' as a seamless extension of their everyday life experience. This leads us to conclude that Freudenthal stands much closer to constructivism than one might gather from his attack on it (Gravemeijer & Terwel, 2000; Ndlovu, 2013a). Ndlovu (2013a) points out while Freudenthal criticised it and tried to distance himself from it, he was ironically neo-constructivist in how he advocated mathematics should be taught.

Freudenthal's (1968) viewpoint is that education should give students the 'guided' opportunity to 'reinvent' mathematics by doing. Freudenthal (1968) emphasises that the focal point should not be on mathematics as a closed system but on the activity, on the process of mathematisation. Freudenthal (1991) argues that students must be exposed to meaningful problems and not given only algorithms, definitions and rules. Freudenthal (1991) adds that we do have algorithmically gifted people who can learn a lot of algorithms. Algorithmically gifted people, according to Freudenthal (1991), seem to form a small minority. He points out that teachers teach for learning first and then to understand afterwards. Even in the early years of RME it was emphasised that if children learn mathematics in an isolated fashion, divorced from their experiences, it will quickly be forgotten and the children will not be able to apply it (Freudenthal, 1971; 1973).

2.4.1 The basic tenets of RME

RME reflects a certain view on Mathematics as a subject, on how children learn Mathematics and on how Mathematics should be taught. These views can be characterised by the following six principles (Van den Heuvel-Panhuizen, 2000; 2010; Van den Heuvel-Panhuizen & Drijvers, 2014, p. 523):

- Activity principle
- Reality principle
- Level principle
- Intertwinement principle
- Interaction principle
- Guided reinvention principle

2.4.1.1 Activity principle

The activity principle refers to the interpretation of mathematics as a human activity (Freudenthal, 1971; 1973; 1991). In RME, students are treated as active participants in the learning process. Freudenthal sees definitions and notations as the start of ready-made mathematics. Instead of students being receivers of ready-made mathematics, they are treated as active participants in the educational process, in which they develop all types of mathematical tools and insights by themselves (Van den Heuvel-Panhuizen, 2000; Van den Heuvel-Panhuizen & Drijvers, 2014). Students' 'own production' is an important factor in RME. Any small discovery made by a student may boost self-confidence, even if the result is well known to others, and aid the student's ability to enjoy Mathematics (Fallstrom & Walter, 2009). Well-designed pen-and-paper activities, or technology enhanced tasks, can be given to students. Students can then use their prior knowledge to formalise Mathematics that is new to them.

In geometry, students can be given graph paper to construct their own similar right-angled triangles in order to find the relationships between the different sides, sketch their own circles to explore the different theorems within circle geometry, or to plot and sketch graphs of functions. Dynamic mathematics technology (software), such as GeoGebra, can be used to complement pen and paper activities and to ensure that students are actively involved in drawing much more accurate sketches and making mathematical explorations.

2.4.1.2 Reality principle

The reality principle emphasises that RME is aimed at having students be capable of applying mathematics (Van den Heuvel-Panhuizen, 2010). Rather than beginning with certain abstractions or definitions to be applied later, one must start with rich contexts demanding mathematical organisation, i.e., contexts that can be mathematised

(Freudenthal, 1968; 1979). While working on context problems, the students can develop mathematical tools and understanding. The context can be from the real-world but this is not always necessary.

An example of De Villiers' (2012) shows how a real world application is used to introduce the concept of rate of change. The activity was created in the software Measure in Motion. Figure 2.1 shows a snapshot of the video clip as water is being poured at a constant rate into a tilted rectangular container. The height of the water is measured and plotted against time in the graph on the right. An activity such as this can also be used to introduce the concept of a function.

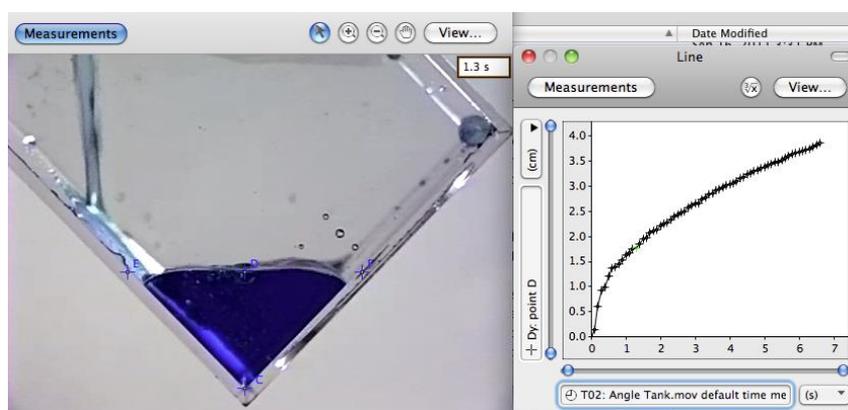


Figure 2.1: Graph of the height of the water measured plotted against time

De Villiers (2012) used the activity to allow students to make a conjecture about the relationship between the height of the water and time. An activity such as this is a good example of how contextual material is used to introduce the concept of a function.

Takács' (2008) research with students shows how the transformations of the trigonometric functions can be explained. Figure 2.2 shows a good example how to introduce periodic functions and is a real-life example of what effect the parameters of the sine functions have on the graph. If the radius of the Ferris-wheel changes, it shows students the vertical stretching or shrinking. When the speed of the Ferris-wheel changes, it will show the students the horizontal stretching or shrinking. Shifting the function left or right with an angle means that the cabin which is already on the way is being investigated.

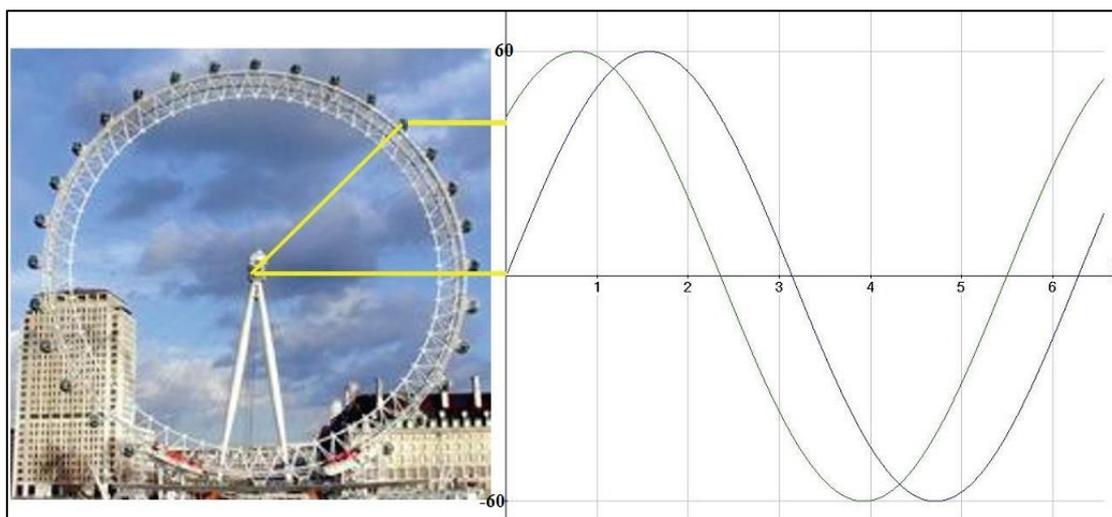


Figure 2.2: Graph of the height of a cabin on the Ferris-wheel plotted against time

According to Van den Heuvel-Panhuizen and Drijvers (2014) the context can also be within the fantasy world of fairy tales. Even the formal world of mathematics can provide suitable contexts for a problem, as long as they are real in the student's mind. Context can be visualised by students with activities, for example, Treffers (1987) who designed an experiment with sixth graders built around the context of Jonathan Swift's well known novel *Gulliver's Travels*. Treffers (1987) created the following fantasy:

In the world of the Lilliputians, all lengths are 12 times as small as in our world, the world of Gulliver. A tree, being 12 m high in our world, would thus be only 1 m high in the world of the Lilliputians. A road, being 12 km long in our world, would be only 1 km long in the Lilliputian world. And, of course, the Lilliputians themselves are also 12 times as short as Gulliver.

As a result of the linear scale factor being imposed by the context, all items involved reductions by factor 12. Students can learn the concept of enlargement or reduction with a story that is now real in their minds. Other examples of making context real in the mind of students are the Land of 8 (Treffers, 1987) where students can explore and discover that calculations in base eight are possible.

Drijvers (2015, p. 10) further points out that "realistic means that problem situations presented in learning activities should be 'experientially' real to students and have meaningful, authentic problem situations as starting points, so that students experience the ... activity as making sense". The importance of using real contexts that are meaningful and natural to learners as a starting point for their learning cannot be overemphasised (Cheung & Huang, 2005; Van den Heuvel-Panhuizen, 2010; Widjaja & Heck, 2003; De Villiers, 2012).

This characteristic of RME will be used in this study as a starting point to teach functions. The researcher in this study developed activities similar to the two abovementioned examples in Figure 2.1 and 2.2, and these were used to introduce the different graphs of functions as prescribed by the CAPS (DBE, 2011a), as well as the transformations of functions. All of these activities (see Appendixes G and H) were done by the students with pen and paper and were made visual by the teacher-researcher with GeoGebra. The purpose of real life activities were that students were able to connect the mathematics to the real-life and also to show them that mathematics will be part of some of their careers.

2.4.1.3 Level principle

Van den Heuvel-Panhuizen and Drijvers (2014) point out that the level principle underlines that learning mathematics means that students pass through various levels of understanding. They move from the ability to invent informal context-related solutions, to the creation of various levels of short cuts and schematisations, and further to the acquisition of insight into the underlying principles and the discernment of broader relationships (formal mathematics).

For Cobb, Zhao and Visnovska (2008) the level principle focuses on the means of supporting the process of vertical mathematisation. It is here that the general approach of building up towards substantial participation in established mathematical practices becomes evident. One of the primary means of support involves activities in which students create and elaborate symbolic models of their informal mathematical activity. This modelling activity may involve making drawings, diagrams, or tables, or it could involve developing informal notations or using conventional mathematical notations. This third tenet is based on the conjecture that, with the teacher's guidance, students' *models of* their informal mathematical activity can evolve through use into *models for* more general mathematical reasoning. This idea of a shift from the '*model of*' to the '*model for*' became a significant element within RME thinking about progress in students' understanding of mathematics (Streefland, 1991; Treffers, 1991; Gravemeijer, 2004; Van den Heuvel-Panhuizen & Drijvers, 2014). This means that designed activities, or tasks, can help students at the start, with informal mathematical reasoning and later, as a result of reflection, with more formal reasoning (Van den Heuvel-Panhuizen, 2003). This reflection can be the result of interaction with physical objects such as building something, or exploring with GeoGebra, or discussion with the teacher, or classmate(s).

First, the students develop strategies closely connected to the context. Students can explore with the pen and paper activities, or activities in GeoGebra. With software such as GeoGebra they move and drag the same activity to see more examples and

“...later on, certain aspects of the context situation can become more general, which means that the context more or less acquires the character of a model and as such can give support for solving other, but related, problems. Eventually, the models give the students access to more formal mathematical knowledge” (Van den Heuvel-Panhuizen, 2000, p. 4).

This third tenet of RME is consistent with Sfard's (1991; 1994) historical analysis of several mathematical concepts including function. Sfard contends that the historical development of mathematics can be seen as a long sequence of reifications, each of which involves the transformation of operational or process conceptions into object-like structural conceptions. She argues that the development of ways of symbolising has been integral to the reification process. The transition from *model-of* to *model-for* coincides with a progression from informal to more formal mathematical reasoning that involves the creation of a new mathematical reality which is thought to consist of mathematical objects (Sfard, 1991) within a framework of mathematical relations. The level of more formal activity is reached when the students no longer need the support of models (Gravemeijer, 2004).

The students were therefore exposed to the method of how the transformation of the graph of functions are applicable to real life in Figure 2.2. They were then exposed to the effect of the parameters on the different prescribed graphs in the curriculum. Students were then expected to be able to move from the reality to symbols and explore, for example, the effect of a , p and q in $y = a(x + p)^2 + q$. This can be done with pen-and-paper but more meaningfully in GeoGebra where students can explore and see that the effect of the parameters in all the other graphs is the same. If students can see this, then the level of more formal activity is reached and they do not need the support models.

Faulkenberry and Faulkenberry (2010) in Figure 2.3 and Figure 2.4 show a pedagogical method for teaching vertical transformations of a function.

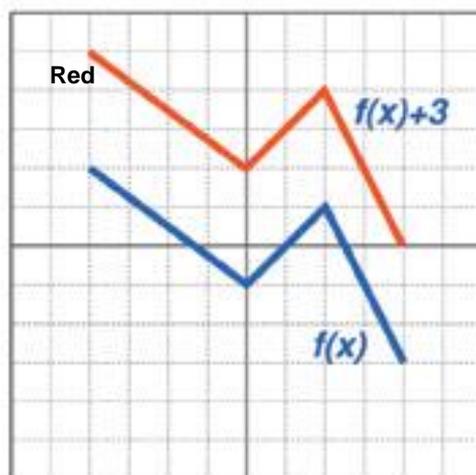


Figure 2.3: The translation of the graph of $f(x)$ 3 units up

Faulkenberry and Faulkenberry (2010) used transparencies to teach this concept. They also view the input and output values as essential to teach the concept of function transformation.

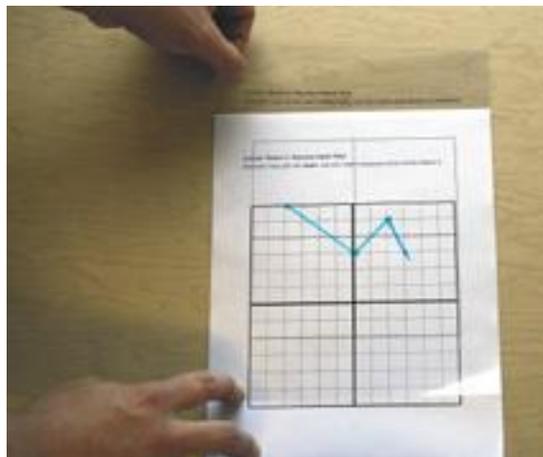


Figure 2.4: The red graph is obtained by shifting the blue graph up 3 units

The activity in Figure 2.3 and 2.4 are a static visual representation of the transformation of $f(x) + 3$. The context is not real but they can create a symbolic model with the exploration of this informal activity. The researcher created a similar approach to all the different function transformations. GeoGebra applets were used by the students to explore the change of the input and output values. Later they could construct and plot their own functions in GeoGebra.

Cheung and Huang (2005) refer to the last level as the formal level that allows pure cognitive thinking or higher level of formal mathematical reasoning, reflection and appreciation. Van den Heuvel-Panhuizen and Drijvers (2014) see that the strength of the level principle lies in guiding the growth in mathematical understanding and that it gives the curriculum a longitudinal coherency. This long-term perspective is characteristic of RME. Therefore, it could help the SciMathUS students to understand (in the long-term) the complex procedures

and equip them with problem solving skills. If students move through these levels it can help them to perform better in NSC examination questions that entail complex procedures and problem solving.

Using this characteristic of RME, this study sought to establish if students could progress from the informal mathematics to formal mathematics, which may well help them with the processes of generalising and formalising of the notation transformation of functions. The activities entailed those with GeoGebra applets, as well as students' own constructions in GeoGebra. This characteristic aims to establish whether the students may have a better understanding of the effect of the parameters of a, b, p and q on the graphs of the following functions: $y = ax + q$; $y = a(x + p)^2 + q$; $y = ab^{x+p} + q$; $y = \frac{a}{x+p} + q$;
 $y = a \sin(x + p) + q$; $y = a \cos(x + p) + q$ and $y = a \tan(x + p) + q$.

2.4.1.4 Intertwinement principle

This principle refers to a mutual relationship between topics or chapters, and also between different concepts within a topic or chapter. According to Van den Heuvel-Panhuizen (2000), Mathematics, as a school subject, cannot be split into separate learning strands. Students need to be exposed to the idea that Algebra, Transformation Geometry, Trigonometry or Euclidean Geometry do not stand alone. For students to understand the transformation of graphs, they need to have a sound understanding of properties of transformation geometry. Instead of recalling algorithms or rules in transformation geometry, students can explore the properties with the different transformations of polygons, especially translations, reflection, reductions and enlargements, and apply that knowledge to the transformation of graphs of functions. As in geometrical transformations, students need to understand that if a graph is translated, its shape does not change, only its position does.

This tenet of RME seeks to develop students' understanding so that topics and concepts in Mathematics are interrelated precisely such as Question 4 in the National Senior Certificate examination (DBE, 2015b) in Figure 2.5. The students were expected to use circle geometry in analytical geometry.

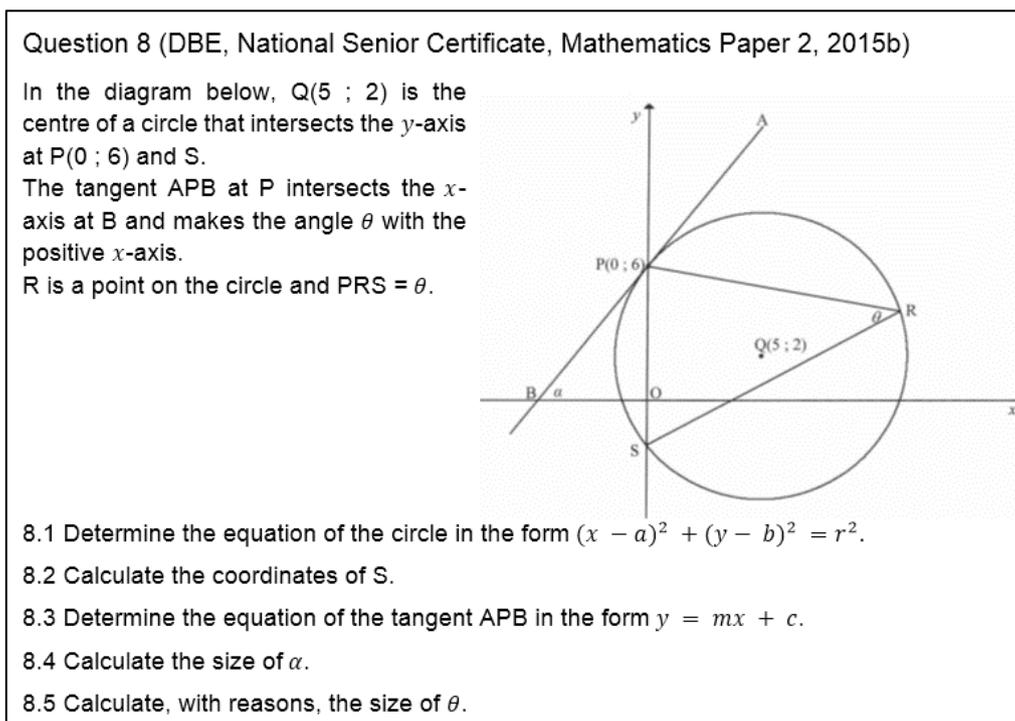


Figure 2.5: Question 4 (DBE, National Senior Certificate, Mathematics Paper 2, 2015b)

2.4.1.5 Interaction principle

The interaction principle of RME signifies that learning mathematics is not only an individual activity but also a social activity (Van den Heuvel-Panhuizen & Drijvers, 2014). RME is in favour of whole-class discussion and group work. In this way students are afforded opportunities to share their strategies and inventions with one another. By sharing, students get varied information which may improve their strategies. The students also have an opportunity to explain mathematics in the way they understand it.

Interaction between students and teachers, as well as among students themselves, is important to RME, and that makes the theory socio-constructivist. Education should offer students opportunities to share their strategies and inventions with one another. By listening to what others discover, and discussing those findings, the students can get ideas for improving their strategies. Moreover, the interaction can evoke reflection, which enables the students to reach a higher level of understanding. RME, according to Van den Heuvel-Panhuizen and Drijvers (2014), recognises individual learning paths and, therefore, also views students as individuals. Students can be divided into small groups or even work together in pairs on an activity or question, and share their answers with each other. They can justify their answer(s) or help someone who doesn't understand the problem (c.f. scaffolding by more knowledgeable others in ZPD). It can also help the students to enhance

their understanding, because they are in a space where they have to explain to one another their method of thinking or solving the problem. This can be linked to Sousa's (2006) argument that when teachers vary their teaching methods the average retention rate of students can move up to 90% if students work in groups and start teaching each other. When students start explaining to each other, or justifying their answers, they also engage at a higher cognitive level. Figure 2.6 adapted from Sousa (2001; 2006), shows the learning pyramid. It shows that if students are taught by lecturing, the average percentage of retention of material after 24 hours is only 5%. However, lecturing "continues to be the most prevalent teaching mode in secondary and higher education, despite overwhelming evidence that it produces the lowest degree of retention for most students" (Sousa, 2001).



Figure 2.6: Average % of retention of material after 24 hours per instructional method (NTL Institute of Alexandria, Virginia).

The interaction principle can, therefore, help students to perform better in questions that require higher order level thinking such as the complex procedures and problem solving.

2.4.1.6 Guided reinvention principle

According to the guided reinvention principle, students should be given the opportunity to experience a process similar to when a given mathematical topic was invented (Freudenthal, 1973). Thus a route has to be designed that allows the students to develop 'their own' mathematics. This process, however, needs guidance from the teacher to help to further develop sensible directions, leave 'dead-end streets' and to ascertain convergence towards the common standards within the mathematical community.

Freudenthal (1991) noted that children who are curious will explore and will not ask for permission to do something. He points out that children who are indifferent and lazy prefer the teacher to guide them. Freudenthal (1991) therefore suggests that children should repeat the learning process of mankind. According to Freudenthal (1991) students can invent something that is new to them but well-known to the teacher. This implies, according to Van den Heuvel-Panhuizen (2010) and Van den Heuvel-Panhuizen and Drijvers (2014), that, in RME, teachers should have a pro-active role in students' learning and that educational programmes should contain scenarios which have the potential to work as a lever to create shifts in students' understanding. To realise this, the teaching and the programmes should be based on a coherent long-term teaching-learning trajectory. The RME guided reinvention heuristic, according to Gravemeijer (2004), is connected with mathematising; the students invent by mathematising. The idea is that the students not only mathematise contextual problems (horizontal mathematisation) – to make them accessible for a mathematical approach – but also mathematise their own mathematical activity, which brings their mathematical activity to a higher level (vertical mathematisation). The reality principle links strongly with horizontal mathematisation whereas the level principle links more closely with vertical mathematisation.

According to the reinvention principle, a route to learning along which students are able, in principle, to discover the envisaged mathematics by themselves has to be mapped out (Gravemeijer & Terwel, 2000). They contend that the curriculum designer, or teacher, should start with a thought experiment, or imagining a route how the students may arrive at a solution. According to Simon (1995) this imagined route is a plan of how the teacher thought the students will learn a particular concept. Simon (1995) refers to this route as a hypothetical learning trajectory (HLT). This research has followed a hypothetical learning trajectory which will be discussed in Chapter 3 (*see paragraph 3.6*) and Chapter 4 (*see paragraph 4.4*). Guided reinvention is therefore an important principle of RME curriculum design. The idea is that teachers create opportunities for students that are consistent with the constructivist and problem-centred approaches. This can be done with paper and pen activities, or with activities on a computer, or software, such as GeoGebra. GeoGebra applets can be given to the students to provide them with the opportunity to explore, or make conjectures. Solving meaningful problems can help students to develop their own problem-solving strategies. It is important to note that reinvention is both a collective and an individual

activity, in which whole-class discussions focusing on conjecture, explanation and justification play a crucial role (Oh Nam, 2002).

2.4.2 Progressive mathematisation

Treffers (1978; 1987) formulated the idea of two types of mathematisation, explicitly in an educational context, and distinguished between 'horizontal' and 'vertical' mathematisation. Mathematising could literally be translated as making something more mathematical. Mathematising includes the processes of generalising, formalising and curtailing; the latter also includes developing algorithms (Gravemeijer & van Galen, 2003). Together, horizontal and vertical mathematisation constitute the process called progressive mathematisation. In a sense Gravemeijer and Van Galen (2003) view progressive mathematisation as the counterpart to reinvention.

Treffers (1978; 1987) and Van den Heuvel-Panhuizen (2010) view horizontal mathematisation as when the students come up with mathematical tools which can help organise and solve a problem located in a real-life situation. Hence, this is the application of the reality principle. It is also the same as context-based problem solving. It is also referred to as mathematical modelling in other sections of the mathematics education community (Biccard & Wessels, 2011; Stillman, Galbraith, Brown, & Edwards, 2007). Vertical mathematisation is the process of reorganisation within the mathematical system itself, for instance, finding shortcuts and discovering connections between concepts and strategies and then applying these discoveries. This is the same idea as that in the level principle of RME. Freudenthal (1991) sees horizontal mathematisation as going from real-life world into the world of symbols, while vertical mathematisation means moving within the world of symbols. By teaching the traditional formal and authoritarian approach, teachers start within this world of symbols. Horizontal mathematising might include, but not be limited to, activities such as experimenting, pattern snoopng, classifying, conjecturing, and organizing. Vertical mathematising may include activities such as reasoning about abstract structures, generalising, and formalising (Rasmussen, Zandieh, King, & Teppo, 2005).

In Figure 2.6 below the guided reinvention model (Gravemeijer, 1994) shows that the learning process starts from contextual problems. Starting to teach functions with

$y = mx + c$ or $y = ax^2 + bx + c$, etc. and also the use of only symbols is an example of vertical mathematization. Therefore, it is starting with formal mathematics language and algorithms (see Figure 2.6). Teaching domain and range of $y = 4a^3 - 120a^2 + 900a$ as $a \in R$ are examples of vertical mathematization. Instead of starting within the world of symbols, contextual material for functions can be used. An example of contextual material utilised is how De Villiers (2012) introduces the concept of rate change with a real world application (see Figure 2.1). Starting with contextual problems, describing and solving it are examples of horizontal mathematization (see Figure 2.7).

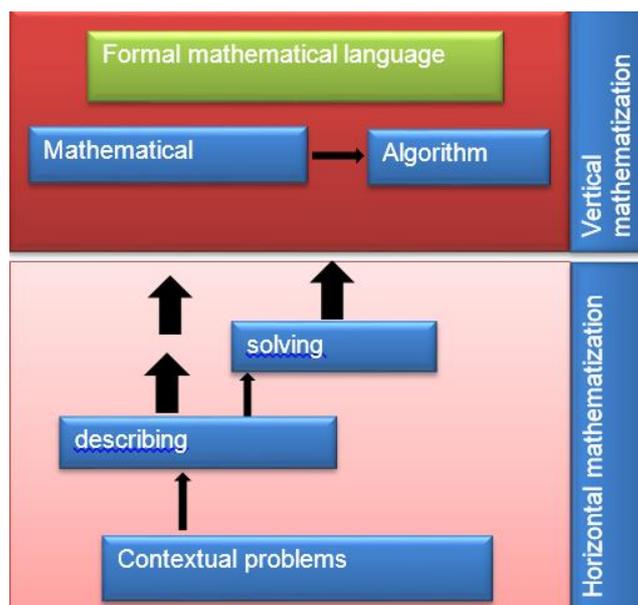


Figure 2.7: Guided reinvention model (Gravemeijer, 1994)

The concepts of domain and range of $y = 4a^3 - 120a^2 + 900a$ as $a \in R$ can also be taught with contextual material (see Figure 2.8). The following activity explains the progressive mathematizing

Activity:

Suppose you have a square piece of cardboard with a side length of 30 cm from which you want to form a box. To do this you cut out identical small squares on each of the four corners and fold the remaining side faces upwards. The capacity of the box depends on the side length a of the small squares. If the side length a changes, the capacity of the box will also change. That means that the capacity (C) of the box is a function of the side length a .

Which values of a can you use to form a box?

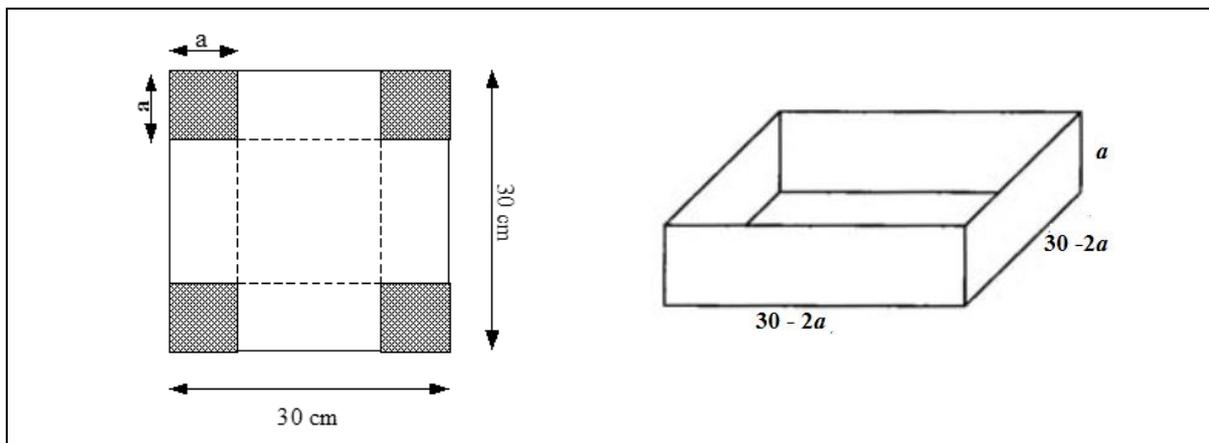


Figure 2.8: Contextual example about a box

Plotting the different heights against volume will give you the graph presented in Figure 2.9.

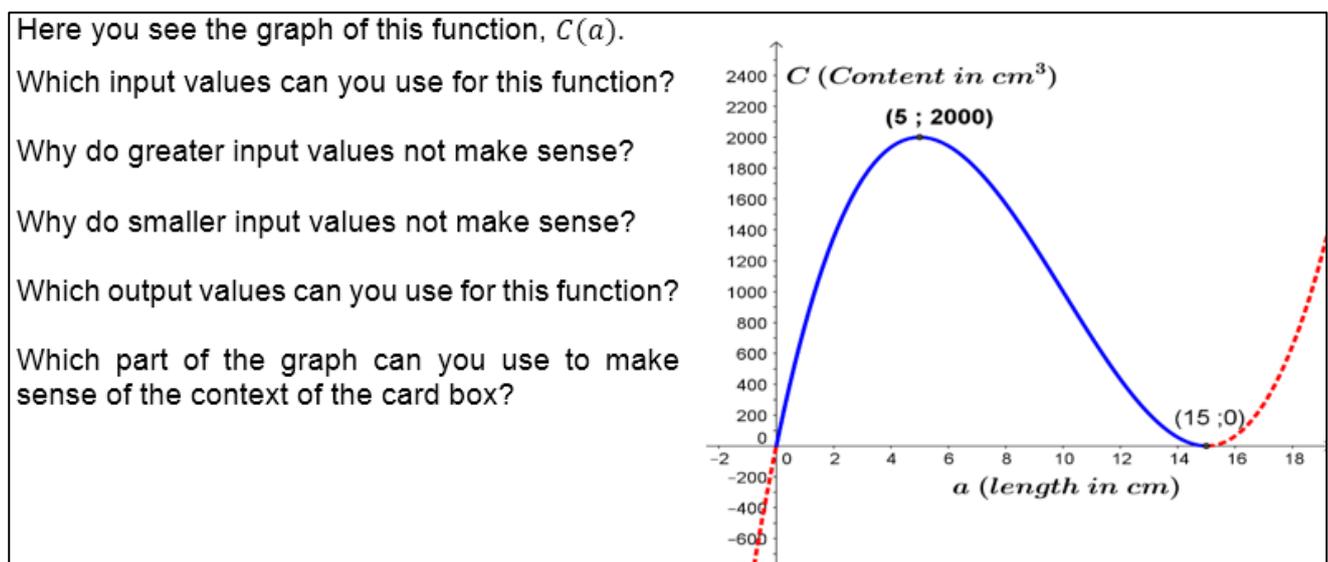


Figure 2.9: Graph of contextual example about a box

From this activity students can derive the formula $C(a) = 4a^3 - 120a^2 + 900a$. Questions on the input and output values can then be linked to the domain, range and continuity. Using the above activity is an example of horizontal mathematisation where students can informally discuss their answers and thereafter move into the world of symbols and concepts, for example, domain and range. As Hadi (2002) describes, the students start with contextual problems, then try to describe the problems using their own language or symbols, and in the process, solve the problems. The student gains an informal or formal mathematical model when using activities in horizontal mathematisation. By implementing activities such as solving, comparing and discussing, the student deals with vertical mathematisation and ends up with the mathematical solution.

Initially, Freudenthal (1991) didn't agree with Treffers' (1987) distinction of horizontal and vertical mathematisation, but later accepted it as an effective tool to characterise various kinds of mathematics education. He used these characterisations to classify mathematics education into four types.

Table 2.1: Four types of mathematics education (Freudenthal, 1991)

	horizontal	vertical
mechanistic	-	-
empiristic	+	-
structuralist	-	+
realistic	+	+

The *mechanistic* or '*traditional approach*', is based on drill-and-practice and patterns, and it treats the student like a computer or a machine (Freudenthal, 1991). It means the activities of students in this approach are based on memorising a pattern, or an algorithm. Mistakes can occur if students are faced with other problems that are different from the one they have memorised. No horizontal and vertical mathematisations are used in this approach. This is exactly what the DBE (2011b) refers to when it says many students are exposed to 'stimulus-response' methods only. This is why students have great difficulty when faced with questions testing the same procedures as previously tested but when asked in a different manner. This approach is also what students are looking for and they have the idea that this is the only way to learn. Widjaja and Heck (2003) see mathematisation in RME as taking place in both directions by means of a reinvention process that is guided by the teacher and by the instructional materials.

Freudenthal (1991) elaborates that the *empiristic approach* sees the world as a reality. In this approach students are provided with materials from the living world. This means students are faced with situations in which they have to do horizontal mathematisation activities. However, they are not provoked to find a formula or to move into the world of symbols. Consequently, students do not find shortcuts or discover connections between concepts and strategies. This is the case when students are given real-life activities but they only complete the activity and the teacher does not abstract the mathematics form of the activity.

The *structuralist*, or 'New Math approach' according to Freudenthal (1991), is based on set theory, flowcharts and games, which are types of vertical mathematisation. This approach originates from an 'ad hoc' created world, which has nothing in common with the student's living world. According to Thornton (2011) a structuralist approach has strong vertical mathematisation, in that it emphasises links between areas of mathematics. However, there is weak horizontal mathematisation, with little attempt to develop the skills of mathematical modelling needed to solve real world problems (Thornton, 2011). Freudenthal (1991) refers to it as mathematics taught in the ivory tower of the rational individual – far from society. Freudenthal argued against structuralism saying that mathematics should be thought of as a human activity of 'mathematising'. Not as a discipline of structures to be transmitted, discovered, or even constructed – but as schematising, structuring, and modelling the world mathematically (Fosnot, 2005).

The *realistic* approach, according to Freudenthal (1991), uses real-world situations or contexts as the starting point of learning mathematics. Then it is explored by horizontal mathematisation activities. This means students organise the problem, try to identify the mathematical aspects of the problem, and discover regularities and relations. Then, by using vertical mathematisation students develop mathematical concepts. Horizontal and vertical mathematisation are both present in the realistic approach and this is what constitutes RME. The main difference with the mechanistic and structuralistic approaches is that realistic mathematics education does not start from abstract principles, or rules, with the aim of learning to apply these in concrete situations (Wubbels, Korthagen, & Broekman, 1997).

Barnes (2004) warns that if students do not go through a process of horizontal mathematisation, a strong possibility exists that they can forget the algorithms they were taught; they do not have a strategy in place to assist them in solving the problem. The students must thus move to vertical mathematising after horizontal mathematising.

Freudenthal (1991) argues that students should be allowed to find their own levels and explore the paths leading these levels with as much and as little guidance as each particular case requires. Barnes (2005) recommends that the theory of RME be considered in the design and implementation of interventions with low achievers. Widjaja and Heck (2003, p. 2) argue that the RME approach affords the students the chance to use their own informal strategies to solve the problems, and to discuss with their teacher and fellow student. In this way the process of understanding is stimulated. Widjaja, Dolk and Fauzan (2010) point out

that RME also helps student to focus, to listen to each other and to investigate the other students' perceptive.

Van den Heuvel-Panhuizen (2001) also emphasises that change is not easy and it takes time:

“If anything is to be learned from the Dutch history of the reform of mathematics education, it is that such a reform takes time. It looks a superfluous statement, but it is not. Again and again, too optimistic thoughts are heard about educational innovations. Our experience is that reforms in education take time. The development of RME is thirty years old now, and we still consider it as “work under construction” (p. 1).

For teachers to change their teaching style, especially after many years of teaching, can be challenging. Even for students to be taught differently from their previous learning approach can be challenging. The researcher is taking cognisance of the challenges, but recognises that change is needed in teaching so that students can develop deeper understanding and perform on a level that gives them access to tertiary institution. The above literature review of six tenets of RME demonstrates the creation of a learning environment in which they construct mathematical knowledge and have possibilities of coming to higher levels of comprehension. The RME theory thus provides possibilities to equip learners with problem-solving skills. Hence the researcher envisages utilising RME as one of the conceptual frameworks in this study.

2.4.3 Criticism against the use of the RME Theory

Reform-attackers in the Netherlands criticise real-life context used in the teaching and learning of mathematics (Van den Heuvel-Panhuizen, 2010). They believe that progressive schematisation leads to a long and unnecessary path (Verstappen, 1991; Keune, 1998; Van de Craats, 2007). They contend students will only understand after training. Van den Heuvel-Panhuizen (2010) contends that these reform-attackers shamelessly state that children do not need to think.

Numerous inaccuracies and misconceptions are suggested by the detractors of reform and the Dutch media (Van den Heuvel-Panhuizen, 2010). The media and reform-attackers posit that the students are not afforded the opportunity to practice in RME. Research done by various researchers (De Jong, Treffers, & Wijdeveld, 1975; De Moor, 1980; Treffers & De Moor, 1990; Van den Heuvel-Panhuizen & Treffers, 1998; Menne, 2001; Van Maanen, 2007)

is a flagrant contradiction of RME's long tradition of including practice. Van den Heuvel-Panhuizen (2010) argues that RME is about practising with understanding and coherence, which is radically different from the isolated drill that the opponents of RME have in mind. This study took the same approach as mentioned by Van den Heuvel-Panhuizen (2010). The researcher believes that it is important for students first to understand the concept before they practice what they conceptually understand.

The media also claims that RME has abandoned algorithmic computations. Van den Heuvel-Panhuizen (2010) points out that the teaching of traditional algorithms differs in each textbook series and gives examples of a few RME textbooks series that contain thousands of the algorithmic calculations. RME is said to teach students as many different calculation strategies as possible, which may confuse them. RME starts teaching following on from what students themselves come up with and do – which has natural variation – and from thereon gradually works towards a standard method, which is not a straitjacket (Van den Heuvel-Panhuizen, 2010). Van der Mee (2008) also points out that RME only works for learners in earlier grades. The reason for this being that learners are only requested to solve problems with 'small' numbers and that makes it very difficult for learners to apply the principles in higher grades where they are requested to solve problems with 'bigger' numbers.

All the RME textbooks were also criticised to be of a low quality (Van den Heuvel-Panhuizen, 2010). The results of some extensive studies done by the National Institute for Assessment in the Netherlands (Cito) into the impact of the RME textbooks contradict the claim. According to Van den Heuvel-Panhuizen (2010) the RME textbooks were among the best textbook series; more often than the traditional ones (Janssen, Van der Schoot, Hemker, & Verhelst, 1999). The oldest RME textbook series, *Wêreld in Getallen* (WIG) and other well recognised RME textbooks were placed in the top 19th of the 24 mathematics textbooks series, while the mechanistic series *Naar Zelfstandig Rekenen* (NZR) were not ranked in the top place in the category best textbook series.

Hickendorff (2013) also points out that, in the Netherlands, research has shown that students who were trained in RME did not necessarily perform better than students who were trained in the traditional approach. This research points out that girls prefer the traditional approach, because according to Hickendorff (2013), it is more systematic. Boys, on the other hand, prefer RME because they are 'free' to pin down the answer (doing the necessary steps mentally).

Van den Heuvel-Panhuizen (2010) indicates that some of the above criticism falls short in several respects. She contends that, as the Periodieke Peiling van het Onderwijsniveau (PPON) results show, the mathematics achievements of Dutch primary school students have not substantially declined in general (some sub-domains show a decrease in scores but others an increase). International comparisons also do not give a cause to indicate that the scores have largely declined. PPON is the Periodic Survey of the Educational Level done in the Netherlands. Van den Heuvel-Panhuizen (2010) points out that based on the Grade 4 data from TIMSS 2007 (Mullis, Martin, & Foy, 2008) it is clear that this statement is wholly unfounded. The Netherlands' Grade 4 students have the highest scores of the nine Western European countries that took part in the TIMSS 2007 (Mullis, Martin, & Foy, 2008). Moreover, the Netherlands finished above the other Western countries that took part, including the United States, Australia and New Zealand. The Netherlands also was ranked eighth for arithmetic in TIMSS 2011 (Mullis, Martin, Foy, & Arora, 2012).

The researcher in this study took some of these criticisms into consideration when he designed the activities. The instructional worksheets and GeoGebra applets were also designed not to waste too much time. The designed activities on the transformations of functions and circle geometry that were done in GeoGebra were not always set in a realistic context, as is required by RME. Students explored the shapes of different graphs of functions as prescribed by the DBE (2003) with contextual questions.

2.5 THE VAN HIELE THEORY OF GEOMETRICAL THINKING LEVELS

Historically, the characteristics of RME are related to Van Hiele's levels of learning mathematics. Van Hiele (1986) emphasises that if the teaching aim is that the student should know how to prove theorems, it is highly improbable that the student's thoughts are aimed directly towards this goal. Van Hiele argues that it is improbable because the student will not be able to grasp the idea of proving a theorem if the proof only shows to him/her.

The Van Hieles were of the view that proving of theorems requires high cognitive levels of thinking (De Villiers, 2010; Mason, 1998). De Villiers (2010) posits that this is a key reason for the failure of the traditional geometry curriculum and why students struggle to understand their teachers. De Villiers (2010) contends that the teacher and students are speaking different languages (not the spoken language) and they think on different levels. The teachers on the other hand do not understand why their students do not understand. Van Hiele contends (1986) that one can only understand the theorems if one knows their

relationships associated with the theorems at the same time. Van Hiele's theory distinguishes between five different levels of thought. (Van Putten, Howie, & Stols, 2010)

At the *Recognition level* (Level 1), originally named by Van Hiele the *Base Level* of geometry, figures are judged by their appearance (Van Hiele, 1984). Students are giving visual definitions at this level. At this level, students can define a rectangle as a quadrilateral that only contains angles 90° and two long and two short sides (De Villiers, 2010). The argument at this level is based on a statement of belief, and not on logical conclusions. This can only be resolved by repeated statement or by claiming authority ("because I say so") (Murray, 1997). Students recognise triangles, squares, parallelograms, and so forth, by their shape, but they do not explicitly identify the properties of these figures. Atebe's (2008) study shows that the majority of the students observed in Nigerian and South Africans schools have low knowledge of school geometry and that most of them are positioned at Van Hiele's level 1. De Villiers (2010) points out that dynamic software plays a role in introducing, for example, the most common quadrilateral visually, and not introducing it via formal definition. He adds that by guided dragging of activity in the dynamic software it is envisaged that a student at Van Hiele 1 may develop easily a dynamic concept image of a rectangle as one that can change into a square when all its sides become equal. Figure 2.10 shows how De Villiers (2010) explains dynamic software can be quite helpful.

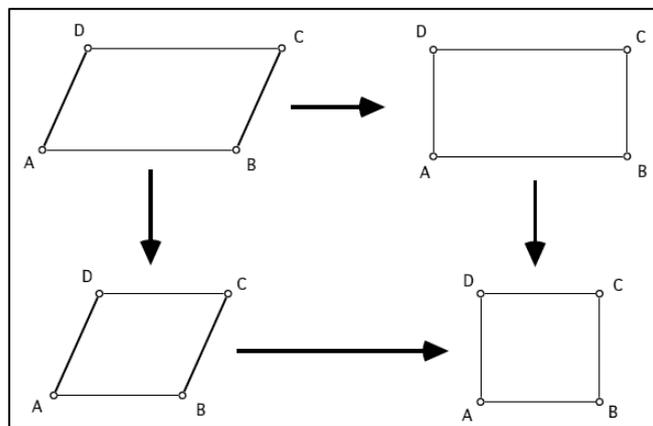


Figure 2.10: Dynamic transformation of a parallelogram (De Villiers, 2010)

Point A, B, C or D of the constructed parallelogram can easily be dragged it into the shape of a rectangle, rhombus or square. This was also done by the teacher-researcher with an interactive lecturing approach (see example no. 7 in Appendix R) with GeoGebra. The geometry of triangles and quadrilaterals was done because the teacher-researcher sees it as building blocks for circle geometry. It was thus part of the HLT for circle geometry.

Consequently the researcher of this study created applets for circle geometry to introduce the theorems visually and not with the formal proof of the theorems.

Lim (1992) contends that it is crucial that students should experience recognising a particular concept, especially in non-standard orientations.

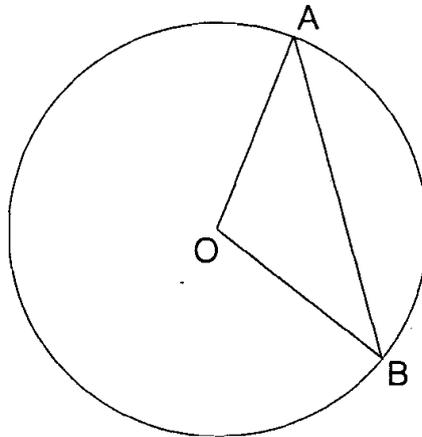


Figure 2.11: Non-standard orientation of an isosceles triangle

If O is the centre circle a student at *recognition level* will be thinking that an isosceles triangle in Figure 2.11 does not look like the one in Figure 2.12

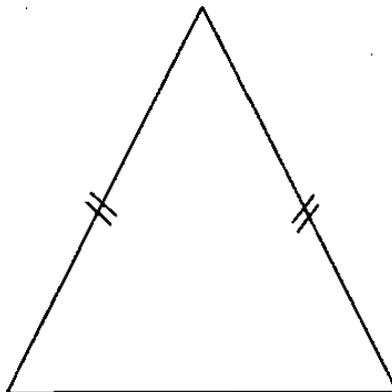


Figure 2.12: Standard orientation of an isosceles triangle

The above example shows (see Figure 2.12) how students can have difficulty in understanding when they are not at the Van Hiele level which is expected for the topic, or at the level at which the teacher is communicating. It is, therefore, necessary for the teacher to be aware of the level of reasoning of his pupils and to work towards bringing pupils up to the expected level for that topic. Lim (1992) emphasises that concrete materials, or transparencies, which can be moved to highlight different orientations, are obviously very useful. Dynamic software, such as GeoGebra, can easily be used to show students different

orientation. Students, and even teachers, according to Lim (1992), are 'conditioned' to seeing one 'standard' orientation by textbooks.

Lim (1992) adds another example that, often, teachers do not spend sufficient time letting pupils recognise "angles subtended at the circumference", "angles subtended at the centre", "angles in the same segment," etc. According to Lim (1992), at times there is a brief introduction of the concept before proceeding (quickly) to the properties. For another example, in introducing the angle properties of circles, teachers often do not spend sufficient time letting pupils recognise "angles subtended at the circumference", "angles subtended at the centre", "angles in the same segment," etc. In most presentations, there is a brief introduction of the concept before proceeding to the properties themselves.

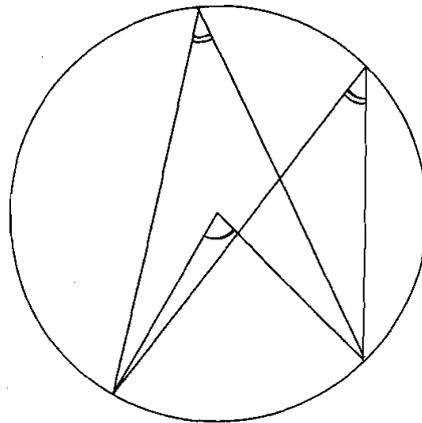


Figure 2.13: Standard orientation of angles at centre and circumference.

Moreover, only one or, at most, two examples are given to introduce the concept and angles at centre, and circumference almost point upwards and are subtended by minor arcs in such introductions like in Figure 2.13 and Figure 2.14.

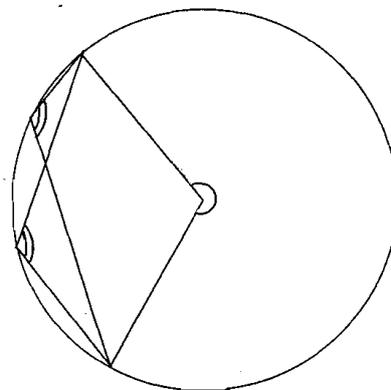


Figure 2.14: Non-standard orientation of angles at centre and circumference.

Lim (1992) suggests that many examples and non-examples in different orientations need to be experienced by the pupils. Students can explore examples with GeoGebra applets and also explore, with the same applet, non-standard orientation of a concept. In this way students are able to see many non-examples as well. Circle geometry was part of this research and students can claim that any angle at the centre is double the angle at the circumference without taking into consideration that both angles have to be subtended by the same chord. GeoGebra applets can be used to visualise the properties of circle geometry.

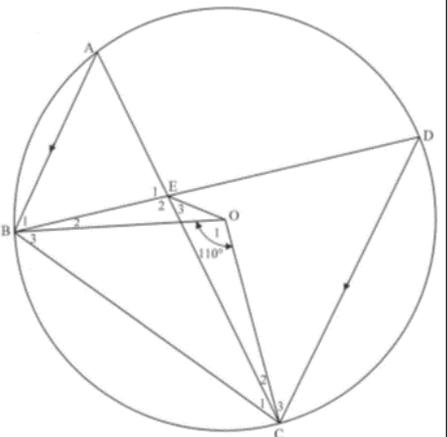
At the *Analysis Level* (Level 2), the figures are bearers of their properties (Van Hiele, 1984; 1999). A figure is no longer judged because “it looks like one”, but rather because it has certain properties. Properties are not logically ordered (Van Hiele, 1999). At this level a learner analyses figures in terms of their parts and the relationships between these parts. They establish the properties of a class of figures empirically, and use properties to solve problems (Fuys, Geddes, Lovett, & Tischler, 1988). Students also have the view that all the properties are important and they see no difference between necessary and sufficient properties (Vojkuvkova, 2012). Arguments, according to Murray (1996; 1997), can be resolved by referring to the definition, for example, “this shape must be a square because it has four sides equal and four right angles”. It can be noted that the definitions are not precise and often include redundancies. De Villiers (2010) sees the definitions as uneconomical. For example, a rectangle is a quadrilateral with opposite sides parallel and equal, all angles 90° , equal diagonals, half-turn-symmetry, two axes of symmetry through opposite sides, two long and two short sides, etc. According to Van Hiele (1984) if a figure with four right angles is badly drawn on a blackboard students will recognise it as a rectangle because of its properties. The properties of a rectangle are, at this level, not yet ordered and a square is not necessarily identified as a rectangle. Students at this level will see all circumference angles are equal, or any angle outside the cyclic quadrilateral, as the exterior angle of the cyclic quadrilateral. De Villiers (2010) says that the students at this level start analysing the properties of figures and learn the appropriate technical terminology for describing them, but they do not interrelate figures or properties of figures. Students can drag the different circumference angles in GeoGebra and explore if it is always equal. Lim (1992) attests that, at this level, students need to experience discussing the properties concerned as often as possible. Vojkuvkova (2012) also points out that students on this level do not see a need for proof of facts discovered empirically.

At the *Informal deduction* or *Ordering Level* (Level 3), the properties are ordered and deduced from each other. The one property precedes, or follows, another property (Van Hiele 1986). The square is recognised as being a rectangle because at this level the definitions of the figures are precise. De Villiers (2010) points out that the definitions are now correct and economical. According to Murray (1996; 1997), students at this level understand the relations within and between figures, for example, “a square has all the properties of a rectangle therefore it is also a rectangle”. Students are capable of ‘if... then’ thinking (but not formal proofs) at this level, so logical reasoning can be developed. Students start realising, in circle geometry, that an angle in a semi-circle is a corollary of the theorem angle at the centre, is double the angle at circumference angle on condition that it is subtended by the same arc or chord. They can also see the link of why circumference angles are subtended by the arc or chord equal. This can be deduced from the theorem that states an angle at the centre is double the angle at circumference, angle subtended by the same arc or chord without any proving it. Activities in GeoGebra can also help to visualise this. Lim (1992) argues that a student has to attain this level at the end of her/his secondary school mathematics. When students attain this, they will be able to follow simple deductions and understand logical deductions between consecutive steps.

At the *Formal Deduction Level* (Level 4), students start developing longer sequences of statements and begin to understand the significance of deduction, the role of axioms, theorems and proof (De Villiers, 2010). Van Hiele (1984) argues that, at this level, thinking is concerned with the meaning of deduction, with the converse of a theorem with axioms, with necessary and sufficient conditions. At this level, students should be able to construct proofs such as those typically found in a high school geometry class (Mason, 1998; Stols, 2012; Van Putten, Howie, & Stols, 2010). Van Putten *et al.* (2010) point out that students that are not on this level, can only do proofs by memorisation.

Students at *Analysis Level* are not able to follow deductive proofs which are at *Formal Deduction Level*. They may be able to follow each individual step under the guidance of the teacher, but are unable to have an overview of the proof. Figure 2.15 is a question from the February/March 2015 National Senior Certificate examination (DBE, 2015b) and is a good example of testing if students are on the analysis level.

In the diagram below, the circle with centre O passes through A , B , C and D . $AB \parallel DC$ and $\widehat{BOC} = 110^\circ$. The chords AC and BD intersect at E . EO , BO , CO and BC are joined.



8.1 Calculate the size of the following angles, giving reasons for your answers:

8.1.1 \widehat{D}

8.1.2 \widehat{A}

8.1.3 \widehat{E}_2

8.2 Prove that $BEOC$ is a cyclic quadrilateral.

Figure 2.15: Question 8 (DBE, 2015b).

This question shows the level at which secondary students must be able to solve circle geometry problems in the National Senior Certificate examination. The experiment in this study with GeoGebra, with the SciMathUS students, focuses on preparing the students with GeoGebra to reach Van Hiele's level 1 to 3 so that they can then move on to level 4.

At the *Rigor Level* (Level 5) the students establish and analyse theorems in different postulation systems. Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. They can understand the use of indirect proof and proof by contrapositive, and can understand non-Euclidean systems (Mason, 1998; Stols, 2012).

Although the Van Hiele theory distinguishes between five different levels of thought, only the first four levels are the most pertinent ones for secondary school geometry (De Villiers, 2010; Lim, 1992; Mason, 1998; Mason, 1998). This study therefore only focused on the first three levels. According to Van Hiele (1984), progress from one level to the next involves five phases namely, inquiry, guided orientation, explication, free orientation and integration. Malati (1999b) adapted the five stages from Fuys, Geddes, Lovett and Tischler (1988) and Presmeg (1991) and used the theory to design student material. Each phase involves a higher level of thinking. These phases are useful in designing activities. These phases will be discussed in more detail in Hypothetical Learning Trajectory (HLT) in teaching of circle geometry with GeoGebra.

The first phase is the *information* or *inquiry* phase: The students will in this phase explore with material presented to them. For example, they can explore examples and non-examples (see Figure 2.12 and 2.13 above). This process causes them to 'discover' a

certain structure. According to Howse and Howse (2014) students develop vocabulary and concepts for a particular task. The students' reasoning and interpretation must be accessed by the teacher. This will inform the teacher how to design future tasks. In the second phase, that of *directed* or *guided orientation*, the student explores the field of investigation using the material, for example by folding, measuring, construction, etc. De Villiers (2010) suggests the use of ready-made software activities for students to be on Van Hiele level 1. Most of the students in this study, as mentioned, were not exposed to circle geometry before and the researcher wanted them to explore the properties of circle geometry and visualise them and then design GeoGebra applets. This made it easier to explore the properties and not to focus on construction and to minimise the possible difficulties that students could experience because of low computer literacy.

In the course of the third phase, *explication* or *explanation* takes place. A student becomes conscious of the network of relations and tries to express them in words, and learns the required technical language for the subject matter. For example, the student expresses ideas about the properties of figures. Students learn to express their opinions on structures during the discussion in the class. During the fourth phase, the *free orientation*, the students are challenged with tasks that are more complex and discover their own ways of completing each task. For example, a student may know the properties of one kind of shape but is required to investigate the properties for a new shape, such as a kite. The tasks should be designed so that they can be carried out in different ways. The fifth phase is that of *integration*: A student summarises all that (s)he has learned about the subject, reflects on his/her actions and subsequently obtains an overview of the whole field that has been explored, for example, summarises properties of a figure.

The researcher of this study assessed, as suggested by Mason (1998), the students' Van Hiele level by analysing the students' responses to specific geometric tasks – in this case pre- and post-test for circle geometry. Lim (1992) states that students are on Van Hiele level 1 for new concepts, but can also be on level 2 in other concepts. Lim (1992) adds that if students are consistently justifying their reasons, students show level 3 understanding. In this study the new concepts are circle geometry and straight-line and 2D shapes – in this case triangle and quadrilateral geometry are the other concepts often required to do riders in circle geometry. Research by Wu and Ma (2006) describes how students' responses in a geometry test, for example, can be at level 3 for triangles and level 2 for quadrilaterals, and

level 1 for circle concepts. Patkin and Barkai (2014) also describe how teachers, for example, are on level 3 for triangles and quadrilaterals concepts, and level 2 for circle concepts. The researcher in this study also assessed students' Van Hiele levels by using straight line, triangle, quadrilateral and circle geometry concepts. Hence the study reports on the students understanding levels of straight line, triangle and quadrilateral geometry as their Grade 8 – 10 Euclidean Geometry Van Hiele levels. Their understanding of circle geometry will be reported as their circle geometry Van Hiele levels. If students show no understanding of circle geometry concepts, they will be assessed as at level 1, but if they used straight line, triangle and quadrilateral geometry, and justifying statement, they will be assessed at level 3 for Grade 8 – 10 Euclidean Geometry. The researcher utilised the criteria to assess students' Van Hiele levels.

Van Hiele	Descriptors based on research
Level 1 Recognition level	<ul style="list-style-type: none"> • Student compares angles and sides visually to determine size (Hoiberg & Sharp, 2001). • Students cannot generalise how properties are interrelated (Malloy, 2002). • Any new concepts (Lim, 1992), for example circle geometry. • Students judged the sketch/figure by its appearance (Van Hiele, 1986;
Level 2 Analysis Level	<ul style="list-style-type: none"> • Students analyse the properties of figures and learn the appropriate technical terminology for describing them, but they do not interrelate figures or properties of figures (De Villiers, 2010). • A figure is no longer judged because "it looks like one" but rather because it has certain properties. Properties are not logically ordered (Van Hiele, 1999). • Cannot recognise which properties are necessary and which are sufficient to describe the object. (Mason, 1998). • Students determine the angle size and give the relationship between two given angles. Student does give generalisation (Hoiberg & Sharp, 2001). Students give angles equal or angles are 90°. Students give the relationship between two (2) given angles, but do not supply sufficient reasons. • At this level pupils, or students, perceive relationships between properties and figures. They create meaningful definitions. They are able to give simple arguments to justify their reasoning (Vojkuvkova, 2012).
Level 3 Ordering Level	<ul style="list-style-type: none"> • Students are able to follow simple deductions and understand logical deductions between consecutive steps. (Lim, 1992) • According to Van Hiele (1986) students start reasoning to justify arguments. • Properties are logically ordered. They are deduced from one another, one property precedes or follows from another property (Van Hiele, 1999). • Students provide reasons for their statements (Hoiberg & Sharp, 2001), e.g. why angles are equal, or an angle is equal to 90°, or sides are equal.

Although many studies all over the world have demonstrated that the Van Hiele theory can help improve geometric understanding, it was not taken into account in the teaching of Euclidean geometry in South Africa. There is no indication in the CAPS (DBE, 2011a) that teachers have to teach Euclidean geometry according to the Van Hiele theory. Teachers believe they express themselves clearly and logically, but their reasoning is not understandable to students at lower levels.

2.6 A THEORETICAL FRAMEWORK FOR THE PROCESS OF COHERENCE FORMATION

Seufert and Brünken (2004) contend that from a cognitive position it is important to know how students operate while learning with multiple representations. The structure mapping theory of Gentner (1983) originates from learning with analogies, but can be modified to the link processes between multiple representations (Seufert & Brünken, 2004). Students often find certain mathematical concepts too abstract to comprehend and teachers can utilise analogies to help them with conceptual understanding (Amir-Mofidi, Amiripour, & & Bijan-Zadeh, 2012). Features of coherence formation are embedded in the structure mapping theory (Seufert & Brünken, 2004).

There are two different levels of helping strategies for coherence formation in learning with multiple representations, i.e. help for coherence formation on surface feature and deep structure level (Seufert & Brünken, 2004). Firstly, the help on the surface level refers to, for example, how students could observe the changes in a graph if they change the parameters in the equation. These were the type of activities that the students in this study were doing. They were viewing graphs with GeoGebra applets and they could see the transformations of the graph if they moved the sliders on the applets. The sliders represented the different parameters, for example, a, b, p and q in $y = a \cdot b^{x+p} + q$, with $b > 0$ and $b \neq 1$.

Secondly, the help for coherence formation on a deep structure level focuses on how the students' attentions should be guided to the significant sections of the representations. Bodemer, Ploetzner, Feuerlein and Spada (2004) point out that the students' deeper understanding of symbolic and pictorial representations can be improved by actively integrating the representations. This was done in this study by affording the students a chance to type in Geogebra, for example, $f(x) = \frac{-4}{x-2} + 3$ and then to check the transformation of $f(x + 1) - 5$ in graphic view in GeoGebra and the algebraic manipulation in the algebraic view. The students gathered from the algebraic view in GeoGebra that the

steps to the final answer is $f(x) = \frac{-4}{x+1-2} + 3 - 5$. The students were guided to link their graphic with the algebraic view. They also had to simplify the notation given in algebraic view to $f(x) = \frac{-4}{x-1} - 2$.

2.7 HYPOTHETICAL LEARNING TRAJECTORY (HLT)

According to Gravemeijer (2004) RME conceptualises learning paths developmentally in the same way as Simon's (1995) learning trajectories. RME's objective is to design support materials by trying to explain learning paths along which students could reinvent conventional mathematics. These learning paths are paved with instructional activities that can function as stepping stones in this conjectured reinvention process. Freudenthal (1973) speaks of thought experiments that are followed by instructional experiments in a cyclic process of trial and adjustment. As mentioned in Chapter 1 (see paragraph 1.3), Stehlíková (n.d.) states that teachers experience challenges daily in presenting certain topics in Mathematics in a way that students can enjoy and also be effective. The challenges are also for teachers to know where their students are. This was also a challenge for the researcher in this study because the participating students were from 37 different schools all over South Africa. The National Diagnostic Report (DBE, 2011b) also helped to inform the researcher where the students generally experience problems with the topics that are covered in this research.

Freudenthal sees mathematising as the core of students' activities in reinventing mathematics. In such an approach, material and visual representations may be used by the students as means of scaffolding and communicating their own ideas while constructing more sophisticated mathematics (Gravemeijer, 2011). However, students are not expected to reinvent all mathematics concepts by themselves. According to Freudenthal (1973), students should be afforded the opportunity to experience the way the concept was discovered. For the researcher of this study, it implies that a learning route should be mapped out that might allow the students to experience the process of guided reinvention. To do so, the researcher starts with envisioning a learning route in which students may engage and develop their thinking and learning, as they participate in the instructional sequence. This sequence is designed to achieve the intended learning goal of understanding transformation of functions and the conditions and conclusion of the circle geometry theorems and corollaries. This envisioned learning route is called a Hypothetical Learning Trajectory (HLT), as proposed by Simon (1995). Simon (1995) refers to HLT as

the teacher's prediction of the route by which learning may proceed. It is hypothetical because the actual learning trajectory is unknown in advance and the teacher anticipates how the student reacts. (Empson, 2011; 2005; Simon, 1995). Although the material was designed at the start of the intervention, it was adjusted according to the students' reaction to the activities. Simon (1995) also indicates that the teacher's learning goal provides a direction for a hypothetical learning trajectory. The hypothetical learning trajectory, according to Simon (1995), is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities.

Clements and Sarama (2004) see a HLT as involving conjectures about both a possible learning route that aims at significant mathematical ideas, and a specific means that can be used to support and organise learning along this route. The trajectory, according to Clements and Sarama (2004), is conceived of through a thought experiment in which the historical development of mathematics is used as a heuristic. Clements and Sarama (2004) describe the original design as a set of instructional tasks with guidelines suggesting an order for the tasks and the types of thinking and learning in which the students can engage as they participate in the instructional tasks. The researcher developed learning trajectories for the teaching of function and circle geometry. The material used for both functions and circle geometry was designed to guide the students. Students were expected to play with the applets, or construct their own activities in GeoGebra and then formalise the mathematics from the activity. Learning trajectories, according to Clements and Sarama (2009), open up windows to viewing young children and mathematics in new ways, making teaching more joyous, because the mathematical reasoning of the student is impressive and delightful. The purpose of this research was also to create an environment in which students can enjoy mathematics but, at the same time, take them to a higher level of thinking. The researcher used guided-reinvention with designed activities and GeoGebra to create an environment in which students could enjoy mathematics. The learning trajectories of how these topics were taught is fully described in Chapter 4. The researcher thus designed a teaching approach based upon what he thought the students should know about these topics before starting with it, and then guided them along a path on which he wished them to be.

2.8 SUMMARY

This chapter has reviewed cognitive constructivism, radical constructivism and social constructivism that are amongst the most prominent and espoused variants of constructivism. RME, viewed by different researchers as a type of social constructivism, with its six main tenets, was also discussed in this chapter. The RME theory provides possibilities to equip learners with problem-solving skills, hence the researcher envisages utilising RME as the main conceptual framework in this study. The chapter has also reflected on the misconceptions and criticism of both constructivism and RME. These criticisms and misconceptions also informed the researcher to take cognisance of these in the designing of the activities for students. The chapter further discussed the five Van Hiele Levels and also the five phases within each level. The researcher starts with envisioning a learning route in which students may engage and develop their thinking and learning as they participate in the instructional sequence designed to achieve the intended learning goal of understanding the transformation of function and circle geometry. In view of this he thus concludes this chapter with a general overview on HTL.

Chapter 3 presents further theories that relate to technology and artefacts which support the constructivist, RME and Van Hiele theories.

The focus of the next chapter will therefore be on the role of Information Technology Communication (ICT), with the focus on GeoGebra in the teaching and learning of circle geometry and transformations of functions.

CHAPTER 3

TECHNOLOGY AS A PEDAGOGICAL TOOL IN THE TEACHING AND LEARNING OF MATHEMATICS

“I’m always ready to learn, although I do not always like being taught.”

– Winston Churchill

3.1 INTRODUCTION

This chapter starts with the role of technology in teaching and learning in mathematics, then proceeds with the nature of dynamic software. A summary of theoretical frameworks that support technology as a pedagogical tool in mathematics education is also presented in this chapter. The chapter is divided into sections: Instrumental approach and Technological Pedagogical and Content Knowledge (TPACK). These theoretical perspectives are supplementary and directly linked to constructivism and some tenets of RME. The instrumental approach is further discussed with a focus on the instrumental genesis of students and the instrumental orchestration of teachers. The chapter also reviews literature on bridging or foundation programmes and GeoGebra as a teaching or learning tool for mathematics, and more specifically focussing on the study of transformations of functions and circle geometry.

3.2 THE ROLE OF TECHNOLOGY IN TEACHING AND LEARNING MATHEMATICS

The use of technology is not new to the teaching of Mathematics. History shows that slide rules, compasses, calculators and, recently, graphic calculators and computers have been used by mathematicians to simplify the doing of Mathematics (Durmuş & Karakirik, 2006). Already in the 1980s the computer demonstrated potential to be used for the teaching of Mathematics. Many countries are in the process of reforming the teaching and learning of mathematics by including technology in mathematics education (Mehanovic, 2009). Ndlovu, Wessels and De Villiers (2011) also allude to the fact that there is increasingly firm motivation to integrate ICTs into teaching and learning of mathematics in various countries of the world today. They add that this global excitement is the result of the global increase of digital technologies. Technology is changing by the day and this will bring more pressure

on schools to integrate it into the curriculum. Technology is not going anywhere, it is here to stay.

South Africa is no exception to this excitement as is shown by the emphasis on the use of available technology in the CAPS (DBE, 2011a). Although the ICT readiness in South Africa (Dutta, Geiger, & Lanvin, 2015) is still very low, there is an emphasis on the inclusion of technology in mathematics education. Preiner (2008) points out that, although many teachers and students have access to computers and appropriate software is available both in schools and at home, technology is rarely integrated into the daily teaching of mathematics. The teacher-researcher therefore added GeoGebra to the teaching of circle geometry and transformations of functions with the view to expose students to technology. GeoGebra is freely available and teachers, or even students, who have access to computers or smartphones, can easily download from the internet and install it. Students can even use it in their free time on smartphones or computers.

Ndlovu (2004) argues that since students have different cognitive abilities, and can work at different paces, the traditional teaching approaches of chalk-and-talk consequently appear less effective in a technology mediated environment. This means that mathematics teaching in a technology rich classroom must take into account different learning styles and the didactic relationship between the teacher, the computer and the student. According to De Villiers (2007) and Ndlovu, *et al.* (2011) the teacher-centred approach of teaching, which emphasises manipulative and computational skills before applications, can strongly be challenged by computer technology. They argue that this new or different teaching practice where a student, or small group of students, has access to a computer demands a renegotiation of the traditional didactic relationships between the teacher, mathematics and the student. Thus it can be concluded that the teacher's and student's role in learning processes need to change if technology is going to be used. Teachers have to be aware of students' cognitive abilities and different learning styles.

A laptop, or a computer, connected to a projector also gives the teacher a much wider range of teaching possibilities than a blackboard or overhead projector (Ndlovu, *et al.* 2011). The teacher can in this way expose the students to a different teaching approach. This approach, however, can become a problem if the technology is used like a blackboard or overhead projector. An example is the use of PowerPoint slides as an alternative way to provide text-based notes (Jones, 2003). The use of a laptop or computer with a data projector with

GeoGebra or any other software or technology can however help students to be involved as a whole class in the teaching and learning process and can challenge the traditional approach of teaching. This is one of the approaches, with GeoGebra, that the teacher-researcher used quite frequently during the teaching of different topics – for example solving of trigonometric equations to show students the periodic nature of the trigonometric functions and why trigonometric equations have general solutions. The static blackboard, or overhead projector presentations do not have abilities to show the oscillating effect of a sine, tangent or cosine function. The dragging and animation functions of GeoGebra also make it very easy for students to explore many examples, or different orientations, of geometric shapes.

The sketching of the graphs of functions is important but the DBE (2011b; 2014) identifies the interpretation of functions, as mentioned in Chapter 1, as the main area where students experience challenges. Drawing graphs by plotting points can be a lengthy and time consuming activity. Kelman, *et al.* (1983) allude to this:

“Illustrations on chalkboard or overhead projector often require artistic talent and/or hours of preparation. In contrast, the computer can create colourful, clear, and precise displays quickly and easily. Interactive computer demonstrations in which shapes are changed or rearranged, or variables are controlled and the resulting images compared, make hidden mathematics visible with the touch of a key. Instead of laboriously plotting a particular function on the chalkboard, the teacher can use the computer” (p. 41).

If technology can help to draw graphs neatly, quickly, easily and accurately then the students can focus on the challenging questions of interpreting functions and graphs, and will have more time to explore in their own time. The effects of parameters in the different functions as prescribed by CAPS (DBE, 2011a) can be investigated by pen-and-paper activities, but students can explore many more activities in the same duration of time with GeoGebra. The stretching, shifting and reflecting of graphs can easily be done using dynamic mathematics software. The DBE (2011b) suggests that topics that require visualisation such as transformation geometry can be done with GeoGebra. The report emphasises that “[f]or students to memorise the rules is dangerous, it is better that they discover, ‘figure it out’ for themselves, so that when they need it, they can make a rough sketch and check; for example, what happens to the coordinate when you rotate a point by 180° , etc.” (DBE, 2011b, p. 109). The students in this study used GeoGebra to sketch their different representations of statistical graphs for their project and this saved them time with the

sketching of the graphs, and they could focus more on the interpretation of these representations to answer their research question for the statistics project.

Oldknow and Taylor (2000) identify advantages of using ICT if computers are used by students individually or in a group. The feedback from computer programme responses acts in a non-judgmental way and this can give students the confidence to try again. Students can also respond in their own time and they can refine their strategies. ICTs such as videos, animations, sounds, etc. can also motivate and hold students' attention. In this research GeoGebra has been selected as the software, because it has the features of visualisation, accurate calculation and measurement which enhance active learning. Lastly the major influencing factor was that it is free open source software.

3.3 THE NATURE OF DYNAMIC SOFTWARE

Dynamic software is not static. Drawing on paper is not the same as dynamic drawing. The dynamic drawing can be transformed while it still conserves the characteristics of the object (Sinclair & Yurita, 2008). In this way students can explore many orientations of the figure or object. Students can be shown in dynamic software, in this case GeoGebra, that not all inverse graphs are functions. The CAPS (DBE, 2011a) requires the students to sketch the inverse graph of the following functions : $f(x) = ax + q$; $f(x) = ax^2$ and, $f(x) = b^x$, where $b > 0$ and $b \neq 1$. Figure 3.1 shows the inverse graph of a parabola.

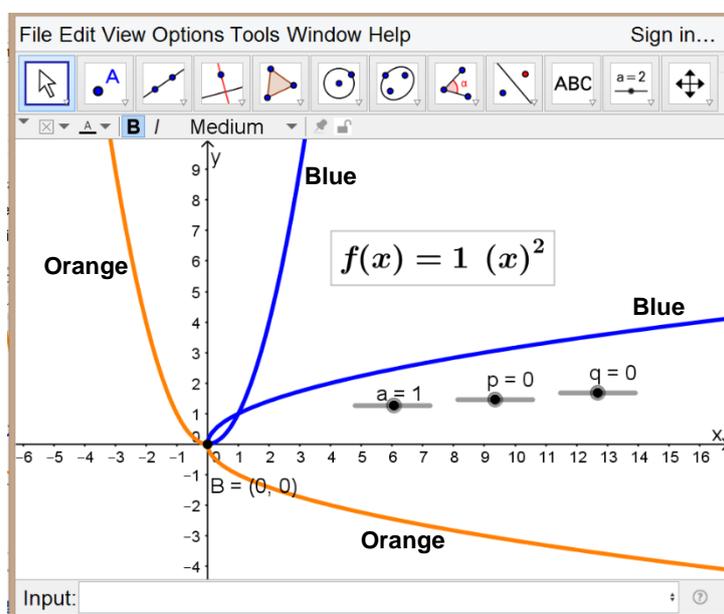


Figure 3.1: GeoGebra applet to teach inverse graphs.

The students also have to know how to restrict the domain to ensure that the inverse is a function. The curriculum requires the students only to know the quadratic function,

$f(x) = ax^2$ and to restrict the domain of $f(x)$ always to be $x \leq 0$ or $x \geq 0$. The curriculum requires students only to change parameter a but p and q in $f(x) = a(x + p)^2 + q$ have to be zero when they do the inverse graphs. The students can just memorise this for the examination, but then have no conceptual understanding of restricting the domain of the function so that the inverse is a function. Figure 3.2 shows a GeoGebra applet that can be utilised to ensure that students have conceptual understanding of restricting the domain of the function to ensure that the inverse is a function.

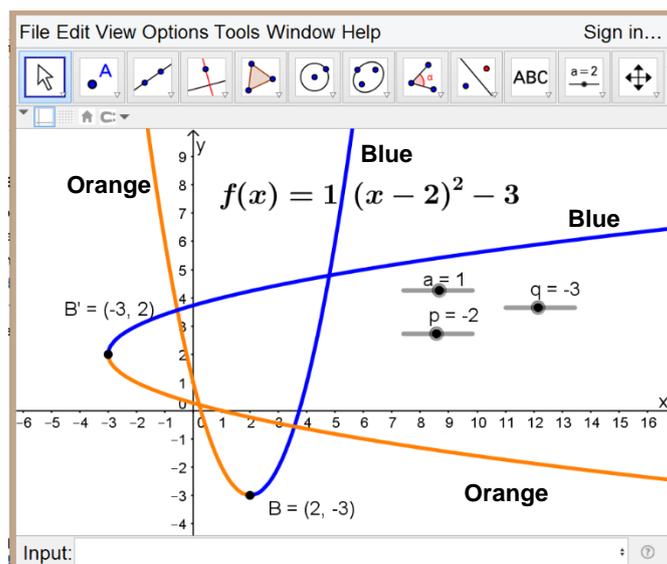


Figure 3.2: GeoGebra applet to teach restricting of the domain of $f(x)$.

By sliding a , p and q the students can explore, or the teacher can show, many orientations of $f(x)$ to ensure that the students can see that the x -value of the turning point of $f(x)$ is determining the domain of the function so that the inverse is also a function.

GeoGebra can be seen as software that creates a Dynamic Geometry Environment (DGE). DGEs allow the user to drag parts of a geometry object, while measurements of the figure change dynamically in an algebra window (Lingefjård, 2015) or the dynamic text in graphics view. Figure 3.3 shows how in the same activity as Figure 3.2 the values of a , p and q can be changed.

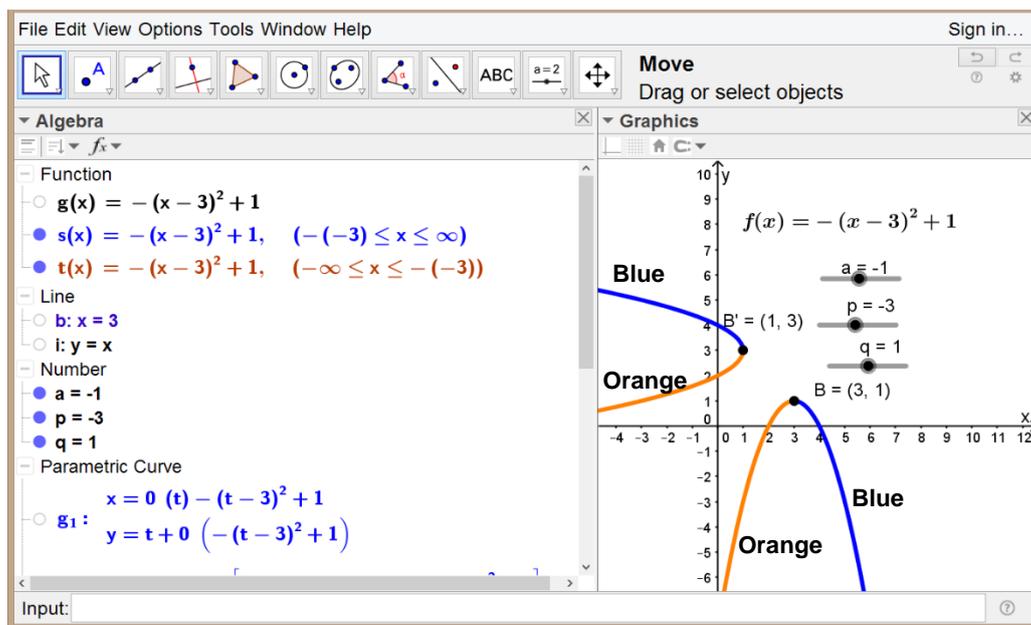


Figure 3.3: GeoGebra applet showing that the graphic view and algebraic view are related. Students can observe how the shapes and equations of the graph change. This can be viewed in the algebraic and graphic window.

DGEs are an interactive environment for visualising and exploring geometry and algebra. GeoGebra has the features of DGE. GeoGebra can be used for geometry, algebra and statistics, etc. and is not created for geometry alone. It can be seen as a dynamic mathematical environment. Hence the researcher views GeoGebra as the favourable tool to use for this research because of its dynamic nature. Students can also view more than one example with one applet.

GeoGebra can help to visualise concepts, explore and discover concepts through activities designed by the teacher. Students can also work together and in that way they can share their knowledge. Özgün-Koca and Meagher (2012) also feel strongly that the virtual environment is much more visually powerful when compared to an environment using only physical manipulatives. Moore (2017) asserts that manipulatives have the ability to help with the cognitive process and also have the advantage of engaging students and increasing both interest in and enjoyment of Mathematics. Xie, Alissa and Nima (2008) emphasize that enjoyment and engagement are integral and essential aspects of children's playful learning experiences. Sutton and Krueger (2002) point out that long-term interest in mathematics transforms to increased mathematical ability. The teacher-researcher of this study therefore wanted the teaching of mathematics to be interesting with the hope to engage students in the learning process. The teacher-researcher subsequently used GeoGebra as the virtual

manipulative to engage students in the learning process. Cockett and Kilgour (2015) found in their research that manipulatives are beneficial for developing mathematics concepts. They further point out that the perception of students is that mathematics manipulatives help them to be more efficient in their work, better understand it and receive greater enjoyment from the learning process. Low levels of student engagement in mathematics have been an area of great concern to mathematics educators and researchers in recent years (Attard, 2012). Mathematical enjoyment is considered particularly significant for addressing student disengagement (Martin, Anderson, Bobis, & Way, 2012). The PISA report of the Organization for Economic Cooperation and Development (OECD, 2004) shows that, within each country, students with greater interest in and enjoyment of mathematics tend to achieve better results than those with less interest in and enjoyment of mathematics. The report does, however, indicate that the strength of this relationship varies per country.

Schroeder (2008) views a virtual environment as a computer generated display that gives the user some sense of being present in an environment different to the one (s)he is actually in. Özgün-Koca and Meagher (2012) point out that hands-on materials and physical manipulatives are helpful, but when exploring higher-level mathematical topics, virtual manipulatives can visualise it much faster and better. Figure 3.4 shows how Özgün-Koca and Meagher (2012) use physical manipulatives to teach the sine-graph.

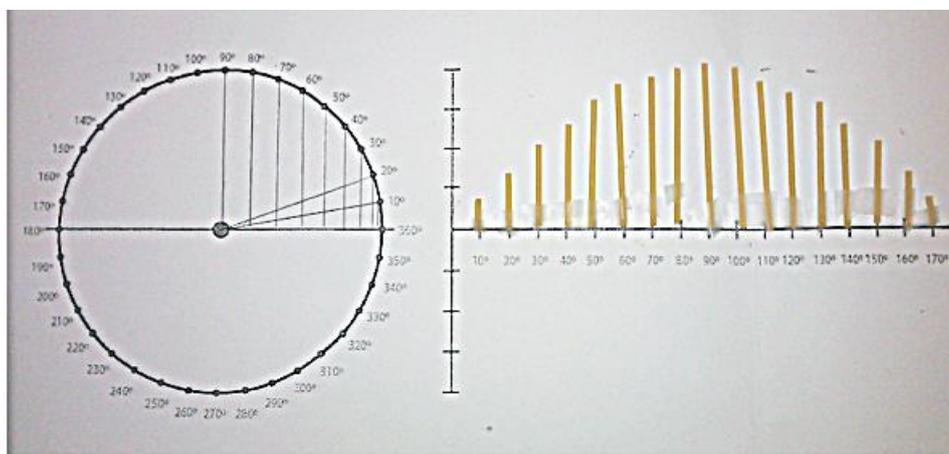


Figure 3.4: Section of sine-graph constructed with spaghetti.

The different lengths of the spaghetti pieces are used to construct the sine-graph. These different lengths of the spaghetti pieces are calculated by finding the opposite side's length of right-angled triangles with different angle sizes. Figure 3.5 shows how Özgün-Koca and Meagher (2012) use the same activity to transform it into a virtual environment.

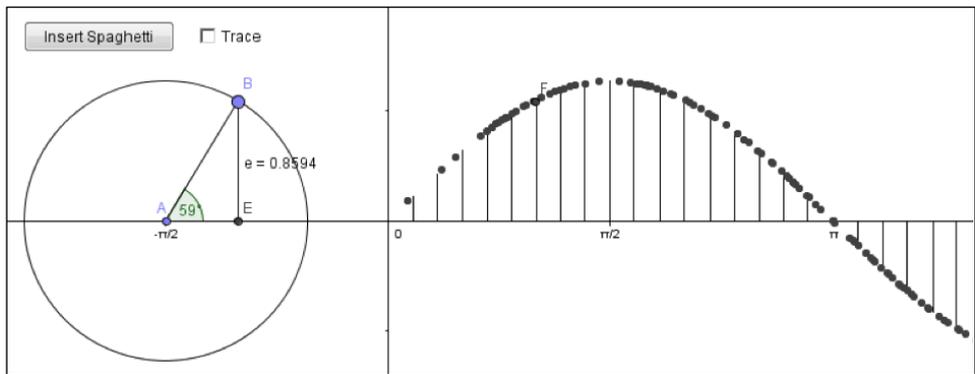


Figure 3.5: Section of sine-graph with tracing tool in GeoGebra.

The lengths of the ‘virtual spaghetti’ pieces, i.e. segment BE, are calculated by dragging point B. GeoGebra visualisation is thus an example of virtual representation of concepts. Students are acting on mathematical objects, obtaining immediate feedback, making observations and testing conjectures in the virtual environment. Shabanova, Yastrebov, Bezumova, Kotova and Pavlova (2014) posit that Dynamic Geometry Software (DGS) will support the development of students' visual thinking but not replace it. The above activity is an illustration of how students can see how the ‘virtual spaghetti’ pieces’ length is changing.

Students can also create and test dynamic models and combine them with pen-and-paper construction activities (Shabanova *et al.*, 2014). GeoGebra was used in some of the designed activities in this study and combined with pen-and-paper activities. The students did not always work on GeoGebra, but the teacher-researcher combined it with the pen-and-paper activities of the students with the expectation of developing students’ visual thinking and reasoning capabilities. Figure 3.6 shows how the researcher in this study utilised pen-and- paper activity for an introduction lesson to trigonometry with GeoGebra.

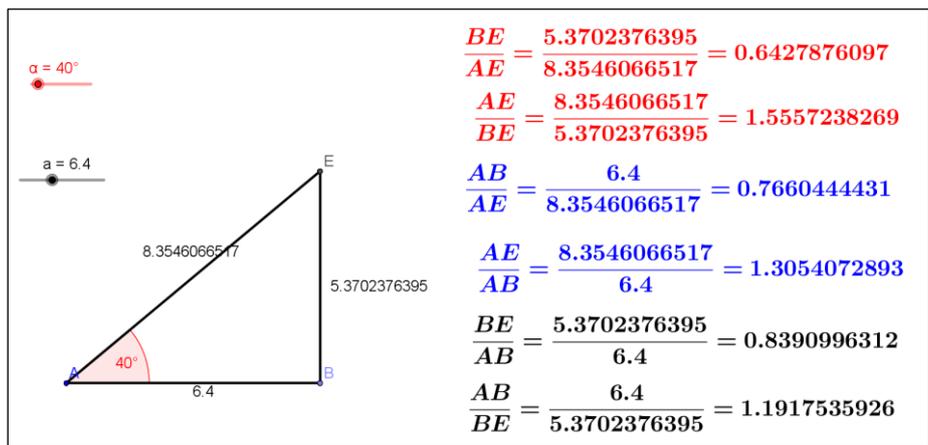


Figure 3.6: Introduction lesson to trigonometry with a GeoGebra applet.

The students constructed their own right-angled triangle on graph paper and calculated each side's length. The teacher-researcher utilised a GeoGebra applet to show the students that their answers were more or less similar to the answers in GeoGebra (see Figure 3.7). When the dynamic figure was dragged, students could see the transformations of the object and see how it happened in real time. That is, the properties of the object 'become obvious' and can also be viewed immediately (Ruthven, 2007). Students can immediately view in the algebraic and graphic windows of GeoGebra what happens to the properties of the object. According to Ruthven (2007) if students are guided towards an intended mathematical conclusion, the students to a great extent discover for themselves without being told. The students thus take ownership of the teaching and learning process. Kimmins and Bouldin (1996) also allude that dynamic software allow students to construct mathematical knowledge rather than memorising theorems and rules. This accords with the guided reinvention principle of RME and constructivism. In this way students can use technology to test conjectures and to construct their own new mathematical understandings.

Lingefjärd (2015) also points out that the dynamic nature of a DGE figure enables the user to observe properties that change and others that never change. The author adds that a DGE provides students an opportunity to measure lengths of line segments and sizes of angles in the virtual space as they manipulate, or drag the mathematical objects back and forth. In this way students can make conjectures by observing the changes in the measurements. One of the fundamental advantages of DGEs is their ability to provide families of diagrams as representing a set of geometrical objects and relations instead of a single static diagram (Lingefjärd, 2015). In agreement with this perspective researchers have described a DGE as a 'microworld' which provides rich opportunities for students to make and test conjectures (Lingefjärd, 2015) and claimed that it also exposes students to an inquiry-based learning experience (Ruthven, Hennessy, & Deaney, 2005). This feature of DGE was, therefore, for the researcher of this study, not only less time consuming but more illustrative than pen-and-paper activities. In this way students could spend time to make conjectures. By doing so they could engage in critical thinking. The focus of the study was to create a learning environment so that students could construct mathematical knowledge and move to higher levels of abstraction.

Ruthven *et al.* (2005) further point out that many researchers cite improvements in the classroom atmosphere, an increase in motivation levels of students and the efficiency in its

ability to show many examples at once as some of the reasons for incorporating dynamic geometry in their classrooms. In a similar way, Petre (2010) states that the purpose of GeoGebra is to stimulate students' imagination and facilitate the transfer of learning to everyday life, both of which often happen when the motivation to learn is high. Surely the mere use of the software will not help us as much if it is not combined with teaching techniques, student-centred teaching approaches and active learning methods such as solving problems creatively, critical thinking, and learning through discovery, practice and experiments. The researcher concurs with Ruthven *et al.* (2005) and Petre (2010) because motivated students and a class atmosphere that is conducive to learning are important factors for effective teaching and learning of mathematics. Hence GeoGebra was also used with the expectation that students would start enjoying what they are doing and increase their motivation levels.

As Gerhäuser, Valentin and Wasserman (2010) point out some of the dynamic software, such as GeoGebra, Cinderella and Geonet can be installed on computers but can also run on web pages because the vendors offer a Java-viewer with their software. Interactive learning, or dynamic worksheets can be embedded in web pages and students can also explore Mathematics in school, or even at home, without necessarily installing the software. GeoGebra is one of the dynamic software vendors that developed their software to be compatible to run on multi-touch devices such as smartphones, e-tablets, Apple ipads, etc.

3.4 COMPUTER TECHNOLOGY SUPPORTS A CONSTRUCTIVIST APPROACH AND RME APPROACHES

Researchers such as Dubinsky (1991), Hooper and Rieber (1995), Jonassen, Peck and Wilson (1999), Malabar and Pountney (2002), Amarin and Ghishan (2013), and many more, attest to the fact that if technology is used correctly and appropriately it supports the constructivist approach of learning. These researchers point out that technology can be a tool that supports students in representing their ideas, articulating what they know, and exploring, manipulating and processing information, while actively collaborating with each other. Technology can be built into activities and that allows students to engage in routine-based and discovery-based activities. Dubinsky (1991) also points out that the constructivist's use of the computer is a powerful way of providing students with experiences that convert the concrete into the abstract. A real-life activity can be brought into the classroom with technology and the mathematics can then be abstracted from it. Figure 3.7

shows an example of a tourist attraction that can be utilised in real-life to a parabolic function (Furner & Marinas, 2013).



Figure 3.7: Picture of a tourist attraction to teach the parabola in real-life.

In this way the students can see that Mathematics is not only abstract as Freudenthal (1973) points out: Mathematics is a human activity. Schmidt, McGee, Scott, Kirby, Norris, & Blaney (2002) allude to the studies done by Howard, McGee, Schwartz and Purcell (2000) and Vannatta & Beyerbach (2000), which found that technology integration may result from exposure to constructivist teaching strategies with technology used as a supplement to enhance these strategies.

Numerous lessons with the computer are often designed for single users but benefits appear to multiply when used collaboratively (Hooper & Rieber, 1995). Students can work on their own on a computer, but it also affords them a chance to interact with one another. Swanepoel and Gebrekal (2010) assert that computer technology can help students to explore and visualise concepts so that they will be able to solve 'more complicated mathematical problems'. Research done by Swanepoel and Gebrekal (2010) in Eritrea on the concept of the quadratic function with MS Excel and RJS Graph shows that the students could explore the nature and the properties of the quadratic function and make conjectures and verify their findings. As mentioned in Chapter 1 (section 1.2.2.3) candidates in the South African NSC Mathematics examinations struggle to sketch graphs from the characteristics of the functions. They add that technology "allows [students] to participate actively and be responsible for their own learning" (Swanepoel & Gebrekal, 2010, p. 404). The students are thus taking responsibility for their own learning, by exploring and making conjectures. In this way students can make decisions applying their critical thinking. This is one of the objectives of the CAPS (DBE, 2011a).

Becta (2003) summarises the key learner-learner benefits of ICT integration by highlighting that it promotes greater collaboration amongst students and encourages communication and the sharing of knowledge. The students have to change their roles in the learning process. Instead of being passive receivers, the students have to interact with the technology and, with the guidance of the teacher, to understand the mathematics that is represented on the computer or software interface. Working together and sharing knowledge are the main tenets of RME and constructivist approaches to teaching and learning. As has already been seen, technology can help students to think critically. Consequently, this observation strengthens the main theoretical perspectives of this study, i.e. constructivism and the three RME principles of learner activity, guided reinvention and learner-learner interaction. Students can listen to what others discover and discuss these findings and, in so doing, improve their strategies. Moreover, the interaction can evoke reflection, which enables the students to reach a higher level of understanding. RME also does recognise individual learning paths and sees students as individuals (Van den Heuvel-Panhuizen, 2000). Students can work on their own computers, or even in a group, or as a whole class.

Keong, Sharaf and Daniel (2005) also confirm that the use of technology provides students with the opportunity to explore and gain understanding of mathematical concepts. They thus agree that ICT supports constructivist pedagogy. They add that this approach promotes higher order thinking and better problem solving strategies. The National Diagnostic reports of the DBE (2011b; 2012a; 2013; 2014) also recommend higher order thinking and better problem solving skills. The fact that technology can promote higher order thinking also ties up with the level principle of RME. This study also seeks to understand how GeoGebra can offer students an opportunity to understand mathematical concepts and move to higher levels of abstraction.

Eskelinen and Haapasalo (2006) discuss how different kinds of approaches and support for reflective communication affect students' conceptions of teaching and learning, group dynamics and interest in ICT support. The results of their report show that the design of a technology-based learning environment within an adequate constructivist theory, linked to the knowledge structure, offers a promising response to the main challenge of student learning: to get students to understand the basic components of learning. Students cannot be given only technology and be expected to use it effectively for them to understand mathematical concepts better. Students must also have an understanding of how technology

will be integrated in the learning of mathematics. It can be a barrier if students are not well acquainted with the technology or constructivist approaches of learning. It is for these reasons that the researcher familiarised his students with how learning happens and showed them sections of the National diagnostics reports on how students performed in certain topics, such as functions, circle geometry and trigonometry, of the curriculum and what the reports recommend. The researcher justified in this way why an active approach to learning is important. As mentioned in Chapter 1 (section 1.2.1) the students who were participants in this study were also doing Academic Literacy and Language, Thinking and Studying Skills, where they were also exposed to how the brain learns, as well as to learning and critical thinking strategies.

Roschelle *et al.* (2000, p. 82) acknowledge that “[c]omputer technology can provide students with an excellent tool for applying concepts in a variety of contexts, thereby breaking the artificial isolation of school subject matter from real-world situations”. Video clips of a rotating wind turbine can be projected to students who have never seen one and the mathematics can then be explored. In other words, technology can be used also to strengthen the reality principle of RME alluded to in Chapter 2 (section 2.4.1.2). These students were also exposed to realistic context activities and the researcher used images of real life situations.

3.5 THEORETICAL FRAMEWORKS THAT SUPPORT TECHNOLOGY AS A PEDAGOGICAL TOOL IN MATHEMATICS EDUCATION

The researcher in this study decided to explore two theoretical frameworks that can underpin successful integration of digital technologies into everyday teaching practice: Trouche’s (2004) Instrumental Approach and Mishra and Koehler’s (2006) Technological Pedagogical and Content Knowledge (TPACK) theory.

3.5.1 Instrumental approach

Trouche (2004) points out that one of the important aspects of the use of technology in education is human/machine interaction. Trouche (2004) discusses the interaction in relation to computerised learning environments and argues that a learning experience is not neutral. The learning environment insists on students’ creativity and action. Mehanovic (2011) uses the instrumental approach in order to understand the processes guiding how the students and the teachers use GeoGebra in their mathematical work, and how the students’ use of the software can be characterised. The instrumental approach explains these processes as the interaction between the tool on the one hand and the student or the

teacher as the user of the tool on the other hand. According to Mehanovic (2011) the tool in the hand of the user, whether a teacher or a student, transforms to a mathematical instrument during the students' mathematical activity. Ndlovu, *et al.* (2011; 2013c) view that the instrumental approach helps teachers to comprehend the impact of tools on the mathematical approach, as well as the building of students' conceptual understanding. The foundation of the instrumental approach is based on Vygotskian and Piagetian perspectives (Drijvers & Trouche, 2008; Maschietto & Trouche, 2010). The instrumental approach links itself to constructivism and is the approach that partly supports this study in terms of the technological approach to teaching and learning.

The instrumental approach acknowledges the complexity of using technology within mathematics education (Artigue, 2002). Drijvers and Trouche (2008) point out that the instrumental approach allows for an analysis of the learning process in technological environments of increasing complexity, and takes into account the non-trivial character of using computerised environments. Moreover, it stresses the subtle relationship between machine technique and mathematical insight, and provides a conceptual framework for investigating the development of schemes in which both aspects are included. They also contend that it is helpful to observe interaction between the students and the computer algebra environment, to interpret the interaction and to comprehend what works and what does not. The teacher-researcher felt that the GeoGebra CAS view was not needed for the purpose of the study. The algebraic and graphic views were utilised. If calculations were needed, the students were obliged to do them by hand. They further contend the instrumental framework helps the teacher to be conscious of technical obstacles that are observed while students are working on a computer. These obstacles observed by the teacher can be turned into learning opportunities. The interaction principle of RME, in this study, investigated how the students interacted with each other but the interaction between the students and GeoGebra were also observed. The students' experiences and gestures whilst working with GeoGebra were also investigated.

Mehanovic (2009) emphasises that an essential starting point in the instrumental approach is the distinction between an artefact and an instrument. Trouche (2004) describes an artefact as a given object and an instrument as a psychological construct:

"The instrument does not exist in itself, it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it with his activity" (Trouche, 2004, p. 285).

Kieran and Drijvers (2006) view the artefact as the object that is used as a tool, while the instrument involves the techniques and schemes that the user develops while using it. This guides both the way the tool is used and the development of the user's thinking. GeoGebra is the artefact that becomes the instrument that will attempt to guide the development of the students' thinking. Ndlovu et al. (2013c) utilised the instrumental approach and the Van Hiele theory to explain participants' proficiencies with technology. Research by Drijvers (2012) shows that the instrumental approach is a perspective that can provide tangible guidelines for both the design of student materials and the analysis of student behaviour when interacting with the artefact. This is the reason why the researcher decided to explore the instrumental approach in this study. Activities with pen-and-paper and technology were designed for this research alongside the GeoGebra ones. The instrumental approach also linked perfectly with some of the sub-questions of this study that investigated the processes that were involved in using GeoGebra as a pedagogical tool; the students' explanations of how GeoGebra afforded them an opportunity to understand mathematical concepts and move to higher levels of abstraction, and what challenges students experienced whilst working with GeoGebra.

3.5.1.1 The theory of instrumental genesis (TIG)

The theory of instrumental genesis (TIG) ascribes a major role to artefacts that mediate the human's activity for carrying out a task (Drijvers, Godino, Font, & Trouche, 2013b). Trouche (2004) posits that instrumental genesis is a complex, time consuming process, linked to the artefact's potentialities and constraints, but also to the user's knowledge and method of working. The students were from 37 schools, as mentioned in Chapter 1, and their previous approach to learning was challenged throughout the year. Students did not in the beginning show understanding of this different approach. It was a challenge for the students to unlearn certain ways of doing mathematics.

Trouche (2004) describes instrumental genesis as a combination of two processes: an instrumentation process and an instrumentalisation process. According to Fahlgren (2015) instrumentation is how the user shapes the artefact by his/her knowledge and previous working approaches. Fahlgren (2015) adds that instrumentalisation is how the artefact

shapes the user by its constraints and potentialities. Ndlovu et al. (2011) also view instrumentation as the process by which the user of the artefact is mastered by his or her tools, or by which the artefact impresses the user by allowing him or her to develop activity schemes within some boundaries. Such limits include constraints which oblige the user in one way and impede in another; enablements which effectively make the user able to do something, and potentialities which virtually open up possibilities and affordances which favour particular gestures or movement sequences (Noss & Hoyles, 1996; Trouche, 2004).

Guin and Trouche (1999, 2002) distinguish between three types of constraints: internal constraints intrinsically linked to the hardware, command constraints linked to the syntax of the various commands, and organisation constraints linked to the interface between the artefact and the user (e.g. symbolic, numeric and/or graphic). Guin and Trouche (1999) suggest that teachers should highlight the constraints and limitations of the software to students in order to support their instrumental genesis. All tools and resources have constraints and limitations (Bretscher, 2009). The instrumentation process in this study was thus how GeoGebra shaped the thinking of the students and how it helped them to better understand concepts.

Trouche (2004) points out that instrumentalisation has various stages: a stage of discovery and selection of the relevant functions, a stage of personalisation and a stage of transformation of the artefact, sometimes in directions unplanned by the designer. The instrumentalisation process in this study was how the students used GeoGebra on their own to be a tool, for example, to validate their answers and test their conjectures. It was also how students viewed GeoGebra as the preferred pedagogical tool: as a teaching/learning tool used by the teacher-researcher, or by students, using applets or constructing their own activities. The study also focused on the processes of how students changed their perceptions on technology as a teaching and learning tool.

Trouche (2004) differentiates between two types of artefact utilisation schemes: usage schemes oriented towards the organisation of the artefact and instrumented action schemes oriented towards the performance of specific tasks. Ndlovu *et al.* (2011) view the turning on of a computer, adjusting the screen contrast, etc. as examples of usage schemes and computing a function's limit as instrumented action schemes. Fahlgren (2015) describes usage schemes as basic and they relate closely to the artefact while instrumented action schemes focus on actions upon objects such as graphs or formulae.

This study used an instrumental approach to understand the influence of GeoGebra on the mathematical approach and on the building of students' knowledge: through a process – instrumental genesis – GeoGebra becomes a mathematical work tool; this process depends on the tool's constraints and potentialities, on students' knowledge, and on the classwork situations. The teacher-researcher thus viewed the instrumental genesis as an appropriate tool to analyse observations of students' behaviour within the dynamic geometry environment and in the context of the RME principles.

3.5.1.2 Instrumental orchestration

The notion of orchestration refers to how the teachers are transforming resources, or artefacts, into orchestration (Drijvers, 2011). Trouche (2004) introduced the metaphor of instrumental orchestration. Trouche (2004) defines instrumental orchestration as the premeditated and systematic use of different technological artefacts by the teacher in a learning environment. According to Drijvers, Doorman, Boon, Gisbergen and Reed (2009) and Trouche (2004) the students' instrumental geneses need to be guided, monitored and orchestrated by the teacher. The theory of instrumental orchestration is meant to answer the question of how the teacher can fine-tune the students' instruments and compose coherent sets of instruments, thus enhancing both individual and collective instrumental genesis (Drijvers, Doorman, Boon, & Van Gisbergen, 2010a; Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010b).

The role of the teacher in supporting instrumental genesis is partly to make the technical and conceptual elements explicit (Bretscher, 2009). Guin and Trouche (1999) suggest that teachers should highlight the constraints and limitations of the software to students in order to support their instrumental genesis. The teacher-researcher explained, while using GeoGebra, how the different viewing windows in GeoGebra work, and also limitations of the software. An example of a limitation in terms of the South Africa curriculum is the notation of coordinates. South Africa still uses the $(x ; y)$ notation and GeoGebra uses the (x , y) notation. Another constraint that was shared with the students was that GeoGebra and many other software do not show discontinuities in graphs and also do not draw the asymptotes. The teacher-researcher also showed the students the possibilities of GeoGebra.

The instrumental orchestration model brings about a double-layered view on instrumental genesis. Firstly, the instrumental orchestration focus is to enhance the students' instrumental genesis. Secondly, the orchestration is instrumented by artefacts for the

teachers, which may not necessarily be the same artefacts that the students use (Drijvers *et al.*, 2009). An instrumental orchestration consists of three features: a didactical configuration, an exploitation mode and a didactical performance. According to Drijvers *et al.* (2009) an instrumental orchestration is partially prepared before the lesson, partially created while the teaching takes place and reflects on the actual and didactical performance. Drijvers *et al.* (2009) define a didactical configuration as the teaching setting and the artefacts involved in it.

They used the musical metaphor of orchestration to describe how the teacher selects the musical instruments (artefacts) for the orchestra and arranges them in such a way that the various sounds result in harmony. The artefacts involved in this study were the pen-and-paper activities and GeoGebra. For most of the students in this study, it was their first time working with GeoGebra and also the first time that they explored concepts in mathematics. It was important for the teacher-researcher to grant the students time to get used to this approach of teaching. Students first explored with pen-and-paper activities while the teacher-researcher used GeoGebra as a teaching tool. They only worked with GeoGebra themselves in transformations of functions and circle geometry. The teacher-researcher used this approach with the expectation that students would not be too unfamiliar with the software. The teacher-researcher used some of the tools in GeoGebra that they needed when they worked with it for the first time.

A teacher's decision to exploit a didactical configuration for the benefit of his/her didactical objectives refers to the exploitation mode of instrumental orchestration (Drijvers, 2011; Drijvers *et al.*, 2009). This decision includes how the task is introduced and approached by the students and the possible roles of the artefacts (Drijvers, 2011; Drijvers *et al.*, 2009). They also add that schemes and techniques that need to be developed and established by the students are included in this decision. Drijvers *et al.* (2009) view these decisions in the exploitation mode as part of a Hypothetical Learning Trajectory (HLT) (Simon, 1995). In terms of the metaphor of orchestration, setting up the exploitation mode can be compared with determining the partition for each of the musical instruments involved, bearing in mind the anticipated harmonies that will emerge. Ndlovu, *et al.* (2011) describe an exploitation mode which involves the teacher's judgments to exploit a didactical setting, or configuration, for the benefit of the teacher's instructional trajectory. These decisions include how the activity, or topic, will be introduced and what the teaching approach will be (whole-class,

group-work, individual, etc.). An HLT for the teaching of transformations of functions and circle geometry is fully discussed in Chapter 4. These learning trajectories explain what activities and artefacts were used and why they were designed in that way. GeoGebra was the artefact used but was not used in all the designed activities.

The didactical performance involves on-the-spot decisions taken by the teacher while teaching on how to actually perform in the chosen didactic configuration and exploitation mode (Drijvers, 2011; Drijvers *et al.*, 2009, Ndlovu, *et al.*, 2011). These decisions involve what questions to ask now, how to deal in a positive way with any particular student's input, how to deal with unexpected aspects of the mathematical activity, or the technological tool, or any other emerging issues. In the metaphor of orchestration, the didactical performance can be compared to a musical performance, in which the actual interplay between conductor and musicians reveals the feasibility of the intentions and the success of their realisation (Drijvers, 2011; Drijvers *et al.*, 2009). The type of questions that were posed are discussed in the data analysis of this study because they depended on the students' reactions and conceptualisations.

The teacher-researcher identified the following types of orchestrations based on the research done by Drijvers *et al.* (2009), Drijvers (2011) and, Drijvers, Tacoma, Besamusca, Doorman and Boon (2013a) : Technical-demo, Explain-the-screen, Link-screen-board, Discuss-the-screen, Spot-and-show, Sherpa-at-work, Guide-and-explain. Drijvers *et al.* (2013a) divide *Technical-demo*, *Link-screen-board*, *Discuss-the-screen* and *Work-and-walk by* orchestrations into whole-class and individual orchestrations. The whole class approach will therefore be predominantly a teacher-centred approach, whereas the individual approaches are more interactive. The influence of the hybridisation of the teacher as a facilitator of learning and student-centred approach was also the focus of this study.

The *Technical-demo* orchestrations are the techniques utilised by the teacher (Drijvers, 2011). A didactical configuration for this orchestration includes access to the applet and the Digital Mathematics Environment (DME), facilities for projecting the computer screen, and a classroom arrangement that allows the students to follow the demonstration. Drijvers (2011) describes a DME as a "web-based environment, which integrates a content management system, an authoring tool and a student registration system, and which already contains content in the form of an impressive amount of applets and modules" (p. 6). A student can thus connect to the internet and access the work from any place. The students'

work can be monitored by the teachers. As exploitation modes, the teachers utilise a technique in an activity, or use a student's work to show new techniques in anticipation of what will follow (Drijvers, 2011). In this study students accessed GeoGebra and all the applets from the shared folder on the University server. GeoGebra does not have a DME. GeoGebra tube is the web-based environment of GeoGebra where anyone can register and download applets. GeoGebra has only a web-based environment, where applets can be downloaded and uploaded. The students had only reading rights on the folders that permitted viewing or accessing of the file and folder contents. They were given the installation file for installation on their own computers and were given the link to the installation file on the GeoGebra website. The teacher-researcher mentioned the GeoGebra tube and how to find it, but did not show them how it works. The students used the portable version of GeoGebra that doesn't require any installation on a computer for the session in the computer lab.

The *Explain-the-screen* orchestration refers to where the teacher explains to the class, guided by what happens on the screen (Drijvers, 2011). Drijvers (2011) adds that the explanation goes beyond techniques, and involves mathematical content. Didactical configurations can be similar to the *Technical-demo* ones. Drijvers (2011) adds that, as exploitation modes, a teacher may present a student's response as a point of explanation, or may start with his/her solution. The didactical configurations discussed earlier in this section will be the same for this type of orchestration (Drijvers *et al.*, 2009). This type of orchestration was used frequently by the teacher-researcher to ensure that students understand what the activities were all about. This was to ensure that the students did not focus too much on GeoGebra, but to extract the mathematics from the activities.

In the *Link-screen-board* orchestration the teacher stresses the relationship between what happens in the technological environment and how this is represented in conventional mathematics of paper, text-book and school board (Drijvers *et al.*, 2009). The teacher-researcher planned to do this as frequently as possible in the computer lab and also in class because it was important that students were making the connections between the pen-and-paper activity or what was screened in the GeoGebra algebraic and graphic window views while teaching, or exploring a concept. In addition to DME access and projecting facilities, a didactical configuration includes a school board and a classroom setting, such that both screen and board are visible (Drijvers, 2011). Each student in this study had access to a

computer for lessons in the computer lab and the teacher-researcher had access at all times to a computer, or his own laptop, and an installed data projector. All the lecture rooms at the university where a computer and data projector are installed had whiteboards, or a separate overhead projector, so that both the screen and the board are visible to the students. The teacher-researcher's exploitation modes in this study were the students' work and also suggested tasks or problem situations. The teacher-researcher did not have access to students' work, but could support them at their computers. The GeoGebra applets, or students' own constructions, were used for the teacher-researcher's exploitation modes.

The *Discuss-the-screen* orchestration concerns a whole-class discussion on what happens on the screen (Drijvers *et al.*, 2009). They add that the goal is to enhance collective instrumental genesis. A didactical configuration includes DME access and projecting facilities, preferably access to student work and a classroom setting favourable for discussion (Drijvers, 2011). As exploitation modes in this orchestration, the students' work is viewed as a point of departure, and having students react to it, as well as a task, problem or approach set in such a way to provoke input from students (Drijvers *et al.*, 2009). Although the teacher-researcher did not have access to the students' work, he utilised this type of orchestration in the computer lab to provoke reaction and interaction from the students.

In the *Spot-and-show orchestration*, students' reasoning is brought to the fore through the identification of interesting digital student work during preparation of the lesson, and its deliberate use in classroom discussion (Drijvers *et al.*, 2009). Apart from previously mentioned features, a didactical configuration includes access to the DME during the preparation of the teaching. As exploitation modes, teachers may have the students whose work is shown to explain their reasoning, and ask other students for reactions, or prompt them to provide feedback to the students' work (Drijvers *et al.*, 2009). In this study, this orchestration was not as frequently used as classroom discussion.

In the *Sherpa-at-work* orchestration the student uses the technology, either to present his/her work, or to carry out operations at the teacher's request (Drijvers *et al.*, 2009). Didactical configurations for this orchestration are similar to the *Discuss-the-screen* orchestration type. The classroom setting should be of such a nature that one student can easily manage the technology, with both Sherpa-student and teacher being easy to follow by all students (Drijvers *et al.*, 2009). As exploitation modes in this orchestration, teachers may ask the students to present their own work using technology, or may pose questions to

the Sherpa-student and ask him/her to carry out specific operations in the technological environment. A Sherpa-student for this study was most of the time a student that followed the instructions of the teacher-researcher and the designed activities.

In the *Guide-and-explain* orchestration (Drijvers *et al.*, 2013a), similar to *Work-and-walk by* (Drijvers, 2011) or *Monitor-and-guide* (Tabach, 2013) orchestration the teachers stroll around the classroom while students are working on the computer activities, take a look, answer questions, provide help or explain, and continue walking around in the class. Students work individually, or in pairs, with computers. The exploitation modes for this individual orchestration refer to the individual discussion between the teacher and (a pair of) student(s), or where the teacher takes the position of the instructor through providing guidance and instruction to the student(s) (Drijvers *et al.*, 2013a). Tabach (2013) utilises this orchestration where students work at the computers for the entire duration of the lessons, while the teacher walks around and provides assistance as needed.

The teacher-researcher used the dynamic nature of GeoGebra to highlight the constraints and limitations of the paper-and-pencil environment, exposed possible misconceptions and thereby gave additional support over and above the students' instrumental genesis in the more traditional approach of teaching.

3.5.2 Technological Pedagogical and Content Knowledge (TPACK)

Shulman (1986) sees Pedagogical Content Knowledge (PCK) as an important part of the teaching and learning process. Mishra and Koehler (2009) view the integration of technology with pedagogical content knowledge as a crucial factor for effective mathematics teaching and learning. This section will focus on the TPACK of the teacher-researcher of this study because he designed the GeoGebra and pen-and-paper activities.

Koehler and Mishra (2009) describe teaching as a complicated practice that requires interweaving of many kinds of specialised knowledge. They further emphasise that there is no perfect approach to integrate technology into the curriculum and it can become a complex task. Teachers have to develop new ways to comprehend and understand this complexity in order to ensure successful integration of technology. The instrumental framework can be of help in this regard, as mentioned earlier in this chapter.

Ruthven (2013) points out that the idea of TPACK was introduced to draw attention to the way in which new technological resources reshape pedagogical knowledge, content knowledge and pedagogical content knowledge. TPACK was introduced to the educational research field as a theoretical framework for understanding teacher knowledge required for effective technology integration and effective student learning in a technology intensive environment (Koehler & Mishra, 2009). TPACK is a framework that emerges from interactions among content, pedagogy and technology knowledge. Underlying truly meaningful and deeply skilled teaching with technology is the fact that TPACK is different from knowledge of all three concepts individually.

TPACK requires teachers to understand how to use technology to represent concepts; pedagogical approaches that use technologies in constructivist ways to teach content; understanding what concepts are difficult or easy to students and how technology can help students who experience problems; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge to develop new epistemologies or strengthen old ones (Koehler & Mishra, 2009).

Although TPACK focuses on the knowledge of the teacher, the researcher viewed TPACK as a critical theoretical framework for this study because the teacher-researcher should be aware of effective teaching with GeoGebra; the understanding of how to present concepts with GeoGebra; what concepts or topics are difficult to learn and how GeoGebra could help with better understanding; and some understanding of teaching and learning theories that can help to lift students to a higher level of understanding. For example, constructivist approaches, the Van Hiele levels of geometrical thought, RME, and the instrumental approach.

Hence the teacher-researcher's TPACK was important for this research because of the nature of the study. The teacher-researcher has been involved in the teaching of Mathematics for secondary school since 1989. The researcher was involved with the Khanya project from July 2004 to February 2009 and worked closely with both primary and high schools (mostly with the latter category of students) in the Cape Winelands and West Coast districts in the Western Cape Province of South Africa. The researcher was responsible for assisting the high schools with training in software from vendors such as Master Maths, Cami Maths and had the opportunity to give Interactive whiteboard training, with an emphasis on how to integrate the technology in mathematics. Schools were provided

with technology and, in most cases, a classroom with an average of 20-25 computers which was to be used for the teaching and learning of Mathematics. The researcher was later involved with the Institute for Mathematics and Science Teaching at Stellenbosch University (IMSTUS), presently known as the Centre for Pedagogy at Stellenbosch University (SUNCEP), from March 2009 to December 2012 in teacher professional development or in-service programmes in five high schools. These in-service training programmes consisted of workshops on Mathematics content and classroom visits. The researcher presented workshops on the use of different software packages to support teachers to teach with technology. Workshops at Khanya were on Microsoft Office software such as Microsoft (MS) Word, MS Excel and MS PowerPoint, mathematics software such as Graph, Graphmatica and GeoGebra. The researcher also presented GeoGebra workshops at the Association for Mathematics Education of South Africa (AMESA) conference; the Cape Teaching and Leadership Institute (CTLI) conference, the Maths, Science & ICT in Education Convention and the GeoGebra conference at Nelson Mandela Metropolitan University (NMMU). The researcher has been using GeoGebra as part of teaching and learning of mathematics at SciMathUS since 2013.

The interactions among content, pedagogy and technology knowledge play out differently across diverse contexts and account for the wide variations seen in the extent and quality of educational technology integration. Figure 3.8 shows seven components included in the TPACK framework. They are defined as:

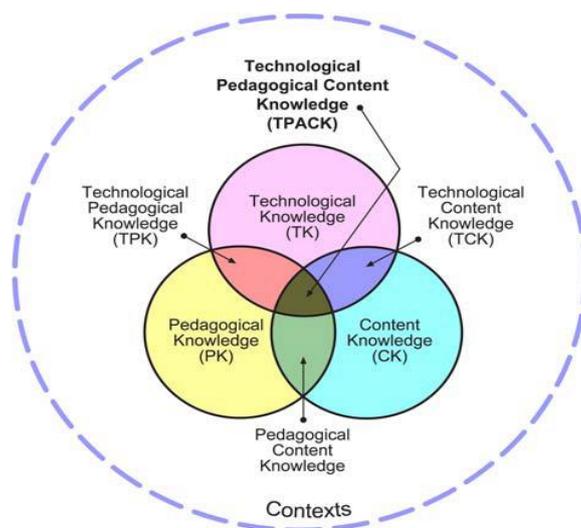


Figure 3.8: The TPACK framework and its knowledge components (Koehler & Mishra, 2009).

Content Knowledge (CK):

Content knowledge refers to knowledge of the actual subject matter – in this case school mathematics that is to be taught and learned (Koehler & Mishra, 2009; Mishra & Koehler, 2006). In this study the content of the subject was transformations of functions and circle geometry. Teachers have to know the content that they are going to teach and how the nature of this content differs from other content areas. Thus the teacher-researcher set up learning trajectories for the teaching and learning of transformations of functions and circle geometry. The fact that HLT's were drawn up shows that the teacher-researcher thought about the nature of the content and how it was to be sequenced or taught. The importance of using real contexts that are meaningful and natural to learners as a starting point for their learning cannot be overemphasised (Cheung & Huang, 2005; Van den Heuvel-Panhuizen, 2010; Widjaja & Heck, 2003; De Villiers, 2012)

Technological Knowledge (TK):

Technological knowledge refers to the knowledge about various technologies, ranging from low-tech or traditional technologies such as pencil-and-paper, books, and chalkboard to digital technologies such as the internet, digital video, interactive whiteboards and software programmes (Koehler & Mishra, 2009; Mishra & Koehler, 2006). The teacher-researcher has been working with GeoGebra since 2006.

Pedagogical Knowledge (PK):

Pedagogical knowledge refers to the methods and processes of teaching, and includes knowledge in classroom management, assessment, lesson plan development and student learning (Koehler & Mishra, 2009; Mishra & Koehler, 2006). It is also important for the teacher to present topics in an interesting way. Sutton and Krueger (2002) assert that interesting contexts stimulate learning and retention. Topics can be presented with an active learning approach to engage the students in the learning process. Students can in this way acquire physical knowledge (Piaget, 1952). Begg (1999) states that good learning activities arouse interest and curiosity. If students value mathematics, they become more skilful, achieve at a higher level, are more persistent problem solvers, and exhibit greater confidence (Sutton & Krueger, 2002). According to Koehler and Mishra (2009) pedagogical knowledge requires an understanding of cognitive, social and developmental theories of learning and how they apply to students in the classroom. Therefore it is important for teachers to know in what way their students are thinking and how they learn. It was important for the teacher-researcher to understand the principles of RME, constructivism, the Van Hiele levels and the instrumental approach in the teaching and learning of transformation of functions and circle geometry.

Pedagogical Content Knowledge (PCK):

Pedagogical content knowledge refers to the content knowledge that deals with the teaching process (Shulman, 1986). Pedagogical content knowledge is different from various content areas, as it blends both content and pedagogy with the goal of developing better teaching practices in the content area (Koehler & Mishra, 2009; Mishra & Koehler, 2006).

Teachers should have the knowledge of representing content knowledge and adopting pedagogical strategies to demonstrate a specific concept, or topic, in such a way that students understand it better. The teaching and learning approach should be appropriate for the topic. Sometimes visualisation, or context, is important to teach a concept. This is why this study used a hybridisation of the student-centred approach and teacher as facilitator of learning approach in order to cater for different student learning styles. Sousa (2001) indicates that lecturing is not always the best way of teaching. However, the researcher in this study refers to interactive lecturing and is of the view that interactive lecturing can be appropriate depending on what the teacher wishes to teach and how the lecture is presented to the students.

Technological Content Knowledge (TCK):

Technological content knowledge refers to the knowledge of how technology can create new representations for specific content (Koehler & Mishra, 2009; Mishra & Koehler, 2006). It suggests that teachers understand, that by using a specific technology, they can change the way students practice and understand concepts in a specific content area.

Technological Pedagogical Knowledge (TPK):

Technological Pedagogical Knowledge refers to the knowledge of how various technologies can be used in teaching by understanding that using technology may in turn change the way teachers teach (Koehler & Mishra, 2009; Mishra & Koehler, 2006).

Technological Pedagogical and Content Knowledge (TPACK):

Technological pedagogical content knowledge refers to the knowledge required by teachers for integrating technology into their teaching in any content area (Koehler & Mishra, 2009; Mishra & Koehler, 2006).

For students to be taught effectively they have to be exposed to teachers who have a sound content knowledge of mathematics and very good understanding of the topics that they are going to teach. Apart from the content knowledge, teachers have to know how to integrate appropriate technologies effectively in the teaching of the topic and they also have to know the type of student they are working with. Students must be exposed to the algebraic and graphical representation of solving of equations, or inequalities, or any other problem that can be solved algebraically or graphically. An example of teachers' TPACK to solve a problem in more than one way was done in the research by Richardson (2009) with the inequality problem, $2(x - 4) \geq \frac{3}{2}(2x + 1)$. During the teachers' discussion, one teacher reflected how she could explain the solution from the graph to her students. She realised that A(-9.5, -27) was the intersection of the graphs (see Figure 3.9 below) and the algebraic solution could also be read off from the graph. She reflected on how she could show her students how to perform the technological procedures and the related solving of inequalities in a coherent way during her teaching (Richardson, 2009). Figure 3.9 shows how this can easily and quickly be done in GeoGebra.

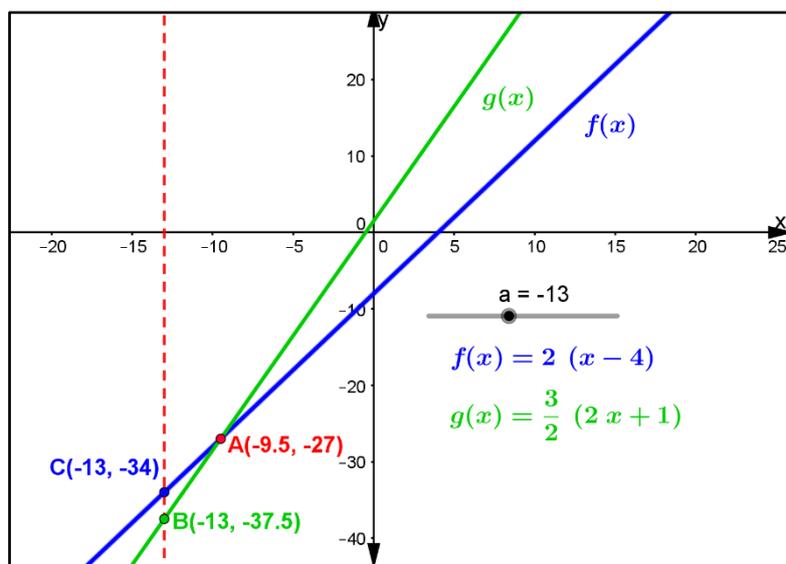


Figure 3.9: Example to solve inequalities with GeoGebra (Adapted from Richardson (2009)).

By moving slider a in Figure 3.11 students can then investigate for which x -values $f(x)$ is greater than $g(x)$. This example shows two linear functions, but this can be done for inequalities and equations such as:

$$\log_3 x \leq -2, \frac{-2}{x-4} = -3x - 5, \cos(x - 30^\circ) = -2 \sin(x + 45^\circ), \text{ etc.}$$

Tripathi (2008) also notes that students are comfortable dealing with functions only when they have equations to represent the functions involved. This means that they are comfortable solving algebraic representations of $f(x) = g(x)$, $f(x) \leq g(x)$ and determining decreasing values over given intervals. It took several sessions for students to accept graphs as being legitimate representations of functions when no equation was given. Tripathi (2008) points out that it takes time and effort before looking at an equation for students to start relating algebraically expressed relations to graphical ones. She also attests that multiple representations can be a powerful tool to facilitate students' understanding. The teacher-researcher exposed students to the algebraic manipulation of transformations of functions and let them visualise it in GeoGebra. Students should have a visual understanding of transformations of functions. GeoGebra was used to help students to see the link between the algebraic manipulation and the graphic representation.

Utilising educational computer software and technology such as Computer Algebra Systems (CAS), Dynamic Geometric Software (DGS) and graphic calculators combined with data projectors, can help the teacher to create better examples and realistic illustrations that can

turn a traditional expository lecture into a more exciting and meaningful class (Baldin, 2002). If the teacher uses technology to transform, or adapt, the traditional expository lecture into something more exciting and interesting (s)he may reduce the boredom of talk-and-chalk. Baldin (2002) adds further that these type of classes where teachers create their own activities to be able to present, requires sound mathematical knowledge and understanding of the technology in order to create excellent teaching material. In other words the TPACK of the teachers is critical for effective teaching and learning of mathematics.

Drijvers (2012) contends firstly, that it is important to investigate the relationship between the use of digital technology, the students' thinking and their pen-and paper activities. Secondly, he adds that the focus has to be on both teaching and learning. TPACK and instrumental orchestration are models that help to understand what is different in the teaching with technology. Ruthven (2013) adds that there is comparison between the TPACK framework and the Instrumental Orchestration framework. Thirdly, the combination of different theoretical frameworks are also prominent in recent studies. The researcher of this study pointed out in Chapter 2 (see section 2.3.4) that effective teaching and learning of Mathematics is more than just active learning. It is not just constructivism or RME. Drijvers (2012) consequently refers to the triangulation of theoretical frameworks. In an effort to answer the research questions, the study focused on the role of GeoGebra in the teaching and learning of circle geometry and transformations of functions, but the researcher also sought to establish what the requirements are for effective teaching of Mathematics.

3.6 RESEARCH ON BRIDGING OR FOUNDATION PROGRAMMES

Snyders (1999) contends that academics at the then University of Port Elizabeth (UPE), now known as Nelson Mandela Metropolitan University (NMMU), would not consider bridging, or foundation programmes, when the universities were requested to consider it in the 1980s or even in the early 1990s. In recent years, universities and colleges across the globe have found that their students do not have sufficient mathematical preparation, or the appropriate mathematical background, to deal with their first year mathematics courses. As a result, universities and colleges have seen an increase in failure rates for these subjects (Rylands & Coady, 2009). Koch and Snyders (2001) also assert that numerous students are still receiving schooling that can be regarded as below standard. Hourigan and O'Donoghue (2007) also similarly note that students are entering mathematics intensive courses with fewer of the basic mathematical skills essential for course success. As a result, students

experience problems, especially with mathematics, physical science and chemistry subjects at university. These problems are often a reflection of their inability to deal with mathematical thinking (Viljoen, 2007). Consequently, universities realised that, even though it is the responsibility of schools to supply universities with well-prepared students, they need to be accommodating. That is why foundation, transition, bridging, or access programmes were established. As mentioned in Chapter 1 (see section 1.7) this study focuses on the SciMathUS bridging programme students. In other countries bridging, or access programmes, are referred to as foundation programmes. A foundation year is an extra year of study at the start of a university course. It enables students, who have not met the entry requirements for the normal course, to fill in the gaps and go on to study a full degree.

Research by Snyders (1999) in a foundation mathematics course at the UPE in the 1990s shows that first year students need skills to enable them to be successful in their degree studies. These skills include much more than mathematical content, and they should also be equipped with general skills such as problem solving, critical thinking, analysing, comparing and categorising. Topics that were eventually included in the course are: Numeracy, Ratio and Proportion, Problem Solving, Basic Algebra, Introductory Trigonometry, Exponents and Logarithms, Introductory Calculus and Elementary Euclidean Geometry. The inclusion of Euclidean Geometry is an example of a topic that was included specifically to be used as a vehicle for teaching some of the abovementioned skills.

The teaching and learning in the Technology Access Programme (TAP) at Tshwane University of Technology (TUT) focused on bridging the gap between secondary and tertiary education (Viljoen, 2007). Viljoen (2007) contends that effective teaching and learning methods greatly enhance the effectiveness of the programme. The aim of the programme is to provide an academically challenging and rewarding course for all our students, to ensure that they are ready to face the challenges of engineering studies with confidence and independence. Activities that are related to the students' real-life experiences are included in TAP to prepare students for an engineering course, aiming to refresh knowledge, correct misconceptions and promote communication in mathematics. The programme promotes the learning of mathematics concepts through social interaction, group discussions and feedback that are provided by the lecturers afterwards.

A survey done by Devesh and Nasser (2014) with a sample of 275 students shows that the students considered the course content of mathematics modules in a foundation program

offered in Majan College, in the Sultanate of Oman, to be very effective. The researchers are of the view that a mathematics foundation programme should support the students to build a strong foundation in computation, numerical sense and problem solving, and the students need to be exposed to critical thinking and apply their knowledge to real world situations. Results of their study show that owing to the students' learning experiences at higher levels, they considered the course content to be highly useful as they accessed their higher education studies at Majan College.

Innovative teaching methods in a mathematics bridging course at the University of South Australia with Excel Spreadsheet Teaching Tools have been devised to enhance students' understanding of mathematics. One of the concepts taught with Excel spreadsheets was the effect of changing the parameters of graphs. The results have shown that study materials developed for this course involving explaining mathematical concepts using Excel spreadsheets, have been rated as successful by the students (Boland, 2004).

Malan, Ndlovu and Engelbrecht (2014) proclaim Problem-Based Learning (PBL) at SciMathUS as a combination of cognitive 'scientific' and social constructivist theories in which learning is viewed as an activity that not only takes place within individuals, but also occurs when they are engaged in social activities. The findings of their study indicate that it is possible to influence student-learning patterns in a favourable direction, if appropriate PBL contextual problems of interest to the students are selected and integrated into the curriculum; if student roles and responsibilities in the collaborative groups are clearly defined and emphasised, and if lecturers de-role from knowledge dispensers to facilitators of knowledge construction and transformation. The mathematics teaching staff of SciMathUS has changed drastically since 2013 when the researcher of this study became part of the programme. The researcher adapted the PBL context of SciMathUS to RME with the focus on guided-reinvention and the use of GeoGebra to be part of the teaching and learning process. This was done because group work is a feature of the PBL approach and is therefore situated in the epistemological paradigm of social constructivism (Ram, Ram, & Sprague, 2007; Kemp, 2011; Roessingh & Chambers, 2011; Deo, 2014). The researcher felt that the students can construct knowledge by working individually, in pairs or in groups. Mayer (2004) and, Kirschner, Sweller, and Clark (2006) conclude that the guided approach to learning is more effective than unguided or minimally guided instructional approaches. The teacher-researcher therefore opted for the RME approach because it recognises

individual and group learning and views guidance as an important factor to the teaching and learning of Mathematics. The activities for this research were designed in such a way that students could explore concepts individually, in pairs or in groups. GeoGebra was merely a tool to visualise concepts such as transformations of functions, trigonometric general solutions, properties of circle geometry, etc.

3.7 GEOGEBRA AS A TEACHING AND LEARNING TOOL FOR FOUNDATION PROGRAMME STUDENTS

Hohenwarter and Fuchs (2004) contend that GeoGebra is a very versatile tool for mathematics education in secondary schools. In teaching mathematics it may be used in many different ways. Hohenwarter and Preiner (2007) reiterate that GeoGebra was created for the teaching and learning of mathematics from middle school through college to university level. Hohenwarter, Hohenwarter, Kreis and Lavicza (2008) add that the basic idea of the software is to join geometry, algebra and calculus (which other packages treat separately) into a single easy-to-use package for learning and teaching mathematics from elementary through university level. GeoGebra is thus suitable for the SciMathUS bridging programme. The CAPS (DBE, 2011a) requires the SciMathUS students to do algebra, geometry, calculus and statistics, all of which fall within GeoGebra's capabilities or affordances.

Hohenwarter, Preiner, and Taeil (2007) point out that GeoGebra is widely used in middle and high schools, especially in Europe, whereas its use at university level is also emerging. They further indicate that there are several teachers in the USA who have published interactive university-level calculus material on the internet. Sangwin (2007) also notes that although GeoGebra has been designed for education in secondary schools, it certainly has uses in Higher Education for demonstrations in lectures, or for students to use in exploring functions and graphs, amongst other things. The researcher found only a few studies done with foundation or bridging programme students exploring GeoGebra. The researcher acknowledges the shortage of research at the level of foundation or bridging programmes using GeoGebra in the teaching and learning of mathematics. The researcher is consequently of the view that this study is important to inform other foundation or bridging programmes locally or abroad about the teaching and learning of mathematics with GeoGebra and for secondary school level.

Research done by Green and Robinson (2009) with 200 foundation programme students report on the responses to the teaching and learning with GeoGebra. Table 3.1 shows the students responses to the question: How useful do you think GeoGebra will be, or has been, for your learning of mathematics?

Table 3.1: Students' views, at the beginning and end of the GeoGebra initiative, on how useful GeoGebra was (Green & Robinson, 2009)

Useful	Session 1	Session 4	Difference (4) – (1)
	At the beginning	At the end	
4 - Very	41%	18%	-23%
3 - Quite	41%	36%	-5%
2 - A little	16%	37%	+21%
1 - Not at all	1%	9%	+8%

The students work on GeoGebra for four two hour tutorials during a semester. The lecture of the Mathematics module did not involve GeoGebra in the tutorials. Table 3.1 shows that although there was some decline between initial expectation and final judgment, the overall impression was that there is at least an indication that it helped with understanding. These are only the students' perceptions of the usefulness of GeoGebra from a pre- and post-intervention questionnaire. The responses of only nine students who attended the focus group interviews were positive about the chance to use GeoGebra. They view the visualising feature of GeoGebra as very helpful to understand Mathematics. They also enjoy working with a tutor and could see the pen-and-paper algebraic manipulations that could be viewed in GeoGebra's graphic and algebra views (Green & Robinson, 2009). They point out that further research has to be done on more integration of GeoGebra in the lectures and assessment, and also on the long-termed usefulness. This study therefore fulfils this appeal. The researcher of this study acknowledges the lesson learnt from it. This study with the SciMathUS students was over eight months and the students were afforded opportunities to work with GeoGebra themselves. GeoGebra was also utilised when a new topic was introduced and for the consolidation of concepts. This study also collected information from different research methods and instruments so that findings could be triangulated.

Findings in the research by Jaworski and Matthews (2011) on the use of GeoGebra show both positive and negative indications. GeoGebra was utilised in the study of Jaworski and Matthews (2011), with the lecture's expository lecturing and also by the students in tutorials.

For example, going into GeoGebra mode in a lecture (from PowerPoint) gained the attention of the student. Students were visibly attentive – they stopped writing, talking or shuffling around. However, students indicated in the interviews that they found too much use of GeoGebra in the module. Ndlovu, *et al.* (2011, 2013c) contend that students, especially in the beginning, feel this way, because they have to master the syntax of the software. Further research by Jaworski, Matthews and Croft (2012), with engineering students, had mixed results. The researchers report that they could see that GeoGebra helped them with a wider range of functions and fitted functions to data, but they indicate that the examination required sketching by hand and more practice of graph sketching without GeoGebra was needed. Findings from the observation notes during the tutorial, show some evidence of conceptual understanding, but it is difficult to quantify or substantiate (Jaworski & Matthews, 2011). They add that some students could not see the link between the graphs represented in GeoGebra and the algebraic representations.

The students also view the use of GeoGebra in lectures as interrupting the lecture flow at times and taking unnecessary time (Jaworski & Matthews, 2011). Students would have preferred more opportunities to practice solving test or examination style questions. Jaworski (2010) further asserts that these responses and reflections of students point to a need for developing an alternative culture in which students are clearer participants in conceptual exploration in mathematics. Collett and Olivier's (2015) pilot project in the Eastern Cape in South Africa for six weeks with Grade 11 and 12 students, exploring functions with GeoGebra applets, shows that students still prefer traditional teacher-centred activities over student-centred, exploratory activities. The results, that could only be confirmed anecdotally, also show that individuals improve conceptual understanding of functions. The results of the pilot study show similarity with the research done by Jaworski and Matthews (2011) that students still prefer a traditional approach of teaching. The researcher in this study foresees this as a challenge for the participants, but agrees with Jaworski (2010) that the students' conception of how they should be taught and learn Mathematics must be addressed, and the only way is to expose them to different ways of teaching and learning. The researcher gave the participants in this study more or less the same type of questionnaires. One was at the beginning of the year and the other at year-end which investigated the changes in the students' perceptions about the teaching of certain topics with GeoGebra

3.8 GEOGEBRA AS A TEACHING AND LEARNING TOOL FOR TRANSFORMATIONS OF FUNCTIONS

Usiskin and Coxford (1972) contend that many good reasons, both mathematical and pedagogical, exist for introducing transformations in the tenth Grade. They see transformations as convenient examples of one-to-one functions and emphasise that the notion of congruence applies to any figures: segments, angles, triangles, circles, polygons, arcs, and so on. Students are also rarely given the opportunity to study the connections between transformations in Euclidean geometry and see the use of it in graphing functions. (Anonymous, n.d.). Hence the researcher decided to explore the teaching and the learning of transformations of functions with transformation geometry. The rationale behind this was that the researcher wished the students to understand the properties of rigid transformation and dilation. Rigid transformation refers to translation, reflection and rotation, and dilation refers to the enlargements or reductions. The researcher focused more on translation, reflection, enlargement and reduction to connect this to the transformation of functions. The researcher viewed the properties of rigid transformations as important for students to realise when a figure is reflected or translated, its shape and size do not change. The original figure and its image have the same shape and size. Rigid transformations are movements without changing the shape or size of two or three dimensional objects. The original shape and its image are thus congruent. The properties of enlargement and reductions are also important because students have to realise that the original shape and its image are similar. Research by Steketee and Scher (2016) shows how students could do the geometric transformations, i.e. dilation and translation that correspond to algebraic parameters of m and c in the linear function, $y = mx + c$.

Research by Kotrikova (2012) shows how the effect of parameters of m and c in $y = mx + c$ can be explored with GeoGebra. Figure 3.11 shows the effect of m on the graph of the linear function.

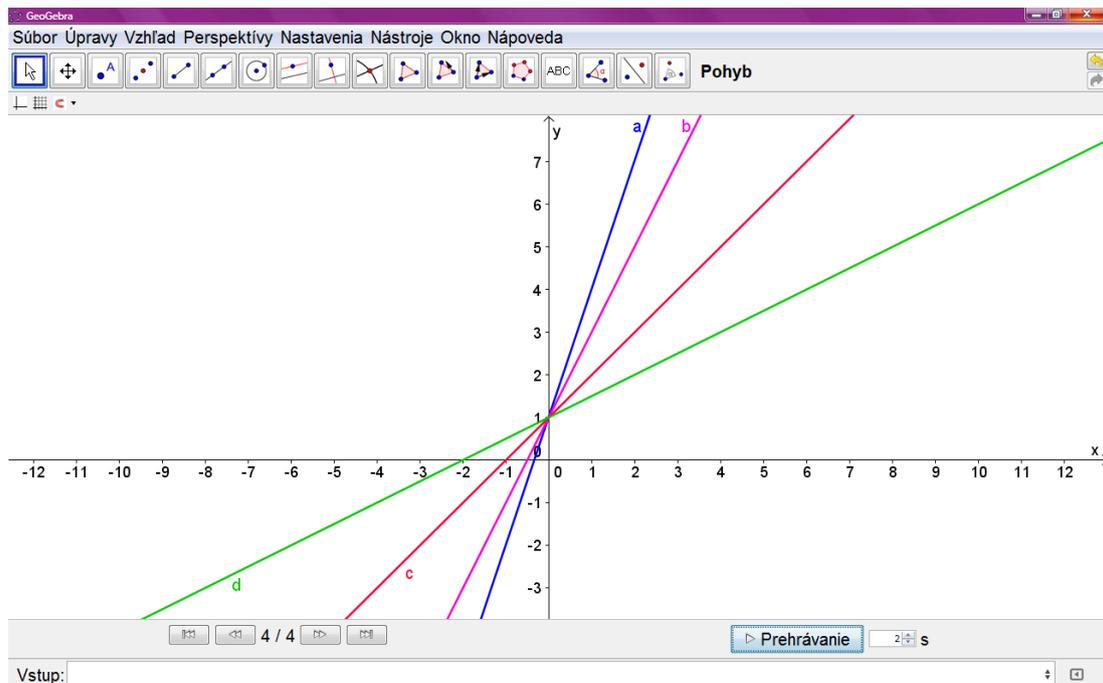


Figure 3.10: Coefficient m in formula of linear function (Kotrikova, 2012).

The change in the value of m can therefore be used to teach the concept of gradient and the meaning m in the equation of the linear function can also be linked to stretching by showing students the linear function in the format $y = a(x + p) + q$.

Figure 3.13 shows how GeoGebra can be used to teach the effect of q on the graph and also the meaning of q in the equation of linear function. Translation can be used if viewing the linear equation in the format $y = a(x + p) + q$.

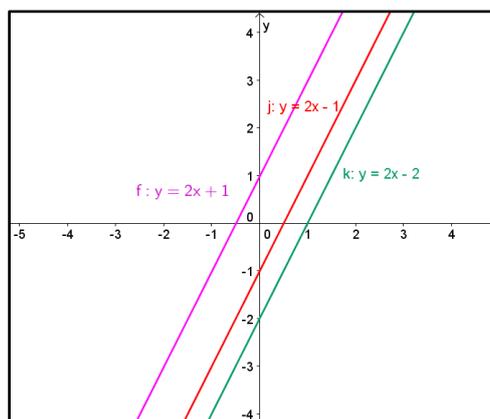


Figure 3.11: Coefficient c in the formula of linear function (Kotrikova, 2012).

Research by Makgakga and Sepeng (2013) also sought to explore benefits of teaching exponential functions through transformation and how this technique can assist students in understanding the concept of exponential functions. The use of transformations to teach and learn exponential functions played a fundamental role in their study. Makgakga and Sepeng

(2013) assert that the approach was effective as it helped students to improve their academic performance in dealing with exponential and logarithmic functions.

Students in this study explored in transformation geometry by writing the algebraic notation and also the understanding of the algebraic notation. This approach intertwined, a tenet of RME, Algebra with Geometry and exposing students to the idea that Algebra, Transformation Geometry, Trigonometry or Euclidean Geometry can't be taught as standalone topics. This also shows how to solve transformation of functions in more than one way and this can be linked to the TPACK of the teacher-researcher. The study explored students' experiences of transformations of functions and transformation geometry with GeoGebra at the start of the topic. Anabousy, Daher, Baya'a and Abu-Naja (2014) suggest that in working with GeoGebra, students generally succeeded with function transformations in their algebraic and graphic representations. This success could be attributed to the properties of the technological tool, as well as to the properties of the given task. The technological tool helped the students to work successfully with transformations because its interface includes both the graphic and the algebraic representations.

Figure 3.13 from Hohenwarter (2006) shows how GeoGebra can be used in the tracing function of GeoGebra to teach transformations of functions, in this case reflection in $y = x$

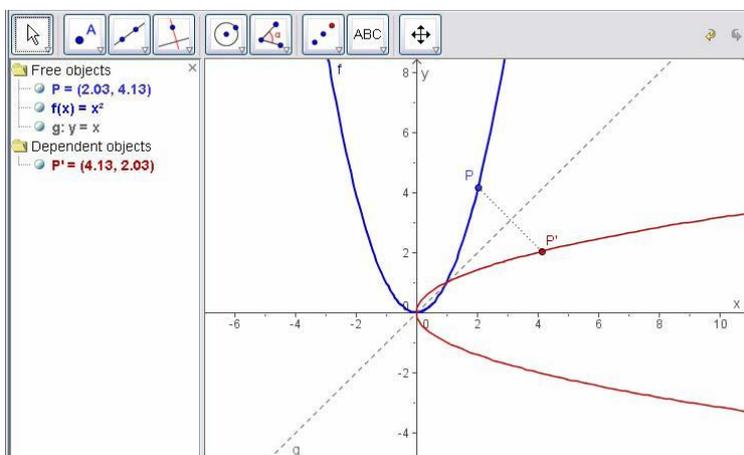


Figure 3.12: Inverse graph $f(x) = x^2$ (Hohenwarter, 2006, p. 5).

Figure 3.12 shows how Hohenwarter (2006) uses GeoGebra to teach inverse graphs with reflections in $y = x$. By moving point P the students can see how the values of x and y are changing. Students can also check these values with substitution in the equation of the function.

Figure 3.13 shows an example of how the researcher in this study used a similar approach with the tracing tool and input and output values to investigate the translation of the functions.

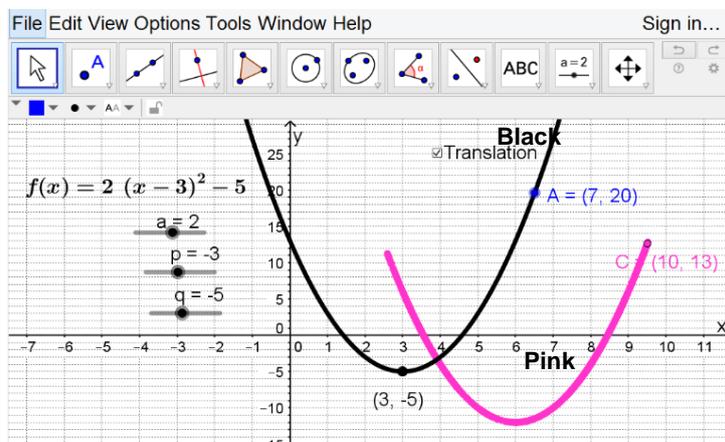


Figure 3.13: Translation of graph of $f(x) = 2(x - 3)^2 - 5$.

The students dragged point A to see the translated path of point C. They then had to change the values of the sliders so that the black graph fitted the path of point C, the pink graph. The idea was that the students could see that the translated graph still has the same shape and that they could see that the algebraic notation can be linked to graphical presentation. The full HLT for teaching transformation of functions with examples is discussed in Chapter 4.

GeoGebra allows students to graph functions rapidly, manipulate the graphs and develop generalisations about the transformation of functions. More time can be spent analysing the graphs and less time on the actual construction of the graphs. Students can also build a deeper understanding of transformation on functions and the graphing of the functions, since less time is spent performing calculations. In this way, students focus on understanding instead of memorising processes and algorithms. This strengthens the level principle of RME since students move through different levels of understanding.

De Villiers (2012) points out that even third and fourth year mathematics education students, “despite knowing and being quite proficient in the ‘rules’ and ‘techniques’ of differentiating and integrating, often have little conceptual understanding of graphs and the slope or rate of change of a function, especially as it relates to a real world context.” Isoda (2007, p. 5) points out that the change of the “parameters is an important approach for exploring invariant families of functions. Before graphing tools existed, only a limited number of students could

imagine those kinds of invariants. In the traditional curriculum, most of the students only learned the quadratic function with the equation of $y = a(x - \alpha)^2 + \beta$ but not the meaning of parameters a , b and c on the equation of $y = ax^2 + bx + c$, even though they learned the meaning of a and b on $y = ax + b$. Before the graphing tools, it was very difficult to imagine the graphs of $\{y = ax^2 + bx + c; a = 1, c = 1, \text{ and } b = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}\}$. The graphing tools allow us to manipulate these parameters” (Isoda, 2007).

De Villiers (2007) is one of the researchers in South Africa that did intensive research on dynamic software, especially Geometer’s Sketchpad. He identifies in his research that children, or students, do not necessarily need to know the software inside out before they can effectively use it to explore, learn, conceptualise, conjecture, etc. Teachers can use the software effectively “by providing students with more or less ready-made sketches that only require dragging, and perhaps clicking, animation or construction buttons” (De Villiers, 2007, pp. 47-48). The implication is that prepared sketches can be given to students rather than letting them waste time on mundane construction procedures, e.g. applets with sliders. On the other hand, teachers could also create self-contained dynamic worksheets prior to the lesson. With such pre-made worksheets, students don’t need to operate GeoGebra, itself but only work with an interactive web page. This saves time in teaching them how to operate the software. By being able to customise the user interface of the integrated interactive applets (e.g. showing or hiding the algebra window, reducing the number of available tools, displaying the toolbar help), teachers can decide beforehand how much freedom or guidance they wish to provide for their students and which features and tools should be available for them. Based on De Villiers’ suggestions the researcher of this study created activities in GeoGebra for certain sections of transformations of functions. The students could still make changes to the activities and save them in their own profiles.

3.9 GEOGEBRA AS A TEACHING AND LEARNING TOOL FOR CIRCLE GEOMETRY

De Villiers (2004) points out that often students at school and university level are not provided with a sense of how new results can or could be discovered or invented. He further explains that quite often, after theorems and their proofs have been presented, students are just given exercises with riders of the type ‘Prove that ...’. This distortion of mathematics can

easily create the false impression that mathematics is only a systematic, deductive science (De Villiers, 2004).

De Villiers (2004) uses a more general term, borrowed from Lakatos (1983), for experimental methods as quasi-empirical methods. Lakatos opined that although objects in mathematics can be abstract and imaginary they can still be subjected to empirical testing. De Villiers (2004) refers to quasi-empiricism as all non-deductive methods involving experimental, intuitive, inductive, or analogical reasoning. De Villiers (2004) asserts that quasi-empirical methods are utilised when mathematical conjectures and/or statements are evaluated in a non-deductive manner; for example, numerically, visually, graphically, through special cases, through construction and measurement, diagrammatically, through physical embodiments, kinaesthetically, by analogy. They are also employed when conjectures, generalizations, or conclusions are made on the basis of intuition, or analogy, or experience obtained through any of the preceding quasi-empirical methods. De Villiers (2004) identifies four quasi-empirical methods that can be useful for curriculum design and teaching. They are (in no specific order) conjecturing, verification, global and heuristic refutation. Conjecturing is looking for an inductive pattern, generalization, analogy, and so on. With verification the student can obtain certainty about the truth, or validity of a statement, or conjecture. Global refutation refers to the disproving of a false statement by generating a counter-example. Heuristic refutation refers to reformulating, refining, or polishing an essentially true statement by means of local counter-examples.

Govender and De Villiers (2004) remark that the teaching approach of construction and measurement to evaluate the correctness of geometric statements (conjectures) before proofs are done is not a new approach but common practice among mathematical researchers. Human and Nel (1989) used a similar approach in the Stellenbosch University Experiment with Mathematics Education (USEME) teaching experiment during 1977/78. Similarly, Smith's (1940) research shows improvements in students' understanding of "if-then" statements by letting them first make constructions to evaluate geometric statements. Smith's (1940) research shows that it enabled students to learn to clearly distinguish between the "given condition(s)" and the "conclusion(s)", and laid the conceptual foundation for an improved understanding of the eventual deductive proof.

Govender and De Villiers (2004) used a similar approach, but with the dynamic software Sketchpad, where a geometric configuration can be continuously dragged into different

shapes to check for invariance. They used ready-made activities in Sketchpad where the students could play step-by-step and observe as the figure was gradually constructed. De Villiers (1998) posits that from the Van Hiele theory, it is clear that understanding of formal definitions can only develop at Level 3, since that is where students start noticing the inter-relationships between the properties of a figure. This study used a similar approach where students were provided with ready-made circle geometry activities in GeoGebra. The students were provided with worksheets with sketches satisfying given conditions similar to those on their screens and they had to find the conclusion based on the conditions. The GeoGebra activities were not used for proof of theorems but just for exploration and conjecturing. Hence this study also focused on the students' justification for their views of empirical evidence. The activities only focused up to Van Hiele Level 3 so that formal proving of the theorems could be done afterwards.

Chimuka and Ogbonnaya's (2015) study shows that GeoGebra software enhanced students' achievement in circle geometry. The results show that students taught with the aid of GeoGebra software performed better compared to students taught through the traditional "talk and chalk" method (Chimuka & Ogbonnaya, 2015). In order to establish an authentic and generalisable effect of GeoGebra software, the study concludes that more research using a much larger sample is necessary (Chimuka & Ogbonnaya, 2015). The researcher was of the view that if students start to believe that they can do circle geometry or any other topic, especially topics they normally struggle with, then improvement in their results can be expected and they will also gain confidence to tackle the higher level questions. The fact that students can drag mathematical objects in GeoGebra activities can also help to show many examples and non-examples (Lim, 1992) so that they can experience different orientations.

Figure 3.17 shows an adapted version of the examples in dynamic software of the research of Ruthven *et al.* (2005).

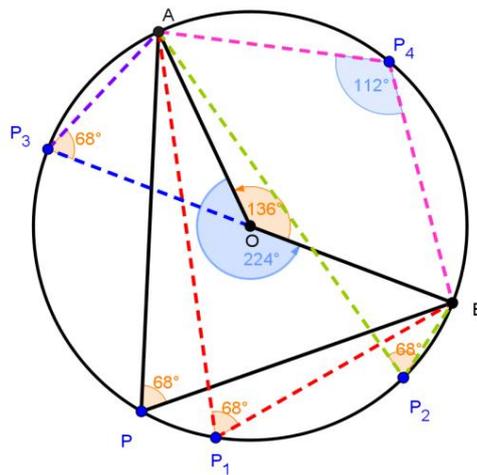


Figure 3.14: Angles at the circumference (Adapted from Ruthven *et al.*, 2005).

It demonstrates how teachers dragged to see the different orientations, or many examples of the angle at the circumference, and a non-example of the angle at the circumference is double the angle at the centre. That is, $\widehat{AOB} \neq 2\widehat{P_4}$. Ruthven *et al.* (2005) points out how teachers explored the size of the angle at the circumference by dragging its vertex, P. Vertices P_1 , P_2 , P_3 and P_4 represent the different positions of vertex P. Students can be given similar activities to make conjectures by dragging vertex P to P_1 , P_2 , P_3 that are lying in the same segment, but will see that it is not the case at P_4 , that is lying in a different segment than \widehat{AOB} .

Consequently students can measure reflex \widehat{AOB} to see what the relationship is between reflex \widehat{AOB} and $\widehat{P_4}$. The property of the opposite angles of the cyclic quadrilateral can also be explored with different orientations and also with non-examples of cyclic quadrilaterals. That is, in this case $AOBP_4$.

Figure 3.18 shows how the relationship to the size of the angle at the centre, \widehat{AOB} , can be explored by dragging the common points, A and B, to generate several pairs of angle measures; and lastly, the special case where A and B are dragged to form a straight angle at the centre.

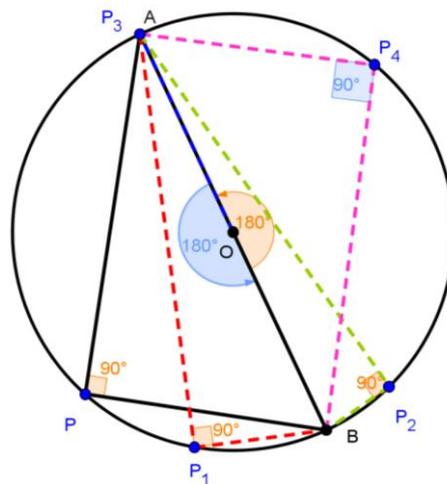


Figure 3.15: Dragging of A and B straight angle at the centre.

The teachers reported that they found this to be an efficient and effective way of covering these topics (Ruthven *et al.* 2005). The teachers also indicated that the technology is of help because their students could actually see, by dragging the circumference that the angle doesn't change on condition that is it subtended by the same chord, or arc, of the angle at centre and they were subsequently much more convinced. One teacher noted how this provoked students into making sense of what was going on:

“I heard one of the boys, for example, saying 'There's something wrong with this, it's always the same angle wherever I move it to'. And then it dawned on him that that was the whole point!” (Ruthven *et al.* 2005, p. 14).

This is the point where they could see why the circumference and centre angle theorem states that the angle subtended by an arc of a circle at the centre is double the size angle subtended by the same arc. This researcher sought responses of this nature because the students are able to see for themselves. The researcher views this as an aha-moment of mathematics where a student understands why mathematics works the way it does. The researcher also views this as a point of reference that he could refer the student to if the student experienced problems with conceptual understanding, or develops a misconception. The students were given 13 different GeoGebra applets to explore all the theorems and corollaries of circle geometry prescribed in the syllabus. Students measured angles; segments and added a construction or two to the given activities. This study was therefore consistent with the views of Freudenthal (1973) and Van Hiele (1984) by not starting with proofs of theorems or definitions but starting with exploration.

3.10 A CONCEPTUAL FRAMEWORK FOR THE TEACHING AND LEARNING OF TRANSFORMATIONS OF FUNCTIONS AND CIRCLE GEOMETRY WITH GEOGEBRA TO ENSURE DEEPER UNDERSTANDING

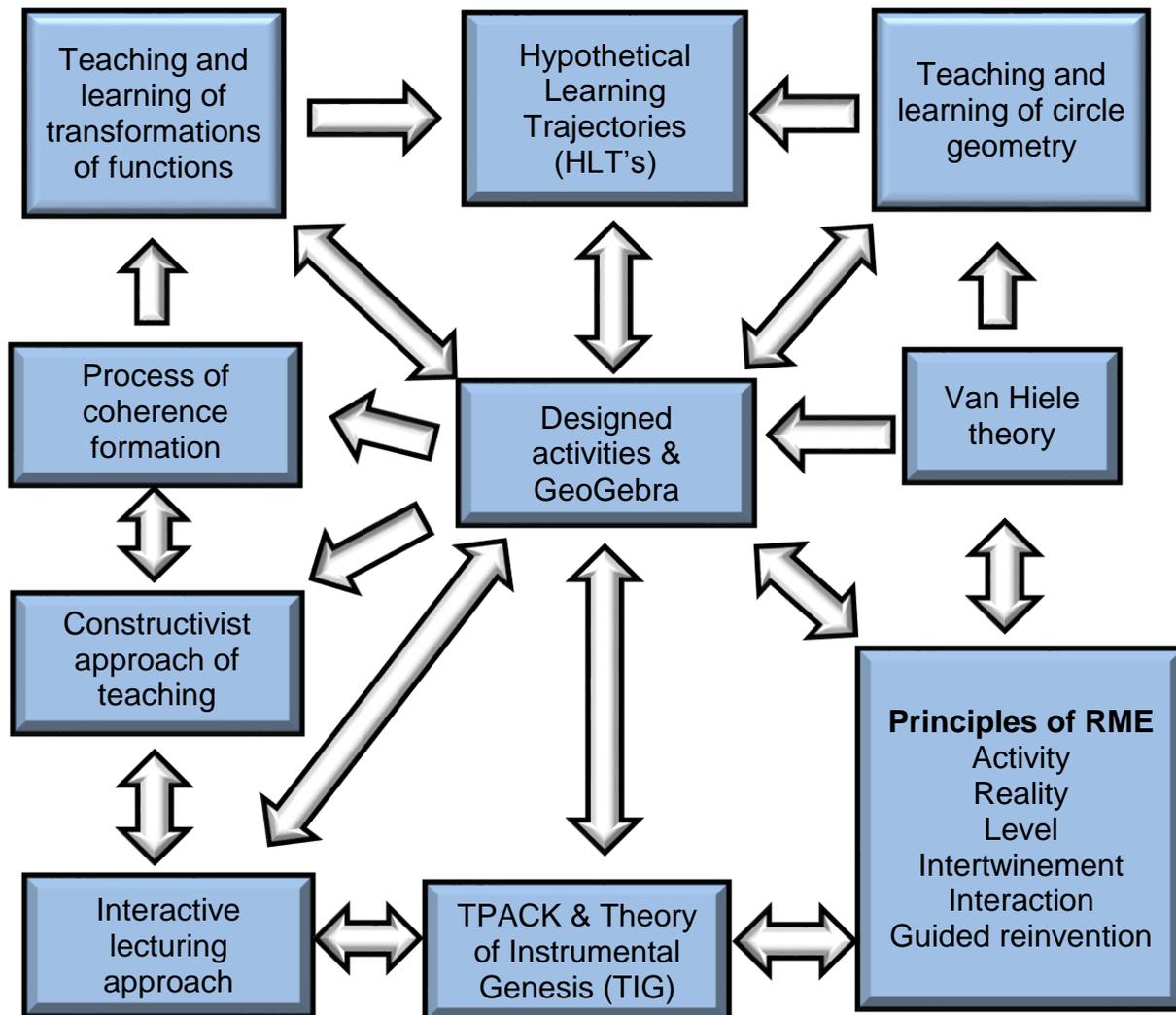


Figure 3.16: Conceptual framework for effective teaching and learning of transformations of functions and circle geometry.

Figure 3.16 shows the conceptual framework that this study adopted for the teaching and learning of transformations of functions and circle geometry for conceptual and deeper understanding. After a critical review of literature on the traditional and constructivist approach of teaching, principles of RME, the Van Hiele theory and the process of coherence formation, the researcher came to the conclusion that the combination of the abovementioned theoretical frameworks were the most suitable approach to teach transformations of functions and circle geometry. The researcher also added theoretical frameworks, such as TPACK and TIG that support technology as a pedagogical tool in the teaching and learning of mathematics.

The diagrammatic presentation (see Figure 3.16) of the adopted conceptual framework model for this study shows that HLTs are crucial for the teaching of transformations of functions and circle geometry. The teacher must have some idea of what learning path needs to be followed to teach any topic, or concept, in mathematics. The designed activities for the topics have to be based on interactive instruction, or constructivist approach, or RME theory, or process of coherence (for transformations of functions), or the Van Hiele theory (for circle geometry). The concept, or topic, or even a section of a topic, can be taught with an interactive lecturing approach, or an active approach. This decision will depend on the teacher, if the topic should be taught with an interactive lecturing approach, or an active approach. The teacher's TPACK is therefore also important because his/her content knowledge of the topic and also knowledge of technology, in this case GeoGebra, will determine what approach will be used. GeoGebra can be used in both interactive lecturing and active learning approaches. The researcher also viewed TIG as an appropriate tool to analyse observations of students' behaviour within the dynamic geometry environment. The levels of Van Hiele and the processes of coherence formation can assist the teacher to see if the students are sufficiently prepared to move to higher levels of abstraction, or understanding.

3.11 SUMMARY

This chapter has reviewed the role of technology in teaching and learning mathematics; the nature of dynamic software; TIG and TPACK as theoretical frameworks that support technology as a pedagogical tool in mathematics education. The role of GeoGebra in foundation programmes and GeoGebra teaching and learning tool for transformations of functions and circle geometry have also been explored.

The researcher acknowledges that technology is not the only solution the effective teaching and learning of teaching and thus combined the theoretical frameworks discussed in this chapter with theoretical frameworks discussed in Chapter 2. Although TPACK focuses on teachers, the researcher points out that the teacher-researcher's TPACK was important in this study. The TPACK of the teacher-researcher is discussed to show what was needed to design activities for the students in this research. The next chapter presents the research methodology for the study.

CHAPTER 4

RESEARCH METHODOLOGY

“Intelligence organizes the world by organizing itself.”

- Jean Piaget

4.1 INTRODUCTION

This chapter presents an overview and elaborates on the research design and the subsequent research methodology for this empirical inquiry. The research design, a mixed methods approach, the specific instruments that were used in the study to generate the data and, data analysis procedures are covered in this chapter. The chapter also discusses the reliability and validity of instruments and statistical procedures such as *t*-tests, effect size, Cohen’s *d*, etc. to describe the instrument reliability and to report data. Moreover the chapter gives a detailed overview of the hypothetical learning trajectories that were used for the teaching of function transformations and circle geometry.

4.2 RESEARCH DESIGN

Research designs are customised to address the diverse types of research questions. They are plans, or blueprints, and procedures for research on how the researcher is going to conduct the research (Creswell, 2009; Mouton, 2011). Research designs can be divided into empirical and non-empirical research design (Mouton, 2011). The primary data of empirical research are normally derived from surveys, experiments, case studies, programme evaluation, ethnographic studies, etc. Non-empirical research designs include philosophical analysis, conceptual analysis, theory building, literature reviews, etc. Empirical research is guided by data obtained from observed and systematic research methods rather than by opinions or authorities (McMillan & Schumacher, 2001). Empirical research is therefore based on actual observation or experiments that can be measured (Empirical, n.d.). The acquired data can be analysed by narrative (qualitative) or / and numerical (quantitative) data.

Valid information and knowledge can be utilised to make informed decisions in education. This information and knowledge can be provided by evidence-based inquiries (McMillan & Schumacher, 2006). The purpose of research is to develop knowledge to improve

educational practices. They see this impact as a process and divide it into five phases as shown in Figure 4.1.

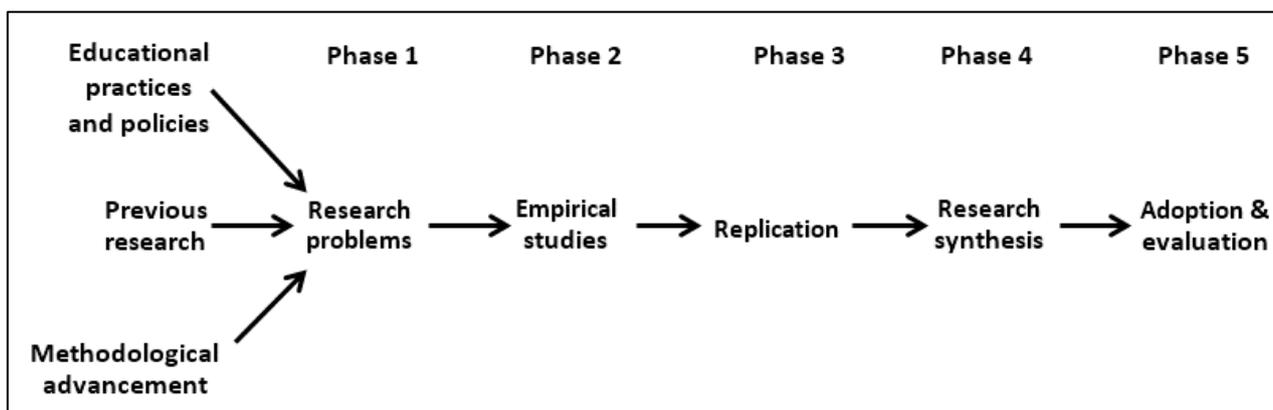


Figure 4.1: Development of Evidence-Based Knowledge to improve educational process (Adapted from McMillan & Schumacher (2006))

Teachers have to do research in order to find best practices to improve their classroom management or the teaching and learning of a new topic to the students (Salem, 2010). In this way teachers can share with the world what works the best and what not. McMillan and Schumacher (2006) point out six reasons for the importance of research in education. The following two are the most outstanding reasons for the researcher. Firstly, teachers need to be informed about the best practices because they are constantly trying to understand educational processes and must make professional decisions. Research is therefore important to help stakeholders such as education departments, schools, teachers, parents and also the students to create awareness of the different approaches to teaching. Secondly, many teachers who are not full-time researchers conduct studies to guide their decisions and to serve as efforts in classroom, school and system accountability. The purpose of this study was to improve the teaching and learning of mathematics by investigating with technology, in this instance GeoGebra, the teaching and learning of transformations of functions and circle geometry.

Chapters two and three focused on literature reviews of learning and teaching, and theoretical frameworks that support technology as a pedagogical tool in mathematics education. The research problems (phase 1) were identified in Chapter 1 and the empirical research (phase 2) will hopefully be replicated (phase 3) by schools or foundation programmes, or teachers, or stakeholders, to improve mathematics teaching in South Africa. For the researcher it would be the ideal situation if this approach to teaching and learning with GeoGebra could be adopted (phase 5) in all schools, where practicable.

McMillan and Schumacher (2001) point out that every research effort entails a logical thinking process. They distinguish between two types of logical reasoning, i.e. deductive and inductive reasoning. Deductive reasoning is primarily employed in quantitative research and inductive reasoning in qualitative research (Creswell, 2009; Harwell, 2011; McMillan & Schumacher, 2001). In a deductive model of thinking the researcher tests theories with hypotheses testing. The researcher collects data with research instruments to approve or disapprove the theory. In an inductive model of thinking the researcher collects detailed information from participants, situations or events and then categorises this information in themes, also called coding. The researcher then generalises these themes and compares them to past experiences or existing literature. (Creswell, 2009; McMillan & Schumacher, 2001). Inductive logical reasoning therefore allows the researcher to explore and discover with an emerging design while deductive logical reasoning tests deductions from theories in a predetermined design (McMillan & Schumacher, 2001). This study utilises both deductive and inductive reasoning.

The focus of this research was on the role of a GeoGebra-focused learning environment in helping students to develop mathematical knowledge. The following main research question and sub-questions guided the inquiry:

What is the role of a GeoGebra-focused learning environment in helping students to develop mathematical knowledge?

The research sub-questions are:

- (i) What are the processes involved in using GeoGebra as a pedagogical tool?
- (ii) How did GeoGebra afford students an opportunity to understand mathematical concepts and move to higher levels of abstraction?
- (iii) What were students' experiences and gestures when taught with GeoGebra?
- (iv) What were the challenges that students experience whilst working with GeoGebra?

To investigate the role of a GeoGebra-focused learning environment in which students develop mathematical knowledge, a literature review was undertaken and an empirical research design was utilised. The empirical research design was an exploratory multiple case design and it comprised a sequential mixed method approach. The data were gathered using a range of methods: pre- and post-test, a quantitative survey, in-depth interviews,

focus group interviews and observations. Both quantitative and qualitative data were used to answer the research questions. This research therefore utilised mixed methods to answer the research questions.

4.2.1 Rationale for the exploratory case study design

Yin (2009) states three conditions for the decision on the case study design. Firstly, a case study has to focus on contemporary events (Benbasat, Goldstein, & Mead, 1987; Schell, 1992; Yin, 2009). The teaching and learning with technology, especially GeoGebra, grows rapidly and minimal reports on the effectiveness of it are available (Arbain & Shukor, 2015). Teaching and learning with GeoGebra are therefore contemporary practices.

Secondly, case studies are considered the best design where the researcher has no or little control over events and subjects (Benbasat, Goldstein, & Mead, 1987; Schell, 1992; Yin, 2009). A control group was unrealistic for the following practical and ethical reasons: The SciMathUS group consisted of 99 students that were divided into four classes; three classes in Stellenbosch and one in Worcester. The researcher was responsible for teaching two of the four classes. The researcher had no control over how the other two facilitators were teaching their students. All the students in the programme are exposed to active learning and problem solving approaches. Although the other two colleagues were not active users of GeoGebra, there was a possibility that they may use it. The students' focus in SciMathUS is to gain access to tertiary institutions. This made it quite difficult and sensitive for students to be part of research and be excluded from any interventions. There was also a possibility that the control group could be contaminated.

Contamination occurs when participants of the control group are inadvertently or intentionally exposed to the intervention (Keogh-Brown, et al., 2007). According to their report this can reduce the estimated effect of an intervention. The two classes taught by the researcher could not be divided into a control and an experimental group because of the nature of the timetable. The timetable was so congested it was not feasible to conduct the study during the students' spare time. The two classes that the teacher-researcher taught, had a class together twice a week and 50 of students on the Stellenbosch campus were accommodated in five houses. There was thus a high risk that students could communicate with each other and share some aspects of the intervention.

Thirdly, the type of research questions or sub-questions will guide the researcher into the type of case study. Yin (2009) categorises case studies into three types: exploratory, explanatory and descriptive. Research and sub-questions with types of “what” questions are exploratory, types of “who” and “where” are descriptive and “how”, and “why” questions are explanatory. This study’s questions focused mainly on “what” questions and therefore according to the researcher opted for exploratory case study.

Case studies are useful for hypotheses generation and testing (Benbasat, Goldstein, & Mead, 1987; Flyvbjerg, 2006; Iacono, Brown, & Holtham, 2011; Yin, 1984). The case study design views also a pilot study as prior field work (Yin, 1984; Zainal, 2007). Yin (1984) views a pilot study a prelude, or the first case of a multi-case study. Three pilot studies were conducted. Two of the pilot studies were in 2012 in two schools and one in 2013 with SciMathUS students. The curriculum for both the secondary schools and SciMathUS are the same because the SciMathUS students write the NSC examinations. The study investigated the case of the role of GeoGebra in the teaching and learning of transformation of functions, and circle geometry.

The last consideration for a case study design by the researcher was that case studies are flexible and can generate various outcomes. It also supports all the different types of philosophical paradigms (Darke, Shanks, & Broadbent, 1998; Iacono, Brown, & Holtham, 2011). Different theoretical frameworks were utilised in this study. They vary from teaching and learning theories to theories that support the teaching of Mathematics with technology.

4.2.2 Research paradigm

A research paradigm is a viewpoint of research based on shared assumptions, concepts, values and practices (Johnson & Christensen, 2004). Poni (2014) adds that research paradigms represent a critical element in the study as they influence both the strategy and the way the researchers construct and interpret the meaning of the reality. According to Poni (2014) research paradigms also have a philosophical underpinning that orientates the researchers’ point of view. There are currently three major research paradigms in education, social and behavioural sciences, i.e. positivist, interpretive and pragmatic.

4.2.2.1 Quantitative research

The quantitative research is strongly associated with the positivist paradigm (Barbie & Mouton, 2009; Mouton, 2011; Muijs, 2011). Researchers in the positivistic paradigm assume

human behaviour in a certain way, and then utilise complex methods to prove the assumption (Wang & Zhu, 2016). According to positivism, reliable measurement instruments are needed to impartially study the physical world. In this way, they try to “understand the truth about how the world works” (Muijs, 2011, p. 4). The researcher is an outsider and the research is not dependent on the researcher (Mouton, 2011; Williams, 2007). According to Mack (2010) the positivist paradigm focuses on hypothesis testing. Quantitative paradigm uses deductive reasoning (Creswell, 2009; Harwell, 2011; McMillan & Schumacher, 2001). Quantitative research is aimed at collecting quantitative data (i.e. numerical data) and testing theories (Johnson, Onwuegbuzie, & Turner, 2007; McMillan & Schumacher, 2001; Williams, 2007). For almost a century quantitative research has been viewed as valid, accurate, and a truth-mirror (Poni, 2014).

The numerical nature of quantitative data help researchers to conduct statistical tests. Statistics is divided into two major categories called descriptive statistics and inferential statistics (Graziano & Raulin, 2000; Johnson & Christensen, 2008; McMillan & Schumacher, 2001). These two types of statistics help researchers to organise and simplify the measurements, scores or number (Graziano & Raulin, 2000). Descriptive statistics summarise, simplify and describe numerical data, while inferential statistics infer the properties of the population from those of the sample. They involve hypothesis or significance testing, effect size measurement, etc. Descriptive statistics includes mean, frequencies, range, standard deviation, histogram, box and whiskers, etc. and inferential statistics includes population, sample, hypothesis testing, t-tests, Analysis of Variance (ANOVA), effect size, Cohen’s d, etc. (Johnson & Christensen, 2008). All the mentioned statistical procedures were utilised in this study to describe the instrument reliability and to report data in a quantitative study (McMillan & Schumacher, 2001).

According to Creswell (2009), and Johnson and Christensen (2008), researchers in quantitative research utilise research questions and hypotheses, and sometimes objectives, to shape and specifically focus on the purpose of the study. They add that the researcher uses quantitative hypotheses to predict the relationship among variables. The independent and dependent variables are measured separately. The aim of hypothesis testing is so that the researcher can make a probabilistic decision about the truth of the null and alternative hypothesis (Creswell, 2009; Johnson & Christensen, 2008). Quantitative data also use effect size to measure the strengths of the relationship between the variables (Creswell, 2009).

The following are strengths of quantitative research or data as pointed out by various authors: It is useful for large numbers of people (Creswell, 2015; Johnson & Onwuegbuzie, 2004). Johnson and Onwuegbuzie (2004) point out that data can be obtained that allows researchers to do quantitative predictions and data can be collected in groups and therefore collected very quickly. Participants can therefore complete surveys, pen-and-paper test, etc. in groups. Quantitative research provides measurements or data that are precise and numerical (Johnson & Onwuegbuzie, 2004) and can therefore efficiently be analysed (Creswell, 2015; Johnson & Onwuegbuzie, 2004). The pre- and post-tests were assessed with a scoring rubric and surveys were done with a Likert scale. The data are therefore precise and numerical. The data analysis was less time consuming than the analyses of the qualitative and were analysed with Statistical Package for the Social Sciences (SPSS). The research results were therefore relatively and independently analysed (Johnson & Onwuegbuzie, 2004), for example, effect size, statistical significance, etc. were utilised and in this way partiality of the researcher was thus controlled (Creswell, 2015).

The limitations of quantitative data are that it is often viewed to be weak in understanding the context and situation of the participants, and the verbal views of the participants cannot be heard (Creswell & Plano Clark, 2011). Quantitative research can also be impersonal and unexciting (Creswell, 2015). The researcher can also miss out on the phenomena occurring because of the focus on theory, or hypothesis testing, rather than on theory or hypothesis generation. (Johnson & Onwuegbuzie, 2004).

The quantitative part of the study was a pre-experimental pretest-posttest design. McMillan and Schumacher (2006) consider pre-experimental designs to be “without two or more of the six characteristics of experimental research” (p. 262). They define experimental research with the following six characteristics: theory-driven research hypotheses; statistical equivalence of subjects in intervention and control and/or comparison groups achieved through random assignment; researcher-controlled intervention independently and uniformly applied to all subjects; measurement of each dependent variable; use of inferential statistics and rigorous control of conditions and extraneous variables. In this study there was no control group to compare with the experimental group because the two classes were ethically supposed to receive equivalent instruction. The experimental groups were not randomly selected but were convenient samples of participants in two SciMathUS bridging programme classes of 25 students each taught by the researcher. There was no rigorous

control of conditions and extraneous variables, and therefore the quantitative part of this study was not a true experiment.

Pre-experimental designs are divided into three types (McMillan & Schumacher, 2006): Single-Group Posttest-Only Design; Single-Group Pretest-Posttest Design and Non-equivalent Groups Posttest-Only Design. There was no comparison group and therefore this study used the Single-group Pretest-Posttest Design as in Figure 4.2.

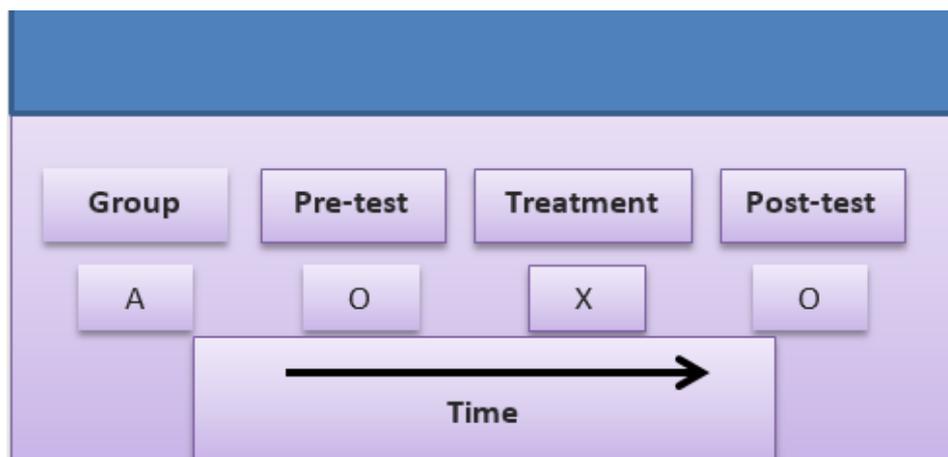


Figure 4.2: Single-group pretest-posttest design (Adapted from McMillan & Schumacher, 2006)

McMillan and Schumacher (2006) point out that maturation is a threat to internal validity of the single-group pretest-posttest design when the dependent variable is unstable, because of maturational changes. They add that the threat is more serious if the time between the pre-test and post-test is too long or increasing. McMillan and Schumacher (2006, p. 265) also point out that “[i]f the time between the pre-test and post-test is relatively short (two or three weeks), then maturation is not a threat.” Researchers should be cautious about controlling threats such as maturation, which become more apparent with a longer period of time between pre-test and post-test (Behar-Horenstein & Niu, 2011). They assert that the impact, or influence of the intervention, can become stronger with a longer period of time between pre- and post-test.

In this study the participants wrote the pre-tests on transformations of functions. The researcher started with the treatment (intervention) for three consecutive days in the computer laboratory (six periods of 50 minutes each), teaching the transformations to participants with GeoGebra and then participants were given the post-test directly after the

treatment was done. Students also wrote a pre-test on circle geometry and then the researcher started with the intervention for two consecutive days in the computer laboratory (two periods of 50 minutes each). Only six out of the 48 students had done circle geometry in Grade 11 and therefore the researcher used more periods to teach circle geometry. The time between the pre-test and post-test for circle geometry was more than that for transformations, because of the fact that this was a new topic. All the fourteen theorems and corollaries had to be covered and there were also school and public holidays. The instruction time between the pre- and post-test was two weeks. Proofs and riders of circle geometry were done during that time. GeoGebra was used as a teaching tool to support students during the learning process. Watson (2006) asserts that research without a comparison group can lack rigour. She points out that the progress of the students in her study from the pre- and post-test is either due to the teaching or is something which happens anyway due to maturation. Maturation was unlikely because the time between the pre- and post-test was not so long. Bell (2010) points out that the longer the time lapse between the pre –and post-test, the more difficult it is to rule out alternative explanations for any observed differences. A study by Subari (2017) also shows how the researcher utilises only a treatment group to identify students' misconceptions with the pre-test and how the post-test was used to identify any reduction of misconception among the students. The quantitative results that will be generated from the pre- and post-tests in this study will not be used to make claims that the teaching and learning with GeoGebra is better than an approach without it. The results will be used to show that the teaching and learning with GeoGebra can help with better understanding of concepts.

The students were also given a pre- and post-intervention questionnaire. The pre-questionnaire was distributed at the beginning of the year and the post-questionnaire at the end of the SciMathUS academic year. The time between the pre- and post-intervention questionnaire was therefore around seven months. The reason for this time gap was that the researcher investigated if there was a change, or not, in the students' perception of how they should be taught.

4.2.2.2 Qualitative research

The qualitative research is linked to the interpretive paradigm (Barbie & Mouton, 2009; Mouton, 2011). In this paradigm, the researcher cannot be an outsider and has to part of the world that is being investigated (Muijs, 2011; Wang & Zhu, 2016). Wang and Zhu (2016)

point out that the interpretive paradigm focuses on understanding of the meaning of the phenomena and human activities, and try to understand individuals. This paradigm is therefore subjective; inductive reasoning is privileged and may lead to theory and/or hypothesis generation.

The researcher acknowledged the strengths of quantitative research and utilised them quite extensively in this research for hypothesis testing, but felt that the quantitative research alone was not sufficient to answer the research questions. The researcher sought the students' responses as to why and how GeoGebra affords them an opportunity to understand mathematical concepts and move to higher levels of abstraction. The researcher also sought the experiences and gestures of students when taught with GeoGebra. For the researcher it was also important to learn what challenges the students experienced whilst working with GeoGebra.

Qualitative research relies on the collection of qualitative data, i.e. non-numerical data such as words, images and categories (Creswell, 2009; McMillan & Schumacher, 2001; Williams, 2007). In qualitative research the participant's viewpoint is the social phenomenon that is being researched (Williams, 2007). According to Williams (2007) qualitative research is a holistic approach that involves exploring the behaviour, views, experiences and emotions of the participants. No objectives or hypotheses are stated as in the case of quantitative research. The research questions are in the form of a central question and associated sub-questions.

Qualitative research utilises a naturalistic approach that seeks to comprehend the phenomena in context-specific settings, such as real world setting where the researcher does not endeavour to control the phenomenon of interest (Patton, 2002). Hiatt (1986) states that qualitative research methods focus on discovering and understanding the experiences, perspectives and thoughts of participants. Harwell (2011) states that a researcher may generate hypotheses, explanations and conceptualisations from data provided by participants. The voices, thoughts and views were important for the researcher of this study, thus quantitative research cannot provide these. Qualitative research was therefore part of this study.

The following are strengths of qualitative research or data as pointed out by various authors: It is useful to describe complex phenomena (Johnson & Onwuegbuzie, 2004); it provides

the participant's viewpoint, or personal experiences of the phenomena, or in the context (Creswell, 2015; Johnson & Onwuegbuzie, 2004) and not of the researcher (Creswell, 2015); it can determine how participants interpret concepts and data are usually collected in natural settings (Johnson & Onwuegbuzie, 2004) for example observation while participants are actively learning or on computers, etc.; and it focuses on the enjoyment of peoples narratives (Creswell, 2015) obtained from interviews or reflection journals.

Johnson and Onwuegbuzie (2004) allude to the following weaknesses of qualitative research or data. It is quite difficult to test hypotheses and theories with qualitative research. Qualitative data are also way more time consuming to collect and analyse the data compared to quantitative data. In-depth interviews can take up a minimum of 30 minutes per interviewee and focus group interviews 50 minutes and more per group. These interviews have to be transcribed and coding has to be done. Although software such as ATLAS.ti can help with the analysis it still takes time to load this into the software. The researcher can also be biased and that can easily influence results. Harwell (2011) contends that embedded in the qualitative approach is the perspective that researchers cannot set aside their experiences, perceptions, and biases and thus cannot pretend to be objective bystanders to the research.

Creswell (2015) points out that qualitative research has limited generalisability and it only provides soft data. He adds that qualitative research is only investigating a limited number of participants and it is highly subjective. Although the analysis and collection of data for this study were time consuming, the researcher still opted for this approach to be part of this study's research design. The students' responses and observations of their involvement in the learning process were crucial for this study because, as mentioned earlier, quantitative research was not sufficient on its own. The researcher therefore decided to triangulate the two research approaches. In the qualitative part of this research participants were observed and video recorded during the teaching lessons. Interviews were also conducted with a sample of participants after being taught with GeoGebra. The interviews were individual in-depth interviews and focus group interviews.

4.2.2.3 Mixed methods research

Creswell and Plano Clark (2011) contend that Campbell and Fiske initiated the use of mixed-methods research in 1959. Campbell and Fiske (1959) used multiple quantitative information in a single study and referred to it as multitrait-multimethod research. Since the early 1990s

mixed-methods research has been progressively articulated and linked to research studies and this contributed to be recognised as the third main research paradigm.

Pragmatism or post-positivism is associated with mixed methods research (Creswell & Plano Clark, 2011; Denscombe, 2008; Johnson & Christensen, 2008). According to Creswell and Plano Clark (2011) researchers in pragmatic paradigm, test hypotheses and provide various views, i.e. inside and outsider perspectives (Johnson & Christensen, 2008). Pragmatism includes researchers with both biased and unbiased perspectives (Creswell & Plano Clark, 2011).

Mixed methods research is also developed in such way that it has been given numerous names. It has been called multimethod research (Hunter & Brewer, 2003; Morse, 2003), multiple methods (Smith, 2006), blended research (Thomas, 2003), integrative research (Johnson & Onwuegbuzie, 2004), triangulated studies (Denzin, 1978; Mathison, 1988; Sandelowski, 2003); ethnographic residual analysis (Fry, Chantavanich, & Chantavanich, 1981), mixed methods research (Creswell & Plano Clark, 2011; Johnson & Onwuegbuzie, 2004; Tashakkori & Teddlie, 2010) and mixed research (Johnson & Christensen, 2008; Johnson, Onwuegbuzie, & Turner, 2007). The use of the terms such as mixed research and integrative research do not limit it to the combining or mixing of methods only (Johnson, Onwuegbuzie, & Turner, 2007). Researchers can therefore integrate methods of collecting, or analysing data, or methodology, or theoretical frameworks, in the same study (Creswell, 2009; Creswell & Plano Clark, 2011; Harwell, 2011; Johnson, Onwuegbuzie, & Turner, 2007). Denzin (1978) also refers to mixed research as a means of triangulation. The researcher adopted a sequential mixed-methods approach to research for triangulation purposes. The researcher opted for the sequential mixed methods research, because the quantitative and qualitative data were collected in different phases of the study. (Creswell, 2009; Creswell & Plano Clark, 2011). This study started with the collecting of quantitative data (pre-intervention questionnaire and pre-test), followed by qualitative data (observations of lessons in computer lab), then quantitative data (post-test and post-intervention questionnaire) and was concluded with qualitative data (interviews). The data were collected from February 2014 to November 2014. Creswell (2009) adds in the sequential mixed methods research that the researchers utilised the qualitative data to strengthen or support the quantitative data or vice-versa in the interpretation phase. In using a mixed methods design the researcher has several options regarding the priority of the research. According

to Creswell and Plano Clark (2011) priority refers to the relative importance, or weighting of the quantitative and qualitative methods for answering the study's questions. The researcher can give priority to both methods or can emphasize one method over the other (Byrne & Humble, 2007). In mixed analyses, either the qualitative or quantitative analysis strands might be given priority or approximately equal priority (Bainbridge & Lee, 2013; Onwuegbuzie & Combs, 2011). In this study, the qualitative and quantitative approaches were triangulated within the data analysis and data interpretation stages, with the qualitative and quantitative phases occurring sequentially and the qualitative phase and data given more weight. Uppercase letters (e.g., QUAN, QUAL) are used to emphasize the priority on the form of data collection, and lowercase letters to imply less emphasis (e.g., quan, qual) (Creswell, Plano Clark, Gutmann, & Hanson, 2003).

The nature of the combination of methods or data, etc. will depend on the research questions and concerns that the researcher is faced with (Johnson & Christensen, 2008). The combining of research methods draws on the strengths of both quantitative and qualitative research methods and it helps to move away from the traditional research approaches, especially those are linked with quantitative methods (Harwell, 2011; Johnson & Onwuegbuzie, 2004). Harwell (2011) adds that the chance of making mistakes can be minimised if two or more research methods with different strengths and weaknesses are combined. The use of different methods will also help to use different types of data and therefore the methods and data triangulation can also be done (Harwell, 2011). Johnson and Onwuegbuzie refer to the strength of mixed-methods research by using the metaphor of fish nets of Lincoln and Guba (1985):

“Perhaps a fisherman has several fishing nets, each one with one or more holes. To come up with one good net, the fisherman decides to overlap the different fishing nets, forming one overall net. All the nets have holes in them; however, when the nets are put together, there will probably no longer be a hole in the overall net” (Johnson & Christensen, 2008, p. 51).

The metaphor emphasises the strength of mixed research methods. There can be numerous weaknesses in the researcher's quantitative and qualitative methods, but taken together a researcher obtains much stronger evidence.

According to Creswell, Shope, Clark and Green (2006) mixed-methods research also involves the collecting, analysing and combining of qualitative and quantitative approaches in a single study or series of studies. Firstly, Greene and Caracelli (1997) add that mixed-

methods research can be utilised for testing the agreement of results acquired from the various research instruments. Secondly, mixed-methods research elucidates and builds on results of one methodology with another methodology. Lastly, mixed-methods research also exhibits how results from one methodology can impact subsequent methodologies or interpretations drawn from results (Greene & Caracelli, 1997).

Denscombe (2008) asserts that researchers use mixed-methods research for the following reasons: to improve accuracy of the data, to produce a more complete picture by combining information from complementary kinds of data or sources, to avoid bias intrinsic to single-method approaches, as a way of developing the analysis and building on initial findings, using contrasting kinds of data or methods and as an aid to sampling with. For example, questionnaires being used to screen potential participants for inclusion in an interview programme. The researcher can collect richer and robust data with a mixed research approach and in this way more complex and broader research questions can be investigated (Johnson & Onwuegbuzie, 2004; Yin, 2009). Case studies usually rely on qualitative methods but both qualitative and quantitative data can be used (Bengtsson, 1999; Johnson & Christensen, 2008). The words, image and narratives in mixing the data, add significance to numbers in a mixed-methods research approach (Johnson & Onwuegbuzie, 2004). The researcher in this study thus wanted to collect rich and robust data and therefore, decided to triangulate the quantitative data and the qualitative data. In this way triangulation was utilised in this study to enhance both validity and reliability of the results and data (Yin, 2009; Zohrabi, 2013).

Johnson and Onwuegbuzie (2004) and Zohrabi (2013) add that different research instruments to collect data can supplement each other and can enhance the validity and reliability of the data. Collecting data with one instrument can be disputed, weak and biased (Zohrabi, 2013). With a variety of resources the results can be trusted and therefore be more valid and more reliable. Johnson and Onwuegbuzie (2004) also allude to the fact that mixed-methods research can also be utilised to increase the generalisability of the results and together qualitative and quantitative research produce more complete knowledge to inform theory and practice. Mixed method approach is therefore important for the researcher so that his findings of this study and interventions can be replicated in order to ensure effective teaching and learning in topics in Mathematics.

There are few weaknesses of the mixed-methods research, identified by Johnson and Onwuegbuzie (2004). It can be a challenge for a single researcher to do both qualitative and quantitative research. For this researcher it was not such a challenge as the time between the collections of data was enough for the researcher to analyse the different data before collecting a new set of data. Figure 4.3 shows the research design with the instruments utilised in this study.

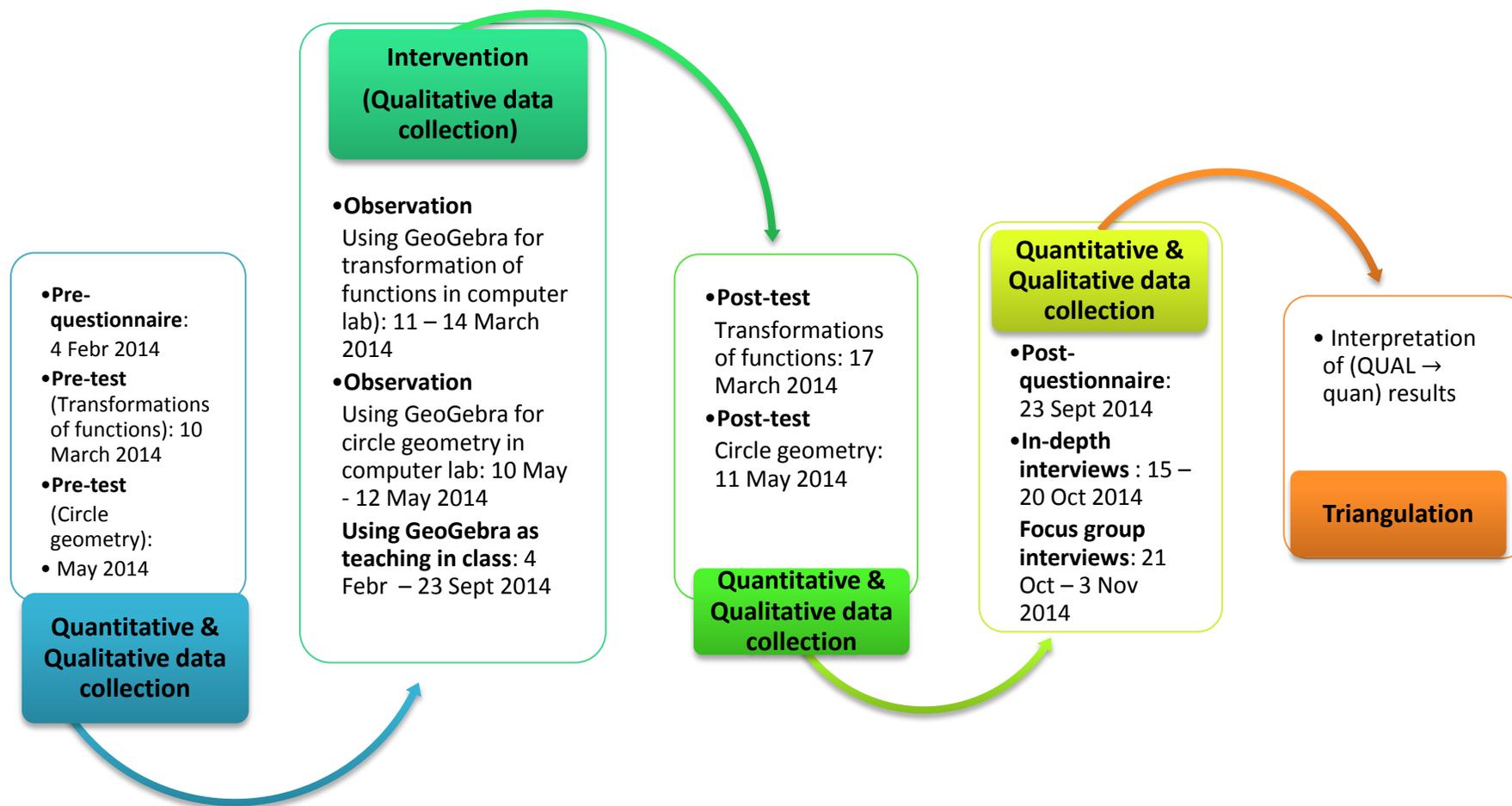


Figure 4.3: Research design of this study

4.3 RESEARCH METHODS, SAMPLING AND INSTRUMENTS

This part of the chapter focuses on the research methods and instruments that were used in the study. According to Mouton (2011) research methodology focuses on the research process and the type of tools and procedures to be used. He also adds that data be collected by various data collection instruments. The researcher fulfilled a dual role, teacher-researcher, during the data collection period of the research. The researcher was also the teacher who conducted the intervention with GeoGebra and pre- and post-tests, pre- and post-questionnaires and interviews and observations. A discussion on the data collection in this study follows now.

4.3.1 Population and sampling procedures

McMillan and Schumacher (2001; 2006) state that a population is a group of elements or cases, whether individuals, objects, or events, that conform to specific criteria and to which we intend to generalise the results of the research. Population in human research is the larger group of all the people of interest from which a sample is collected (Graziano & Raulin, 2000). As mentioned in chapter four (see section 4.2.1) there was a chance of contamination of the control group and therefore the researcher did not select one or the other. The two classes received teaching from the researcher were therefore a convenient sample in this research. In convenience sampling a group of participants is selected on the basis of availability or expediency (Johnson & Christensen, 2008; McMillan & Schumacher, 2006).

This researcher also used simple random sampling for selection of the students to participate in the in-depth and focus group interviews. In simple random sampling participants are selected from the population so that all members of the population have the same probability of being selected (Graziano & Raulin, 2000; McMillan & Schumacher, 2001; 2006). This study also utilised purpose sampling to select the students' responses for the qualitative analyses from the students' answers in the pre- and post-tests. In purpose sampling the researcher selects specific features from the population or samples that were informative about the phenomena investigated (McMillan & Schumacher, 2006). The researchers investigated the learning gains of the students from the pre-test to the post-test.

4.3.2 Quantitative data collection and analysis strategies

The quantitative data for this study were collected through questionnaires and paper-and-pencil achievement tests. The analysis of the quantitative (quan) data were given less priority and weight.

4.3.2.1 Survey method and questionnaire instrument

A questionnaire is an instrument utilised in survey research (McMillan & Schumacher, 2006). The researcher selects a sample of respondents to gather information on the phenomena investigated. Surveys are research methods, utilised to investigate participants' beliefs, values, opinions, ideas, etc. (McMillan & Schumacher, 2006). This study utilised pre-and post-intervention questionnaires and were completed by all 48 students, who were part of the intervention. Convenient sampling was therefore administered.

Questionnaires are the most commonly used instruments for obtaining information from participants in a research project. A questionnaire is relatively economical, has the same questions for all the participants, and can ensure anonymity (McMillan & Schumacher, 2006). Muijs (2011) views a pencil-and-paper questionnaire as the most common method in educational research. The main advantage according to him is that "it allows [participants] to complete at their own convenience and provides them with some time to think about their answers" (Muijs, 2011, p. 39). The often low response rate and time-consuming follow-up are some of the disadvantages of using questionnaires. In the pilot data analysis (Chapter 5) the results show how the response rate increased from the pilot studies to the actual study. This was as a result of granting the students enough time to complete it in class and not to take it home and submit the next day.

The pre-intervention questionnaire (see Appendix C) was first given to the students at the start of the programme for both the pilot and main study in February 2013 and 2014 respectively. The purpose of the pre-intervention questionnaire was to elicit students' perceptions of how they thought they should be taught and how they should learn Mathematics; students' perception on the use of computers in the teaching of Mathematics; if students liked doing Mathematics; why they were doing Mathematics and how they perceived their exposure to computers and if they ever did Mathematics on a computer before. The administration of the pre- and post-intervention questionnaires was a

longitudinal design. A longitudinal survey helped the researcher of this study to investigate the change of the participants' perceptions over time (Graziano & Raulin, 2000).

The post-questionnaire (see Appendix D) was also given to students in September. The purpose of the post-questionnaire was to elicit students' perceptions of how they thought they should be taught and to learn Mathematics; students' perceptions of the use of GeoGebra in the teaching and learning of transformations of functions and circle geometry; students' perspective of problem solving; students' perspective of the effectiveness of GeoGebra as a teaching/learning tool; how they viewed the process involved in using GeoGebra as a teaching tool and participants' experiences of the GeoGebra activities.

The first two "themes" in both questionnaires were the same. It was done this way to establish students' views on how they should be taught and how they should learn Mathematics and also obtain their views on the use of computers in the teaching of Mathematics before and after the intervention. Rating scales produce numerical (quantitative) data rather than qualitative data (Johnson & Christensen, 2004). The Likert rating scale was used in both questionnaires (see Table 4.1).

Table 4.1 is an example of items that were used in the questionnaire (see Appendix D) after the intervention.

Table 4.1 Likert scale used in post-questionnaire

Students' perspective of how they think they should be taught and to learn Mathematics	Strongly agree	Agree	Neutral	Disagree	Strongly disagree
I am capable of finding out myself how Mathematics works.					

The Likert scale is an ordered, one-dimensional scale from which respondents choose one option that best aligns with their view. A five-point Likert scale was used in both questionnaires.

Fisher's exact test was utilised to test if the relationship between the pre- and post-intervention questionnaires were statistically significant. This test is administered when a researcher wishes to conduct a chi-square test, but one or more of cells has an expected frequency of five or less (Morgan, Gloeckner, & Barret, 2013). Fisher's exact test can be

used regardless of how small the expected frequency is (IDRE, n.d). There were frequencies from both the questionnaires' data that were less than five. Fisher's exact test for 2 x 5 table (two questionnaires x five-point Likert scale) was thus utilised to test if the relationship between the pre- and post-intervention questionnaires were statistically significant. The researcher did the analysis with an online calculator. (SISA, n.d)

This study used Cramer's V to measure the effect size. Cramer's V is a measure of association, independent of the sample size. According to Morgan *et al.* (2013) this statistic is a modification of the *Phi* statistic, so that it is appropriate for larger than 2 x 2 tables. Both Cramer's V and *Phi* provide information about the strength of the association between two categorical variables (Morgan *et al.*,2013). Cramer's V ranges between 0 (no relationship) and 1 (perfect relationship). Botsch (2011) indicate the following interpretation of Cramer's $V \in [0; 0.1)$ as very weak; $[0.1; 0.2)$ as weak; $[0.2; 0.3)$ as moderate and $[0.3; 1]$ as strong.

4.3.2.2 Paper-and-pencil achievement test

According to McMillan and Schumacher (2006) a paper and pencil test is "a standard set of questions [that] is presented to each participant (subject) in writing (on paper or computer) that requires the completion of cognitive tasks. The responses or answers are summarised to obtain a numerical value that represented the participant's content knowledge of mathematics on the twin topics of transformation of functions and circle geometry. The cognitive task can focus on what the person knows (achievement), is able to learn (ability or aptitude), choose or selects (interests, attitudes, or values), or is able to do (skills). McMillan and Schumacher (2006) identify the standardised tests; norm- and criterion-referenced interpretation; aptitude tests and achievement tests and alternative assessments: performance based assessment and portfolio assessment.

The researcher constructed the criterion referenced achievement tests in the form of a pre- and post-test which were used in this study. Achievement tests are more closely tied to school subjects and are restricted to topics in the subject (McMillan & Schumacher, 2006). They measure more recent learning than aptitude testing. The purpose of it is not to predict future performance. Participants wrote the pre-tests before the start of the intervention programme and the post-tests were written after the completion of the intervention programme to measure the learning gains in transformations of functions (see Appendix A) and in circle geometry (see Appendix B). The purpose of the pre- and post-tests as data

collection instrument were to test the null hypothesis that the post-test scores for the experimental groups will not be different from those for the pre-test.

The quantitative part of this research made use of Analysis of Variance (ANOVA) to test the null hypothesis (H_0); researcher-controlled intervention (the teaching style with GeoGebra) was independently and uniformly applied to all participants; the dependent variable (learning gains) was measured and, both descriptive and inferential statistics used to analyse the results. The two-tailed null hypothesis (H_0) for this research was that there would be no significant difference between pre-test and post-test scores of students. The two-tailed alternate hypothesis (H_1) for this research was that post-test scores of the students would be significantly different from the pre-test scores. The null hypothesis was tested by the use of inferential statistics. The Statistical Package for the Social Sciences (SPSS) software was used to conduct ANOVA, calculate descriptive statistics and perform the t -tests.

The t -test is designed to show the probability that the means of two groups are different (McMillan & Schumacher, 2001). The paired t -test (dependent t -tests) was utilised in this study. A paired t -test is utilised in a pre- and post-test design to compare two groups that were matched. These groups were dependent (McMillan & Schumacher, 2001). In this study each student's pre- and post-tests were compared. The 48 students in the study were also divided into one group who entered the SciMathUS programme with less than 50% in NSC Mathematics and the second group who entered with 50% or more in NSC Mathematics. For the students in the group with less than 50% in NSC Mathematics the pre- and post-test results were compared and the same was done with the group with more than 50% in NSC Mathematics.

The statistically significant difference, p -value <0.05 , indicates the difference between two or more means large enough that is unlikely to be a chance occurrence (Graziano & Raulin, 2000). The observed difference between the means appears to be reliable (Graziano & Raulin, 2000). The p -value informs the researcher that an effect exists, but does not indicate the magnitude of the effect (Sullivan & Feinn, 2012). Effect size was measured using Cohen's d in this study. According to Sullivan and Feinn (2012) the effect size, also referred to as a substantive significance, is the main finding in quantitative research. The effect size is the magnitude of the difference between the groups. The absolute effect size is the difference between the mean in two different intervention groups (Sullivan & Feinn, 2012). Cohen's d is one of effect indices that was used in this study.

The p -value was calculated by SPSS and Cohen's d was calculated with the following formulae (Cohen, 1988; Sullivan & Feinn, 2012):

$$d = \frac{Mean_A - Mean_B}{Standard\ deviation_{pooled}}$$

Fritz, Morris and Richler (2012) point out when the standard deviations differ but the sample sizes for each group are very similar, then the following formula is used for standard deviation pooled.

$$SD_{pooled} = \sqrt{\frac{SD_A^2 + SD_B^2}{2}}$$

The researcher utilised the suggestions for effect size interpretation of Cohen's d provided by Sullivan and Feinn (2012) and Fritz *et al.* (2012) to interpret the effect sizes in this study. Ellis (2009) indicate the following interpretation of Cohen's d 0.2 as small; 0.5 as medium; 0.8 as large and, 1.3 and above as very large.

4.3.3 Qualitative data collection and analysis strategies

“Qualitative research is interactive, face-to-face research, which requires a relatively extensive amount of time to systematically observe, interview and record processes as they occur naturally. Data collection strategies focus on what the phenomenon means to participants” (McMillan & Schumacher, 2006, p. 340). McMillan and Schumacher also see strategies as sampling and data collection techniques and these strategies are “refined throughout the data collection process to increase the validity” (p.340).

The challenge for the teacher-researcher with the analysis of the qualitative data was to select the data from all the qualitative data gathered to support the research's theoretical stance and to answer the research questions. Brown (1992) points out that a methodological concern is that a researcher could select portions of edited transcript, or interviews, to illustrate a theoretical point. Brown (1992) asserts that the problem is how to avoid misrepresenting the data, however unintentionally. She refers to the problem of data selection as the Bartlett Effect. The analysis of the qualitative data in this study focused on the students' positive experiences (see section 5.4) and challenges (see section 5.5) whilst working with GeoGebra. Although the qualitative analysis of the pre- and post-test (see section 5.3) were only on a purposeful sample, it also focused on students' correct and

incorrect responses. The students' responses presented showed that not all of them understood transformation geometry better, and not all moved up to Van Hiele Level 3 for circle geometry. The qualitative data were also triangulated with quantitative data to support the teacher-researcher selection of qualitative data. The following sub-sections present how the different qualitative and quantitative data were collected.

4.3.3.1 Observations, observation schedules and techniques

Observation in this study was a qualitative research method (Kawulich, 2005). Observation can be viewed as the observing of behavioural patterns of people in certain situations to gather data about the phenomena investigated (Driscoll, 2011; Johnson & Christensen, 2008). They add that observation is an important approach of gathered information, because they sometimes do not do what they say they do. Le Grange (2001) argues that the purpose of observation is to give the researcher direct, first-hand experiences of the phenomena under study. This research method can therefore be triangulated by the participant's responses from survey methods, such as questionnaires and interviews.

The researcher's role during the observation can be a complete participant, participant-as-observer, observer-as-participant or complete observer (Johnson & Christensen, 2008). They view a complete participant as a member of the group being studied. The complete observer takes the role of an outsider. The participant-as-observer role is similar to the complete participant but the group is informed that they are being studied. Johnson & Christensen (2008) further add that the observer-as-participant takes the role of observer much more than the role of participant. The researcher was the teacher during this intervention and therefore took the role of participant-as-observer. The researcher in this study made video recordings of all lessons presented to participants. Activities were recorded in an unstructured way.

The observations also focused on the students' involvement in the learning process whilst working on GeoGebra. The observations focused on the instrumental genesis of the students and instrumental orchestration of teacher-researcher. The observation focused on the students' *Guide-and-explain orchestration* how the teacher-researcher walked around in the computer lab while students are working on the computer activities. In the *Explain-the-screen orchestration* it was observed how the teacher-researcher explained to the whole-class what happens on their screen. The observation of the *Link-screen-board orchestration* was on how the teacher-researcher stressed the relationship between what happens in the

technological environment and how this is represented in conventional mathematics of paper, book and school board. The *Discuss-the-screen orchestration* was on how the teacher-researcher used the screen to evoke a whole-class discussion.

McMillan and Schumacher (2006) point out a few strengths of observations. They assert that observations capture natural behaviours, mitigates social desirability and is relatively unobtrusive. McMillan and Schumacher (2006) also posit the following weaknesses: it is costly, time consuming, observer bias and usually not anonymous. The cost was reduced, because two video cameras were placed in the computer lab recording what students were doing. Anonymity and confidentiality were respected and guaranteed by using pseudonyms and faces of students were blurred out on the images presented in the data analysis chapter.

An observation schedule (see Appendix E) was completed from the videos. The schedule was divided into the following categories: gestures and instrumental orchestration and exploitation modes. Scherr (2004) points out that gestures are the spontaneous hand or body movements that normally accompany face-to-face conversations and are potentially an important component of interpersonal communication. Alibali and Nathan (2012) assert that gestures are often taken as evidence that the body is involved in thinking and speaking about the ideas expressed in those gestures. Piaget (1959) emphasises that gestures play an important role in learning, development and communication. Okrent (2017) points out that when people are speaking, they put their thoughts into words, and when they gesture, they place their thoughts into one's hands. She asserts further that gestures do not just show what people are thinking, they actually help people to think. Evans and Rubin (1979) also contend that gestures provide cognitive support when children attempt to talk about difficult tasks. Scherr (2004) also points out that students' gestures are part of how they articulate their ideas, and can be of use to the researcher in diagnosing students' thinking and forming effective pedagogical responses. The videos were analysed to observe the dominant gestures. Gestures can specify aspects of the mathematical object referred to and by this, enrich the verbal utterance (Krause, 2015). According to Goldin-Meadow (2004) students' gestures often provide insight into the way they represent the problem in a problem-solving situation. The purpose of observing gestures in this study was to investigate the students' thinking and explanation, and how they were used during social interaction. Krause (2015) points out that mathematical knowledge can be acquired with gestures during the social

interaction. In general, the gestures that a child produces in a problem-solving situation often provide insight into the way that child represents the problem (Goldin-Meadow, 2004).

The researcher also observed how frequently the teacher-researcher utilised different instrumental orchestrations and what exploitation modes were utilised. A complete summary of the different instrumental orchestrations and exploitation modes are presented in the data analysis chapter. Some students' conversation and discussions were recorded while busy with an activity on their computers. The information was also used in the data analysis chapter to explain the students' gestures. The data from the observation were compared (or triangulated) with the in-depth and group focus interviews, to check if there were similarities between what was observed and what students experienced when taught with GeoGebra.

The teacher-researcher was aware that students would change their normal behaviour, often positively, when observed. The tendency of people, in this case students, to increase their work pace and performance when they sense they are being observed is referred to as the "Hawthorne effect" (Harrell, Gladwin, & Hoag, 2013). Informed by Gaskel's (2012) suggestions to mitigate the Hawthorne effect, the teacher-researcher informed students that the test scores would not be used for their formal assessment marks (see Appendixes O, P and Q). Students were assured that the study was to see how GeoGebra can be used in the teaching and learning of Mathematics and, not to judge them. The students were observed during the teaching of transformations of functions for three sessions (5 hours) and two sessions (3 hours and 20 minutes) for circle geometry period of time. It was not just a once-off observation. It was done in March and May. Students also gave written feedback after their first time exposure to GeoGebra. Observation was not the only qualitative research method and was triangulated with other qualitative methods such as in-depth and focus group interviews, and content analysis of students' pre and post-test. The results of the study were therefore not based only on the observations. The *Guide-and-explain* orchestration was also utilised by the teacher-researcher with exploitation modes during the sessions in the computer lab. The teacher-researcher also engaged with students whilst busy on GeoGebra.

4.3.3.2 In-depth, semi-structured interviews, interview schedules and data analysis

An interview is a type of survey that can be administered face-to-face or by telephone, etc. (Graziano & Raulin, 2000). They add that the best findings and information are gathered when a survey is administered face-to-face.

In-depth interviews were one of the type interviews utilised in this study. Barbie and Mouton (2009) refer to in-depth interviews as basic individual interviews. According to them, open interviews allow the participant to raise his or her views. In-depth interviews are organised approaches to collect data by having discussions with individuals (Kajornboon, 2005). In-depth interviews are therefore approaches to collect data with regards to participants' perceptions and interpretation of a given situation (Kajornboon, 2005; McMillan & Schumacher, 2006); gather insights on participants' attitudes, thoughts and actions (Kendall, 2008). Also it affords participants a chance to clarify and elaborate on their ideas in their own words (Richman, Keisler, Weisband, & Drasgow, 1999). The researcher selected in-depth face-to-face interviews, because of the advantages of the social and gestural cues. Opdenakker (2006) adds that the benefit of social cues, such as voice, intonation, body language, etc. of the participant during the interview, can give extra non-verbal information to the participant's response to a certain question. Another advantage of in-depth interviews is that the interviewer can immediately react to the interviewee reactions or responses to seek more information or clarification. The responses of participants are also more spontaneous, without an extended reflection (Opdenakker, 2006). The participant therefore reacts immediately to the question and it becomes more spontaneous or naturalistic.

Interviews can be structured, unstructured or semi-structured (Bryman, 2008). Bryman (2008) and Corbetta (2003) and many other authors state that structured or standardised interviews are interviews in which all respondents are asked the same questions with the same wording and in the same sequence. Questions are usually very specific and very often the interviewee gives a fixed range of answers. The questions in this type of interview are often called closed, close-ended, pre-coded or fixed questions (Bryman, 2008; Corbetta, 2003). The strengths of structured interviews are that the researcher has control over the topics and the format of the interview. This type of interview is quite rigid and makes probing very difficult. Gray (2004) views probing as a way to explore new paths which were not initially considered when setting up questions for the interviews. The researcher did not opt for this type of interview because the study was investigating the students' experiences whilst working with GeoGebra and the researcher wished to probe the students during the interview. The researcher also wanted to add any new questions because the students' response could be of such a nature that the researcher had to know more about their specific or unique experiences.

According to Bryman (2008) and Kajornboon (2005) unstructured interviews are the type of interviews that are non-directed and are a flexible method. There is no need to follow a detailed interview guide or schedule. Each interview is different. Interviewees are encouraged to speak openly, frankly and divulge as much detail as possible. Kajornboon (2005) adds that there are no pre-set topics to pursue in non-directive interviews and can deal with a range of topics (Bryman, 2008). Questions are usually not pre-planned. The interviewer listens and does not take the lead. The interviewer follows what the interviewee has to say and the interviewee leads the conversation. The researcher did not utilise this type of interview, because the study investigated specific research and sub-research questions. The researcher felt that in-depth and semi-structured interviews would be more appropriate to answer these research questions.

This study thus utilised both in-depth and semi-structured interviews. Semi-structured interviews are non-standardised and are frequently used in qualitative analysis and the interviewer does not do the research to test a specific hypothesis (David & Sutton, 2004). The researcher has a list of key themes, issues, and questions to be covered. In this type of interview the order of the questions can be changed depending on the direction of the interview. An interview schedule is also used, but additional questions can be asked. Kajornboon (2005) indicates that semi-structured interviews are more of an open-ended questions' nature and lend themselves to probing. Kajornboon (2005) points out that if the respondent is uncertain about the question, the researcher can explain or rephrase the question. The researcher decided on semi-structured interviews because of their flexible nature. The semi-structured interviews were most beneficial because probing was possible to understand students' experiences when working with GeoGebra. They also helped with regards to students' perspective on the effectiveness of teaching and learning to improve their marks to access higher education. The semi-structured interview schedule was validated and moderated by piloting it. In this way the researcher ensured that the in-depth interview schedule measured what it is supposed to measure.

Simple random sampling was administered to select a sample of 10 students; 5 per class to conduct in-depth interviews with. All the students selected for the in-depth interviews agreed to participate in the interviews. These interviews were transcribed and coded in ATLAS.ti, which is Computer Aided Qualitative Data Analysis Software (CAQDAS). Saldaña (2009) views a code in a qualitative investigation as "often a word or short phrase that symbolically

assigns a summative, salient, essence-capturing and/or evocative attribute for a portion of language-based or visual data (2009, p. 3).” Richards and Morse (2007) refer to coding not just labelling, but rather it leads the researcher from the data to an idea, and from the idea to all the data relating to that idea. The responses of the students gave the researcher an idea or code and then linked all the rest of the ideas.

Figure 4.4 shows the coding of students’ responses from in-depth interviews.

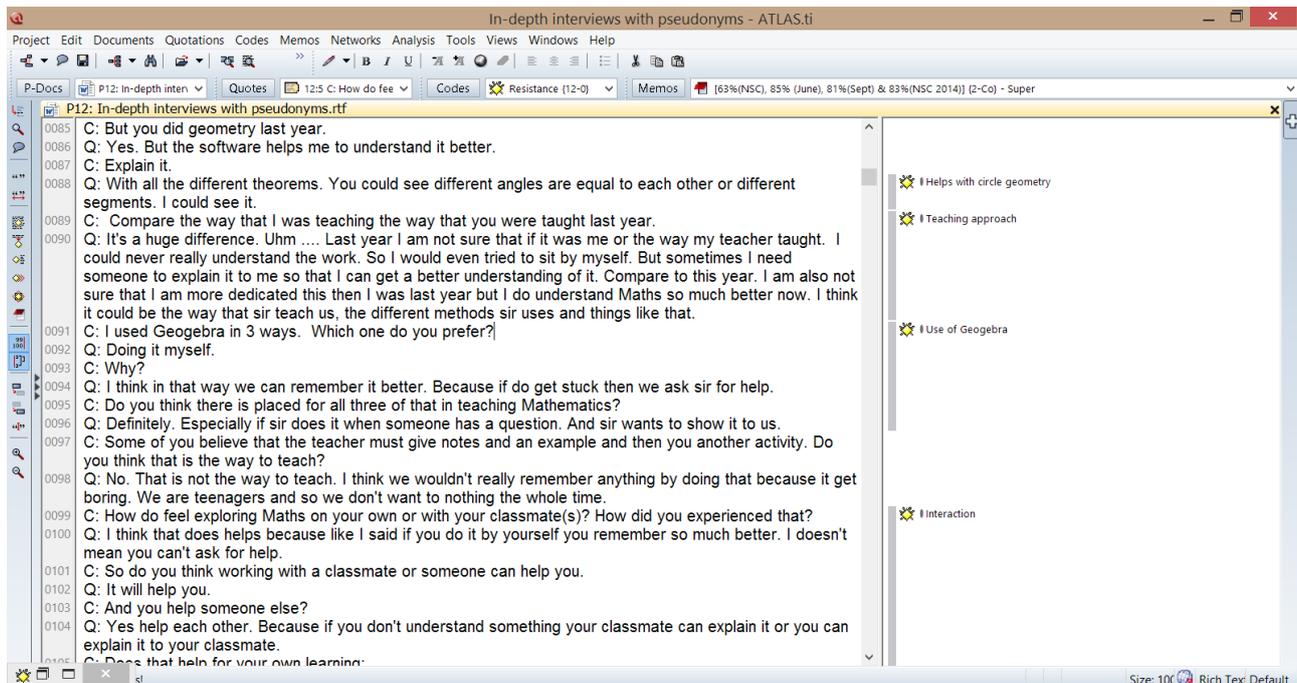


Figure 4.4: Coding of students’ responses in ATLAS.ti

The themes identified from the in-depth interviews were interaction, enjoyment, visualisation, challenges, real-life, resistance, etc.

Figure 4.5 shows the summary of the code manager in ATLAS.ti.

The screenshot shows the ATLAS.ti Code Manager interface. The window title is "Code Manager [HU: In-depth interviews with pseudonyms]". The interface includes a menu bar (Codes, Edit, Miscellaneous, Output, View), a search bar, and a table of codes. The table has columns for Name, Grounded, De..., Author, Created, Modified, and Famil. The codes listed are: Difficulty (11), Enjoyment (5), Help with other topics (1), Helps with circle geo... (7), Helps with transform... (8), Interaction (8), Real-life (4), Resistance (12), Teaching approach (15), Training in Geogebra (8), Transformation geom... (3), Use of Geogebra (10), and Visualization (9). The status bar at the bottom indicates "13 Codes" and "[1] Resistance".

Name	Grounded	De...	Author	Created	Modified	Famil
Difficulty	11	0	Super	2015-1...	2015-10-26 20:50:21	
Enjoyment	5	0	Super	2015-1...	2015-10-26 20:54:10	
Help with other topics	1	0	Super	2015-1...	2015-10-26 15:07:43	
Helps with circle geo...	7	0	Super	2015-1...	2015-10-26 21:08:09	
Helps with transform...	8	0	Super	2015-1...	2015-10-26 20:55:30	
Interaction	8	0	Super	2015-1...	2015-10-26 20:43:42	
Real-life	4	0	Super	2015-1...	2015-10-26 15:35:22	
Resistance	12	0	Super	2015-1...	2015-11-04 13:03:38	
Teaching approach	15	0	Super	2015-1...	2015-10-26 21:13:46	
Training in Geogebra	8	0	Super	2015-1...	2015-10-26 21:04:59	
Transformation geom...	3	0	Super	2015-1...	2015-10-26 15:35:53	
Use of Geogebra	10	0	Super	2015-1...	2015-10-26 21:12:08	
Visualization	9	0	Super	2015-1...	2015-10-26 21:10:53	

Figure 4.5: Summary of themes in ATLAS.ti code manager

Some themes such as *GeoGebra Helps with transformations of functions* and *Helps with circle geometry* were combined as a theme. For example, these two themes were in the analysis chapter referred to as the theme *Use of GeoGebra as pedagogical tool*.

There are limitations with in-depth interviews. The responses of participants can sometimes be inconsistent with other students' responses or deliberately not true (Bryman, 2008). Interview responses might be biased because participants want to prove the interventions are working (Boyce & Neale, 2006). The researcher therefore used focus group interviews, and achievement tests, surveys and observation to minimise partiality. The interviewer must be trained appropriately in interviewing techniques (Boyce & Neale, 2006). That is why the researcher also piloted the interviews so that he could become more aware of how to conduct interviews more effectively. Although in-depth interviews provide valuable information the results cannot usually be generalised (Boyce & Neale, 2006).

Interviews can generally be time-consuming, because of the time it takes to conduct them, transcribe them, and analyse the results. The researcher was aware of this, but felt that interviews are important for this study and wanted to triangulate the findings with quantitative results. The ten in-depth interviews were used in this research to triangulate the quantitative data that was obtained from the post-intervention questionnaire. The semi-structured interviews were therefore helpful to explain the students' experiences and what their

gestures were when taught with GeoGebra. Students also gave the researcher insight into what challenges they experienced whilst working with GeoGebra. These interviews also gave the teacher-researcher feedback on how to improve the implementation of GeoGebra in future with other students in the SciMathUS programme.

4.3.3.3 Focus group interviews, schedules and data analysis

Focus-group interviews were the other type of interviews utilised in this study. Bryman (2008) and Johnson and Christensen (2008) view focus group as a form of group interview. Focus group interviews are viewed as a research technique that collects data through interaction and how the participants view and think about the topic determined by the researcher (Bryman, 2008; Johnson & Christensen, 2008; Kitzinger, 1994). This interaction is facilitated by a moderator or facilitator that leads the discussion (Bryman, 2008; Johnson & Christensen, 2008). Group interviews are often utilised as a convenient approach to gather data from a number of participants at the same time (Kitzinger, 1994; Naukusha, 2015). With group interviews the responses are between the interviewer and interviewee. Within focus group interviews, the participants are encouraged to comment on each other's views, ask questions and interact with each other.

The researcher therefore also decided to utilise focus group interviews to obtain more feedback of participants and also the responses of participants within a group context. Twenty five students of the 38 that were not part of in-depth interviews were also randomly selected to conduct focus group interviews with. Three groups had six participants and one group had seven participants. Two students declined to participate in the focus group interviews. There were therefore four focus group interviews conducted with 23 students. The other reasons for using the focus group interviews were because they allow participants to challenge or differ from each other. In this way the group discussions can provide evidence of similarities and different opinions.

Morgan (1996a; 1996b) and, Barbie and Mouton (2009) point out that the main advantage of focus groups is that the researcher can collect data in a limited period of time with more participants. The data collected in a focus group is qualitative and in the words of the group participants (Johnson & Christensen, 2008). According to Williams and Katz (2001) focus groups have the potential to generate data that may not surface in individual interviews or survey research. Williams and Katz (2001) allude to the fact that focus group interviews can

provide feelings, attitudes and beliefs from participants that can prove to be enlightening and productive.

Interviewees in conventional in-depth interviews do not challenge each other because of the one-on-one nature of the arrangement, but in focus groups the participants will often differ from each other and challenge each other's views (Bryman, 2008). Bryman adds that these arguments or disagreements can be more realistic accounts of the participants' thoughts. The researcher can therefore collect data from individuals collectively to help with understanding of the topic that is the subject of the research.

One disadvantage of the focus group interviews is that individual dynamics in the group can outweigh the group dynamics, if the moderator does not take that into consideration and control it (Barbie & Mouton, 2009). The researcher in this study experienced this problem of focus groups during the piloting phases and was more cautious during the data collection of the actual study. The researcher's ability to assemble and direct the focus group can also be a disadvantage or a threat to the validity of the focus group interviews because it can in some sense create an unnatural social setting (Barbie & Mouton, 2009). Morgan (1998) also adds that focus group interviews should be avoided if participants are too uneasy with each other to express their opinions openly.

The researcher or moderator has less control over the data produced and interaction (Gibbs, 1997; Morgan, 1998). The semi-structured interview schedule (see Appendix F) was validated and moderated by piloting it. In this way the researcher ensured that the focus group interview schedule measured what it is supposed to measure. During the focus group interviews the researcher guided the participants to ensure that there was no domination by a few individuals and that everyone's opinion was respected.

According to Gibbs (1997) anonymity and confidentiality can be a limitation because participants are sharing their views and feelings. The nature of this study did not focus on sensitive issues and personal information, and was therefore not a serious problem. Although the students differed in their views on teaching and learning with GeoGebra they did it in a positive and constructive way.

The data of the focus group interviews were analysed in the same approach as the data of the in-depth interviews in ATLAS.ti. The quantitative data that were obtained from the post-intervention questionnaire will be triangulated with the in-depth and semi-structured

interviews. The focus groups interviews were also triangulated with responses from the in-depth interviews and the post-intervention questionnaire. It therefore tests consistency of findings with the different instruments. The focus group interviews were therefore also helpful to explain the students' experiences and what their gestures were when taught with GeoGebra. Students also gave the researcher insight into what challenges they experienced whilst working with GeoGebra.

4.4 RELIABILITY AND VALIDITY OF INSTRUMENTS

Reliability and validity are the two fundamental elements that are used to measure and evaluate instruments (McMillan & Schumacher, 2001; McMillan & Schumacher, 2006; Tavakol & Dennick, 2011). An instrument such as a questionnaire, or test, is considered reliable if the same results are measured consistently, or reproduced, when the instrument is re-administered (Graziano & Raulin, 2000; Tavakol & Dennick, 2011). The focus of reliability is therefore on producing the same results, regardless by whom or when the instrument is administered. Validity is concerned with the extent to which the instrument such as a questionnaire, or test, measures what it is intended to measure (Tavakol & Dennick, 2011).

The validity of the pre- and post-test for both transformations of functions and circle geometry were determined by face validity and content validity. Content validity determines whether the questions that are being examined in the tests are relevant and appropriate, or cognitive processes, that they are intended to test (Beanland, Schneider, LoBiondo-Wood, & Haber, 1999; Polit & Hungler, 1999) and the adequacy of the content covered in the instruments (Kimberlin & Winterstein, 2008). The anticipated qualities of the instruments (Streiner & Norman, Health measure scales: a guide to their development and use, 1989) and judgments on the appearance of the instrument are determined by face validity (Cook & Beckman, 2006).

In the absence of statistical testing both content and face validity were done by experts in the field of mathematics teaching, as recommended by Streiner and Norman (1989), and Kimberlin and Winterstein (2008). Both the pre- and post-tests content were validated by two colleagues that have been involved with the teaching of Mathematics at secondary school level for more than 20 years. One of the colleagues has been involved in-service training of mathematics teachers for almost ten years. The questionnaires and interview schedule for both in-depth and focus group interviews were also evaluated by the

Stellenbosch University's Research Ethics Committee (REC). To ensure validity of the instruments and procedures in this study, the researcher also piloted the intervention, questionnaires, interviews and the pre- and post-tests with two schools in 2012 and SciMathUS students in 2013. Cohen, Manion and Morrison (2007) point out that pilot results can also be analysed to ensure the reliability of a survey or questionnaire. The researcher started with a piloting phase in two schools that were part of a bigger project at SUNCEP in 2012. The schools agreed to pilot the intervention with the Grade 10 mathematics learners, on condition that all the students were exposed to the intervention. Principals felt that they wanted all the students to be exposed to this way of teaching and the teachers had to attend the lessons to experience the teaching of functions with GeoGebra.

Cronbach alpha was used to measure the internal consistency reliability (Kimberlin & Winterstein, 2008; Peer & Gamliel, 2011; Tavakol & Dennick, 2011) for both the pre- and post-intervention questionnaire. Tavakol and Dennick (2011) point out that Cronbach alpha is a number between 0 and 1 and measures the internal consistency of a test or scale. It describes the inter-relatedness of items within a test or survey. Streiner (2003) points out that Cronbach alpha can sometimes be negative. This occurs mainly when some of the items are negatively correlated with others in the scale.

The closer Cronbach's alpha coefficient is to 1.0, the greater the internal consistency of the items in the scale. George and Mallery (2003) indicate the following interpretation of Cronbach alpha: $\alpha \in [0; 0.5)$ as unacceptable; $\alpha \in [0.5; 0.6)$ as poor; $\alpha \in [0.6; 0.7)$ as questionable; $\alpha \in [0.7; 0.8)$ as acceptable; $\alpha \in [0.8; 0.9)$ as good and $\alpha \in [0.9; 1]$ as excellent. Various authors on statistics indicate that any values ranging from 0.7 and 0.95 are acceptable (Bland & Altman, 1997; DeVellis, 2003; Nunnally & Bernstein, 1994). Streiner (2003) and Tavakol and Dennick (2011) point out that alpha values higher than 0.9 point to redundancy and show that the test length should be shortened. According to Tavakol and Dennick (2011) a longer test increases the reliability of a test regardless of whether the test is homogenous or not. In this study Cronbach interpreted within a range of 0.7 and 0.95, as recommended by Bland and Altman (1997), DeVellis (2003), Nunnally and Bernstein (1994). The Cronbach alpha was calculated with SPSS in both the pilot studies and the main study.

The researcher was initially satisfied with a Cronbach alpha above 0.9, but with more intense research into the interpretation of the Cronbach alpha, the researcher realised during the actual study that this can be a concern. Nunnally and Bernstein (1994) suggest that in

principle alpha should be calculated for each of the concepts rather than for the entire test or scale. The researcher therefore divided the post-questionnaire into 12 questionnaire constructs and recalculated the Cronbach alpha reliability coefficient for each theme/construct in the post-intervention questionnaire. With this recalculation there were some constructs that yielded negative Cronbach alpha values. Recoding was used during the final analysis of the actual data. Recoding is one of a transforming data procedure to reverse scale items so that they can correlate with other items of the scale (Sarantako, 2007). According to Sarantako (2007) transforming of data is a method that allows the analyst to convert the SPSS data into a different format that allows more accurate analysis. Recoding was done to six items of the 88 items in the post-questionnaire used in the pilot phase. Recoding was also done for six of the 96 items in the post-questionnaire in the actual research.

The tests were assessed by a scoring rubric that was also validated by one of my SciMathUS' colleagues. He was also involved with external marking of Grade 12 Mathematics for more than five years. In the scoring rubric (see Appendixes A and B) are descriptors given to allocate marks. This is done so that the allocation of marks to students' answers is consistent. If a student made a mistake in a question and he/she used the incorrect information in the next question or continued working with the mistake, it was marked positively. The DBE refers in the memorandums to this as consistent accuracy and it is utilised in the marking of the NSC examination.

The interrater reliability of the researcher's scores for both post-tests were compared to the scores of his colleague's scores. The interrater reliability can be computed by Cohen's kappa or intra-class correlation (ICC). Cohen's kappa is utilised for nominal or categorical data and ICC for continuous data (Rankin & Stokes, 1998). The intra-class correlation (ICC) was utilised to calculate the interrater reliability as recommended by Iacobucci (2001). An ICC is a correlation coefficient that assesses the consistency between the ratings of judges on a set of objects (Field, 2013). Reliability is considered to be acceptable if the ICC is greater than 0.75. Reliability is considered to be very good if the ICC is greater than 0.9 (Morgan, Leech, Gloeckner, & Barret, 2013; Portney & Watkins, 2000; Waninge, Rook, Dijkhuizen, Gielen, & Van Der Schans, 2011; Rankin & Stokes, 1998). The ICC for the post-tests for transformations of functions and circle geometry were calculated in SPSS and were found to be 0.987 and 0.992 respectively. The p -values for both tests were $p < 0.05$. This

result shows there were no significant differences between the scores of the researcher and his colleague.

As mentioned earlier, (see paragraph 4.2.2.3) this research study used a mixed methods approach to answer the research questions. Mathison (1988) suggests that good research practice obligates the researcher to triangulate methods to enhance the validity of research findings. This research used quantitative and qualitative data and different data sources (achievement tests, interviews, observation and questionnaires) to validate the findings. Denzin (1978) distinguishes four categories of triangulation: data triangulation, investigator triangulation, theoretical triangulation and methodological triangulation. This study utilised three of these types of triangulation. Firstly, data triangulation was utilised to collect data through various types of instruments or sources, i.e. tests, surveys and interviews. The in-depth and focus group interviews were also used to triangulate to validate findings. Secondly, different theoretical frameworks mentioned in the HLT for both transformations of functions and circle geometry, were also utilised and therefore theoretical triangulation was used to interpret data, especially in the observations of lessons and students' feedback from questionnaires and interviews.

Thirdly, methodological or method triangulation was also utilised, because quantitative and qualitative methodologies were used to collect data. This research used method triangulation to validate the findings. Cohen, Manion and Morrison (2007) note that methodological or method triangulation uses either the same method on different occasions or different methods on the same object of study. Thurmond (2001) shows that methodologic triangulation has the potential of exposing unique differences, or meaningful information, that may have remained undiscovered with the use of only one approach or data collection technique in the study. Denzin (1978) divides the method triangulation into the within- and between-method type of methodological triangulation. Lamnek (1995) points out that the within-method triangulation applies to multiple techniques within one method for data collection and interpretation. The within-method triangulation therefore refers to the use of either multiple quantitative or multiple qualitative approaches. This approach was not used in this study, because it utilised both qualitative and quantitative approaches.

The between-method triangulation or across-method triangulation involves combining and utilising both qualitative and quantitative methods in studying a single phenomenon (Hussein, 2009). Hussein (2009) asserts that the between-method triangulation has been

used for the aim of achieving convergent validity and testing the degree of external validity. Denzin (1978) and Lamnek (1995) acknowledge that the use of appropriate multiple methods will result in more valid research findings. Denzin (1978) notes that the rationale for the between-method triangulation strategy is that the flaws of one method are often the strengths of another. Lamnek (1995) supports the idea that the between-method triangulation usually combines qualitative and quantitative data, e.g. results of quantitative research could be illustrated and supported by qualitative data. This research therefore specifically used across-method triangulation.

4.4.1 Pilot study: phase 1

One difference between pilot case report and the actual case study reports is that the pilot reports should be explicit about the lessons learned for both research design and field procedures (Yin, 2009). If more than a single pilot case is planned, the report from one pilot case also indicates the modifications to be attempted in the next pilot case. If enough cases are done in this manner, the final agenda may actually become a good prototype for the final case study protocol (Yin, 2009, p. 94).

As mentioned earlier in this section, the phase 1 piloting was done in two schools. School A had 4 Grade 10 classes that were doing Mathematics and school B had two Grade 10 classes. In total it was 270 participants that were part of the piloting phase. Before showing what changes were made from the pilot phase 1 to pilot phase 2 it is important first to show what lessons were learnt from pilot phase 1 and why the changes later mentioned were made. Lessons with real life context on linear, quadratic, hyperbolic, exponential and periodic functions were planned. Lessons were also created in GeoGebra and all the applets of the activities were loaded on the school's server and were made available to any users on the network. In school A there were 35 computers in the computer laboratory and the class sizes ranged from 43 – 48 participants per class. Most of the participants therefore did not share a computer. Only about ten participants had to share a computer. In school B there were 25 computers and the class sizes ranged from 43 – 48 participants per class. In this school two students shared a computer.

There were a few lessons learnt from piloting in both schools. Students were not exposed to constructivist approach of teaching and struggled to do Mathematics on their own. The time spent on these activities was too short to get students into the active learning approach. Activities were too many for participants. Participants worked very slowly on the computers

and they also struggled with the type of contextual questions as well as interpreting them. The contextual lessons on linear, quadratic, hyperbolic, exponential and periodic functions were covered as planned, but not all activities on the different transformations of functions with GeoGebra were covered within in the two weeks that was granted to the researcher by the schools. Only transformations of quadratic functions were covered and only the pre-test and post-test were written on this topic. Some students also struggled with the English and this was only noticed during the interviews with some of them. Their home language is Xhosa and the teaching was in English. Two of the students that were interviewed could not answer the researcher in English. As mentioned earlier in this section (see paragraph 4.4) the schools granted the researcher permission on condition that the teachers had to learn from it.

Changes were made and the changes were guided and influenced by the lessons learnt during pilot phase 1 which were implemented in pilot phase 2. The researcher wished to investigate the effect of GeoGebra with RME on the teaching of Mathematics and therefore wanted to implement this approach over more topics and also over a longer period of time so that students could get used to technology-enhanced teaching. The first changes were the addition of designed activities to cover more topics such as transformations of functions (see Appendixes G, I and J) and circle geometry (see Appendixes K, L and M) so that GeoGebra could be used more and students could see the effect of it over a longer period of time. The researcher was also interested to investigate functions as a complete topic in the Further Education and Training (FET) phase. Pilot phase 1 focused only on the effect of parameters a and q in, for example, $f(x) = a(x + p)^2 + q$ and not on the effect of p . The design material therefore changed in order to include the effect of changes in parameter p and also to include the reflection of functions in the line $y = x$.

Secondly the pre- and post-tests (see appendix B) for functions therefore changed and pre- and post-tests for circle geometry were included in pilot phase 2. Thirdly, more questions were added in the post-intervention questionnaire to investigate students' views on how they experienced the teaching of functions and properties of quadrilaterals. The fourth change to instruments was that the interview questionnaire also changed because of the material added to activities. The researcher added focus group interviews to the procedures of gathering data. Lastly the researcher changed the way he administered the pre- and post-intervention questionnaire. Students were given the pre- and post-intervention

questionnaire, but the researcher was not satisfied with the response rate during phase 1. The response rate of the post-intervention questionnaire dropped by about 10% in comparison to the pre-questionnaire. The students in the pilot phase 2 were given the time to complete the questionnaires during class time.

4.4.2 Pilot study: phase 2

Changes from pilot phase 1 were implemented in 2013. The researcher started in 2013 to teach at SciMathUS, the bridging programme at Stellenbosch University. Fifty-four students were exposed to teaching with GeoGebra in a computer laboratory throughout the year. All 54 students wrote a pre-test and post-test on transformation of functions and completed two questionnaires; one at the beginning of the academic year and the last one was administered in September 2013. Interviews with 15 students and three focus group (six students in one group and seven students in each of two groups) were conducted. The interviewees were randomly selected using a table of random numbers.

The Cronbach alpha reliability coefficient for pre-questionnaire was 0.409. There were only 13 items in the pre-questionnaire. According to Tavakol and Dennick (2011) a low value of alpha could be due to a low number of questions, poor interrelatedness between items or heterogeneous constructs. Tavakol and Dennick (2011) also affirm that alpha is affected by the test length and dimensionality. The pre-questionnaire was not changed, because the researcher only wanted to investigate the participants' perceptions at the start of the SciMathUS programme. The Cronbach alpha reliability coefficient for the post-questionnaire was 0.943 and was not of concern after piloting of the questionnaires. As mentioned earlier in this chapter (see section 4.4) the post-questionnaire was divided into 12 questionnaire constructs. The Cronbach alpha reliability coefficients were recalculated for each theme in the post-intervention questionnaire. As mentioned before some constructs that showed negative Cronbach were recoded. The items affected were SPTLM1, SPTLM3, SPTLM4, SPCTM1, SPCTM2 and SPCTM4 (see Appendix D).

The post-intervention questionnaire only changed because circle geometry was added to the intervention's activities. The pre- and post-test items for functions did not change but circle geometry items were added to the pre- and post-tests for the main study.

4.5 HYPOTHETICAL LEARNING TRAJECTORY (HLT)

As mentioned in previous chapters (e.g. section 3.6) this study made use of a hypothetical learning trajectory (HLT). The mathematics teaching cycle of Simon (1995) shows (see Figure 4.6) a schematic model of the cyclical interrelationship of aspects of the researcher's knowledge, thinking, decision-making and designed activities. The HLT in this study follows a mathematical route to give students a deeper understanding of transformations of functions and circle geometry.

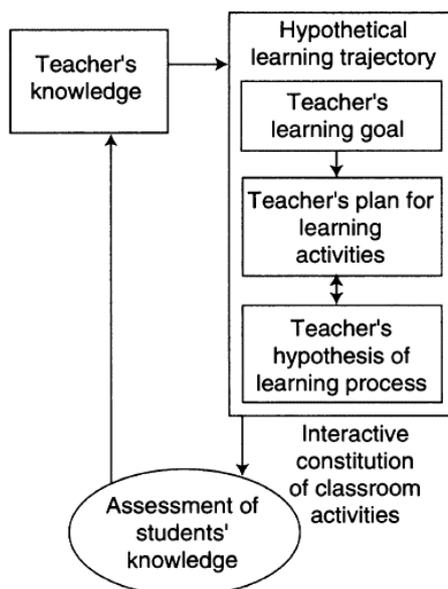


Figure 4.6: Mathematics teaching cycle (Simon, 1995)

The researcher used the following HLT to support and organise students' learning of transformations of functions. The route used in this study describes the mathematical knowledge a student needs to understand transformation of functions. As mentioned in Chapter 3 (see paragraph 3.6) the 48 students in the main study came from 37 different schools in South Africa. The researcher's knowledge as in Figure 4.6 was based on interactive teaching and learning theories discussed in chapters 2 and 3.

The researcher's knowledge of the challenges that students experience with the learning of functions was derived from (see section 1.2.2.3) the National Diagnostic Reports on Learner Performances (DBE, 2003) and his experiences of teaching functions and circle geometry for the past 20 years, and also the use of GeoGebra as a teaching tool during his facilitation of the in-service training of teachers in ICT integration. This is consistent with Mishra and Koehler's (2006) perspective of TPACK which, as discussed in Chapter 3, refers to the

researcher's knowledge of mathematical representations, materials and activities in technology enhanced classrooms. The researcher's knowledge of how students learn, as well as learning retention rates after 24 hours of teaching (Sousa, 2001) informed the development of the HLT.

4.5.1 The HLT for the teaching and learning of transformations of functions

The HLT for the teaching and learning was presented as a mathematical route within six phases. The students in SciMathUS followed the CAPS syllabus (DBE, 2011a) but were not taught using the pacesetters (syllabus aligned Learning and Teaching Materials) from DBE. The teaching and learning of functions and graphs were taught in SciMathUS, starting with Grade 8 work moving up to Grade 12 work. Some of the grades in which work was covered were not part of the experiment, but the researcher viewed it important to give the students a better background for understanding of functions and graphs and to help them with transformations of functions.

The hypothetical learning trajectory for the teaching of functions was done in a student-centred approach of RME. That is, the guided-reinvention principle was applied with design instructional material and GeoGebra, and also the traditional approach of teaching by interactive lecturing. Interactive lecturing was often complimented with the use of GeoGebra (see Appendix R). The researcher decided on this approach because the students had already been exposed to functions when they were at school. According to the CAPS (DBE, 2003) students should have studied shape, domain (input values), range (output values) and also real-life applications of the linear, quadratic, hyperbola, exponential and periodic functions. The researcher was of the view that to learn function transformations effectively the students had to know the shapes of the functions mentioned above and to understand what a function, domain, range and the critical value for each function is.

Table 4.2 shows the HLT for teaching and learning of transformations of functions.

Table 4.2: Overview of the HLT for the teaching and learning of transformations of functions

Phase	Designed activities	Appendix	Addressing / investigating the following theories	Data obtained with	Role of teacher-researcher (TR) and student
1	Amazing relations	G & examples in R	RME: Activity / Reality / Level / Intertwinement / Interaction / Guided reinvention Instrumental orchestration: Explain-the-screen / Link-screen-board	Observation (QUAL) Questionnaire (quan)	Students work in groups of five and complete activities. Students will explore the internet and build boxes to answer some of the activities. TR: Interactive lecturing with GeoGebra will be used to verify students' answers in the activities.
2	Every graph tells a story	H & examples in R	RME: Activity / Reality / Level / Intertwinement / Interaction / Guided reinvention Instrumental orchestration: Explain-the-screen / Link-screen-board	Questionnaire (quan)	Students work individually, or in pairs, and complete activities. TR: Interactive lecturing with GeoGebra will be used to verify students' answers in the activities.
3	What is a function?	Examples in R	RME: Activity / Reality / Level / Intertwinement / Interaction / Guided re-invention	Questionnaire (quan)	Students work individually, or in pairs, and complete activities. TR: Interactive lecturing with GeoGebra will be used to verify students' answers in the activities.
4	Students exploring the properties transformation geometry with GeoGebra	I	RME: Activity / Level / Intertwinement / Interaction / Guided reinvention Instrumental genesis Instrumental orchestration: Explain-the-screen / Link-screen-board / Discuss-the-screen / Guide- and-explain	Observation (QUAL) Questionnaire (quan) In-depth interviews (QUAL) Focus group interviews (QUAL)	Students create own activities in GeoGebra in computer lab.
5	Transformation of functions with GeoGebra	J	RME: Activity / Level / Intertwinement / Interaction / Guided reinvention Instrumental genesis Instrumental orchestration: Explain-the-screen / Link-screen-board / Discuss-the-screen / Guide- and-explain	Observation (QUAL) Questionnaire (quan) Pre- and post-test (quan & QUAL) In-depth interviews (QUAL) Focus group interviews (QUAL)	Students work with GeoGebra applets (see list in Appendix J) and type in own functions in GeoGebra in computer lab.

4.5.1.1 Description of the activities and rationale for the activities in the phases

The six tenets of RME were built into the first three phases of the HLT. There were no realistic contexts utilised in Phases 4 and 5, but the activities were ‘experientially’ real to students (Drijvers, 2012) because GeoGebra was used to support the students in grasping the mathematical concepts (Widjaja & Heck, 2003). The students’ instrumental genesis and the types of instrumental orchestration the teacher used, were investigated and analysed during the observations in phases 4 and 5. The lessons during phases 1, 2 and 3 were not video recorded but the teacher-researcher had photos of students’ work and interaction during phase 1. The lessons during phases 4 and 5 were video recorded.

Phase 1 involved an activity (see Appendix G) where all the students in SciMathUS on the Stellenbosch campus were divided into groups of four or five and the whole group was together in one lecture. Students were given five activities to explore the different shapes or graphs of different functions, as well as the domain and range within contextual activities. These five activities focused on the following functions or graphs: linear, quadratic, cubic, hyperbolic, exponential and periodic functions (sine or cosine graph). The plotting and drawing of the graphs were first done on graph paper. The rationale for this was for the students to understand the shape of the graph and that they could see the purpose of Mathematics, and also realise that mathematics is a human activity. Below is an example of the one of the activities used in phase 1.

Worksheet excerpts: Activity 3 in phase 1 (Metal weights)

Investigate the displacement of water on addition of each weight up to a total of 7 weights.

- Draw a table with your number of weights and the volume. Make sure that your measurements are accurate.
- Draw a graph. Select an appropriate scale for your graph.
- Find the equation of the functions.

This activity was very practical and hands-on because the students had to measure, build and do research on the internet, etc. and plot and draw the graphs. The nature of this activity was that students might measure incorrectly. For example, the students had to measure the displacement of the water if metal weights (with the same weight) were added. Incorrect measurements would not give students a perfect linear graph.

The information that the students needed for drawing the graphs of the activity *Amazing Relations* was done in a 2 hour 30 minute tutorial and was also done as a group activity.

The discussion, interpretation and analysing of the activities were done the next period for 50 minutes. The teacher-researcher did the plotting of points in GeoGebra and used sliders to check whether the equation fitted the points. During this 50 minute lesson the teacher-researcher utilised the *Explain-the-screen* and *Link-screen-board* instrumental orchestration. This lesson was thus an interactive lecture where pen-and-paper activities were explained by the teacher-researcher with GeoGebra. The teacher-researcher made sure that students understood what happens on the GeoGebra screen, especially the algebraic and the graphic views.

The activities (Appendix H) in phase 2 were more or less the same as phase 1 activities, but the students did not measure or build anything. They just had to complete the table, plot the points, draw the graph on graph paper, find the equations of the functions and answer a few questions. In this way students could find the exact graph. Below is an example of one of the activities used in phase 2.

Worksheet excerpts: Activity 4 in phase 2 (Growing water plant)



In various places in Belgium one finds a certain type of pond which they call vijers. In one particular pond there is a fast growing water plant. On 1 January this year the water plant covered 5m^2 of the surface of this pond. The area covered by the plant doubles every year.

- Set up a table and draw the graph of it.
- What happens (see table) to y -values if the x -values increases?
- What shape is this graph?
- What possible values for x can we use to draw this graph?
Hint: Can x be negative numbers?
- Write down the equation of the graph that represents the above story.

The activities in phase 2 were done in a 1 hour 40 minutes session. During this lesson the teacher-researcher utilised the *Explain-the-screen* and *Link-screen-board* instrumental orchestration. The activity below was done by the teacher-researcher in GeoGebra. In this way the teacher could change the information in the activity with ease and by using sliders in GeoGebra the students then could see how the equation changed. This activity is also important, because it exposed the students to change of the parameters in the different function and also to the realistic (RME) approach of transformation of functions. The teacher-researcher in this activity did not use any transformation terminology but pointed out that graphs changed (positions, etc.) if the parameters changed.

Activities in phase 3 (Appendix H) were to formalise to the students the concepts of a function, domain and range. The focus here was to show students that the domain is determined by the context of the situation, for example the building of a box, and constraints or conditions in the formula of a function. For example, $f(x) = \frac{2}{x}; x \neq 0; y \neq 0$. The researcher viewed this activity as important for the students to understand the domain and range for transformation of functions, especially for the reflection of the graph in $y = x$.

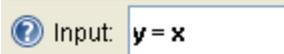
The activities in phase 4 (Appendix I) were the first part of the pre-experimental design of this study. The students wrote the pre-test on transformations of functions at the start of this intervention. These activities were done with GeoGebra in the computer lab. The students were given a worksheet and had to complete it by creating the activities on the worksheet in GeoGebra. These activities were on the properties of transformation geometry. The researcher viewed this as part of the learning trajectory, because students had to understand the properties of rigid transformations and reduction or enlargement. As stated earlier, (section 3.9) rigid transformations refer to rotation, reflection and translation. The properties of reflection and translation were important for this study, because CAPS requires students to translate and reflect functions. GeoGebra uses the term dilation for enlargement or reduction. Students have to understand that reflection and translation change only their positions and not their size. Dilation is a transformation that changes the original object to the same shape as the original, but is different in size. A larger image is called an enlargement and a smaller image a reduction. Dilation also refers to stretching and shrinking of the original figure (Dilations, 2012-2016). The properties of transformations and the concept of congruency and similarity were for the researcher the foundation for students to

understand the transformations of functions. Below is an example of the one of the activities used in phase 4.

Worksheet excerpts: Activity 1 in phase 4 (Reflections)

Reflect a polygon in GeoGebra on a set of axis about a line

- Draw any polygon.
- Reflect your object in the following lines and complete the table as you progress:
 - ✓ y -axis (complete table below)
 - ✓ x -axis (complete table below)
 - ✓ $y = x$. Click in the Input box (bottom left of your screen) and type in $y = x$.

 Input: $y = x$

and press Enter on keyboard

- ✓ $y = -x$. Click in the Input box and type in $y = -x$.

What are the co-ordinates of the image points of the vertices of this triangle after the different reflections?

Co-ordinates of your original polygon	Co-ordinates of image when reflected y-axis	Co-ordinates of image when reflected in x-axis	Co-ordinates of image when reflected in $y = x$	Co-ordinates of image when reflected in $y = -x$
A	A'	A'	A'	A'
B	B'	B'	B'	B'
C	C'	C'	C'	C'

- What do you notice about the co-ordinates of the image when reflected in: (give rules to explain your answers)

- ✓ y -axis : $(x; y) \rightarrow$
- ✓ x -axis : $(x; y) \rightarrow$
- ✓ $y = x$: $(x; y) \rightarrow$
- ✓ $y = -x$: $(x; y) \rightarrow$

Investigate the properties of reflection (discuss them with a classmate)

Move the vertices of the original polygon, as well as the line of reflection. Check whether the following is true:

- The mirror image of a figure always lies on the opposite side of the mirror line.
- The line that connects the original figure to its mirror image is always perpendicular to the axis of symmetry (mirror line) and is bisected by the axis of symmetry.

This activity was done in 1 hour 40 minutes. During this lesson the teacher-researcher utilised the *Explain-the-screen*, *Link-screen-board*, *Discuss-the-screen* and *Guide-and-explain* instrumental orchestration. Students' instrumental genesis gestures were also observed during this phase.

The activities in phase 5 (Appendix J) were also part of intervention of this study. This activity explored visually the transformation when the function is given algebraically. Students were given nine prepared applets in GeoGebra and designed instructional material to explore the effect of the parameter of the different functions. In each GeoGebra applet students used the tracing tool to draw the graphs of the different transformations as shown in Figure 4.7.

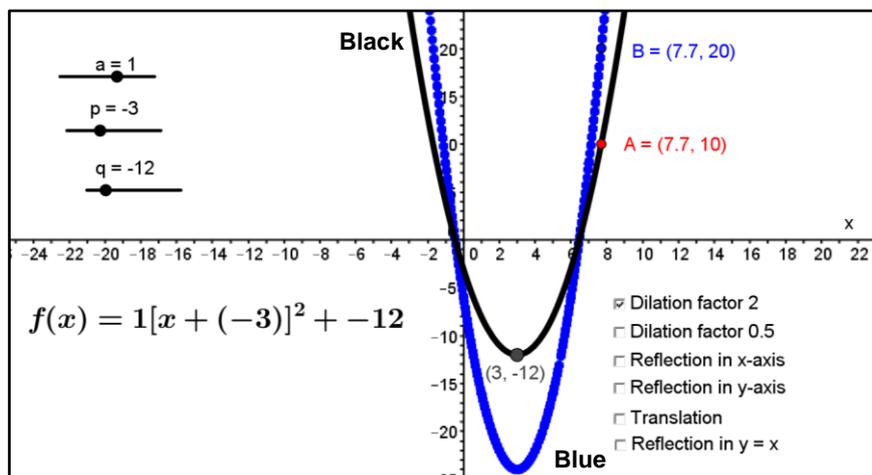


Figure 4.7: Designed GeoGebra applet: Function transformations

The graph in black in Figure 4.5 represents the graph of $f(x) = (x - 3)^2 - 12$. The students ticked the type of transformation; in this case the dilation factor was selected. The graph in blue represents the graph of the dilation of $f(x)$ with scale factor 2. The students then used sliders a, p and q to find the equation of the transformed function and completed the first row of the table below. Students then continued to dilate by factor 0.5 and completed the first row of the first table (see worksheet excerpt below). All the functions with a transformation (in this case dilations with scale factor 2) were in a table so that students could see that the effect of the transformation is the same to all the functions. Below is an example of one of the activities used in phase 5.

Worksheet excerpts: Activity 1 in phase 5

Activity 1: Transformation: Dilations with scale factor 2 (see full worksheet in Appendix J)

Graph type	Equations of graph		Co-ordinates of point		Co-ordinates of TP/ Equations of asymptote(s)	
	Original	Transformed	Original	Transformed	Original	Transformed
Parabola						
Hyperbola						
Exponential						
Sine						
Cosine						
Tangent						

The activity above was followed by Activity 2: Transformation of functions (see Worksheet excerpts below). After completing the activity the researcher ensured that the students understood the properties of translation, rotation, reflection and dilation (stretching or shrinking of the graph). Students were also expected, when asked, to understand the type of transformation that a function undergoes $-f(x)$, $f(x) - 2$, $f(x - 3)$, etc.

Worksheet excerpts (see the full worksheet in Appendix J)

$f(x) = 2(x + 4)^2 - 3$. Type in Input bar $2(x+4)^2 - 3$ and then Enter.

Write down the turning point co-ordinates:

Type the following into the input bar. Also write down each time the new equation, then simplify and leave in turning point form, describe transformation and the co-ordinates of the new turning point.

(a) $-f(x)$	(b) $-3f(x)$	(c) $f(x) - 2$
(d) $f(x) + 3$	(e) $f(x - 3)$	(f) $f(x + 3)$
(g) $f(x + 4) + 2$	(h) $f(x + 1) - 5$	i) $f^{-1}(x)$ (find it algebraically)

Figure 4.8 shows a screenshot of the graph of $g(x)$ that is the answer to the question of $-f(x)$. The aim of this activity was that the students could see the meaning $-f(x)$ and also how GeoGebra does not give the simplest form of the answer. In the Algebra window it gives the steps so that students can see how to solve the function algebraically and see what the notation for $f(x)$ looks like graphically.

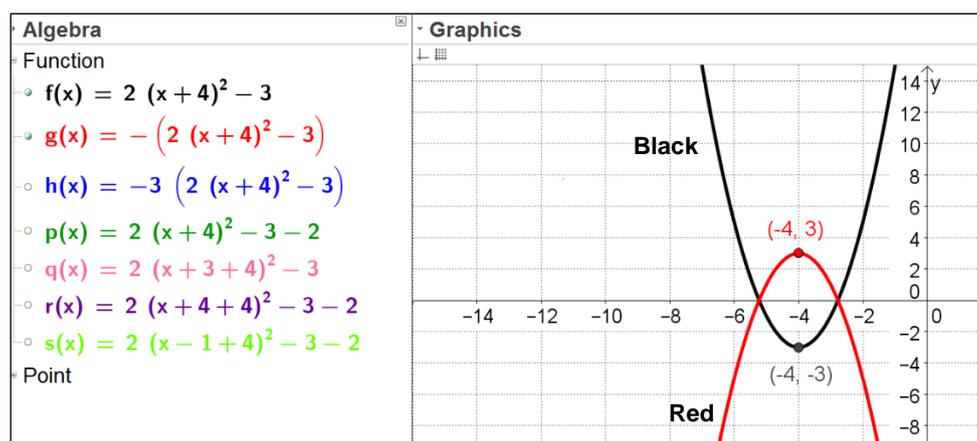


Figure 4.8: Screenshot of $-f(x)$ in GeoGebra

After the last lesson the students wrote the post-test on transformations of functions. This was then followed by consolidation with questions that tried to prepare students to answer questions from cognitive levels 1 to 4 as required by the DBE (2015a). Questions, for

example, $(x) \geq 0$; $f(x).g(x) \leq 0$; $f(x) < g(x)$; etc. were then covered with lecturing with GeoGebra. Although this approach of teaching became lecturing, the researcher used GeoGebra interactively to visualise the answering of this question. It was now also easier for the researcher to discuss a question. For example, $3f(x - 1) + 4$. To answer a question such as $3f(x - 1) + 4$ was now easier to discuss because the researcher could now refer to the lessons on the properties of transformations and the critical values of the function $f(x)$.

This activity was done in 3 hours 20 minutes. During this lesson the teacher-researcher utilised the *explain-the-screen*, *link-screen-board*, *discuss-the-screen* and *guide-and-explain* instrumental orchestrations. Students' instrumental genesis gestures were also observed during this phase.

4.5.2 The HLT for the teaching of circle geometry

Only 13% of the students in this research had been exposed to circle geometry at school. The researcher's teaching route for circle geometry was influenced by guided reinvention of RME, instrumental approach and the Van Hiele theory (1986). The HLT trajectory for the teaching and learning of circle geometry functions was also done with a student-centred approach, i.e. guided reinvention with instructional design materials involving GeoGebra and traditional approaches of teaching by interactive lecturing. Interactive lecturing was at times complimented with the use of GeoGebra.

Table 4.3 shows the HLT for teaching and learning of circle geometry.

Table 4.3: Overview of the HLT for the teaching and learning of circle geometry

Phase	Designed activities	Appendix	Addressing the following theories	Data obtained with	Role of teacher-researcher (TR) and student
1	Straight lines, triangles and quadrilaterals	Appendix K Only on quadrilaterals Also Appendix R	RME: Level / Interaction / Guided re-invention Van Hiele: Level 1 , 2 , 3 Instrumental orchestration: Explain-the-screen / Link-screen-board	Was not part of study	Students work individually, or in pairs TR: Interactive lecturing with GeoGebra will be used to verify students' answers in the activities.
2	Circle geometry with GeoGebra	Appendix L	RME: Level / Interaction / Guided reinvention Van Hiele: Level 1 , 2 , 3 Instrumental genesis Instrumental orchestration: Explain-the-screen / Link-screen-board / Discuss-the-screen / Guide-and- explain	Observation (QUAL) Questionnaire (quan) Pre- and post-test (quan & QUAL) In-depth interviews (QUAL) Focus group interviews (QUAL)	Students work with GeoGebra applets (see list in Appendix L) in computer lab.
3	Exploring theorems with riders, proving of theorems by using riders and basic riders using theorems	Appendix M & R	RME: Level / Interaction / Guided reinvention Van Hiele: Level 1, 2, 3 Instrumental orchestration: Explain-the-screen / Link-screen-board	Questionnaire (quan) In-depth interviews (QUAL) Focus group interviews (QUAL) Pre- and post-test (quan & QUAL)	TR: Interactive lecturing with GeoGebra
4	Exam type of question (Higher order level)		Van Hiele: Level 1, 2, 3, 4	Was not part of study	

4.5.2.1 Description of the activity and rationale for the activity in the phases

The level, interaction and guided reinvention principles of RME were built into the first three phases. The students' Van Hiele levels were also assessed during phase 2 and in the pre- and post-test. The students' instrumental genesis and the types of instrumental orchestration the teacher utilises, were investigated and analysed during the observations in phase 2. Only the lesson during phase 2 was video recorded.

Some of activities in phase 1 were not part of the study but the researcher included them in the HLT for the teaching and learning of circle geometry, because most of the students in SciMathUS did not do circle geometry. The geometry of lines, triangles and quadrilaterals is a pre-requisite to understanding circle geometry. Circle geometry is thus another topic where there is extensive intertwinement of topics which evokes the intertwinement principle of RME. Straight line geometry includes corresponding, alternate, co-interior, vertically opposite angles and with properties of triangles were isosceles triangles, equilateral triangles, congruency and similarity important. The straight line geometry was done by interactive lecturing by the teacher-researcher with GeoGebra. The concepts of congruency and similarity were revised again together with rigid transformation and dilation in GeoGebra. The four cases of congruency were done with a pen-and-paper activity by the students. The students had to cut out constructed triangles to see which ones were fitting. The purpose of the activity was for students to see that two triangles with the corresponding angles equal, are not necessarily congruent; two triangles with two angles and a corresponding side equal, are congruent and that triangles with two angles and a side are not congruent, etc.

The teaching and learning of the properties of quadrilaterals were done with an interactive lecture in GeoGebra. The students were given the worksheets (see Appendix K) with different properties and quadrilaterals listed. They had to tick the properties while the teacher-researcher showed the applets of a parallelogram, square, rectangle, rhombus, kite and trapezium. In this lesson students were exposed to give their own definition or uneconomical definitions, but were then guided by the teacher-researcher to give economical definitions (De Villiers, 2010).

Before the start of phase 2 the pre-test for circle geometry was written. Phase 2 presented the second pre-experimental design of the study. During this phase the students were taken to a computer laboratory to explore geometry with GeoGebra applets. Each student again

worked on a computer, and they were encouraged to discuss and work with a classmate. Students had to click tick boxes and then drag B, D and C as shown in Figure 4.9.

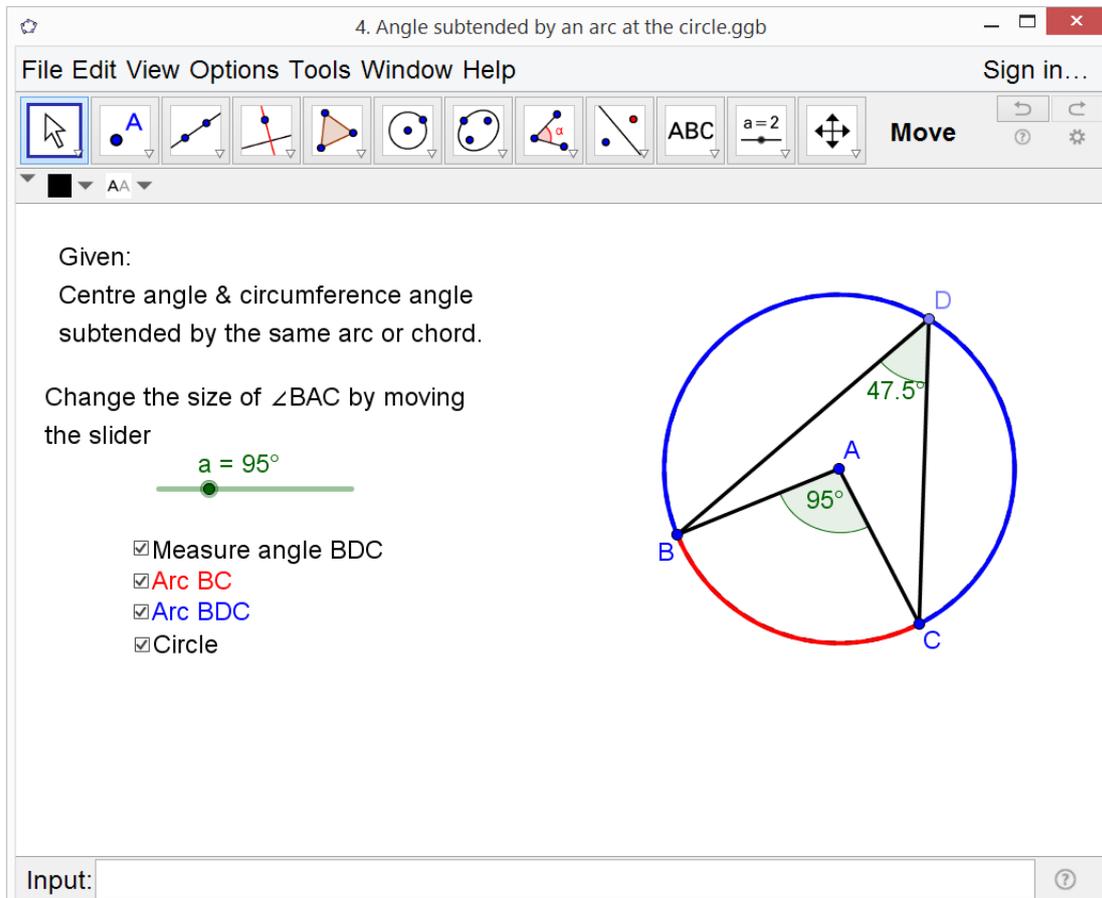


Figure 4.9: GeoGebra applet of the angle at centre theorem

Students were also encouraged to drag B, D and C in such way that different orientations of angles at the centre and circumference were obtained. The students then had to complete the activity on the following worksheet excerpt:

Worksheet excerpt (see the full worksheet in Appendix L)

The angle subtended by an arc of a circle at the centre

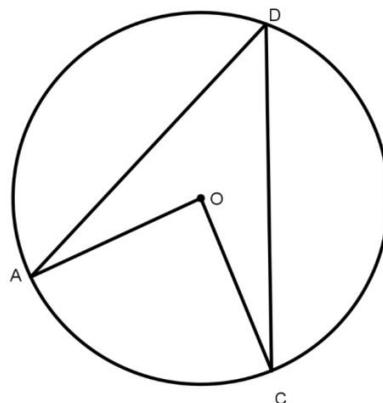


Figure 4.10 shows an activity that the researcher planned to give the students to test their conjecture about the angle at the centre and the circumference angle.

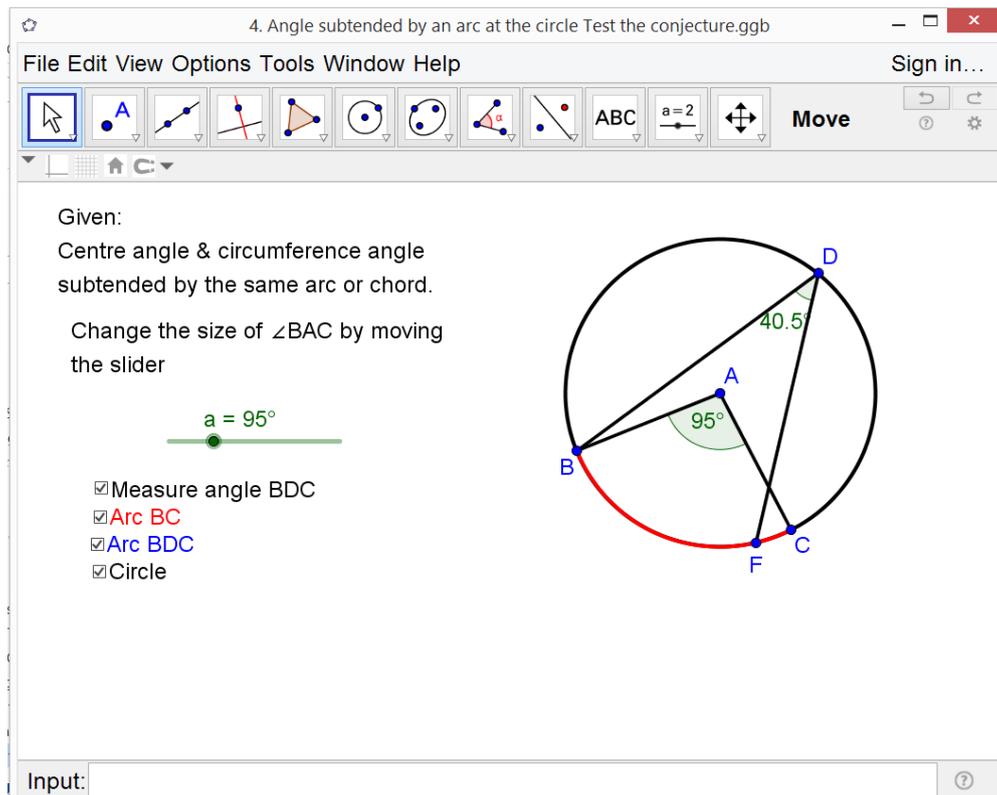


Figure 4.10: GeoGebra to test students' conjectures about the angle at the centre

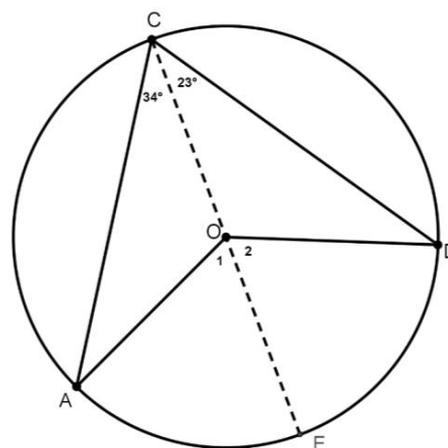
The students continued using the same approach to complete the activities in Appendix L. They completed this activity over two double sessions in the computer laboratory. The students thus explored the properties of circle geometry in GeoGebra for 3 hours 30 minutes. During that time students explored on their own, but the teacher-researcher utilised the *Discuss-the-screen* orchestration to make sure that students were aware of what was happening on their screens

The following rider in the worksheet excerpt below (see the full worksheet in Appendix M) was given to the students using other theorems and axioms to explore the following theorem before proving it: *The angle subtended by an arc of a circle at the centre is double the size angle subtended by the same arc at the circumference.*

Worksheet excerpt (see the full worksheet in Appendix M)

You cannot use the theorem that states that: Angle at the centre is = 2 x circumference angle.

O is the centre of circle and A, B and C are on the circumference of circle. Determine the size of:



- (a) \widehat{ACD}
- (b) \widehat{A}
- (d) \widehat{O}_1
- (e) \widehat{D}
- (f) \widehat{O}_2
- (g) \widehat{AOD}

Students were given two riders similar to the one above, but with different angle sizes and also different orientations and sizes of the angles at the centre and circumference. After this activity students had to prove the theorems (see worksheet excerpt below) by following the same route, but had to add the construction to prove the theorem that states: *The angle subtended by an arc of a circle at the centre is double the size angle subtended by the same arc at the circumference.*

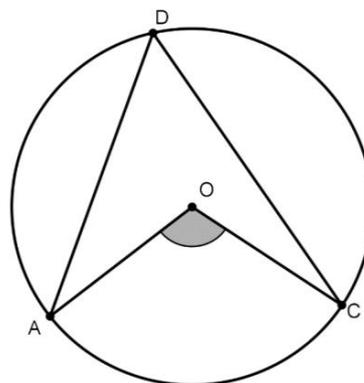
Worksheet excerpt (see the full worksheet in Appendix M)

Given: A, C and D are on the circumference of the circle with centre at O.

To prove: $\widehat{AOC} = 2\widehat{ADC}$

Construction:

Proof:



The students also had to prove the theorem where the angle of the centre and circumference angles are of different sizes and orientations (see Appendix M). After proving the theorems students were given basic riders of the theorem proven. The rest of the other theorems were proven in the same way and followed by basic riders on the theorems and their corollaries. The students were doing these activities in phase 4 for a double session that equates 6 hours and 40 minutes. The students then wrote the post-test.

Phase 5 involved activities to consolidate with activities from old examination papers and textbooks. No data during this phase were collected. Although in some phases data were collected the researcher included all the phases in HLT, because he views a phase important for students to understand circle geometry better.

4.6 ETHICAL CONSIDERATIONS

The Western Cape Education Department (see Appendix N) gave written approval for conducting the research in the two pilot schools. Although the piloting in the schools was only on transformations of functions, it was very informative to the main research, because SciMathUS students are also writing the NSC examinations. The pilot data were used by the researcher to inform what changes were needed to conduct his research with the SciMathUS students at Stellenbosch University. The researcher also obtained ethical clearance from Stellenbosch University's Research Ethics Committee (REC) (see Appendixes P and Q) to conduct the research with the students that the researcher was also teaching Mathematics to in the SciMathUS bridging programme.

The active learning approach and interactive lecturing are an integral part of the researcher-teacher way of teaching. Exploring mathematics with GeoGebra and designed activities were part of the teacher-researcher's signature teaching approach. At the beginning of the year the programme manager informed the students that active learning would be a major part of the SciMathUS teaching philosophy. Informed consent (see Appendix O) was obtained for the data collection from the students and was used for the sole purpose of this research. Anonymity and confidentiality were respected and assured. Pseudonyms were used for interviewees and the names of the students observed in the computer lab in the data analysis chapter. The faces of students were also blurred out on the images presented in the data analysis chapter.

4.7 CONCLUSION

This chapter discussed the research design and methodology which were used in this study. The choice for using the mixed methods approach, in this particular study, was the focal point in Chapter 4. The chapter gave a detailed overview of the hypothetical learning trajectories that were used for the teaching of function transformations and circle geometry. The researcher also pointed out that these trajectories utilised traditional approaches of teaching and with GeoGebra and contextual material in topics throughout the year. In addition to that, issues of population and sampling procedures were also elaborated on in this chapter, and then procedures and the different instruments for data collection were also discussed, together with issues of validity and reliability for each instrument used in this study. Apart from that this chapter also presented the ethical issues which were considered before the study was undertaken.

Chapter 5 will present the analysis of the data collected from the various data collection procedures of the study.

CHAPTER 5

DATA ANALYSIS AND INTERPRETATION

If we teach today as we taught yesterday, we rob our children of tomorrow.

(John Dewey)

5.1 INTRODUCTION

The aim of this chapter is to provide an analysis of quantitative and qualitative data in order to answer the primary research question, which is: What is the role of a GeoGebra-focused learning environment in helping students to develop mathematical knowledge?

In order to achieve this, the following four sub-questions are addressed:

- (i) What were the processes involved in using GeoGebra as a pedagogical tool?
- (ii) How did GeoGebra afford students an opportunity to understand mathematical concepts and move to higher levels of abstraction?
- (iii) What are students' experiences and gestures when taught with GeoGebra?
- (iv) What are the challenges that students experience whilst working with GeoGebra?

In the quest to answer the above research questions this chapter presents quantitative and qualitative data in four different sections: The processes involved in using GeoGebra as a pedagogical tool; students' explanations of how GeoGebra afforded them an opportunity to understand mathematical concepts and move to higher levels of abstraction; students' experiences; and gestures when taught with GeoGebra and the challenges that they experienced whilst working with GeoGebra. In each section the quantitative and qualitative data were triangulated in an attempt to answer the research questions. The qualitative and quantitative phases occurring sequentially and the analysis of qualitative (QUAL) were given more weight.

5.2 PROCESSES INVOLVED IN USING GEOGEBRA AS A PEDAGOGICAL TOOL

This section presents the triangulation of both quantitative and qualitative data that will attempt to answer the following first research sub-question:

What are the processes involved in using GeoGebra as a pedagogical tool?

GeoGebra on its own will not suffice and be the solution to effective teaching. The researcher investigated a hybridisation of the student-centred approach and teacher as a facilitator of learning approach. Constructivist approaches to teaching, mixed with interactive lecturing, are what the researcher suggested and investigated in this research. The researcher used the RME as an instructional design theory as point of departure, with TPACK, to teach transformation geometry, transformations of functions, circle geometry, properties of quadrilaterals, etc. The researcher also utilised the TIG, where GeoGebra was the technology used in this regard. The researcher thus used instructional activities with GeoGebra where students were engaged in activities designed to invoke powerful informal understandings or strategies (Weber & Larsen, 2008). GeoGebra was not investigated alone, but also how students experienced the instructional material that was designed for the teaching and learning of some of the topics mentioned above.

The teaching of transformations of functions was not to teach the topic in isolation because students had to know the shape of the graphs of the different functions and had to understand the critical values linked to the graphs. Students were therefore given activities with real life or realistic contexts to help them understand the shape of the graphs. The teacher-researcher started with the worksheets on *Amazing relationships* (Appendix G) and *Every graph tells a story* (Appendix H), and also used GeoGebra as a teaching tool in class. The researcher viewed this as an important process that built up towards the teaching of function transformations. The teacher-researcher found that from the pre-questionnaire, students had certain conceptions about how they should learn and be taught. Hence the teacher-researcher started with the processes of changing students' conceptions of how they should be taught, and learn, in order for them to see that mathematics is about teaching for understanding and not just algorithms or recipes. At the beginning, GeoGebra was therefore used by the teacher-researcher in the classroom and not by the students themselves. The students were only later exposed to use GeoGebra themselves for exploration. One of the processes that the teacher-researcher investigated, was how students responded to instructional design theory, i.e., RME.

The following sub-sections will present and analyse data regarding the types of instrumental orchestrations that the teacher-researcher utilised and how the students viewed the teaching and learning based on some of the principles of RME. It will thus focus on the reality, guided reinvention, interaction and activity principles of RME (Freudenthal, 1973).

5.2.1 Summary of different types of instrumental orchestrations in the computer lab

The researcher analysed the video material of the classroom observations and observed the different types of instrumental orchestrations utilised by the teacher-researcher. These types of instrumental orchestrations, as described in Chapter 3 (see 3.5.1.2), of the teacher-researcher were also the processes involved in using GeoGebra as a pedagogical tool. Table 5.1 shows the summary of the different types of orchestration observed during the teaching and learning of transformations of functions in the computer lab.

Table 5.1: Summary of the frequencies of orchestrations used by the teacher-researcher during the teaching and learning of transformations of functions

Orchestration type	First 50 minutes computer lab (1 st session): Frequency	Next 50 minutes computer lab (2 nd session): Frequency	Next 100 minutes computer lab (3 rd & 4 th session): Frequency	Next 100 minutes computer lab (5 th & 6 th session): Frequency
Technical demo	3	2	2	1
Explain-the-screen	4	5	2	1
Link-the-screen	5	3	2	2
Discuss-the-screen	6	3	2	2
Guide-and-explain	Most of the time	Most of the time	Most of the time	Most of the time

The first two sessions show that the teacher-researcher used all five types of orchestrations. Technical issues, as well as whole-classroom explanations and discussions, were initially important to make students comfortable in front of the computer. It was also important for the students to be familiarised with the software and to understand how to explore the mathematics. During the *Technical demo* orchestrations, the teacher-researcher explained how the portable- and installation-versions of GeoGebra work. The interface and how certain icons in GeoGebra work, were explained. Only the icons that students needed during the sessions were explained. During the first session the teacher-researcher explained to the students how to hide the grid, axes, to save, to open a new page and how to open applets from a shared folder. Other technical support given to the students was how to draw different polygons, segments, measuring of segments and angles in GeoGebra. The teacher-researcher demonstrated the moving of sliders and dragging of points when needed.

Table 5.1 also shows how the teacher-researcher utilised less frequently the *Technical-demo*, *Explain-the-screen*, *Link-the-screen* and *Discuss-the-screen* orchestration in sessions three to six. The teacher-researcher thus supported the students more frequently on their computers, i.e. the *Guide-and-explain* orchestration. In this orchestration, students were supported when they needed help with something on the computer.

The support varied from technical to mathematical content. Students were most of the time following instructions from the activities or from the teacher-researcher. Consequently, the students worked more independently during session three to six. For students to work independently in constructing knowledge is consistent with cognitive and radical constructivism. RME also recognises individual learning paths.

Table 5.2 shows the frequencies of the different orchestration utilised in the four sessions in the computer lab with circle geometry.

Table 5.2: Summary of the frequency of orchestrations used by the teacher-researcher during the teaching and learning of circle geometry

Orchestration type	First 100 minutes computer lab (1 st & 2 nd session): Frequency	Next 100 minutes computer lab (3 rd & 4 th session): Frequency
Technical demo	5	1
Explain-the-screen	4	3
Link-the-screen	Most of the time	Most of the time
Discuss-the-screen	6	3
Guide-and-explain	All the time	All the time

During the first session in the computer lab, the teacher-researcher explained to the students how to open applets from the shared folder and measuring of angles and segments, i.e. *technical-demo* orchestration. The *Link-the-screen* demo was one of the orchestrations most frequently utilised, because the teacher-researcher made sure that the students could link the screen with the pen-and-paper activities. The *Guide-and-explain* orchestration was the type of orchestration that was utilised most of the time.

The *Guide-and-explain* by orchestration was the type of orchestration, which was highly dominant in both the sessions. The teacher-researcher also utilised exploitation modes during the sessions in the computer lab. The exploitation modes were in whole-class context and with individuals for both the transformations of functions and circle geometry sessions.

Here are a few of the questions the teacher-researcher posed during the session on transformations of functions:

- What is happening to the coordinates of your object in the algebra window when you move any point?
- Can you link these answers in the algebra window to the rules of reflection and translation?
- What happens to the values of the vector in the algebra window when you drag the point of the vector in the graphic view?
- What is the meaning of doubling the size of the rectangle?
- How do you enlarge or reduce the size of a picture in MS Word?
- What is the equation that fits the newly transformed graph?

The questions during the circle geometry were mostly based on given information of the applet. Here are a few of the questions that the teacher-researcher posed during the circle geometry session:

- What is given on the applets or sketch? Measure the angle. The teacher-researcher referred here to first activity 1 on the worksheet on Appendix L.
- What can you conclude?
- Where is the angle formed? The teacher-researcher referred here to the angle at the centre and circumference.
- By what chord or arc is the angle at centre subtended? Does the same chord subtend the circumference angle?
- What is the size of angle B? What is AC? This referred to activity 5 in Appendix L.

Hence the researcher is able to conclude that the different types of orchestration utilised supported the students to explore mathematics with GeoGebra. The above type of questions posed by the teacher-researcher were intended to create a disequilibrium or cognitive conflict in the students' minds and afford them the opportunity to contend with their own solutions (Waite-Stupiansky, 1997). In this way the teacher-researcher was teaching logico-mathematical knowledge (Piaget, 1952; 1954). The different types of the teacher-researcher's instrumental orchestration focused on enhancing the students' instrumental genesis (Drijvers *et al.*, 2009). The researcher will further refer to different types of individual orchestration and the exploitation modes in the other analysis sections in this chapter.

5.2.2 Students' perceptions of contextual activities, guidance and exploration

This section presents and analyses data on how the students viewed real life activities, guidance and exploration.

5.2.2.1 Qualitative analysis of students' responses to real life activities, guidance and exploration

This section presents qualitative findings regarding the students' perceptions about real-life contexts, guidance and exploration. The students were asked the following questions in the in-depth and focus group interviews: *How did you experience lessons taught with GeoGebra and worksheets?* The purpose of this data set was to hear how students benefited from worksheet activities at the start of a new topic while working on GeoGebra. The students were given pseudonyms in-depth (ID) and focus group (FG) interviews.

Student ID8: Vir my op 'n skaal van 1 tot 10, was dit 7 tot 8 uit 10. Die werkkaarte het ooreengestem met die werk wat meneer gedoen het. Ek praat van hoe meneer sekere goed op GeoGebra uitgelig het. Die goed op die werkkaart wat ons moet sien. Dit het lekker ooreengestem met mekaar. Ek kon duidelik sien wat meneer op die rekenaar probeer wys het. (*For me, on a scale of 1 to 10 it was 7 to 8 out of 10. The worksheets were in line with the work done with us. I'm talking about how you highlighted some things on GeoGebra. The things that you wanted us to see, was on the worksheet. It nicely corresponded with each other. I could clearly see what sir tried to show on the computer*).

The above student could see the relevance and the importance of the worksheets. She could link the worksheets with GeoGebra. She also could see what the teacher-researcher wished them to understand. This also supported the Hypothetical Learning Trajectory (HLT), as discussed in Chapter 4 (section 4.5), for transformations of functions and circle geometry. She also gave an example of how she linked the worksheet (Appendix L) and the applets given in GeoGebra.

Student ID8: Ek het nie altyd gesien hoe die middelpuntshoek 56 grade en ander hoek 28 grade is nie. Ek het nooit daardie koord (sy verwys eintlk na die boog) gesien nie en meneer het dit gewys. Dit was op die werkkaart. Dit was in pers of blou toe meneer in GeoGebra gewys het hoe dit werk. (*I could not always see how the angle at the centre is 56 degrees and that other angle 28 degrees. I've never been able to see the chord (she refers actually to the arc) and the teacher showed it to us. It was on the worksheet. It was purple or blue in GeoGebra when the teacher showed how it works.*)

The student was referring to the GeoGebra applet below (see Figure 5.1) given to the class during a session with GeoGebra in the computer lab.

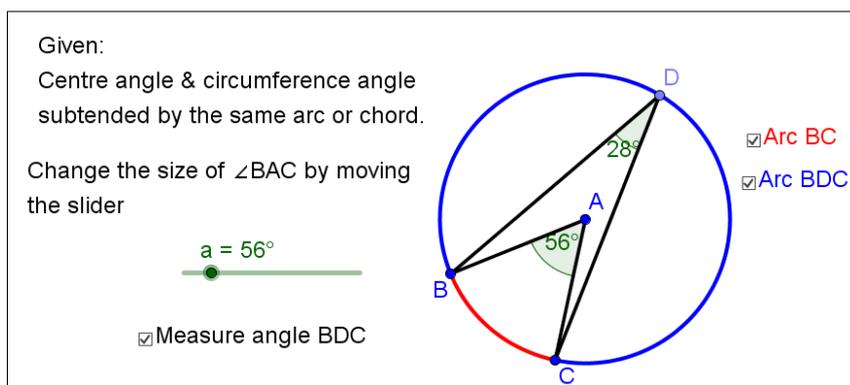


Figure 5.1: GeoGebra applet: Angle at centre and circumference angle theorem

The response of Student ID8 in the in-depth interview referred to how she understood that there is a condition to conclude that the angle at the centre is double the angle at the circumference. This physical knowledge was obtained through the hands-on interaction and discovery by making mathematical conjectures in a non-deductive manner (De Villiers, 2004) with the provided GeoGebra applets. The student was evaluating, or justifying, her conjecture with empirical reasoning (De Villiers, 2004) derived from what she visualized in GeoGebra. At this stage the student did not prove the theorem, but based her conjecture on intuition and numerical investigation (De Villiers, 2004). This is how the teacher-researcher used the whole class discussion; it was instrumental orchestration (Trouche, 2004). The orchestrations utilised here were the *Explain-the-screen* and *Discuss-the-screen* (Drijvers, Doorman, Boon, & Van Gisbergen, 2010a). The students were not exposed to the formal theorem, but to dynamic software (De Villiers, 2010), in this case GeoGebra. This also accords to Van Hiele level 1.

Students in the focus group interviews gave the following responses with reference to how they experienced the relevance and the visualisation of Mathematics:

Student FG6: For me the worksheets were very interesting, especially the beginning of the topics. We always like to know where this fits in. We were just doing it our own way and then afterwards we realised that we were actually working out the gradient. This is nice, because we could see how it actually works. It also actually shows what it is about, where it comes from, and how it can be applied in different places in today's life outside.

Student FG4: It helps us with the question most children were asking last year: Why must I do this? Why is this even necessary in Maths? Why must we do this part of Maths and these kinds of questions?

Student FG14: For me it was better to understand if we first worked with GeoGebra and then with the worksheets. In this way you can see the stuff, visualise it, actually understand why it is that way; then apply it; and get onto the topic.

Student FG13: I liked it when you showed us real life examples because last year it was basically just theory; you just have to know this comes after this and that and you really don't know why it is like that. But sir, you showed us the sunrise and all those things. I could then link the two things together and by linking it, I understood it better when we were working it out.

The responses above show how students felt realistic or experiential contexts helped to clarify the relevance of mathematics and that mathematics is not only about theory and algorithms, or procedures. They could see where mathematics can be applied in real life. Student FG6 referred to the activity of Introduction of Calculus (Appendix R activity 12) where students worked with different examples of rate of change and the students could link it to what they see as gradient. This also refers to level principle, where the students moved the informal contextual activities to the concept of rate change and then ended up with the limit concept. Student FG13 referred to the sunrise activity in Amazing relations (Appendix G) that ended up being the cosine function. Both these examples also refer to horizontal and vertical mathematising. These students' responses also showed how some of the worksheets helped them to obtain clarity on questions that they always asked: Where do we need this and why?

Students also felt that they wished to be more involved in their learning and not just always be listening to the teacher's voice. They wanted to visualise certain concepts for themselves. The students started to realise that GeoGebra helped them to visualise and also make mathematics come alive, and that is why some felt as follows:

Student FG10: Ek was mal daaroor. Partykeer sit 'n mens in 'n klas en dit word later twee ure en die onderwyser staan net. Daar was eenkeer 'n les wat meneer vir twee ure net aaneen gepraat het. Ek het gedink: kan die man nie ophou en eerder GeoGebra gebruik om te wys waarom nie? Jy raak moeg van luister en niks doen nie, maar waar ons in die rekenaarlokaal gesit het en meneer gedemonstreer het wat ons moet doen, was jy die hele tyd besig. Jy kon sien wat gebeur, jy hoef nie net die hele tyd te luister nie. GeoGebra maak dit als 'n bietjie lekkerder. *(I loved it. Sometimes when I was in a class you are sitting for two hours. There was a lesson where the teacher was just talking for two hours and I thought, can't this man just stop and use GeoGebra and display stuff? We were getting tired of listening and not doing anything. When we were in the computer lab, you demonstrated what we should do. You were always busy. You can see, we did not just have to listen to you all the time. GeoGebra made it a little nicer.)*

Student FG10 refers here to the times that the teacher-researcher, during expository lessons without any GeoGebra, was just talking, and found it very tiring and boring. It shows that this student wants to be actively involved with a lesson or that concept should be visualised with Geogebra. The student was therefore open to acquire knowledge from a physical-knowledge perspective and as Kamii and Ewing (1996) point out, not just from a social-knowledge perspective.

Student FG11: Ek het daarvan gehou as die dataprojektor aangaan, dan het ek geweet hier kom iets. (*I loved it when you switched on the projector, then I knew something interesting was to follow.*)

Student FG10: Ons kon met GeoGebra rondspeel. Ons kon die hoeke skuif, vormpies ontwerp en ons eie hoeke insit. (*Because we could play around, we could move the angles, create shapes, put in our own angles.*)

Student FG20: It was more illustrative, because you could see things, not like on a piece of paper where you can't see the things move. If we maybe used graphs or triangles in GeoGebra, you could see what will happen to angles if you maybe move or extend the lines. When you do it on paper it wouldn't be that accurate.

Student FG10 refers to how they executed the instructions given to them. The student was actively busy and this refers to the activity principle of RME (Freudenthal, 1973). Student FG11 refers to how the teacher-researcher utilised GeoGebra to transform the traditional expository lectures into interactive lectures. In this way, a teacher can create excitement and enhance learning (Eison, 2010). Student FG20 and FG10 emphasised the benefit of the dynamic nature of GeoGebra versus the static representations of concepts with pen-and-paper activities. The fact that GeoGebra allows the students to move the objects, afforded them an opportunity to empirically verify, or test their results and statements (De Villiers, 2004). The movement of the objects also shows how GeoGebra afforded the students with organising and exploring. The organising and exploring are consistent with horizontal mathematising (Rasmussen *et al.*, 2005).

The above responses just emphasised again that traditional teaching is not always the best approach to teaching and learning. These responses also show that the students wish to see the relevance of mathematics and that the worksheets helped in that, respect, especially when introducing a new topic. Studies by Cobb (2000) and Weber and Larsen (2008) support this notion. According to Cobb (2000) one of the three central tenets of RME is that instructional sequences should contain activities in which students create and elaborate

symbolic models of their informal activity. Weber and Larsen (2008) also allude to the fact that students must be engaged in activities designed to invoke powerful informal understandings and strategies. GeoGebra activities, supplemented with instructional material or worksheets, can guide the students to focus on the informal understanding of concepts so that they can later formalise them. This material has to ensure that students focus on the intended mathematics and not on the use of GeoGebra. Hence, exploration with guidance was one of the processes to ensure that GeoGebra was used successfully as a pedagogical tool.

This section shows the qualitative analysis of students' interviews on how the worksheets and GeoGebra afforded them the opportunity to explore concepts and helped them with conceptual understanding. The next section shows the quantitative data collected from the survey questionnaire to see how all the students experienced the worksheets given to them at the start of a new topic.

5.2.2.2 Quantitative analysis of students' responses to real life activities, guidance and exploration

The following quantitative data were collected from the survey questionnaire given to the students after the year of teaching them. The purpose of this section was to see how students experienced worksheets given to them at the start of a new topic, and worksheets given while working on GeoGebra. The Cronbach alpha coefficient reliability for the 12 items in the construct on RME is 0.766. This section reports on seven of the 12 items and the other five items are presented in section 5.2.3. A Cronbach alpha of $\alpha \in [0.7; 0.8)$ is acceptable (George & Mallery, 2003). Forty-six (95.8%) of the 48 students completed the post-intervention questionnaire.

Figure 5.2 shows the students' perceptions of guidance and exploration from their responses to the post-intervention questionnaire.

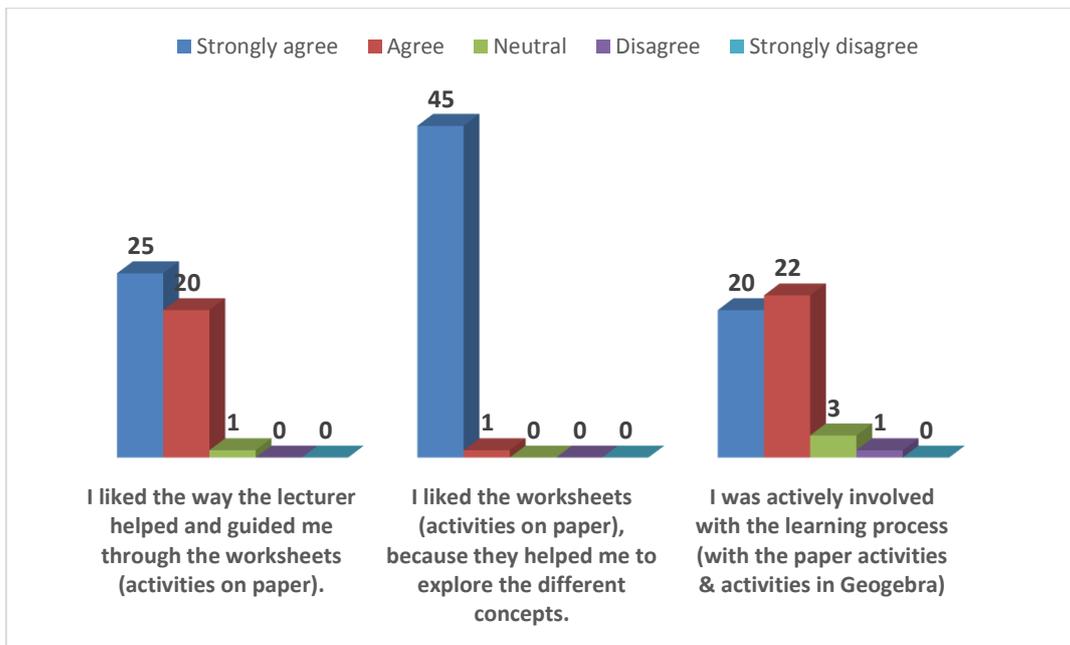


Figure 5.2: Students' perception of guidance and exploration (n= 46)

Figure 5.2 shows that 45 (97.8%) of the students experienced the support and guidance given by the teacher-researcher. Forty-six (100%) of the students felt that the worksheets helped them to explore concepts. Forty-two (91%) students wish to be actively involved with the learning process. Students thus viewed guidance with worksheets as important. They viewed that activities given to them were designed in such way that they could explore concepts and could be guided by the teacher. Results in the next section will show how the students responded in the in-depth and focus group interviews, how the designed activities help them to explore concepts and how the teacher guided them.

Figure 5.3 shows students' views on designed activities from their responses to the post-intervention questionnaire

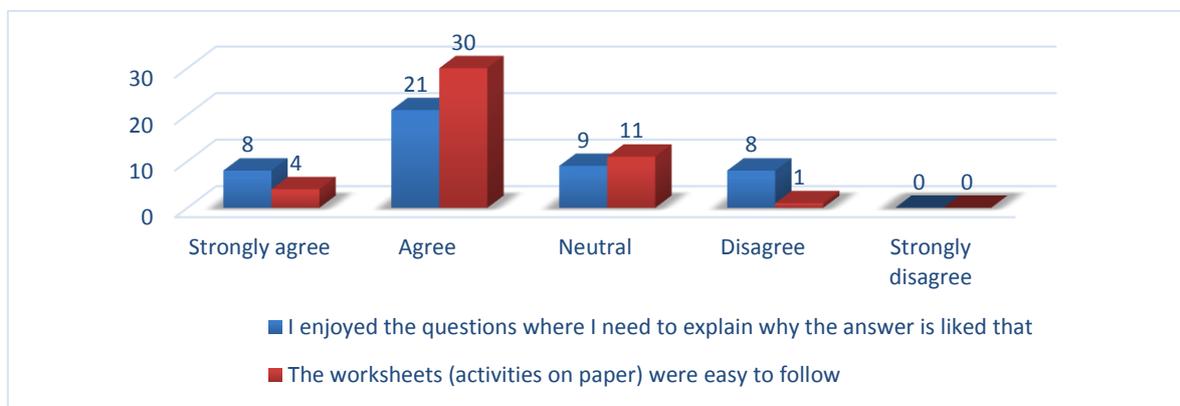


Figure 5.3: Students' views on designed worksheets (n= 46)

Figure 5.3 shows that 29 (63%) of the students enjoyed working with questions asking them to explain their answers and 34 (73.9%) felt that the worksheets were easy to follow. Students were therefore of the view that if they explain their answers and that can lead to their conceptual understanding.

Figure 5.4 shows students responses to experiential and realistic contexts that were part of the designed activities and interactive lecturing given to them.

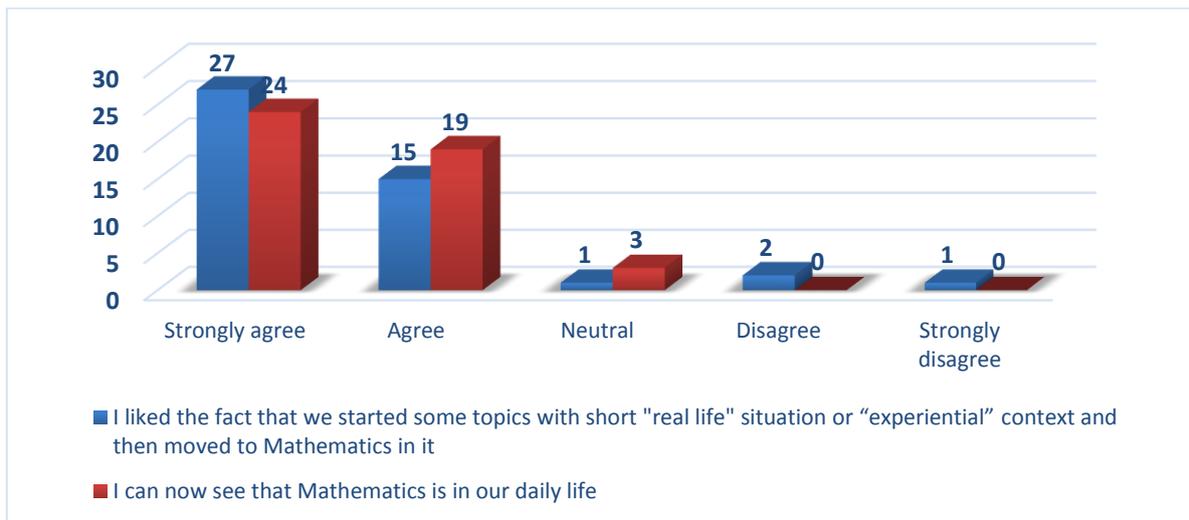


Figure 5.4: Students' responses to experiential and realistic contexts (n = 46)

Figure 5.4 shows that 42 (91.3%) of the students liked the fact that new topics started with realistic or experiential contexts. Experiential learning takes events and learning out of the realm of abstract thought and brings them into the realm of concrete exploration, discovery and invention (Hull, 1999). Forty-three (93.5%) of the students were able to see that Mathematics is not an abstract subject and that it is part of daily life. The teacher-researcher frequently used realistic (see Appendixes G and H and, no. 1, 2 and 3 in Appendix R) and experiential contexts (see no. 4, 5, 6 and 7 in Appendix R) when starting a new lesson or concept. Most of the time, GeoGebra was used in these activities. One of the examples used as an activity with realistic context was the Moses Mabhida stadium (see Figure 5.5).

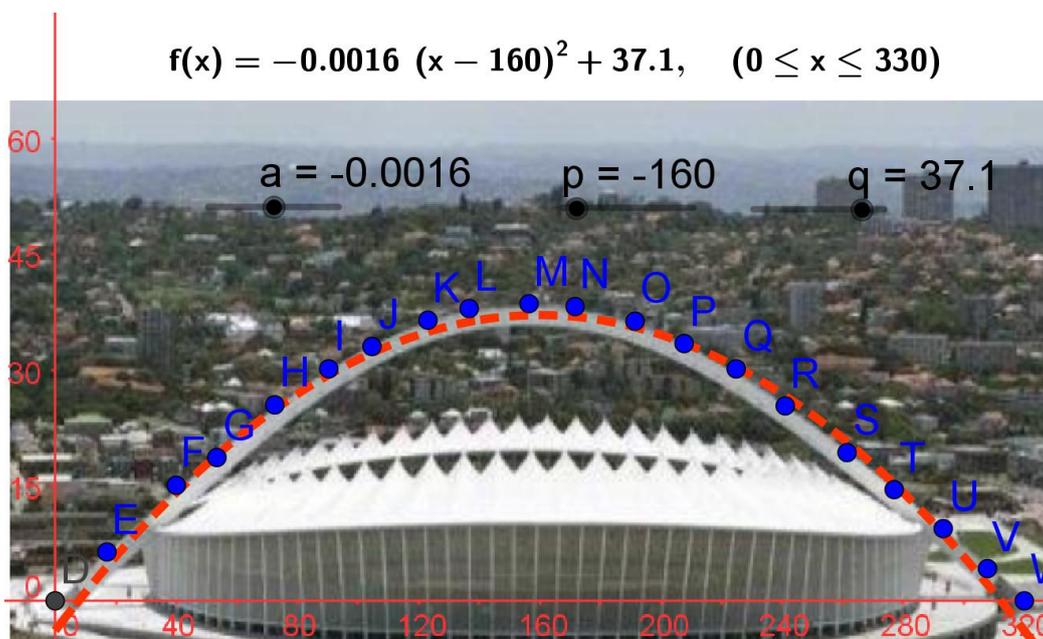


Figure 5.5: GeoGebra used with a picture of the Moses Mabhida Stadium

The teacher-researcher used GeoGebra to show the students how a parabola fits on a graphical representation of the stadium's arc. The responses of the students in Figure 5.4, thus show the need for starting a new topic in mathematics with realistic or experiential contexts as pointed out by many researchers (e.g., Barnes, 2005; De Villiers, 2012; Freudenthal, 1991; Julie, 2004; Ndlovu, 2014; Van den Heuvel-Panhuizen & Drijvers, 2014). Real-life activities can, as pointed out by De Villiers (2012), motivate students. This also accords with the reality principle of RME and how students could acquire physical knowledge through discovery with the environment and by exploration (Piaget, 1952; 1954). Real-life contexts, guidance and exploration, therefore afforded the students an opportunity to learn and acquire physical knowledge that according to Piaget (1952; 1940) can lead to the construction of knowledge in their minds, i.e., logico-mathematical knowledge.

The quantitative (quan) data in this section are triangulated with the qualitative (QUAL) data in section 5.2.2.1. The triangulation of the data exemplifies how the students viewed contextual activities, guidance and exploration as important in the teaching and learning of Mathematics. This accords with the activity, reality and guided reinvention principles of RME. The activities that the students (see section 5.2.21.) referred to in the in-depth and focus group interviews also shows how the students acquired physical and logico-mathematical knowledge. This also accords to level principle of RME.

5.2.3 Students' perceptions of interaction

This section presents an analysis of qualitative and quantitative data on how the students viewed interaction.

5.2.3.1 Qualitative analysis of results with regards to interaction

5.2.3.1.1 Results from observations

This sub-section presents the students' interaction during classroom observations. Pseudonyms were used for the students during the observations (OG) of the activities.

Students were divided into groups of five and had to complete the five activities in the Amazing relationship (Appendix G) activity. Students had to draw graphs of all the activities and then indicate what type of graph or function it is. They had then to present it the next day to the rest of the class. Figure 5.6 shows how the students were interacting with each other while busy with an activity.

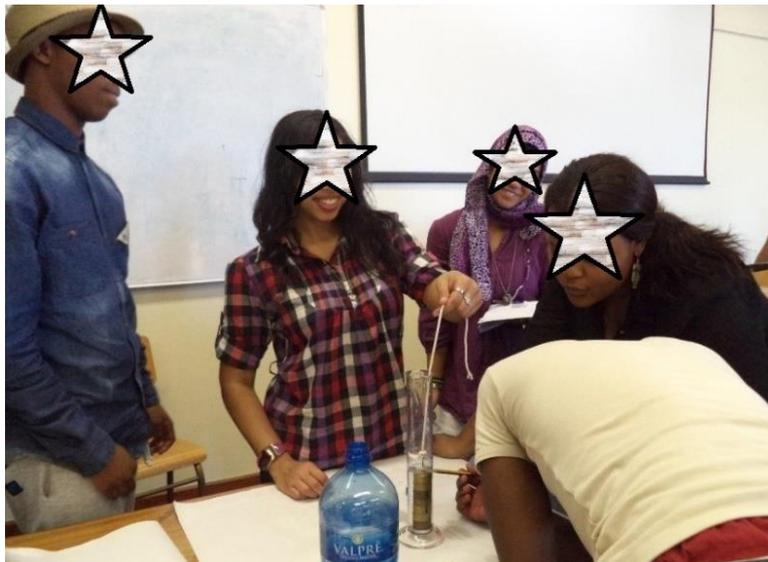


Figure 5.6: Students interacting as members of a group

The group in Figure 5.6 was busy measuring the different water levels (see activity 3 in Appendix G), when they added the same weight in a cylinder with water. One student was measuring, while the other one was adding the weights and there was another one recording the measurement. The other students of the group also had the opportunity to measure. The students were actively involved in the learning process and interacted with one another. This refers to the reality, interaction, activity and level principle of RME (Freudenthal, 1973). The students were at this stage measuring, recording and organising the measurements in a table. They were therefore transforming a situation into a mathematical problem. The activity

in Figure 5.6 shows how the students reacted to the instructions of the activity in Appendix G. Treffers (1987) refers to this as horizontal mathematising or the *models-of*.

The teacher-researcher only used GeoGebra after the students had presented their graphs. Figure 5.7 is the graph Activity 1 in Appendix G where students only plotted a few points.

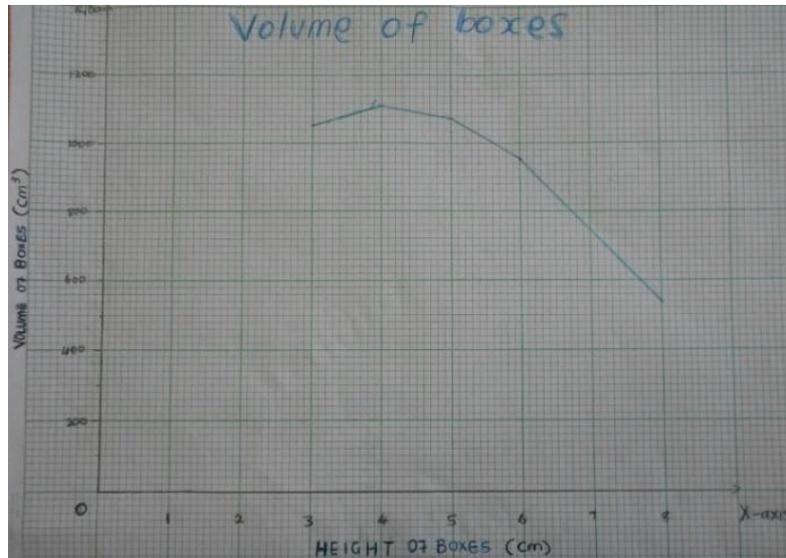


Figure 5.7: Students graph on activity 1 (see Appendix G)

Most of the groups assumed that the graph in Figure 5.7 was a parabola. They were asked to find the equation of the function by using the diagram in Figure 5.8 (see Appendix G). The students acquired physical knowledge through the building of the different boxes and by completing a table for different heights and volumes and could therefore see the sizes of the boxes were different. The students start with more formal mathematical activities by drawing the graph and conjecturing what type of graph it was. They could see that the volume depends on the height of the box and also started to acquire logico-mathematical knowledge because they could see the relationship between height and the volume. This was further enhanced by the next activity to find the equation of the function. This is consistent with view of Treffers (1987) and Rasmussen *et al* (2005) of vertical mathematising or *models-for*. The student therefore dealt with vertical mathematisation and ended up with the mathematical solution (Hadi, 2002).

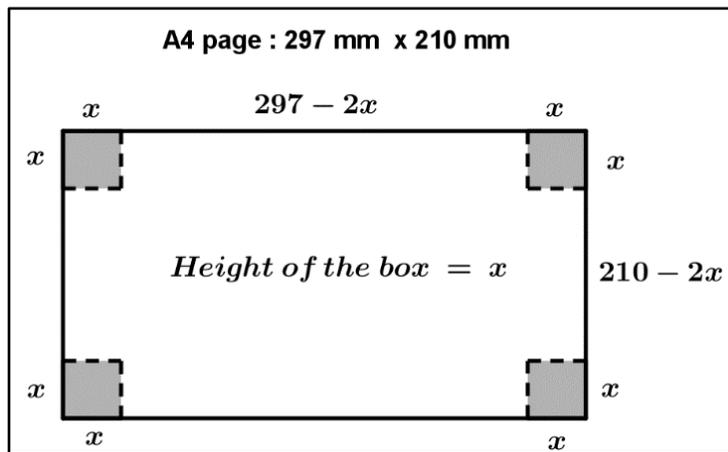


Figure 5.8: Measurements of the box in terms of the height

The students could see with their algebraic computation that the function is cubic. This activity was also used to discuss the concept of domain. The students also had to answer the following questions in their groups:

- (a) What values do you think are the independent or dependent?
- (b) What input values can be used in all the activities?
- (c) What output values can be used with all the activities?
- (d) Can the plotted points on your graphs be connected?
- (e) What type of graph does each activity represent?

The above type of questions posed by the teacher-researcher was intended to create a cognitive conflict in the students' minds and give them the opportunity to deal with their own solutions (Waite-Stupiansky, 1997). This type of question also showed the exploitation modes of the teacher-researcher. In this way the teacher-researcher was teaching logico-mathematical knowledge (Piaget, 1952; 1954). The teacher-researcher posed questions such as (b), (c) and (d) to help the students to construct their own understanding of continuity, domain and range. The students were given the opportunity to formulate their own explanation on what they thought the answers were. Terminologies, such as continuity, domain and range, were the social knowledge used by the teacher-researcher after the students answered the above questions. This also shows how students were exposed to move from informal to formal mathematical reasoning about the concept of domain, range and continuity.

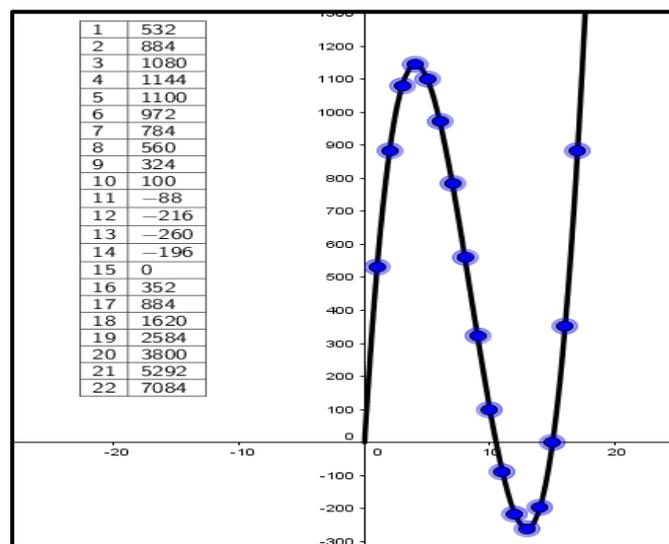


Figure 5.9: GeoGebra applet used for activity 1 (see Appendix G)

The teacher-researcher verified with GeoGebra (see Figure 5.9) with the equation of the function, that it was a cubic function, restricted to a certain domain, because of the context. The teacher-researcher applied the verification quasi-empirical method (De Villiers, 2004) by using GeoGebra to show that the relationship between the height of the box and its volume is cubic. The numerical values (see table in Figure 5.9) that the teacher-researcher used, showed that the points formed a cubic graph. The students also had to discuss the input and output values that were relevant for this activity and how the graph should look like.

The students also asked for the teacher's support and were helped as Figure 5.10 shows. The student in the figure was busy with the reflection of the quadratic function in the line $y = x$ and the following conversation ensued:



Figure 5.10: Student OG1 interacting with the teacher-researcher

The discussion between teacher-researcher and the student was on a GeoGebra applet, shown in Figure 5.10.

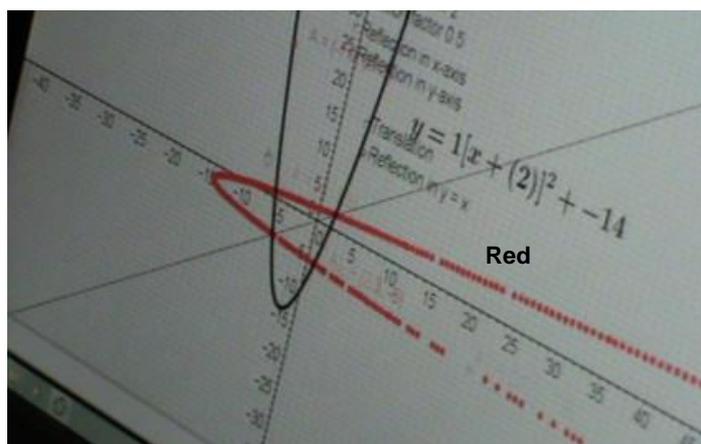


Figure 5.11: Student's discussion of the y -intercepts of the parabola reflected in $y = x$

Teacher: What do the x -intercepts of the quadratic functions become in the reflected graph?

The above question refers to the teacher-researcher's individual exploitation mode and the teacher-researcher strategy to create understanding that the x -intercepts of the parabola (black graph) are mapped to the y -intercepts of the inverse (red graph) of the parabola.

Student OG1: x and y swop around and therefore become the y -intercepts. But you have two x -values (she meant here two x -intercepts for quadratic function) so you also have two y -values (she meant two y -intercepts for the reflected graph).

The student was referring to the inverse graph of a parabola (red graph) in Figure 5.11. GeoGebra afforded the students an opportunity to see that the inverse of the parabola was not a function. Student OG1 acquired physical knowledge because she could see it on GeoGebra. The response of the student shows the logico-mathematical knowledge that she constructed through GeoGebra. Students worked out algebraically the equation of their inverse graph and the teacher-researcher showed them how to type in the input bar of GeoGebra. The students could therefore evaluate visually and graphically with GeoGebra. The syntax for this student's inverse graph was: $y = \text{sqrt}(x + 14) - 2$ and $y = -\text{sqrt}(x + 14) - 2$. The teacher-researcher showed the students the following applet (see Figure 5.12) to demonstrate to them how to restrict the domain of the function so that the inverse is a function.

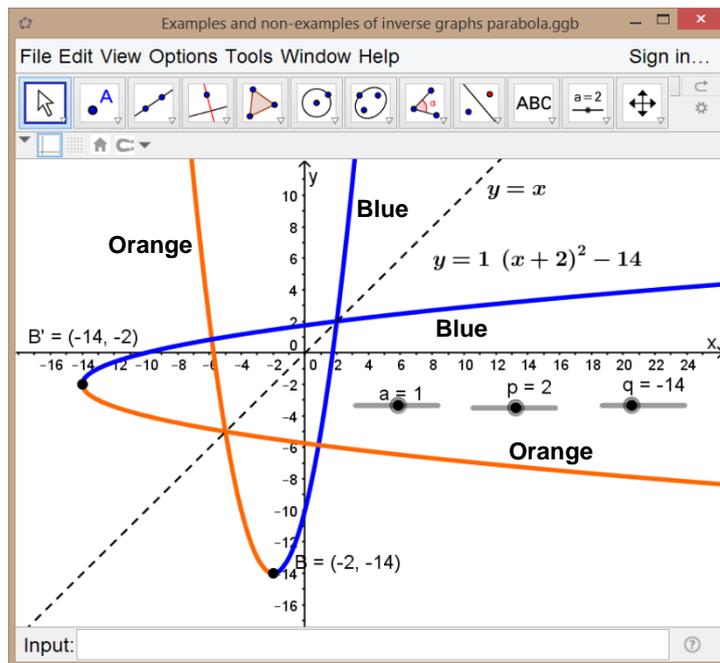


Figure 5.12: Geogebra applet to teach the restricting of the domain of parabola

By sliding a , p and q the teacher-researcher showed the student that the x -value of the turning point of the parabola can be used to restrict the domain of the function so that the inverse is a function. This activity also shown, shows how the intertwinement principle of RME was applied in this activity. Properties of reflection in transformations geometry were applied to understand the reflection of functions of function in $y = x$. The student could drag a point, did the algebraic manipulation to find the new equation of the transformed graph and typed in the equations in GeoGebra. She could subsequently link the algebraic and graphic representations. This shows how GeoGebra helped for coherence formation on deeper structure level of the notation $f^{-1}(x)$ (Seufert & Brünken, 2004). The dragging of slider p afforded the students an opportunity to acquire physical knowledge that the restriction of the domain of the parabola is so that its inverse is also a function and relies on the x -value of the turning point. The students could now make generalisations and abstractions (Lutz & Huitt, 2004) and could therefore acquire logico-mathematical knowledge that the domain of the functions can be written as $x \leq -p$ or $x \geq -p$ if given $f(x) = a(x + p)^2 + q$.

This refers to the teacher-researcher's individual *Guide-and-explain* orchestration. The question posed by the teacher-researcher was the exploitation mode, to check if the student could see that the black graph's x -intercepts were mapped to the y -intercepts of the red graph. This interaction with the student shows the teacher-researcher's TPACK how he

demonstrates to the student graphically that there are two y -intercepts for the inverse of the parabola. She could also gather from the notation if $x = 0$ that $y = \pm\sqrt{x + 14} - 2$ has two values.

Figure 5.13 shows students interacting with each other.

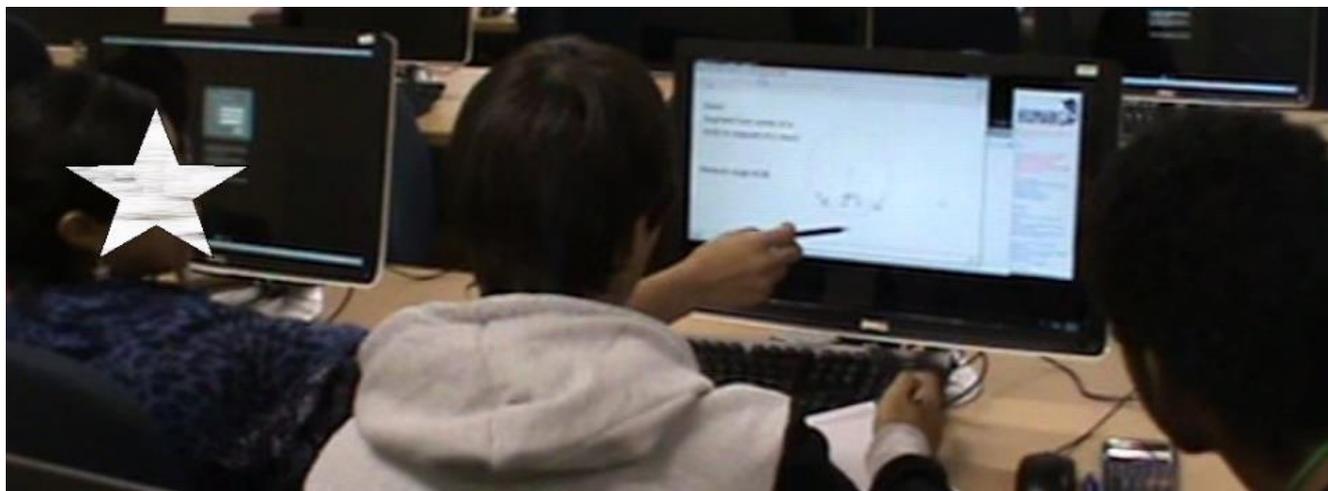


Figure 5.13: Students interacting with each other on circle geometry

Figure 5.13 shows how students explained things to each other. The three students were discussing what they could deduce from the diagram (see Figure 5.14 below).

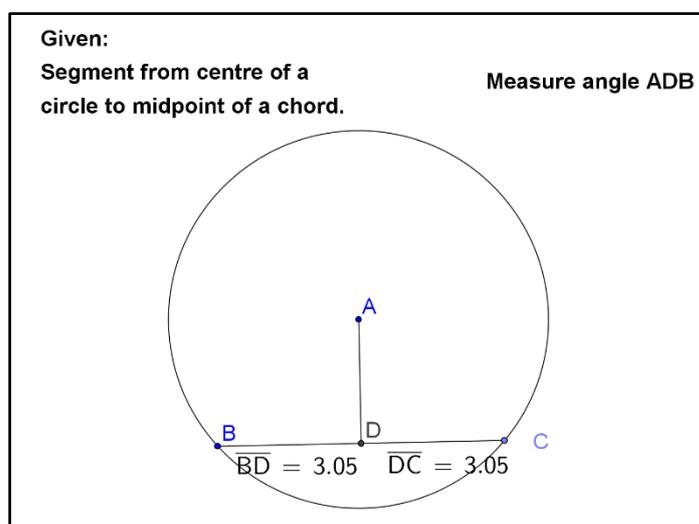


Figure 5.14: GeoGebra applet of segment from centre to midpoint of chord

Student OG2: AD is perpendicular to BC. (Student on the left).

Teacher: But that is your finding. What was given?

Students: BD is equal DC

- Teacher: Put it in words. What is given? If the line segment from the centre bisects the chord, then ...
- Students: At its midpoints.
- Teacher: And so what?
- Students: It cuts in 90°

The teacher-researcher left, but the students continued discussing their conclusion (camera was still on them). Figure 5.13 shows how the students used GeoGebra to carry out operations that the teacher-researcher requested and they presented them to each other. This shows the individual *Guide-and-explain* orchestration, the individual exchange between the teacher-researcher and a group. The instrumentation process here was how GeoGebra shaped the thinking of the students and how it helped them to understand and visualise the theorem. The instrumentalisation process here was how the students used GeoGebra on their own to be a tool to test their conjectures and to validate their answers. They tested and validated their answers by the dragging of point A, B or C on applet (see Figure 5.14). All the questions asked by the teacher-researcher showed the exploitation modes of teacher-researcher. The students acquired physical knowledge because they explored the properties of circle geometry with GeoGebra. The students made a conjecture on the basis of intuition, numerical investigation and measurement (De Villiers, 2004). GeoGebra provided the students visual images that contributed to their growing mathematical understanding (De Villiers, 2004). This shows how the students constructed their own understanding of the theorem that states the line from the centre of a circle to the midpoint of a chord is perpendicular to the chord. They acquired logico-mathematical knowledge by showing understanding that the segment is only perpendicular if it is drawn from the centre of the circle to the midpoint of the chord. The students' interaction and discussion with each another, and with the teacher-researcher, show how they negotiated meaning of the mentioned theorem. Although the theorem was not proved at this stage the students' discussions moved them from informal to formal mathematical reasoning. This shows how the students moved to higher order thinking. This also alludes to the level principle of RME and Van Hiele Level 3.

Figure 5.15 below shows another example of how students were interacting with each other.

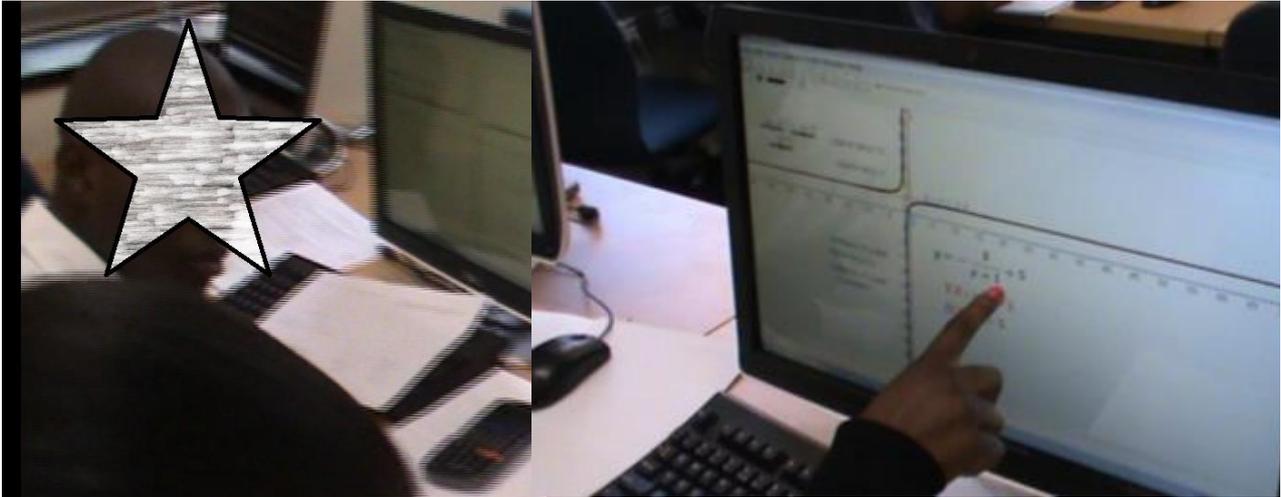


Figure 5.15: Students interacting on transformations of functions

The two students were discussing what they could deduce from the diagram (see also Figure 5.16 & 5.17) below. Student OG3 (student on the left) and Student OG4 (pointing at the screen) were working together and Student OG3 showed her the answers. The student next to her was doing the activities without using GeoGebra. He was relying on his prior knowledge and was able to do it. Student OG4 could also see the answer but was excited to check her answer. She commented to him as follows:

Student OG4: I am double-checking, it is interesting ... it is interesting. It is the equation but I just want to figure out.

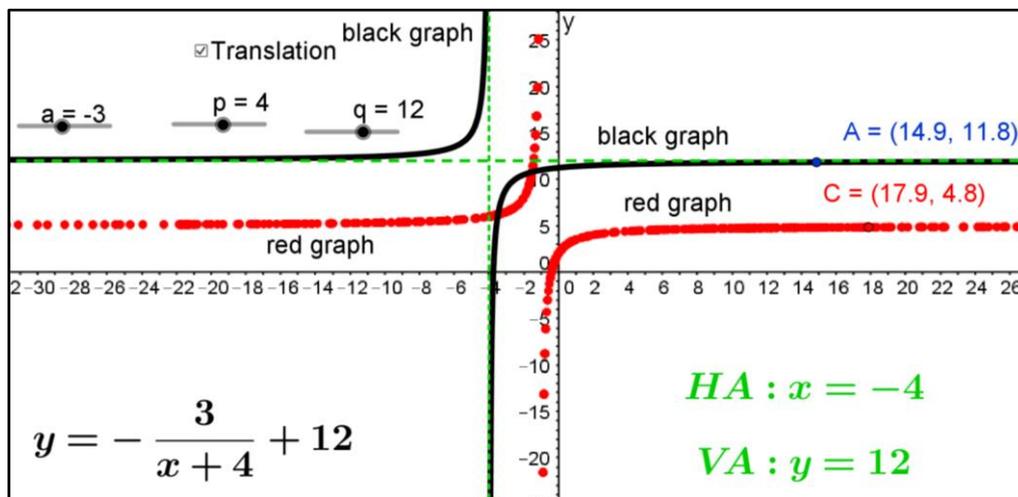


Figure 5.16: GeoGebra applet that Student OG4 used to explore translation

Figure 5.16 shows how the Student OG4 dragged the blue point (point A) to obtain the red graph. The activity was designed so that the students could see from the coordinates of A and C that the red graph represents a translation of the black graph.

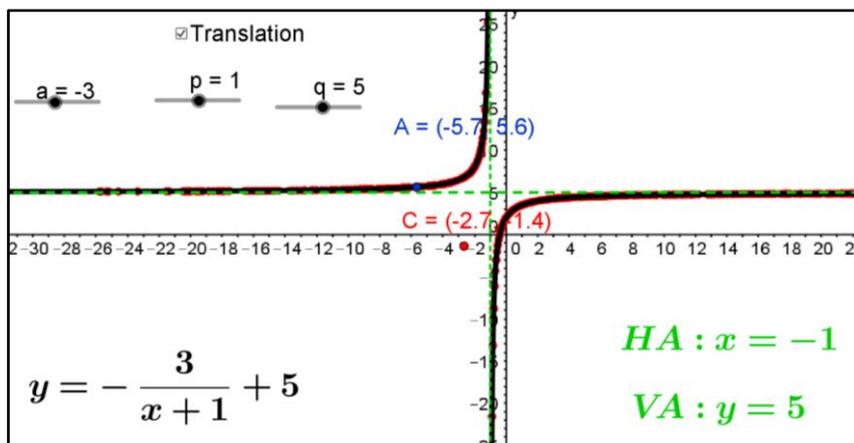


Figure 5.17: GeoGebra applet that represents Figure 5.16 of Student OG4

Figure 5.17 shows how Student OG4 changed sliders a , p and q to find the equation that fits the red graph. Student OG3 did not change the sliders because he could see the translation was three units to the right and seven down from the coordinates of A and B and therefore just wrote down the equation.

Student OG3 entered SciMathUS with 77% in the NSC Mathematics examination. Student OG4's comment shows how interesting she found the activity and verified that her answers were correct. Student OG4 acquired physical knowledge because she explored it with GeoGebra. She verified the answer of the student next to her with GeoGebra and obtained certainty about the answer (De Villiers, 2004).

The students also worked independently (see Figure 5.18).



Figure 5.18: Students working independently.

During the classroom observations, the students mostly started an activity on their own and then later began to consult or support each other. Figure 5.18 shows the four students were actively busy and this refers to the activity principle of RME (Freudenthal, 1973). Kamii and

Russell (2012) recommend that students should be allowed to do their own thinking to enable the construction of logic-mathematical relationships. Many of the students worked independently and later started discussing, or consulting with a classmate, or with the teacher-researcher. This also referred to the *Guide-and-explain* orchestration that was the type of orchestration, which was highly dominant during the computer lab sessions.

5.2.3.1.2 Results from students' responses to interviews

This sub-section presents the students' responses from the in-depth (ID) and focus group (FG) interviews. One of the questions asked in both the in-depth and focus group interviews was: *How did you experience interaction between yourself and your classmate(s) when you were doing mathematics?*

Students alluded to the fact that interacting with someone helped them to remember concepts better, because explaining it to someone helps one to learn from one's mistakes; build's one confidence, etc. Below are some of the students' responses:

Student ID9: Uhm ... I rather work on my own if I understand something. Everything is faster and smoother. If I don't understand, I would rather honestly speak to the teacher than someone else. I am that kind of person. If someone doesn't understand and they ask me for help and then we start talking and I start explaining, I enjoy that, because when I start speaking I understand it more. That is how I am.

Student ID10: I think the best way to understand stuff, is to work with someone. If we work and then do stuff from a student perspective ... maybe we do circle geometry ... If I do like this and he uses another method, then I can explain my method to him, he can explain his method to me and then we understand both of our methods.

The above students' responses during the in-depth and focus group interviews were similar to what Sousa (2001; 2006) considers to be promoted by interaction in the learning pyramid. As seen in Chapter 2 (section 2.1.4.5) the learning pyramid shows that the average percentage of retention of material after 24 hours when students work in discussion groups is 50% while that of teaching each other is 90%. These responses also referred to how the students negotiated meaning of concepts during their interaction. Student ID9 preferred to interact with the teacher as the primary source to determine the correctness of the answer. The teacher is seen as a source of authority (Frid & Malone, 1995). He indicated that he was prepared to help others. Student ID10 realised the importance of interaction to make sense of concepts and he viewed his peers as a source of mathematical knowledge (Frid &

Malone, 1995). Some students used GeoGebra outside the classroom. The student below interacted with her roommate using an activity done in GeoGebra to show her why she was thinking that way.

Student FG4: I actually used GeoGebra to prove a point. I have a roommate in my house: we were arguing about calculus and you printed it from GeoGebra and I still had it. I showed her to prove a point.

Student FG2: When maybe you are doing something and then try to explain it to someone and you realise that “oh I also made a mistake somewhere” you then correct yourself while helping someone. You won't forget it.

Student FG5: When you are struggling to do something, you always remember it better when somebody explains it to you or you explain it to someone.

Student FG2 contended that by interacting with classmates, it also helped her to see her own mistakes while she was explaining something to them. This refers to the student's instrumental genesis, i.e., the instrumentalisation process. Student FG4 used GeoGebra to validate her answer. Student FG4 therefore used GeoGebra to empirically verify statements (De Villiers, 2004). Although some of the students' preferences were to work independently (e.g. Student ID9) they realised that interactions are beneficial because they could learn from each other. Interaction therefore, according to the students' responses, was very important. This next session presents quantitative data (quan) that will be triangulated with the above qualitative data (QUAL).

5.2.3.2 Quantitative analysis of students' responses to interaction

The following were quantitative data collected from the survey questionnaire given to the students after the year of teaching them. This section presents five of the 12 items in the construct on RME. The Cronbach alpha for this construct, as mentioned in 5.2.2.1, is 0.766. A Cronbach alpha of $\alpha \in [0.7; 0.8)$ is acceptable (George & Mallery, 2003). The purpose of this section is to see how students experienced interaction. Students interacted in the following way: (a) with each other (b) with the computer on its own and (c) with the teacher.

Figure 5.19 shows that 42 (91.3%) students assisted each other. Thirty-six (78.3%) of the students indicated that they helped their classmates to understand certain concepts better and 43 (93.5%) learned from their classmates.

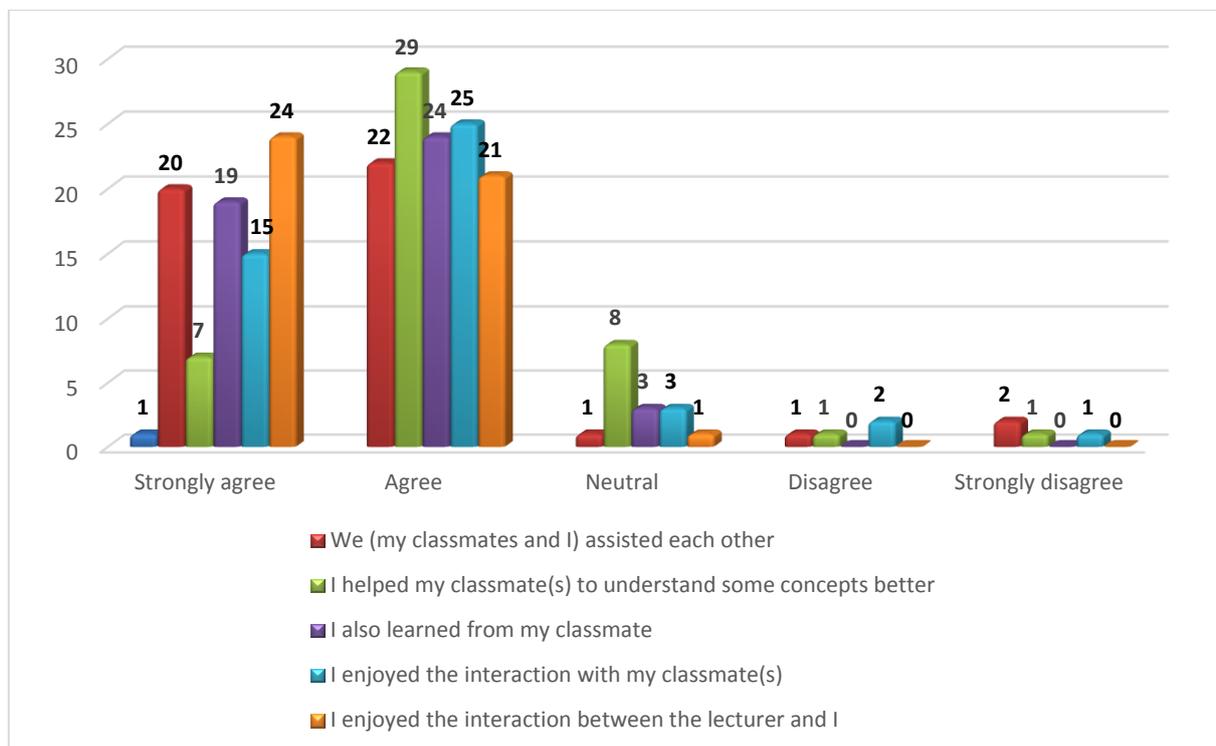


Figure 5.19: Students' views on interaction

Forty (87%) of the students indicated that they enjoyed working with their classmates and 45 (97.8%) also enjoyed the interaction between themselves and the teacher.

Thus, based on the responses, it could be inferred that the students wished to be actively involved in their learning by assisting each other, learning from their classmates and were keen to interact with their teacher. All of the abovementioned gave the researcher an indication that the students view interaction to be important during their learning process. These results show that students were open and keen to obtain knowledge through social interaction. The students were interacting with each other during the sessions in the computer lab. This accords with the view of Vygotsky (1978) of social constructivism that could help students to negotiate meaning based on their experiences, individually or socially (Narayan *et al.*, 2013). Constructivist approaches are considered supporting deeper understanding and internalising material or concepts (Narayan *et al.*, 2013). The researcher thus saw interaction as the second process to ensure that GeoGebra is used successfully. Freudenthal (1973) promoted this as the interaction principle in RME.

5.2.4 Students' perceptions of the pedagogical use of GeoGebra

GeoGebra was utilised by the teacher-researcher as a pedagogical tool for the entire year to explain concepts throughout the year and was also part of interactive lesson presentations

to the students. Topics covered in this way were the properties of quadrilaterals, general solutions of trigonometric equations, etc. The teacher-researcher also used GeoGebra in some cases to explain exercises and questions from the textbooks or past examination question papers used in class or to visualise answers. Students were also afforded the opportunity to explore with GeoGebra on their own. They explored in the computer lab on GeoGebra transformation geometry, transformations of functions and circle geometry.

5.2.4.1 Qualitative analysis of students' perceptions of the pedagogical use of GeoGebra

One of the questions from the in-depth and focus group interview schedules was for students to indicate which of the approaches used they preferred among the following: the use of GeoGebra in the class; students working on prepared GeoGebra applets, or students creating their own activity in GeoGebra.

5.2.4.1.1 Students' reasons why they preferred the teacher using GeoGebra

This section focuses on how the teacher-researcher used GeoGebra in the computer lab and classroom as a teaching tool. Figure 5.20 shows how the teacher-researcher utilised GeoGebra in the computer lab.

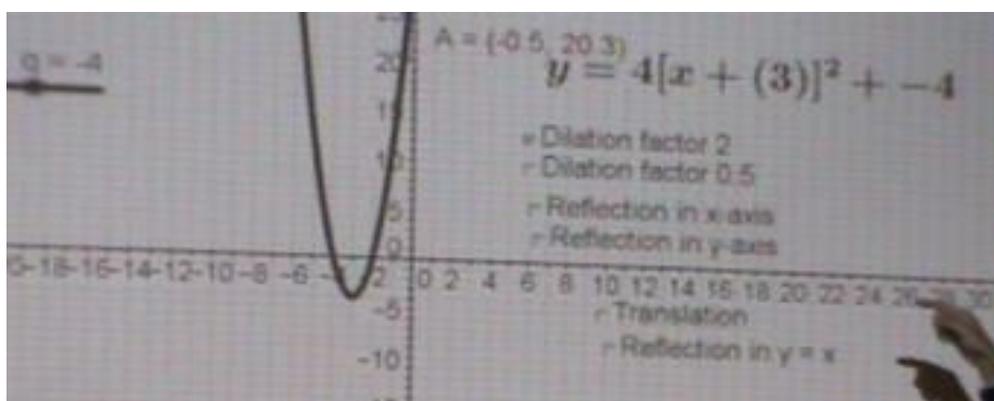


Figure 5.20: Teacher-researcher using GeoGebra in computer lab.

Figure 5.20 shows an example of the teacher-researcher's *Link-the-screen*, *Discuss-the-screen* and *Explain-the-screen* orchestration in the computer lab. That is, the teacher-researcher also used GeoGebra with interactive lecturing. The following are responses of the students from the interviews on the interactive lecturing where GeoGebra was utilised to involve them in a discussion on what they observed from the screen. Below are some of the responses as to why students preferred this approach of using GeoGebra as a pedagogical tool.

Student ID7: It is better for me. I am more focused on how you explain the work to the class and how you showed us how it is done.

It may seem that this student sees the teacher as the source of knowledge and that the teacher is the only person who can explain. This was only his preference, but he indicated during the interview that he saw the benefit of GeoGebra because he liked the visualisation capabilities of the software. He indicated that he also liked working with someone else, but found exploring mathematics with GeoGebra a bit challenging. His computer skills were also not good. He pointed out that his knowledge of the topic at that stage was not good.

Student ID7: Because my knowledge about the topic at that moment was not that good it helps you to find out yourself and when you work it out on your own.

Student ID7 came into the SciMathUS programme with 54% and achieved 81% in the NSC examination at the end of 2014. He also achieved 42% for the pre-test in transformations of functions and 91% in the post-test. In the pre-test for circle geometry, he scored 0% and 76% in the post-test. He had not done circle geometry in Grade 11.

Student FG16's response below shows that she did not like using computers and was anxious from day one in the computer lab. She indicated that she preferred the teacher to use GeoGebra as a teaching tool.

Student FG16: I like it when you are doing it on the board, because at least then I know how you are doing it. You know what you are talking about, but at the same time I like it when I am told this is how something is supposed to be done and if it is like that, I can carry on myself doing it that way, because I know you said it.

She was relying on the teacher. She liked to visualise but the teacher had to work on GeoGebra for most of the time. She mentioned during the interview that:

Student FG16: So they (her teachers) did like cut out shapes and stuff to demonstrate how things should be and sometimes my Maths teacher would use GeoGebra ... but he himself was using it showing us. But then, I think the difference was this year that I could also do it myself in the computer lab.

Although her high school teacher taught with visualisation and teaching aids, he did not afford them the chance to explore. Although her teachers attempted to teach in a constructivist approach, she was not exposed to exploring. This is why she saw the teacher as the person with the knowledge and relied on the teacher to explain. It seems that she did not want to be confused when starting to learn a new topic.

Student FG16: The first day was fine, but if I must be honest with myself especially with transformation geometry, I think the whole GeoGebra confused me a bit. I was feeling anxious and when I couldn't get something I just switched off. But then it helped me in terms of circle geometry and that we could actually see that the angle at the centre is twice the circumference angle.

The students were given GeoGebra circle geometry applets. Transformation geometry and functions were taught in early March of the SciMathUS year and circle geometry in late April and May. Student FG16 enjoyed circle geometry with GeoGebra. It could be argued that this student gained more confidence working with GeoGebra, or started to adapt to different approaches of teaching used by the teacher-researcher.

Student FG16: I like it and sometimes I don't like it. I like it because it is something different and it's a challenge. I don't like it, because I don't want to be challenged – only sometimes. For me, I didn't enjoy it that much. I don't know. I am used to one method in a certain way and then I just wish to continue in that way. Now, when you introduced a new thing and you like GeoGebra; and then I can't put the two things together.

She was familiar with computers and had worked on them before, i.e., internet and MS Word. She continued with the abovementioned statement and this clearly indicated that she was one of the students that did not wish to be taken out of her comfort zone. Her comfort zone was that the teacher teaches and the student just absorbs the knowledge. This student came into the programme with 62% and only achieved 66%. Her mark for the pre-test for transformations of functions was 47% and for post-test 69%. She did transformations of functions in her Grade 11 and 12 years, but managed only to score 69% in the post-test. She managed to increase her pre-test score from 36% to 92% in the post-test for circle geometry. She also did circle geometry when she was in Grade 11, but did not write the NSC Paper 3 in Grade 12.

5.2.4.1.2 Students' reasons why they preferred using GeoGebra themselves

This section shows how the students used GeoGebra themselves. They had to create their own GeoGebra applets from the worksheet on transformation geometry. They were also given prepared GeoGebra applets that the teacher-researcher designed for transformations of functions and circle geometry.

Figure 5.21 shows how they created their own activities in GeoGebra and had to complete worksheets.

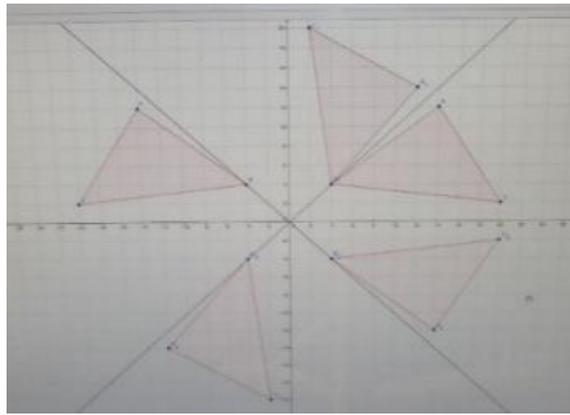


Figure 5.21: Student created activity in GeoGebra.

This activity was the first lesson (see Appendix J) in which the students used GeoGebra. This activity required students to construct a triangle and then to apply the different constructions. The students did not just construct the triangles in GeoGebra, but had to explore the properties of transformation geometry. It was also expected from them to complete questions on the worksheet (see Appendix I). The students therefore had an opportunity to generalise and make conclusions of rules of transformation on the basis of intuition or experience obtained through GeoGebra. They made deductions from special cases. De Villiers (2004) refers to this as quasi-empirical methods.

Figure 5.22 shows how the students had to measure the sizes of the different angles in the diagram with the angle-measuring tool in the designed GeoGebra applet and then had to complete the worksheet (see Appendix L).

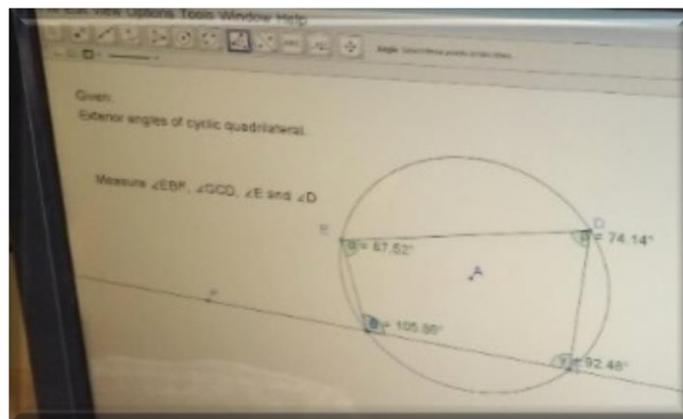


Figure 5.22: GeoGebra applet on circle geometry.

The students responded to this way of using GeoGebra with the following responses:

Student ID2: I think in that way we can remember it better because if we are stuck we ask you for help.

Student ID6: Dit wat klaar gemaak was kon ons nog steeds op werk. Ek sal nie altyd weet hoe om dit self so te bou nie en dit sal baie tyd neem. (*Those that are already created we can still work on. I will not always know how to create it myself and it will take some time.*)

Student ID9: Preferred circle geometry in the computer lab, so that I can see for myself.

Student FG13: Enjoy working on my own as well as figuring things out. When I figured it out and if I see it, then I understand. But, if you are working on the board (screen) it speeds up the process. Compared to working on my own, it will take a while for me to understand something. Maybe I am not getting it right there, but if you are doing it, it speeds up the process of my understanding.

The above students' responses from interviews show they wished to work by themselves on GeoGebra and indicated that the software helped better with understanding concepts.

5.2.4.1.3 Students' reasons why they preferred all three ways of using GeoGebra

Some of the students felt that only the teacher should use GeoGebra. Other students indicated that the students themselves could use it. They can create their own activities from worksheets in GeoGebra to explore, or when they are given GeoGebra applets.

Student ID10: It won't be that much beneficial only listening to the teacher without doing it ourselves – we were not sure if we do or don't understand. However, if we do it practically on our own, we do understand. However, I think it is still more of the teacher teaching than more of the students are doing it on their own.

Student FG5: All three and it depends on the topic because if you look at the first one where I have to construct, we didn't know much of Euclidean Geometry. So we couldn't draw anything, but when you gave the applets we at least had our drawing and played around with it.

Student FG12: That was also fun and interactive, but for me personally, I like to do stuff on my own so that I know that I actually understand what I am doing. It's fine to watch you doing it, or using the applets and playing around with them, seeing your friends playing with them, but it helps me more when I'm doing it myself.

Student ID10 preferred the use of GeoGebra when the teacher used it as a teaching tool. This refers therefore to use of GeoGebra with interactive lecturing. He acknowledged that working with GeoGebra on his own can help him to understand concepts, but the teacher has to use it more than the students do. Student FG5 indicated that the topic that the teacher wished to teach would determine how GeoGebra would be utilised. He mentioned the example of circle geometry and is of the opinion that the prepared GeoGebra applets were easier to work with, instead of creating or constructing them. He thus preferred all three ways of using GeoGebra, but it depended on the topic. Student FG12 clearly preferred doing

it herself even though she tolerates the other approaches. She also enjoyed and found it very interactive. The abovementioned students wish to explore and do not only want to listen to the teacher. This accords with the constructivist approach of teaching and the activity principle of RME.

Seven (70%) of the ten students in the in-depth interviews (although they had their preferences), while 15 (65%) of the 23 students in the focus group interviews indicated that all three ways of using GeoGebra have a place in the teaching of mathematics. One (10%) student in the in-depth interviews indicated that only the teacher must use it, while 5 (22%) students preferred it in the focus group interviews. Hence, the conclusion can be drawn that the majority of the students in the interviews indicated that all three ways of using GeoGebra have a place in the teaching and learning of mathematics.

5.2.4.2 Quantitative analysis of students' perceptions of the pedagogical use of GeoGebra

This section presents the quantitative analysis of the students' responses from the post-survey questionnaire on the use of GeoGebra. Figure 5.23 below shows the students' views from the post-survey questionnaire on the use of GeoGebra. The Cronbach alpha reliability coefficient for the six items in this construct is 0.704. A Cronbach alpha of $\alpha \in [0.7; 0.8]$ is acceptable (George & Mallery, 2003).

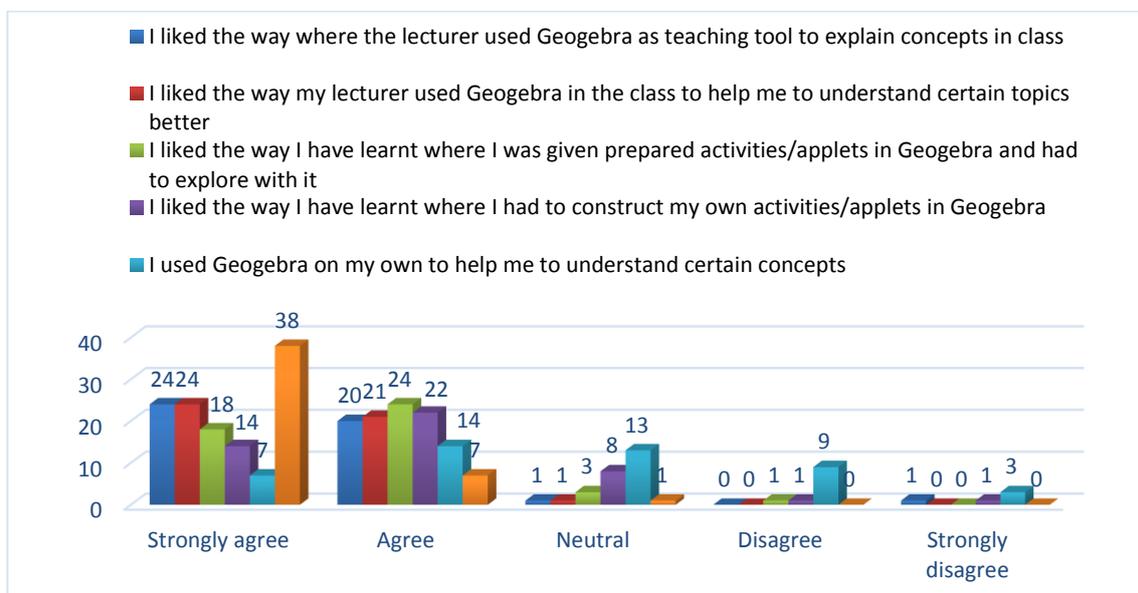


Figure 5.23: Students' views on the use of GeoGebra (n = 46).

Figure 5.23 shows that 24(52%) students strongly agreed and 20 (43%) agreed that they liked the way the teacher-researcher used GeoGebra as a teaching tool during class to

present his interactive lessons. Twenty-four (50%) students strongly agreed and 21 (45%) students agreed that they liked the way that the teacher-researcher used GeoGebra as a teaching tool during class to explain a concept. Thus, respectively, 44 (95.7%) and 45 (97.8%) of the students indicated that they liked the way the teacher-researcher used GeoGebra as a teaching tool during class to present his interactive lessons or explain a concept. Figure 5.23 further shows that 42 (91.3%) students indicated that they liked to explore with prepared GeoGebra applets, while 36 (78.3%) indicated that they preferred to construct their own activities or applets in GeoGebra. An average of 42 (91,3%) students therefore indicated that they liked the way the teacher-researcher used GeoGebra as a teaching tool during class to present his interactive lessons, or explain a concept, or to explore with prepared GeoGebra applets, or they preferred to construct their own activities, or applets, in GeoGebra.

Figure 5.23 continues to show that only seven (15%) students strongly agreed and 14 (30%) agreed that they used GeoGebra at home or after class to draw graphs. Thirty-eight (83%) students agreed strongly and seven (15%) agreed that they used GeoGebra to draw the graphs and to do statistical calculations in GeoGebra. A low percentage of 45.7% of students utilised it at home or on their own. Forty-five students (97.8%) utilised GeoGebra to help them with the Statistics project. The use of GeoGebra by the students refers to students' genesis, i.e., instrumentalisation. They utilised GeoGebra to sketch their graphs so that they could analyse the data to answer their research questions in the Statistics project.

The abovementioned quantitative (quan) analysis are triangulated with the in-depth and focus group interviews (QUAL) in section 5.2.4.1 on how the students preferred the use of GeoGebra. The fact that students preferred to work for themselves affirms Freudenthal's (1991) observation that first knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others. This therefore accords with the activity principle of RME.

The triangulation of the analysis of the qualitative (QUAL) and quantitative (quan) data show that students viewed contextual activities, guidance, exploration and the pedagogical use of GeoGebra as important processes, or considerations, when teaching Mathematics. The instrumental orchestrations of the teacher-researcher showed how the students started consulting and asking for support but later on worked more independently. This accords with the view of Amineh and Asl (2015) on how facilitators in social constructivism first provide

support and help for students, and this support is gradually decreased as students begin to learn independently. The exploitation modes were also playing a role in the teaching and learning of Mathematics because they helped to create cognitive conflict in the minds of the students, and assisted them to move from informal to formal mathematical reasoning.

5.3 HOW GEOGEBRA HELPS STUDENTS TO UNDERSTAND MATHEMATICAL CONCEPTS AND MOVE THEM TO HIGHER LEVELS OF ABSTRACTION

This section presents both qualitative (QUAL) and quantitative (quan) data that will attempt to answer the second sub-question: *How did GeoGebra afford students an opportunity to understand mathematical concepts and move to higher levels of abstraction?*

This section presents how GeoGebra afforded the students an opportunity to understand mathematical transformations of functions and circle geometry concepts, and move to higher levels of abstraction. The analysis of the section on the teaching and learning of transformations of functions and circle geometry will commence with the students' responses during the in-depth and focus group interviews, followed by a comparative analysis of students' responses in the pre- and post-tests, and then by a quantitative analysis of pre- and post-tests. This section then concludes with the students' perceptions before and after the intervention. The quantitative (quan) data are triangulated with the qualitative (QUAL) data to answer this abovementioned sub-question.

5.3.1 Analysis of data on transformations of functions

The following section analyses the students' responses during the in-depth (ID) and focus group interviews (FG), after being taught for the first time with GeoGebra in the computer lab and after first being taught in a combination of an active learning approach and teacher as a facilitator of learning approach. Pseudonyms AT were used for the students' individual feedback after being taught for the first time with GeoGebra in the computer lab and after first being taught in a hybridisation of an active learning approach and a teacher as a facilitator of learning approach. It then continues to analyse the pre- and post-tests on transformations of functions qualitatively and quantitatively.

5.3.1.1 Students' responses during the in-depth and focus group interviews on how GeoGebra afforded them an opportunity to understand transformations of functions

The researcher used purposeful sampling to select specific students' responses that were informative about the phenomena investigated (McMillan & Schumacher, 2006). This

qualitative data will be triangulated with quantitative data of students' responses from the post-questionnaire, and the pre- and post-tests, for the transformations of functions. These responses will be triangulated to validate the data mentioned to answer the sub-question on how GeoGebra afforded them an opportunity to better understand concepts and move to higher levels of abstraction.

The following show students responses during in-depth interviews (ID) and during feedback after their first time exposure to GeoGebra (AT) and specifically to the question if they thought that GeoGebra had helped them with certain concepts and topics. The following responses of the students show the instrumentation process of how GeoGebra made an impression on them (Ndlovu, Wessels, & De Villiers, 2013c). The following responses of students show the *affordances*, *potentialities* and *enablements* of GeoGebra. Most of the students responded positively and stated that GeoGebra had helped them with better understanding of concepts in transformations of functions and circle geometry.

Some of the students responded as follows on how GeoGebra helped them to understand transformations:

- Student ID1: I could see [*visual affordances*] the mother graph. I could see how the transformations of it occur. I could see if the formula changes [*potentialities*], then the mother graphs shifts or transforms.
- Student ID6: Dit het ook baie gehelp. Ek het nooit geweet wat die a (parameter) beteken het in transformasies nie. Ek het dit gesien en ek kan dit nou onthou. (*It also helped a lot. I never knew what a (parameter) means in transformations. I saw it [*visual affordances*] and now I remembered it.*)
- Student FG9: For those ones like $f(x + 30^\circ)$ maybe it was initially very confusing because it is plus 30° . So it is plus 30 and you think it is $x + 30^\circ$ so you add 30 then move that to the right, but $x + 30^\circ$ is actually moving to the left. Therefore that was confusing. Then you could see there will be a shift ... you could actually see how it was shifted [*visual affordances*]. That was actually helpful. There were many questions put to us, whether we were doing exercises, or in the exams, where we had to draw that graph or solve the graph.
- Student AT1: I could see how the function's graph looks like [*visual affordances*] and that the changing or adding of values to the equation [*enablements*] will change the equation of the graph. I did not know how graphs changed, that was until I used GeoGebra.
- Student AT2: It was different doing movements of graphs on the computer (student is referring to GeoGebra) as it gave a clearer picture of what certain changes

will do to a graph. Next time when I am doing it manually, I will remember the picture of it. Making graphs different colours [*enablements*], made them easier to compare with the original. It made me realise that things are not as complicated as I make them. Certain changes will always have the same effect.

Student ID1, FG9, AT1 and AT2 refer to how they could see that if parameters in the equation changes, then the graph showed a type of transformation. This suggests that GeoGebra helps the students to move from a surface feature level to a deeper structure level of understanding (Seufert & Brünken, 2004). The following students mentioned these specific examples of how GeoGebra helped them to understand transformations of functions better:

Researcher: Do you think GeoGebra helped you to understand transformations better?

Student ID9: I knew from last year that if you reflected about the y -axis, the x -value is going to change to positive or negative. You were just doing it mechanically, but with GeoGebra you could see what is going on [*visual affordance*] and it is making sense. If I move this here and this is going to happen. I could see it and it must be easier in that sense.

Figure 5.24 shows how GeoGebra helped Student ID9 to see the meaning of reflection of a point in the y -axis.

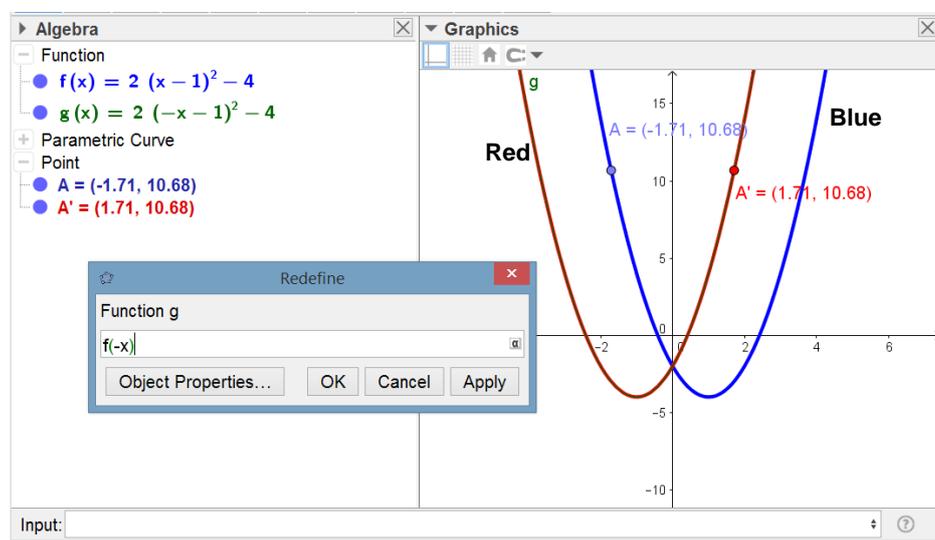


Figure 5.24: GeoGebra applet used to teach reflection in y -axis

Student ID9's response referred to how she could see that point A is mapped to A' if reflected in the y -axis. She observed that the coordinates of A' can be obtained only by changing the x -value positive. GeoGebra therefore afforded the student an opportunity to acquire physical knowledge about the rule for reflection in the y -axis and gave her the understanding of

negating x -value, i.e. acquiring logico-mathematical knowledge. She understood now why rule for the reflection in y -axis is: $(x; y) \rightarrow (-x; y)$. This activity was based on Faulkenberry and Faulkenberry's (2010) study to teach transformations of functions with input and output values.

Researcher: Did GeoGebra help you with the algebraic notation of the transformation of functions?

Student FG16: If am being honestI didn't like the dilation part of the transformation but with the shifting and reflection part it really helped because especially with the exponential graph ... if they want the inverse of the graph. If I can't see it now, then I draw the mirror image of it [*enablements*], so all the time now when I do it algebraically it actually now make sense ... okay this point is now to this point.

Student FG16 referred to how she was able to sketch the inverse graph of the exponential function by just using points. The students were shown to use dots or critical points to sketch the graphs of the transformed functions. Figure 5.25 shows how students were exposed to the teaching of the inverse graph of the exponential functions with GeoGebra.

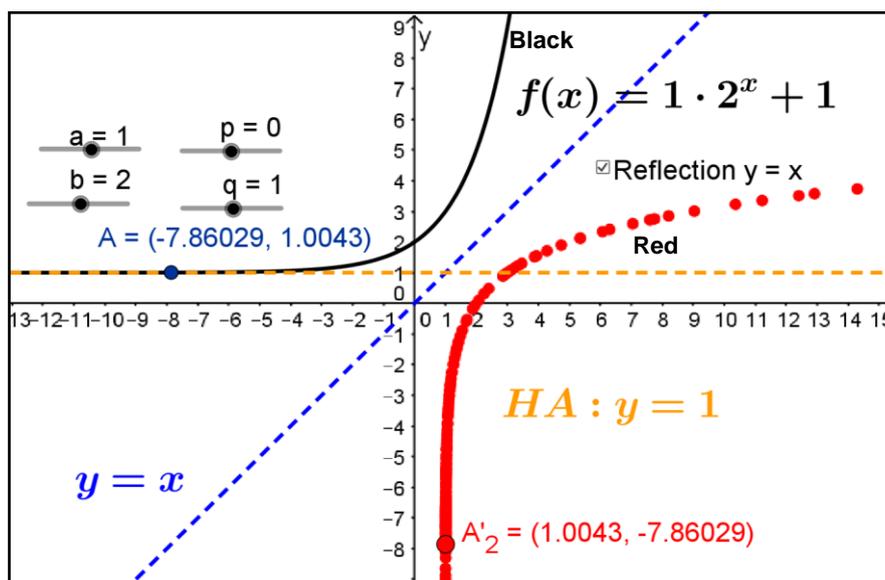


Figure 5.25: GeoGebra applet used to sketch the inverse graph of exponential function

Student FG16 could see that the red dots formed the inverse graph of the exponential function. She could see that point A was mapped to point A'_2 . The dynamic nature of GeoGebra therefore helped the student to see the path that the red dots formed. The student thus acquired in this way physical knowledge about the shape of the inverse graph of the exponential. The students could also see that y -value of point A stays above 1. This was done by moving A to the left and so they could see that the x -value of point A'_2 was above

1. Students could also in this way explore how the domain of the exponential function (f) is the same as the range of the inverse of exponential function, also known as the logarithmic function (f^{-1}) and the range of f is the same as the domain of f^{-1} . The student acquired logico-mathematical knowledge and understood how to sketch the inverse graph of an exponential function.

The following student explained how GeoGebra helped her with the reflection of a function in the x -axis.

Researcher: With which of transformations of functions did GeoGebra help?

Student ID8: Die notasies (*opgewondenheid in haar stem*) het ek baie beter verstaan. (*The notations (excitement in her voice) I understood much better.*)

Researcher: What about the notations?

Student ID8: Ek kon die tekens sien. Ek verstaan nou beter waarom die teken as $g(x)$ gereflekteer word in die x -as... dan weet ek dat die negatiewe teken staan voor aan die $g(x)$ en dit beteken dan dat die nuwe grafiek $h(x)$ is. Ek het dit so gesien [*potentialities and enablements*]. (*I could see the signs. I understand now better why the sign if $g(x)$ is reflected in x -axis then I know the negative sign has to stand in front of $g(x)$ and it also mean that new graph is $h(x)$. [*potentialities and enablements*]*)

Figure 5.26 shows Student ID8's explanation of how GeoGebra helped her to understand the notation of transformations.

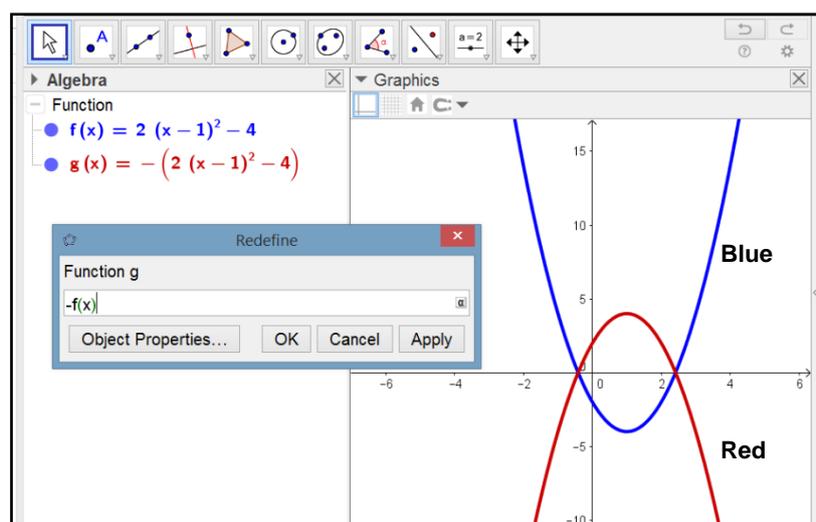


Figure 5.26: GeoGebra applet to support Student ID8 explanation

Student ID8 explained how GeoGebra helped her to link the graphic and algebraic representations. She also explained how the dialog box helped her to understand that $f(x)$

is redefined as g and $g(x) = -f(x)$. All the abovementioned students' responses informed the researcher that GeoGebra helped them to understand the effect of the parameters as the DBE (2010, 2011, 2012, 2013 and 2014) recommends that functions and graphs should be taught in a way that leads to an understanding of the effect of the different parameters. The activities in Appendix J were designed for students to visualise graphic representations with the algebraic notations and to understand the transformations in words from the notation format. This student could interpret from the GeoGebra applet (see Figure 5.26) and link the notation with the type of transformation. This referred to the students' vertical mathematical reasoning. Using the options of the graphic window allowed the students to interpret problems graphically and link them to the algebraic window in GeoGebra. This shows how GeoGebra helped with coherence formation at deeper structure level of the notation $-f(x)$ (Seufert & Brünken, 2004). This also shows how the teacher-researcher's TPACK afforded the students an opportunity to link their graphic observations to the algebraic notation of the answer.

Thirty-five (73%) of the 48 students indicated in the pre-questionnaire they preferred the teacher to give them notes when starting a new topic. It showed that students were used to start a new topic with notes without exploring and were taught to memorise rules and algorithms. Some students, during the in-depth (ID), explained how they were taught in school:

Student ID9: It was just given ... like the formulas were just given ... you don't know what you are doing. You just memorised the formulas and then just when to write it?

Student ID6: Ek het die formules gememoriseer. As dit in $y = x$ gereflekteer word, dan moet jy onthou om dit om te draai. (*I memorised the formules. If it is reflected in $y = x$ then you must remember to swop it around*)

The students knew during the computer session that the x - and y -values swop around when working with the inverse of the function. They could not explain why and most of them showed in the pre-test no understanding of the meaning of the transformation of the notation $f^{-1}(x)$. The fact that the students could not explain why the x - and y -values interchange showed, could be the result that this was taught as an algorithm or procedure to follow.

Both students indicated that they just memorised the rules. There was no understanding of how it should be like that. This is an example of how knowledge that could be explored, was just shared as social knowledge. The teacher-researcher pointed out in the HLT for the

teaching of transformations of functions (see section 4.5.1) that the understanding of the properties of transformations is important. Some students, during the focus group interviews (FG), explained how they were taught in school:

Student FG4: Last year, you came into class, sat down, a new topic, for example trigonometry, these are the questions, this is how you simplify it and this the answer. It was just spoon-feed, spoon-feed, spoon-feed...

Student FG13: As compared to last year we were just going like ... we were just solving things because you need to solve them. You don't know why you are solving it and this was the way that you had been taught to do it.

Both Student FG4 and FG13 acknowledged that they were just taught without any understanding of the concept. The responses of the students showed that certain concepts and topics were taught using the traditional approach to teaching. These concepts were therefore taught as social knowledge and students were just taught to memorise the rules and algorithms. This could be the reason why students answered the pre-test because they could not remember that it was shared with them as social knowledge. Kamii and Ewing (1996) point out that teachers sometimes teach all Mathematics with a social knowledge approach. The teaching of rules for transformations of functions by just memorising it, can be seen as an example of a social knowledge approach.

The responses of the purposefully sampled students during the in-depth and focus group interviews show how visual affordances and enablements of GeoGebra afforded the students an opportunity to acquire physical knowledge and logico-mathematical knowledge. The next section is the qualitative analysis of the pre- and post-tests on transformations of functions.

5.3.1.2 Comparative analysis of students' responses in the pre- and post-tests on transformations of functions

The purpose of this section is to investigate how students answered differently in the post-tests and what their learning gains were. The pre- and post-test results presented in this section were purposefully sampled (McMillan & Schumacher, 2006). They were divided into groups based on their incoming NSC Mathematics marks: Group_{<50%} and Group_{≥50%}. Answer sheets of three students from each group, were analysed, because the researcher felt that their responses provided the best information to address the purpose of a specific research question (McMillan & Schumacher, 2001). The purpose was to see the students'

learning gains. The students were given the pseudonyms TF for pre- and post- test transformations of functions.

The students had some prior knowledge of transformations of functions from their school years. The pre-test therefore, was testing what the students could remember from their school Mathematics. The post-test was written after the sessions in the computer lab and they were not informed when they would be writing the post-test. The students therefore did not prepare or study for the post-test and their results represent how they understood transformations of functions immediately after the invention. This was done to establish the students' learning gains after the invention. This section will use qualitative data to explain the learning gains of students after being taught with GeoGebra activities. Consistent accuracy (CA) marking was applied in the scoring rubric (see Appendix A). If a candidate made a mistake in a question and used the incorrect information in the next question, or continued to work in a mathematically correct way, s(he) was marked positively.

Student TF1 (see Figure 5.27) first simplified the parenthesis and then applied the transformation of $-f(x)$. In this question the students were given as $f(x) = 2(x + 3)^2 + 5$.

$f(x) = 2(x + 3)^2 + 5$ is given.
 1.1 Write down:
 (a) the equation of $-f(x)$.
 $= -2(x + 3)^2$
 $= -2(x^2 + 6x + 9)$
 $= -2x^2 - 12x - 18$
 $= x^2 + 6x + 18$ X

Figure 5.27: Student TF1's response to question on equation of $-f(x)$ in the pre-test

The concept tested here was algebraic manipulation of $-f(x)$ and two marks were allocated for this question. The student's first error was to exclude 5 in his expression, but he knew that the change in sign applied to the entire expression as shown by step 4. The student's second error was to divide by -2, because he equated the expression $-2x^2 - 12x - 18$ equal to zero. Consequently the student scored no marks in this question. The student unsuccessfully tried a longer method to solve this problem. The student did not answer question 1.1(c) that asked to explain the transformation in words. The student thus did not understand what the meaning of $-f(x)$ was, but understood partially the notation of $-f(x)$ by multiplying with -1, but did not include 5. Student TF1 entered the SciMathUS programme

with 34% in NSC Mathematics. He scored 9% in the pre-test and 39% in the post-test for transformation in functions.

Figure 5.28 shows how Student TF1 responded after intervention and solved the question with shorter methods although the answer was partially correct.

$$f(x) = 2(x + 3)^2 + 5 \text{ is given.}$$

1.1 Write down:

(a) the equation of $-f(x)$. (2)

$$f(x) = -2(x+3) - 5 \quad \checkmark$$

Figure 5.28: Student TF1's response to question on equation of $-f(x)$ in the post-test

He scored only the mark for -5. He neglected to raise the parenthesis to the power of two. This student also explained in question 1.1c that $-f(x)$ means a reflection in the x -axis. The student's steps to compute the answer in post-test were shorter than his attempt in the pre-test.

Figure 5.29 shows student TF2's answer to question 1.1 in the pre-test.

$$f(x) = 2(x + 3)^2 + 5 \text{ is given.}$$

1.1 Write down:

(a) the equation of $-f(x)$. (2)

$$\begin{aligned} -f(x) &= -[2(x+3)^2 + 5] \quad \checkmark \\ &= -(2(x^2 + 6x + 9) + 5) \\ &= -(2x^2 + 12x + 25) \\ &= -x^2 - 6x - 14 \quad \checkmark \end{aligned}$$

Figure 5.29: Student TF2's response to question on equation of $-f(x)$ in the pre-test

She started by multiplying $f(x)$ with a minus one. She started with the correct approach but then went on to simplify the parenthesis ending up with an incorrect answer. This student also treated the function as an equation and divided by 2. Student TF2 entered the programme with 69% for NSC. The student scored 1 out of 2 marks. This student thus obtained high marks in NSC Mathematics, but could not remember what to do. She scored 20% in the pre-test and 73% in the post-test. It shows that although this student obtained high marks in the NSC Mathematics, she still equated the expression

$-(2x^2 + 12x + 28)$ to zero. From the way she wrote out her answer it can be assumed that this student viewed the function $-f(x) = -(2x^2 + 12x + 28)$ and $-f(x) = -x^2 - 6x - 14$ as the same.

Figure 5.30 shows how student TF2 responded in the pre-test in question 1.1(b) to calculate the turning point as $(-3 ; 5)$. In this question the students were given $f(x) = 2(x + 3)^2 + 5$.

(b) coordinates of the turning point of $-f(x)$. (2)

$$x = \frac{-b}{2a} = \frac{-6}{2(-1)} = \frac{6}{-2} = -3$$

$$y = -(-3)^2 - 6(-3) - 14 = -9 + 18 - 14 = -5$$

\therefore turning point $(-3 ; 5)$

Figure 5.30: Student TF2's response to the question on the turning points of $-f(x)$ in the pre-test

She made a calculation error by adding up incorrectly but continued in question 1.1(c) (see Figure 5.31 below) to give the correct explanation for the transformation. The concept tested here was if the students could read off the turning points coordinates from the format $f(x) = a(x + p)^2 + q$. They could also calculate turning points from $f(x) = ax^2 + bx + c$ with the formula $x = \frac{-b}{2a}$. The researcher was interested if students could see from

$f(x) = 2(x + 3)^2 + 5$ that the turning point is $(-3 ; 5)$ and if asked for $-f(x)$ that the negating of $f(x)$ on $(-3 ; 5)$ is mapped to $(-3 ; -5)$. Student TF2 changed her approach of answering question 1(b) differently in the post-test. She read off the turning point of $-f(x)$ as $(3 ; -5)$. Although her answer was incorrect she was able to find the answer without any computation such as her attempt in question 1(b) in the pre-test.

(c) in words, the explanation of the transformation $-f(x)$. $f(x)$ was reflected in the x -axis. (2)

Figure 5.31: Student TF2's response to type of transformation for $-f(x)$ in pre-test

This student knew that $-f(x)$ means reflection in the x -axis. Although the turning point of $f(x)$ was $(-3 ; 5)$ she calculated the turning point of $-f(x)$ as $(-3 ; 5)$. This shows that this student was able to calculate the turning point and explain the transformation, but could not see that the turning point of $-f(x)$ should be $(-3 ; -5)$. This proves that this student did not fully understand the properties of reflection of a quadratic function and was probably just taught an approach to memorise the reflection from the notation $-f(x)$. This student

response could be the result of rote manipulations of symbols and memorising of formulas and procedures (Faulkenberry & Faulkenberry, 2010). Faulkenberry and Faulkenberry (2010) emphasise that the focus must be on the input and output values when teaching transformations of functions so that students may be aware of why procedures are working. The researcher utilised GeoGebra prepared applets for students to see the effect of transformations on input and output values. In the post-test Student TF2 followed the same steps (see Figure 5.32) as in the pre-test but simplified it without first simplifying the parenthesis.

QUESTION 1

$f(x) = 2(x + 3)^2 + 5$ is given.

1.1 Write down:

(a) the equation of $-f(x)$.

$$-f(x) = - [2(x+3)^2 + 5] \quad (2)$$

$$= -2(x+3)^2 - 5 \quad \checkmark$$

Figure 5.32: Student TF2's response to question on equation of $-f(x)$ in the post-test

The GeoGebra algebraic windows (see Figure 5.33) shows the same method as the one that Student TF1 and TF2 used in their answers.

File Edit View Options Tools Window Help

Algebra

Function

- $f(x) = 2(x + 3)^2 + 5$
- $g(x) = -(2(x + 3)^2 + 5)$
- $h(x) = 2(x + 3 + 3)^2 + 5 - 6$
- $p(x) = 2(2(x + 3)^2 + 5)$

Figure 5.33: GeoGebra algebra window showing the algebraic manipulation

The way that the students answered this question in the post-test shows how GeoGebra helps for coherence formation on deeper structure level of the notation $-f(x)$ (Seufert & Brünken, 2004). Student TF2 was also Student FG7 in the focus group interviews. She indicated that GeoGebra helped because it visualized concepts and assisted her to remember better.

During the intervention, students used GeoGebra for five hours for the transformations of functions. During that time, they explored the reflection, translation, rotation and dilation

properties of polygons before embarking on the transformations of functions in GeoGebra activities that were designed by the researcher. They also explored transformations of functions using GeoGebra applets created by the teacher-researcher and by creating their own activities in GeoGebra.

Student TF3 (see Figure 5.34) showed no understanding in the pre-test of how to find the equation, but he knew that the transformation was a translation. $f(x)$ was the same for all the sub-questions in question 1.

1.2 Skryf neer: (a) die vergelyking van $f(x + 3) - 6$. $f(x - 3) - 6 = 0$	1.2 Write down: (a) the equation of $f(x + 3) - 6$. (2)
(b) koördinate van die nuwe draaipunt van $f(x + 3) - 6$. $(-3; -6)$	(b) coordinates of the new turning point of $f(x + 3) - 6$. (2)
(c) in woorde, die die verduideliking van transformasie $f(x + 3) - 6$. This is a translation	(c) in words, the explanation of the transformation $f(x + 3) - 6$. (3) 1-unit down

Figure 5.34: Student TF3's response to question on $f(x + 3) - 6$ in the pre-test

Question 1.2(a) tested if students could do the algebraic manipulation of $f(x + 3) - 6$. Two marks were allocated for this question 1(a) and (b) and three marks for 1(c). Student TF3 scored only one mark for he knew $f(x + 3) - 6$ means a translation. Figure 5.35 shows Student TF3's response to question 1.2 in the post-test.

1.2 Write down: (a) the equation of $f(x + 3) - 6$. (2) $2(x + 3 + 3)^2 + 5 - 6$ $2(x + 6) - 1$	
(b) coordinates of the new turning point of $f(x + 3) - 6$. (2) $(-6; -1)$ CA	
(c) in words, the explanation of the transformation $f(x + 3) - 6$. (3) 3 units to the right and 6 down	

Figure 5.35: Student TF3's response to question $f(x + 3) - 6$ in the post-test

He showed that he understood the vertical translation, but still experienced problems with the horizontal translation. The student was noticeably confused with the horizontal

translation as evident in part 1.2c of Figure 5.35. He presented the horizontal translation as three units to the right and scored only one mark out of three for six units down. He interpreted $f(x + 3)$ as horizontal transformation to the right. Faulkenberry and Faulkenberry (2010) contend that students are sometimes given a procedure to apply and they suggest that transformations of functions with input and output-values be taught. The students were exposed, with GeoGebra, to the same approach of the input and output values to see that $f(x + 3)$ means that $(4 ; -3)$ is mapped to a point three units to the left (see Figure 5.36) , i.e. $(1;-3)$.

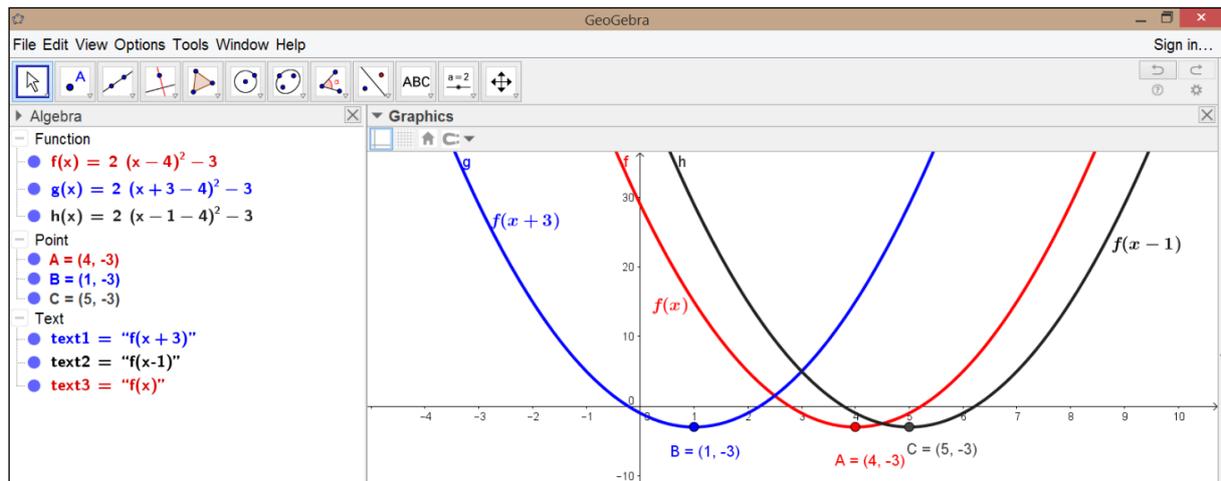


Figure 5.36: Meaning of notation of $f(x + 3)$

This shows that Student TF3 was possibly trying to recall the procedures, but was confused about whether the graph shifts left or right. Student FG8 indicated in the interviews that his knowledge was still influenced by how he was taught at school. Student ID6 and ID9 (see p.209) also mentioned that they were just taught at school to memorise how to write it, or the formulas, down. This can still be the result of rote manipulations of symbols and memorising of formulas and procedures (Faulkenberry & Faulkenberry, 2010). It was thus taught as social knowledge. The unlearning of memorisation of methods taught in school was one of the challenges that the teacher-researcher experienced with SciMathUS students during this study.

Student TF4 entered the programme with 62% in NSC Mathematics and showed in the pre-test in question 1.3a (see Figure 5.37) that he knew that he had to multiply $f(x)$ by 2, but also multiplied the three in the parenthesis by 2. $f(x)$ was given as

$$f(x) = 2(x + 3)^2 + 5.$$

1.3 Skryf neer:	1.3 Write down:
(a) die vergelyking van $2f(x)$.	(a) the equation of $2f(x)$.
	$f(x) = 4(x+6)^2 + 10$
(b) koördinate van die nuwe draaipunt van $2f(x)$.	(b) coordinates of the new turning point of $2f(x)$.
	$(-6; 10)$
(c) in woorde, die verduideliking van die transformasie $2f(x)$.	(c) in words, the explanation of the transformation $2f(x)$.
The graph is shifted vertically upwards by double and horizontally left by double.	

Figure 5.37: Student TF4's response to the question on $2f(x)$ in the pre-test

The student indicated that the turning point of $2f(x)$ is $(-6 ; 10)$. This was marked correctly, because the student read it correctly from the incorrect answer in question 1.3b. The student showed he knew how to compute the coordinates of the turning points from the format $f(x) = a(x + p)^2 + q$. Figure 5.38 shows that Student TF4 understood to some extent how to do the algebraic manipulation, but did not show any conceptual understanding of the vertical stretch of the function. He scored 33% in questions 1.3a, 1.3b, 1.3c in the pre-test, but managed to score 83% for the same questions in post-test (Figure 5.38).

1.3 Write down:	
(a) the equation of $2f(x)$.	(2)
	$2f(x) = 4(x+3)^2 + 10$
(b) coordinates of the new turning point of $2f(x)$.	(2)
	$(3; 10)$
(c) in words, the explanation of the transformation $2f(x)$	(2)
The graph is enlarged by scale factor of 2.	

Figure 5.38: Student TF4's responses to question on $2f(x)$ in the post-test

Figure 5.38 shows Student TF4's responses to question 1.3 in the post-test. He answered question 1.3 almost correctly. The only mistake was the x -coordinate of the turning point that he said was 3 and not -3. This may have been a careless mistake because he gave the coordinates for the same question in the pre-test (see Figure 5.37) correct. The way that the students answered this question in the post-test shows how GeoGebra helps for coherence

formation on deeper structure level of the notation $2f(x)$ (Seufert & Brünken, 2004). During the intervention with GeoGebra, the students could drag a point and then find the new equation of the transformed graph. They could link the algebraic and graphic representations. The responses of some students in the in-depth and focus group interviews show how GeoGebra (see section 5.3.1.1) helped them to understand transformations on functions. The responses of the students suggested that the visual affordances and enablements of GeoGebra help them see the link between the notation of $-f(x)$, $f(x + a)$, etc. and the graphic representation of the notation. The activities in Appendix J were designed for them to visualise graphic representation with the algebraic notation and to understand the transformations in words from notation format.

Figure 5.39 shows the same student's response in the pre-test to sketch the graph of $-2f(x)$ if $f(x) = \cos x$. It was expected of the student to sketch $f(x + 30^\circ)$ and $-2f(x)$ from the properties of transformations without using a calculator.

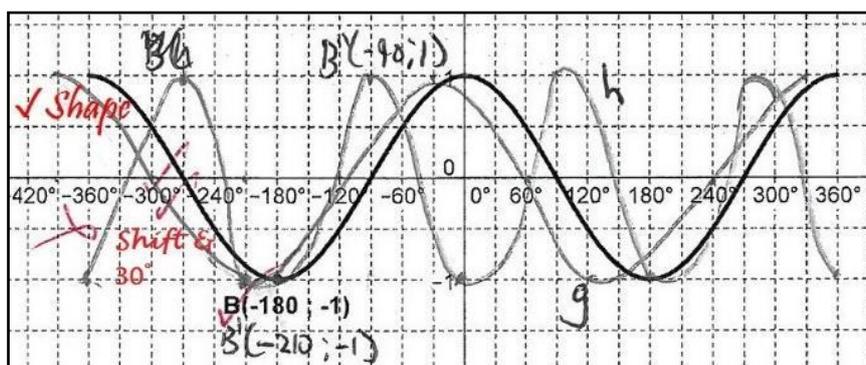


Figure 5.39: Student TF4's response to sketching of $f(x + 30^\circ)$ and $-2f(x)$ in the pre-test. The h graph (see Figure 5.39) shows that this student did not understand the notation $-2f(x)$. He knew that the y -values must be multiplied by two in question 1.3a (see Figure 5.38), but did not do it in question 2b (see Figure 5.39). This suggested that he remembered how to determine the equation of the transformed function, but did not understand what the transformation meant. He understood the notation of $f(x + 30^\circ)$ if $f(x) = \cos x$.

Figure 5.40 shows Student TF4 scored full marks in question 2a in the post-test. He showed a better understanding of question 2b that required sketching the graph of $-2f(x)$ if $f(x) = \cos x$.

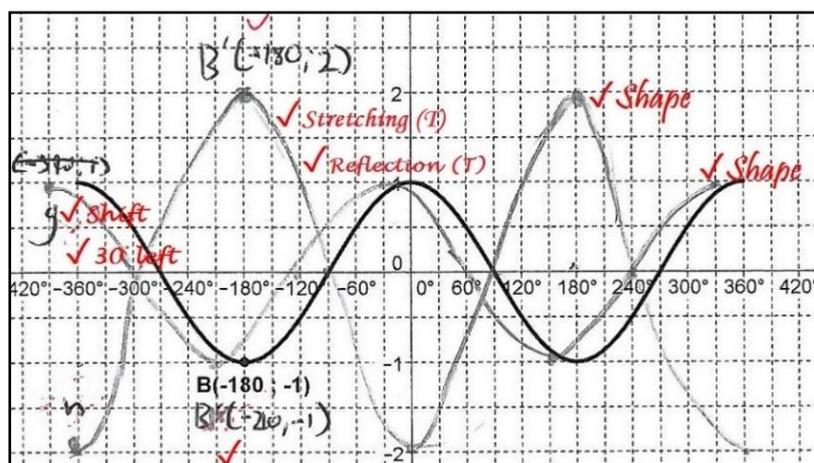


Figure 5.40: Student TF4's response to sketching of $f(x + 30^\circ)$ and $-2f(x)$ in the post-test

This shows that Student TF4 understood the meaning of the transformation from the notation $-2f(x)$. Trigonometry had not yet been taught when they wrote the pre- and post-test of the study. Trigonometry was only covered after the interventions. This response in the post-test thus shows that Student TF4 understood the properties of transformations, because he could not use a calculator to sketch the cosine graph or find values to plot, nor did he receive any formal teaching on how to sketch the graph of a trigonometry function in SciMathUS. It was expected from the student to sketch $f(x + 30^\circ)$ and $-2f(x)$ from the properties of transformations. The student could sketch this graph after intervention with GeoGebra. The student could therefore apply his knowledge of what he experienced with GeoGebra to answer the trigonometric question without the use of a calculator, or formally taught how to sketch a trigonometric graph in the SciMathUS programme. The way that the student answer this question in the post-test shows how GeoGebra helps for coherence formation on deeper structure level of the notation $-2f(x)$ (Seufert & Brünken, 2004). Students explored with GeoGebra that the notation $af(x)$, where $a \in \mathbb{R}$ means the same transformations for any function. The student could therefore link the type of transformation to the notation. He acquired in this way logico-mathematical knowledge. Student TF4 was also Student FG1 in the focus group interviews. During the focus group interviews he shared the knowledge that he used GeoGebra frequently to draw graphs on his mobile phone. It helped him to visualise the graphs. He consequently acquired physical knowledge with GeoGebra app on his mobile.

Further analysis of question 2a and 2b of all the students' pre- and post-test results shows the following:

Question 2a:

Twenty-six (54%) and 38 (79%) of the 48 students in pre-and post-test, respectively, computed that the shape of $f(x)$ and $f(x + 30^\circ)$ were the same, i.e., rigid transformations. Eight (17%) and 21 (44%) of the 48 students in pre-and post-test, respectively, computed the correct translation of $f(x + 30^\circ)$. Nineteen (19%) and 31 (65%) of the 48 students in pre-and post-test, respectively, computed the correct coordinates of point B of $f(x + 30^\circ)$.

Question 2b:

Twelve (25%) and 31 (65%) of the 48 students in pre-and post-test, respectively, knew that the shapes of $f(x)$ and $-2f(x)$ change. Nine (19%) and 19 (40%) of the 48 students in pre-and post-test, respectively, knew the correct transformation of reflection and stretching of $-2f(x)$. Eighteen (25%) and 28 (58%) of the 48 students in pre-and post-test, respectively, computed the correct coordinates of point B of $-2f(x)$.

Figure 5.41 shows the response of Student TF5 to question 1.4. In this question $f(x)$ is given as $f(x) = 2(x + 3)^2 + 5$. Question 1.4a required the students to find the coordinates for two marks for $f^{-1}(x)$ without any algebraic computations. The DBE (2011a) only requires the students to work with $f(x) = ax^2$. The format of $f(x) = a(x + p)^2 + q$, assessed in the pre- and post-test, was therefore to test students' understanding of transformations without any algebraic manipulations. Question 1.4b (two marks) was merely to test if students showed understanding of the inverse as equivalent to a reflection in the line $y = x$.

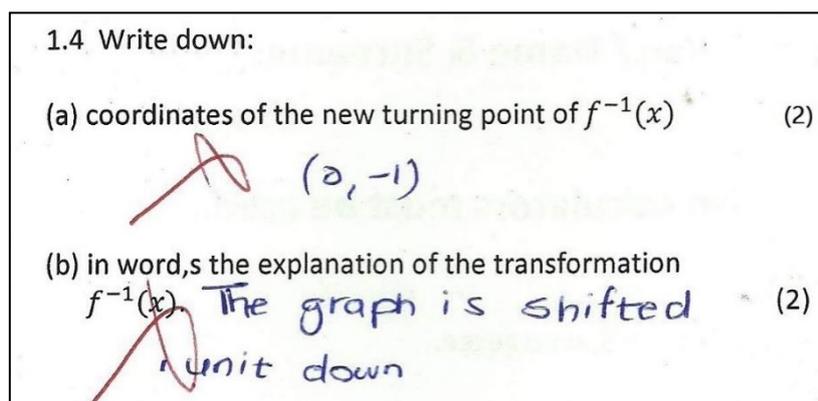


Figure 5.41: Student TF5's response to the inverse of $f(x)$ in the pre-test

Question 1.4a expected the students to write the coordinates of the turning point of $f^{-1}(x)$ if $f(x) = 2(x + 3)^2 + 5$ and question 1.4b to explain the transformation of $f^{-1}(x)$. Student TF5's responses show that she did not have an understanding of the inverse of $f(x)$.

Figure 5.42 shows Student TF5's response to the same question in the post-test.

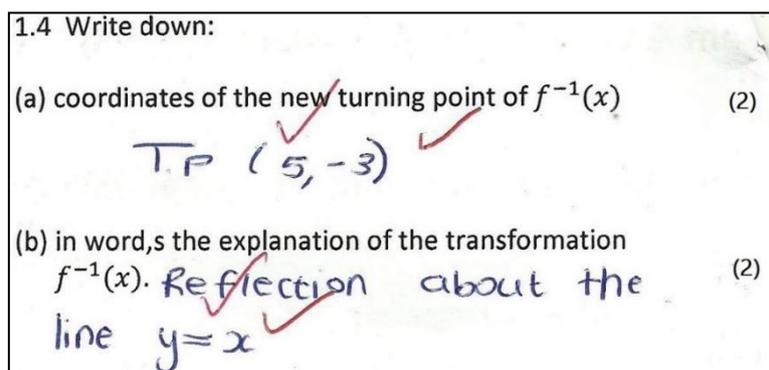


Figure 5.42: Student TF5's response to the inverse of $f(x)$ in the post-test

Student TF5 showed understanding of how to find the coordinates of the turning point of the inverse of $f(x) = 2(x + 3)^2 + 5$. The student took $(-3 ; 5)$ and re-wrote it as $(5 ; -3)$. The researcher was seeking this type of response because he wished students to read the turning points of the inverse from the turning point of $f(x)$. The way that the students answer this question in the post-test shows how GeoGebra helps for coherence formation on deeper structure level of the notation $f^{-1}(x)$ (Seufert & Brünken, 2004). Responses from students such as the student in section 5.2.3.1.1 (see Figure 5.11) and Student FG16 (see section 5.3.1.1) show how students could see in GeoGebra that the inverse graph is a mirror image of the given function. During the intervention with GeoGebra, the students could drag a point and with their own algebraic computation find the new equation of the transformed graph and type in the equations in GeoGebra. They could link the algebraic and graphic representations.

Forty-four (92%) and 33 (69%) students scored no marks in the pre-test in question 1.4a and 1.4b respectively, while only 23 (48%) and 10 (21%) students scored no marks for the questions in post-test. Most of the students did not score high marks in this question in the pre-test. In the pre-test students scored for question 1.4a an average of 5%; for question 1.4b an average of 19% and for question 4c an average of 27%. This changed drastically in the post-test to 42%, 68% and 75% respectively. In the pre-test, many students left out the answers, or were supplying answers such as the one in Figure 5.41.

After students had been taught the inverse functions these results showed that many of the students still did not know that the graph of an inverse function is geometrically a reflection in $y = x$. The diagnostic reports of the DBE (2013; 2014; 2015c) show that the candidates

in examinations understood the algorithm of deriving the inverse equation, but still struggled with concepts of domain and range, and the asymptotic behaviour of functions and interpretations of the inverse graph.

Figure 5.43 shows Student TF5's response to the question on the asymptotic behaviour of the exponential of functions.

QUESTION 4

Given:

$$f(x) = (5)^x$$

(a) Give the equation of the asymptote of $f(x)$ if reflected in the y -axis.

$y = 0$

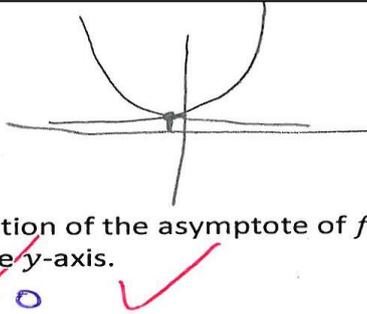


Figure 5.43: Student TF5's response to equation of asymptote of $-f(x)$ in the post-test

Figure 5.43 shows Student TF5's response to question 4a in the post-test. The student had to reflect $f(x) = 5^x$ in the x -axis and then had to find the equation of the horizontal asymptote of $-f(x)$. Another student could also find the equation of the horizontal asymptote of $f(x)$ and reflect the equation in the x -axis. She scored full marks in question 4a in both the pre-test and post-test. Student TF5 was also the same student (Student ID9) that indicated earlier (see section 5.3.1.1) how GeoGebra helped with visualization of reflection in the y -axis and that it assisted her to understand the concept better. This student used a technique, or heuristic, to solve this problem. She sketched the graph of $f(x)$ and the graph of $f(x)$ when reflected the in y -axis. The sketching shows that the student understood the reflection of the function in the y -axis. This student showed she did not apply rote manipulations of symbols and memorising of formulas and procedures (Faulkenberry & Faulkenberry, 2010), but conceptual understanding of reflections of functions. The student also acquired the understanding of this concept through physical knowledge with GeoGebra (see section 5.3.1.1) and to answer Question 4(a) it required that the student reason in an abstract manner that was beyond the physical interaction with GeoGebra. The student thus acquired logico-mathematical knowledge in this way (Lutz & Huitt, 2004). The students indicated in the in-depth and focus group interviews (see section 5.3.1.1) how GeoGebra helped them understand transformations of functions better.

Figure 5.44 shows Student TF6's response to the hyperbolic functions in the pre-test.

QUESTION 3

Given:

$$f(x) = \frac{8}{x-2} + 3$$

(a) Give the equations of the asymptotes of $f(x)$ reflected in the x -axis. (2)

$x = 2$ ✓

(b) Give the equation of the graph if $f(x)$ is translated 3 units to right and 5 units down. (2)

$$f(x) = \frac{8}{x+3} + 6$$

(c) Describe, in words, the transformation of f to g if $g(x) = \frac{-8}{x+1}$ (3)

I + was shifted

Figure 5.44: Student TF6's response to the hyperbola in pre-test
 Student TF6 only gave the equation of the horizontal asymptote of $f(x) = \frac{8}{x-2} + 3$ reflected in the x -axis.

Figure 5.45 shows student TF6's response to the hyperbolic functions in the post-test.

QUESTION 3

Given:

$$f(x) = \frac{8}{x-2} + 3$$

(a) Give the equations of the asymptotes of $f(x)$ reflected in the x -axis. (2)

$x = 2$ ✓
 $y = -3$ ✓

(b) Give the equation of the graph if $f(x)$ is translated 3 units to right and 5 units down. (2)

$$f(x) = \frac{8}{x-3-2} + 3 - 5$$

$$= \frac{8}{x-5} - 2$$

(c) Describe, in words, the transformation of f to g if $g(x) = \frac{-8}{x+1}$ (3)

A reflection in the y -axis then a translation 1 unit to right and 3 units down.

Figure 5.45: Student TF6's response to hyperbola in the post-test

Student TF6 scored 14% in the pre-test in question 3 but 100% in the post-test. She also sketched graphs on the right hand of her answer sheet (see Figure 5.45). The first sketch is for the graph of $f(x)$ and the second one is for the graph of $f(x)$ when reflected in x -axis.

This student used a technique, or heuristic, to solve this problem. This shows, as discussed in Chapter 2 (see section 2.6), the student's deeper understanding of transformations of functions. Instead of doing the problem algebraically, she did it graphically. This confirms how the teacher-researcher's activities with GeoGebra in the computer lab helped with coherence formation on a deep structure level (Seufert & Brünken, 2004). Student TF6 was also Student FG20 (see section 5.2.2.2) that indicated that GeoGebra made concepts more illustrative, i.e. the visual affordance of GeoGebra. The fact was that she could see the objects moved and pointed out that it was better to do it on GeoGebra than on paper. The students were asked to give the equations of the asymptotes from the transformed function. Student TF6 did not sketch any graphs in her pre-test (see Figure 5.44), but did so in her post-test. She understood the transformation that was asked and sketched the graphs to see where the graph and its asymptotes lie. Although her second graph did not present the reflection in the x -axis, she produced correct answers. The approach that she used to reach her answer is what was important. She showed by this approach that she understood the properties of reflection. This student came into the programme with 45% for NSC Mathematics and obtained 24% in the pre-test but attained 78% in the post-test.

All the abovementioned qualitative data analyses in this sub-section show the learning gains of the purposefully sampled students in the pre- and post-tests on transformations of functions. This analysis shows how these students understood transformation of functions in the pre-test and how they altered their solutions in the post-test. Some of their answers reflected how GeoGebra could have influenced their attempts. Examples of this were how Student TF1 and TF2 presented their answers with shorter steps in the post-test (see Figure 5.28 and 5.32) than the pre-tests (see Figure 5.27 and 5.29). The activities in Appendix J with GeoGebra required the students to understand that if a function's equation is given in the format $f(x) = a(x + p)^2 + q$ there is no need to re-write it in the format $f(x) = ax^2 + bx + c$ is to determine the coordinates of the turning point of the transformed graph. The algebraic and graphic window of GeoGebra helped to visualise this. Examples of this were responses of the interviews of students (see Figure 5.24, 5.25 and 5.26) and how students answered the questions in similar manner as in GeoGebra (see Figure 5.32 and 5.33).

The activities in Appendix J (no. 5 – 7) required the students to sketch the graphs of the trigonometric functions with GeoGebra. The students at this stage had not been taught the

sketching of trigonometric functions. The activities focused on the transformations of all the functions in the CAPS (DBE, 2011a). This activity was designed to help the students to understand that all transformations are the same for any graph and to understand that reflection and translation mean the same graph. However, on a different position and dilation it means the same graph but it stretches or shrinks. More students were able to sketch the cosine function (see Question 2a and 2b) by just understanding the concept of translation, reflection and dilation. The responses (see section 5.3.1.1) from the students in the in-depth (Student ID1 and ID6) and focus group (Student FG9 and FG16) interviews and, feedback from students (Student AT1 and AT2), show how students referred to the visual affordances and enablements of GeoGebra. The pre- and post-tests results also show more students were able to answer Question 2a and 2b correctly. These responses also show how students acquired physical and logico-mathematical knowledge through the designed activities with GeoGebra. The students' explanations show how GeoGebra afforded them an opportunity to understand mathematical concepts and move to higher levels of abstraction.

The next sub-section will focus on the pre- and post-test results of all the students in the intervention. The abovementioned qualitative data analysis was only on purposefully sampled students in the pre- and post-tests on transformations of functions. It will investigate if there were significant learning gains for all the students in transformations of functions.

5.3.1.3 Quantitative analysis of pre- and post-test in transformations of functions

The teaching of transformations of functions requires students firstly to understand what happens to functions, or graphs, when given the transformation in words. Secondly, to understand what happens to functions or graphs when given the transformations in algebraic notation, e.g., $(x - 2) - 3$. Thirdly, to determine the equation of a transformed function when given the notation of the transformation; fourthly sketching the graph of the function when given the transformation in notation format, or in words, and lastly to determine the critical points of the functions, or graphs, when given the transformation in words or in notation format. The pre- and post-test were consequently set in this way to assess the participants' understanding of transformations of functions. The following type of questions were in both the pre- and post-test and divided into different domains. The domains represent the different types of questions students should know in the transformations of functions:

Domain 1 (D1): Determining the equation of a transformed function from the given algebraic notations;

Domain 2 (D2): Determining the critical values (turning points, asymptotes, etc.) of the transformed function from the given algebraic notations;

Domain 3 (D3): Explanation of the transformation in words from the given algebraic notations; and

Domain 4 (D4): Sketching the transformed graph of the cosine graph from the given algebraic notations.

A paired t-test was also conducted to test the significance of the differences between the students' means of the pre- and post-test in the different domains. The qualitative analysis of the pre- and post-test was only for a purposeful sample. The paired t-test was conducted to show that there were not just learning gains for the purposeful sample of students, but that there were significant learning gains for all the students in the study. The t-test sought to test the following hypothesis:

H_0 : There is no difference between the pre-test mean and the post-test mean of transformations of functions for the different domains for the students in the study.

$$H_0: \mu_{\text{pre-test}} = \mu_{\text{post-test}}$$

H_1 : There is a difference between the pre-test mean and post-test mean of transformations of functions for the different domains for the students in the study.

$$H_1: \mu_{\text{pre-test}} \neq \mu_{\text{post-test}}$$

Table 5.3 shows a paired t-test to test the significance of the differences between the students' means of the pre- and post-test in the different domains.

Table 5.3: Paired sample t-test shows the students' means of the pre- and post-test in the different domains

	Paired Differences					<i>t</i>	df	Sig. (2-tailed)	Cohen's <i>d</i>	Effect size
	Mean difference	SD	SE	95% Confidence Interval of the Difference						
				Lower	Upper					
Pre- & post-domain1	33.333	20.456	2.953	27.394	39.273	11.290	47	.000	1.69	Very large
Pre- & post-domain2	36.625	17.663	2.549	31.496	41.754	14.366	47	.000	2.05	Very large
Pre- & post-domain3	42.521	25.017	3.611	35.257	49.785	11.776	47	.000	1.95	Very large
Pre- & post-domain4	27.417	25.618	3.698	19.978	34.855	7.415	47	.000	0.95	Large
Domains combined	36.542	13.350	1.927	40.418	32.665	18.965	47	.000	2.33	Very large

Table 5.3 shows all the p -values (0.000) are smaller than the α -value (0.05). All the null hypotheses (H_0) which stated that there is no significant difference between pre-test and post-test scores of students in the different domains for transformations of functions were thus rejected in favour of the alternative hypothesis. It is observed that in all the different domains presented above, the means for pre- and post-test were statistically significantly in favour of the post-test.

The effect sizes very large for three domains, namely 1, 2, 3 (Ellis, 2009). The effect size of domain 4 is large. Although there was no comparison group for this particular task, the results show that the students improved their understanding of transformations of functions. The mean differences for all the domains (see Table 5.3.) also show that the students' understanding of higher order level questions in transformations of functions improved. The national diagnostic reports of the DBE (2013; 2014; 2015c) show that students lack the understanding of the behaviour of functions. Questions as in Domain 2 and 3 show a greater improvement than the questions in Domain 1 and 4. Although Domain 4 showed the lowest improvement it was still very promising to see students improving their understanding of the sketching of the transformed graph of trigonometric functions without being taught formally, by the teacher-researcher, the sketching of trigonometric functions. The students were up to this stage not formally taught the sketching of any functions by the teacher-researcher and were only exposed to the properties of the transformations of all functions in the CAPS curriculum. Watson (2006) did a study with low-achieving students without any comparison group and concluded that the progress of the students can be the result of the teaching, or

is something that happens anyway due to maturation. This study did not investigate if the teaching and learning with GeoGebra is better than an approach without it, but showed that it can help with the understanding of transformations of functions.

This quantitative (quan) analysis can be triangulated with the students' perceptions of how GeoGebra helped them with transformation geometry and transformations of functions (see section 5.3.3.2), the qualitative data (QUAL) analysis of in-depth and focus group interviews (see section 5.3.1.1) and the qualitative (QUAL) data analysis of a purposefully sampled students' pre-and post-tests on transformations of functions (see section 5.3.1.2). All the qualitative responses from the students show that GeoGebra illustrated how they thought about transformations of functions and how they would approach questions on it. The quantitative results of the pre and post-test corroborated claims the students made during the pre- and post-questionnaires and interviews. The triangulation of the analysis of the qualitative (QUAL) and quantitative data (quan) show how students indicated that GeoGebra helped them with visualizations of transformations of functions. It also showed the students' learning gains in transformations of functions. The next sub-section presents the students' performance in circle geometry.

5.3.2 Analysis of data on circle geometry

The questions assessed in the pre- and post-test were mostly on knowledge and at a routine level (DBE, 2015a). The post-test was written during phase 3 (see HLT for the teaching circle geometry) and students were not informed when they would be writing the post-test. This was done to see the students' learning gains from the invention. The pre- and post-tests were marked with a scoring rubric (see Appendix B). The scoring rubric for this study was set in the same way as those of the DBE's NSC examination. The students' tests were assessed with consistent accuracy (CA). This means that if a student made a mistake in a question and then used the incorrect information in the next question it was marked positively (see Appendix B). The first six questions were of the two-tier multiple-choice type. The respondents had to select an answer first and then give reasons.

Students were tested on two different occasions on circle geometry. The purpose of this section is to present and analyse how students performed before and after being exposed to teaching with GeoGebra. The section starts with the content analysis of purposefully sampled students' pre- and post-test scripts, followed by the students' responses during the in-depth and focus group interviews, and concludes with pre-and post-tests statistics. A

paired t-test was conducted within SPSS to test the significance of the differences between the participants' means of the pre- and post-test.

5.3.2.1 Students' responses during the in-depth and focus group interviews on how GeoGebra afforded them an opportunity to understand circle geometry

The researcher used purposeful sampling to select specific students' responses that were informative about the phenomena investigated (McMillan & Schumacher, 2006). This qualitative data will be triangulated with quantitative data of students' responses from the post-questionnaire, and the pre- and post-tests, for circle geometry. These responses will be triangulated to validate the data mentioned to answer the sub-question on how GeoGebra afforded them an opportunity to better understand concepts and move to higher levels of abstraction. Some students responded as follows on how GeoGebra helped them to understand circle geometry:

Student ID2: With all the different theorems, you could see which angles are equal to each other or different segments. I could see them [*visual affordances*].

Student ID7: It helped. Especially to see them visually [*visual affordances*]. Like the chords, the angle subtended by the same chord to show that they are equal.

Student FG13: I did not do circle geometry last year. When you started with it this year, I could see on GeoGebra that this angle is equal to that and because of that. I normally learn while visualising it. When I see something I understand and then I can do it. GeoGebra made circle geometry very understandable and much easier. Maybe if we had started without GeoGebra, I don't think I would understand it.

Student FG21: Ek het dit geniet om te sien hoe die hoeke verander as 'n mens 'n koord op die omtrek van die sirkel beweeg en te sien dat twee hoeke wat deur dieselfde koord onderspan word, gelyk is. (*I enjoyed seeing how the angles change if you move a chord on the circumference of the circle and saw two angles subtended by the same chord are then equal.*)

The abovementioned students' responses are proof that GeoGebra helped them to visualise the theorems and corollaries. They emphasised that GeoGebra also made it easier for them to understand the theorems and the corollaries of circle geometry. Students' ID21, FG13 and FG21 responses on the chords, or equal angles, refer to how these students analysed the figures in terms of their parts and the relationships between these parts (Van Hiele, 1986). This shows how these three students moved to Van Hiele level 2.

The students (see Figure 5.46) used colour while they themselves were working on GeoGebra and they also used colour when they were doing the riders in their tests and examinations.

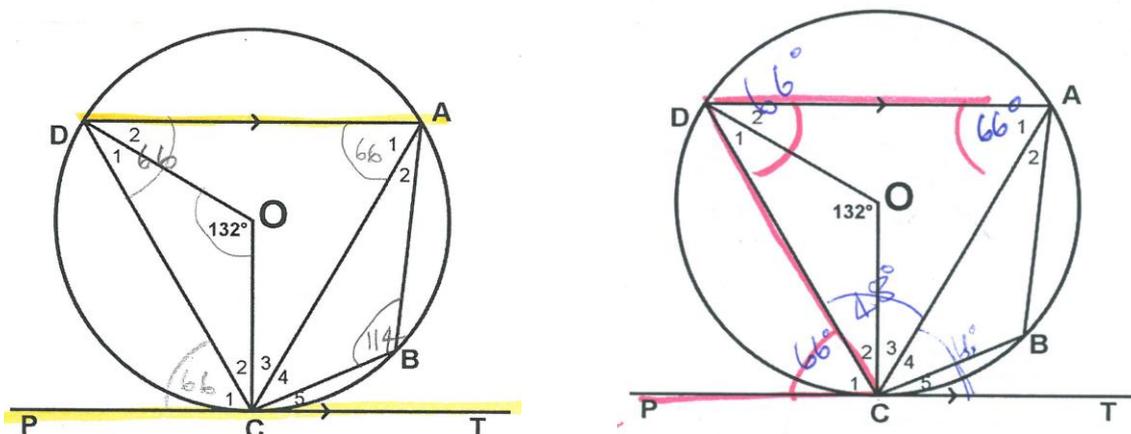


Figure 5.46: Students' answer sheets.

This visual affordance of GeoGebra helped the students with concept formation, or instrumental genesis, and assisted learners to progress to higher Van Hiele levels of understanding. This also enabled students to visualise the given information in geometry problems and helped them analyse the riders.

5.3.2.2 Comparative analysis comparison of students' responses in the pre- and post-tests in circle geometry

This section will use qualitative data to explain the learning gains of students after being taught with GeoGebra activities. Consistent accuracy (CA) marking was applied in the scoring rubric (see Appendix B). When a candidate made a mistake in a question and used the incorrect information in the next question, or continued to work in a mathematically correct way, s(he) was marked positively.

The researcher assessed, as suggested by Mason (1998), the students' Van Hiele level by analysing the students' responses to specific geometric tasks – in this case the pre- and post-test for circle geometry. The students were assessed with a rubric, as discussed in Chapter 2 (see section 2.5).

Forty-two of the students did not do circle geometry in Grade 11 and the researcher was aware of the fact that they would score low marks in the pre-test. The tests consisted of six multiple questions and two questions in which they had to find unknown angles. The

researcher decided to allocate marks stating reasons (see Appendix B) for the multiple choice questions' answers. These questions required the students to make deductions, understand logical deductions between consecutive steps (Lim, 1992) and they had to justify their answers with reasons. The questions in the tests were thus on Van Hiele level 3. When they were writing the pre-test, they worked through the test paper as if they knew something about circle geometry. The researcher was actually most amazed at the effort the students came up with, and it was interesting to see them answering the pre-test. Some of them just ticked answers at the six multiple choice type of questions and just gave up on the last two questions where they had to write out their steps to solve the riders. Some students recognised some of the theorems, axioms and corollaries that they knew from their Grade 8 to 10 Euclidean Geometry (DBE, 2011a; 2011c). The Curriculum Assessment Policy Statement (CAPS) (DBE, 2011a; 2011c) refers to Grade 8 to 10 Euclidean Geometry as the geometry of straight-lines and geometry of 2D shapes. A student had to give all the reasons that he/she used to explain his/her answer (see Appendix B). No steps or calculations were required in the multiple-choice questions. The qualitative analysis of the pre- and post-test were only for a purposeful sample and students were not interviewed on their responses. The students were given pseudonyms CG for the pre- and post- test on circle geometry.

Figure 5.47 shows Student CG1's response to question 1.1 in the pre-test. She entered the SciMathUS programme with 45% for NSC Mathematics and had not done any circle geometry in Grade 11.

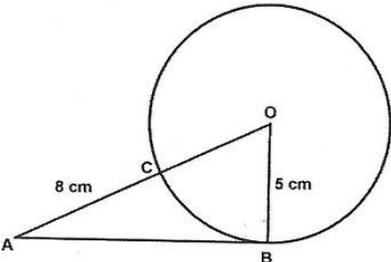
<p>1.1 Punt A is 8 cm vanaf die omtrek van die sirkel O, met radius 5 cm. Wat is die lengte van raaklyn AB?</p>	<p>1.1 Point A is 8 cm from the circumference of circle O, with radius 5 cm. What is the length of tangent AB? (4)</p>
<p>(a) 13 cm (b) $\sqrt{194}$ cm (c) 12 cm (d) $\sqrt{89}$ cm</p>	
<p><i>phythagoras theorem</i></p>	

Figure 5.47: Student CG1's response to rider on the radius and tangent in the pre-test

This student assumed that $\hat{B} = 90^\circ$, because it looks like $OB \perp AB$ and that is why she used the theorem of Pythagoras. She judged the sketch by its appearance (Van Hiele, 1999) and compared angles visually to determine size (Hoiberg & Sharp, 2001), i.e., Van Hiele level 1

for circle geometry and assumed $OB \perp AB$. Any new concept to students is also viewed as Van Hiele level 1 (Lim, 1992). The new concept of circle geometry was the theorem that states that the tangent to a circle is perpendicular to the radius or diameter of the circle at the point of contact. She used a correct reason (Pythagoras theorem) to determine the length of AB, but arrived at an incorrect answer. She calculated

$AB = \sqrt{8^2 + 5^2}$ incorrectly, instead of $\sqrt{(8+5)^2 - 5^2}$. This student operates for Grade 8 – 10 Euclidean Geometry (DBE, 2011a; 2011c) on Van Hiele level 2, she used the theorem of Pythagoras, although incorrectly, and did not mention that $OC = OB = 5$ cm.

Figure 5.48 shows that Student CG1 used the tangent and chord theorem in circle geometry to support the theorem of Pythagoras to answer question 1.1 in the post-test.

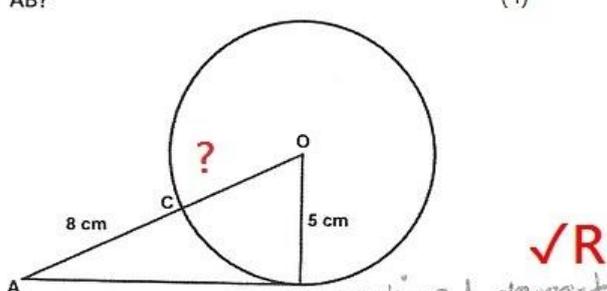
<p>Vier moontlike antwoorde is voorsien in vrae 1.1 – 1.6. Kies die korrekte een. Gee redes vir jou antwoord.</p> <p>1.1 Punt A is 8 cm vanaf die omtrek van die sirkel O, met radius 5 cm. Wat is die lengte van raaklyn AB?</p> <p>(a) 13 cm (b) $\sqrt{194}$ cm (c) 12 cm (d) $\sqrt{89}$ cm</p> <p><i>pythagorus (OB ⊥ AB)</i></p>	<p>Four possible answers have been supplied in questions 1.1. – 1.6. Choose the correct one. Give reasons for your answer.</p> <p>1.1 Point A is 8 cm from the circumference of circle O, with radius 5 cm. What is the length of tangent AB? (4)</p> 
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Figure 5.48: Student CG1's response to rider on the radius and tangent in the post-test

In the pre-test Student CG1 used the theorem of Pythagoras without proving $\hat{B} = 90^\circ$ (see Figure 5.47). In the post-test this student responded as shown in Figure 5.48. The student gained the knowledge that $OB \perp AB$ from the GeoGebra activities. In the post-test, this student moved up to Van Hiele (1986) level 3. According to Van Hiele (1986) students start using mathematical reasons to justify arguments at Level 3. Student CG1 could still neither show that $OC = 5$ cm, nor provide sufficient reasons to conclude her answer. She was only able to follow simple deductions between consecutive steps (Lim, 1992).

Figure 5.49 shows the response of Student CG2 to question 1.2 in the pre-test. He entered the SciMathUS programme with 43% for NSC Mathematics and also had not done circle geometry in Grade 11.

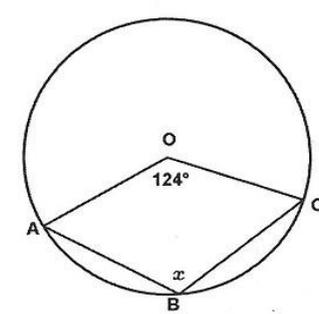
<p>1.2 O is die middelpunt van die sirkel ABC en $\widehat{AOC} = 124^\circ$. Die grootte van $x = \dots$</p> <p>(a) 56°</p> <p>(b) 124° (vertoppel)</p> <p>(c) 118°</p> <p>(d) 236°</p>	<p>1.2 O is the centre of the circle and $\widehat{AOC} = 124^\circ$. The size of $x = \dots$ (3)</p> 
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Figure 5.49: Student CG2's response to rider angle at centre theorem in the pre-test

This student claimed that $\widehat{ABC} = 124^\circ$ and gave the reason: vertically opposite angles. Both the answer and reason were incorrect. Students are from the teacher-researcher's experience sometimes confused with opposite, vertically opposite and corresponding angles. The assumption of the researcher was that this student possibly assumed that the opposite angles of a quadrilateral are equal. The student could also compare \widehat{ABC} with \widehat{AOC} visually (Hoiberg & Sharp, 2001), or judge the sketch by its appearance (Van Hiele, 1999). Hence, this student operated on Van Hiele level 1.

Figure 5.50 shows the evidence of Student CG2's learning gains after the GeoGebra intervention in question 1.2 in the post-test for circle geometry.

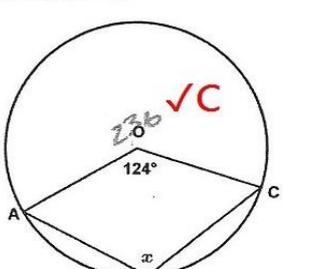
<p>1.2 O is die middelpunt van die sirkel ABC en $\widehat{AOC} = 124^\circ$. Die grootte van $x = \dots$</p> <p>(a) 56°</p> <p>(b) 124°</p> <p>(c) 118°</p> <p>(d) 236°</p> <p><i>at centre $\rightarrow 2 \times$ CIRCUM \leftarrow</i></p>	<p>1.2 O is the centre of the circle and $\widehat{AOC} = 124^\circ$. The size of $x = \dots$ (3)</p> 
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Figure 5.50: Student CG2's response to rider angle at centre theorem in the post-test

The student obtained full marks for question 1.2 in the post-test. He did not give all the reasons, but the scoring rubric indicated that any calculation shown on the sketch is a mark, on condition that it is not mentioned in other calculations. The student did not calculate the reflex angle AOC in the pre-test, but calculated it in the post-test. In the post-test, the student had moved up to Van Hiele (1986) level 3. This student was able to follow simple deductions and understood the logical deductions between consecutive steps (Lim, 1992).

Figure 5.51 shows the responses of Student CG3 to question 1.3 in the pre-test. She entered the SciMathUS programme with 63% for NSC Mathematics and had not done circle geometry in Grade 11.

1.3 In die sirkel middelpunt O, $\widehat{AOB} = 68^\circ$ en $AB = BD$. Die grootte van $\widehat{C}_2 = \dots$

1.3 In the circle with centre O, $\widehat{AOB} = 68^\circ$ and $AB = BD$. The size of $\widehat{C}_2 = \dots$ (3)

(a) 68°
 (b) 34°
 (c) 44°
 (d) 56°

$\widehat{AOC} = (180^\circ - 68^\circ) = 112^\circ$ (str L) 56°
 ~~$\widehat{ABO} = 63^\circ$ (isos Δ ; $AB = AO$) radus.~~

Figure 5.51: Student CG3's response to rider with equal chords in the pre-test

She judged the sketch by its appearance and assumed $\widehat{D} = 90^\circ$, i.e. that Van Hiele level 1 (Van Hiele, 1986; 1999) for circle geometry. The theorem that states that the angle subtended by the diameter at the circumference of the circle is 90° , was also a new concept for the student. She obtained two marks for finding $\widehat{AOC} = 112^\circ$ and $\widehat{C}_1 = 34^\circ$. The labelling of \widehat{AOC} as \widehat{O} was also incorrect, but she showed the size of it on the sketch. She made an attempt to find the size of \widehat{C}_2 , but could not find it, because of her limited knowledge regarding properties of circle geometry. This student operates on Van Hiele level 3 for Grade 8 – 10 Euclidean Geometry.

Figure 5.52 shows how Student CG3 used circle geometry theorems and corollaries after the HLT for circle geometry with GeoGebra.

1.3 In die sirkel middelpunt O, $\widehat{AOB} = 68^\circ$ en $AB = BD$. Die grootte van $\widehat{C}_2 = \dots$

1.3 In the circle with centre O, $\widehat{AOB} = 68^\circ$ and $AB = BD$. The size of $\widehat{C}_2 = \dots$ (3)

(a) 68°
 (b) 34°
 (c) 44°
 (d) 56°

$\widehat{C}_1 = 34^\circ$ (centre L = 2x circum L)
 $\therefore \widehat{C}_2 = 34^\circ$ (cords of equal length)

Figure 5.52: Student CG3's response to rider with equal chords in the post-test

Student CG3 did not use any of the geometry applied in her pre-test to answer the same question in the post-test. She answered the question in the shortest possible way and obtained full marks. She moved up to Van Hiele (1986) level 3 for circle geometry in the post-test. She started to use reasons to justify her arguments. She was able to follow simple deductions and understood the logical deductions between consecutive steps (Lim, 1992).

Figure 5.53 shows how Students CG4, who knew nothing about circle geometry, used her Grade 8 – 10 Euclidean Geometry to answer question 1.4 in the pre-test. Student CG4 was one of those students who did not do circle geometry in Grade 11 and joined SciMathUS with 69% for NSC Mathematics.

1.4 In die figuur is O is die middelpunt van die sirkel en AD is 'n raaklyn aan die sirkel. As $AB = BC$ en $\widehat{BAD} = 30^\circ$, dan is $\widehat{O}_1 = \dots$

(a) 60°

(b) 30°

(c) 120° ✓

(d) 90°

isosceles triangle
corresp. \angle s X

✓R

1.4 In the figure, O is the centre of the circle and AD is a tangent to the circle. If $AB = BC$ and $\widehat{BAD} = 30^\circ$, then $\widehat{O}_1 = \dots$ (5)

Figure 5.53: Student CG4's response to rider with tangent chord theorem in the pre-test

Student OG4 found the correct answer, but her reasons were not completely correct. She noticed that $AB = BC$ and subsequently concluded that $\triangle ABC$ was an isosceles triangle and showed that $\widehat{O}_1 = 120^\circ$. She also obtained a mark for the correct answer and also one for using one of the four correct reasons. She could have found the correct answer by assuming that the diagram was drawn to scale and that \widehat{O}_1 was an obtuse angle, and based her answer on that assumption. She listed a few Grade 8 – 10 Euclidean Geometry reasons, but they were not sufficient to explain the correct answer. The student's Van Hiele level for Grade 8 – 10 Euclidean Geometry is on level 2. The teacher-researcher also made the Grade 8 – 10 Euclidean Geometry part of HLT for circle geometry and covered it before the students wrote the pre-test. The assumption of the researcher was that she assumed this was possibly a parallelogram and that is why she indicated $\widehat{O}_1 = \widehat{B} = 120^\circ$. She thus operated on Van Hiele level 1 for circle geometry. She assumed that $\widehat{ACB} = \widehat{BAD} = 30^\circ$ and judged the sketch by its appearance (Van Hiele, 1999). This student is therefore on Van Hiele level 1, because circle geometry was unknown to her.

Figure 5.54 shows Student CG4's response to question 1.4 in the post-test, after being taught with the designed activities and GeoGebra.

1.4 In die figuur is O is die middelpunt van die sirkel en AD is 'n raaklyn aan die sirkel. As $AB = BC$ en $\widehat{BAD} = 30^\circ$, dan is $\widehat{O}_1 = \dots$

(a) 60°

(b) 30°

(c) 120° ✓

(d) 90°

1.4 In the figure, O is the centre of the circle and AD is a tangent to the circle. If $AB = BC$ and $\widehat{BAD} = 30^\circ$, then $\widehat{O}_1 = \dots$ (5)

$\widehat{BAD} = 30^\circ$ (tangent/chord theorem) ✓R

$\widehat{A_2} = 30^\circ$ (ABC isosceles Δ) ✓R

$\widehat{B} = 120^\circ$

$\widehat{O_2} = 2\widehat{B} = 240^\circ$ ✓R

$\therefore \widehat{O_1} = 360^\circ - 240^\circ = 120^\circ$

(revolution) ✓R

Figure 5.54: Student CG4's response to rider with tangent chord theorem in the post-test

The student even showed all the steps she followed to conclude that $\widehat{O}_1 = 120^\circ$. She gave $\widehat{O}_2 = 2\widehat{B} = 240^\circ$, but did not provide a reason. The fact that she mentioned $\widehat{O}_2 = 2\widehat{B}$, was sufficient to give her the mark. She only gave one of the correct reasons in her response in the pre-test (see Figure 5.53). Her response in this question in the post-test, showed that she had moved up to Van Hiele (1986) level 3. She utilised reasons to justify her arguments. This student was able to follow simple deductions and understood the logical deductions between consecutive steps (Lim, 1992).

Figure 5.55 shows the responses of Student CG5 to question 1.5 in the pre-test. He entered the SciMathUS programme with 51% for NSC Mathematics and had also not done circle geometry in Grade 11.

1.5 In die sirkel is A, B, C en D op die omtrek van die sirkel. Die grootte van $\widehat{ABC} = \dots$

(a) 40°

(b) inligting nie genoeg om \widehat{ABC} te bereken / information not enough to determine \widehat{ABC}

(c) 28°

(d) 24°

1.5 In the circle is A, B, C and D on the circumference of the circle. The size of $\widehat{ABC} = \dots$ (3)

Figure 5.55: Student CG5's response to rider with subtended by same chord (arc) theorem in the pre-test

He calculated certain angle sizes (see students' response on sketch), but did not give any reasons why $\widehat{C\hat{O}D} = \widehat{A\hat{O}B} = 100^\circ$, $\widehat{D} = 24^\circ$ and $\widehat{A\hat{O}C} = 80^\circ$. He was awarded one mark for any calculation on the sketch. His calculations were correct, showing that he knew Grade 8 and 9 Euclidean Geometry, but did not see the importance of giving reasons why the angles were those sizes. He concluded that he needed more information to continue to find $\widehat{A\hat{B}C}$. The researcher concluded that he was on Van Hiele level 2 in terms of what he knew about Grade 8 – 10 Euclidean Geometry (DBE, 2011a; 2011c), because he did not give sufficient reasons. This student was able to follow simple deductions and understood the logical deductions between consecutive steps (Lim, 1992). He made no assumptions about any of the circle geometry theorems and judged the sketch on its appearance. He argued that the information was not sufficient to continue with calculating $\widehat{A\hat{B}C}$.

Figure 5.56 shows how Student CG5 responded after the intervention with GeoGebra in question 1.5 in the post-test.

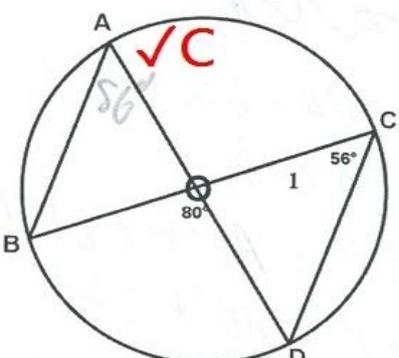
<p>1.5 In die sirkel is A, B, C en D op die omtrek van die sirkel. Die grootte van $\widehat{A\hat{B}C} = \dots$</p> <p>(a) 40°</p> <p>(b) inligting nie genoeg om $\widehat{A\hat{B}C}$ te bereken / information not enough to determine $\widehat{A\hat{B}C}$</p> <p>(c) 28°</p> <p>(d) 24°</p>	<p>1.5 In the circle is A, B, C and D on the circumference of the circle. The size of $\widehat{A\hat{B}C} = \dots$ (3)</p> 
--	---

Figure 5.56: Student CG5's response to rider with subtended by same chord (arc) theorem in the post-test

This student was awarded only a mark for showing $\widehat{A} = 56^\circ$ on the sketch, but did not give any reasons. He did not use Grade 8 – 10 Euclidean Geometry, as it was the case in his pre-test (see Figure 5.55). He still believed that he could not find $\widehat{A\hat{B}C}$, because the information was insufficient. Had he continued with his calculation that he did in the pre-test, he would have answered it correctly. The researcher's assumption is that the student used the reason that the circumference angles subtended by same arc or chord are equal. For the researcher, this student is on Van Hiele level 2 for circle geometry. The only difference was that he possibly applied the properties of circle geometry, i.e., circumference angles are

equal if subtended by same chord or arc. If the student assumed that $\hat{A} = \hat{C}$, then he was on Van Hiele level 1 with regards to circle geometry.

Figure 5.57 shows how Student CG6's answer to question 1.5 is similar to Student CG5. She entered the SciMathUS programme with 56% for NSC Mathematics and did not do circle geometry in Grade 11.

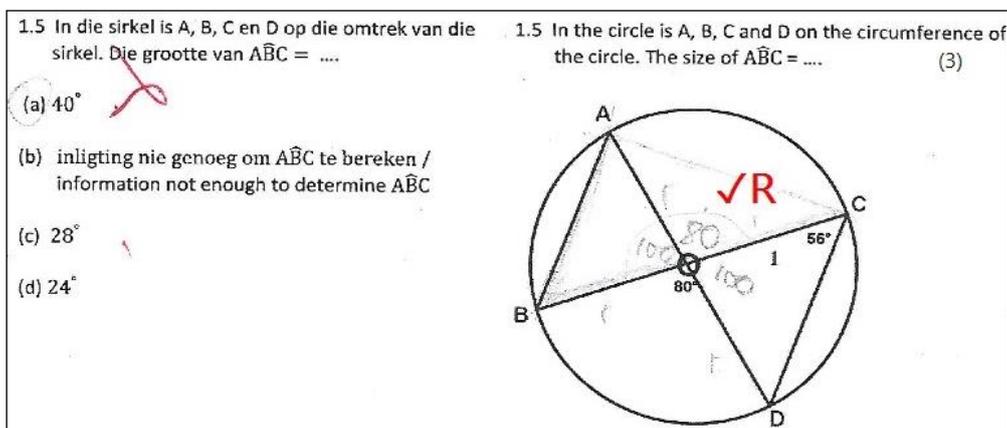


Figure 5.57: Student CG6's response to rider with subtended by same chord (arc) theorem in the pre-test

She gave her answer $\hat{A}BC = 40^\circ$ and was awarded a mark for the calculations on the sketch (see scoring rubric in Appendix B). She did not provide any reasons, although they were required. She calculated $\hat{A}OB = \hat{D}OC = 100^\circ$ and $\hat{A}OB = 80^\circ$, without stating reasons. The student probably assumed O to be the centre of the circle and deduced that $\hat{A}BC = \hat{B}AD$, because $AO = BO$ (radii). She saw the relationship between the angles, but did not give sufficient reasons, i.e. that Van Hiele level 2 (Van Hiele, 1986) for Grade 8 – 10 Euclidean Geometry. She did not assume any circle geometry properties.

Figure 5.58 shows how Student CG6 did not assume O to be the centre of the circle in the post-test.

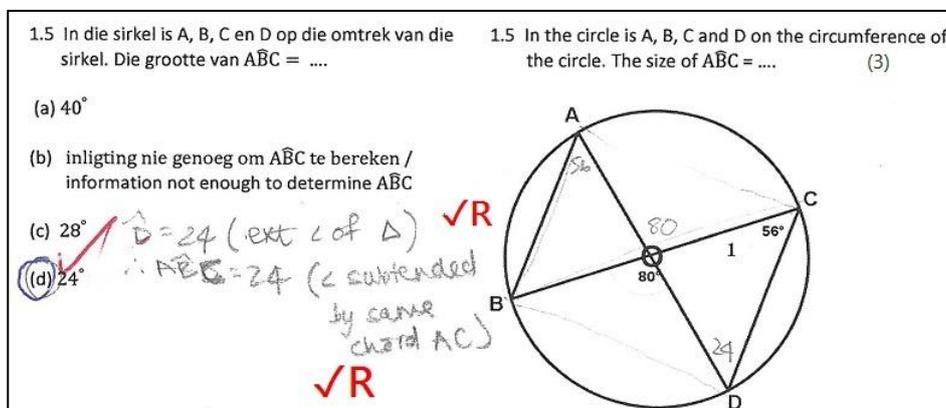


Figure 5.58: Student CG6's response to rider with subtended by same chord (arc) theorem in the post-test

This student showed learning gains (see Figure 5.58) with regard to circle geometry in the post-test. Her attempt in the post-test showed an improvement on how she answered the question, although she still did not mention a reason why $\widehat{AOC} = 80^\circ$. She actually stated the two reasons that she needed to conclude that $\widehat{ABC} = 24^\circ$. In the post-test she had moved up to Van Hiele level 3 for circle geometry. According to Van Hiele (1986; 1999), students start using reasons to justify arguments at Level 3 and also state sufficient reasons to conclude their answers. This student was able to follow simple deductions and understood the logical deductions between consecutive steps (Lim, 1992).

Figure 5.59 shows Student CG7 assuming $AD = BD$. Student CG7 came in with 39% in the NSC Mathematics exam and had also not done circle geometry in Grade 11.

QUESTION 2

Helpunt O in die middelpunt van die sirkel. AB is 'n tou van die sirkel met O die sentrum in die diagram hieronder. OC sny tou AB by D. $AB \perp OC$. Indien $OA = 15$ cm en $DC = 6$ cm, bereken met behulp van redenasie die lengte van AB. (5)

Handwritten student work:

$AB = 15 \times 2$
 $\approx 30 \text{ cm}$

AB is

$$\sqrt{AB^2} = \sqrt{AO^2 - OD^2}$$

$$AB = \sqrt{(15)^2 - (9)^2}$$

$$= 12 \quad \text{S/R}$$

$AB = 12 \times 2$
 $= 24 \text{ cm}$

With the theorem of Pythagoras, the length of AD can be calculated and the length of AD is half the length of AB.

Figure 5.59: Student CG7's response to the rider on the perpendicular segment from centre to chord in the pre-test

She managed to calculate AB correctly to find $AD = 12 \text{ cm}$. This again showed that she was able to use her knowledge of Grade 8 – 10 Euclidean Geometry, but had no knowledge of circle geometry theorems. This student's understanding of Grade 8 – 10 Euclidean Geometry is on the Van Hiele level 2. She could have visually determined the size of AB (Hoiberg & Sharp, 2001) and made certain assumptions about the length of $AD = \frac{1}{2} AB$. This assumption was correct, but was based on the judgment of the sketch that $AD = DB$ (Van Hiele, 1986, 1999). She also could have seen that $OB = OC = AO = 15 \text{ cm}$ and therefore concluded that $\triangle AOD$ would have the same dimensions as $\triangle BOD$, but she did not show evidence that the triangles are congruent. Consequently she was on Van Hiele level 1 for circle geometry.

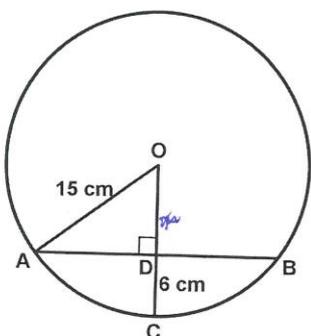
Although Student CG7's pre-test (Figure 5.59) and post-test (Figure 5.60) answers were similar, the student's post-test answer was more complete.

QUESTION 2

niddelpunt O in die koord AB in D. $\angle C = 6$ cm, bereken met redensies die lengte van AB.

AB is a chord of the circle with O the centre in the diagram below. OC intersects chord AB at D. $AB \perp OC$. If $OA = 15$ cm and $DC = 6$ cm, calculate with reasons the length of AB.

(5)



Since $OA = OC$ (radii)
 $OD = 15 - 6$
 $= 9$ cm

Using the theorem of Pythagoras ($AB \perp OC$)
 $OA^2 = OD^2 + AD^2$
 $(15)^2 = (9)^2 + AD^2$
 $AD^2 = (15)^2 - (9)^2$
 $\sqrt{AD^2} = \sqrt{225 - 81}$
 $AD = 12$ units

Since $AD = DB$ ($AB \perp OC$)
 \therefore length of $AB = 2(12)$
 $= 24$ units

Figure 5.60: Student CG7's response to the rider on the perpendicular segment from centre to chord in the post-test

In the pre-test she could not explain why $AD = DB$. In the post-test she showed all the steps with the necessary and sufficient reasons to conclude that $AB = 24$ cm. In the post-test, she had moved up to Van Hiele (1986) level 3. She began to apply reasons to justify her arguments and did not merely make assumptions. According to Lim (1992) this student was able to follow simple deductions and understand the logical deductions between consecutive steps).

Figure 5.61 shows Student CG8's response to question 3 in the pre-test.

VRAAG 3

In die diagram hieronder is O die middelpunt van die sirkel en $\widehat{D\hat{O}C} = 132^\circ$. PCT is 'n raaklyn aan die sirkel en $DA \parallel PCT$.

Bereken, met redes die grootte van:

a) $\widehat{A_1}$ (2)
 (b) $\widehat{C_1}$ (2)
 (c) $\widehat{A\hat{D}C}$ (2)
 (d) \widehat{B} (2)

QUESTION 3

In the diagram below, O is centre of the circle and $\widehat{D\hat{O}C} = 132^\circ$. PCT is a tangent to the circle and $DA \parallel PCT$.

Calculate with reasons the size of:

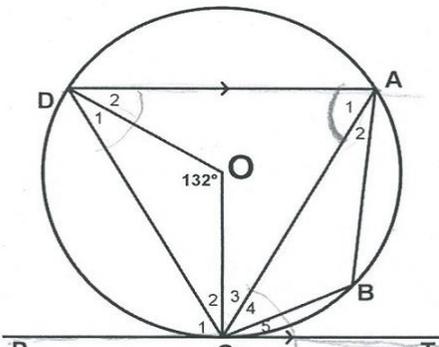


Figure 5.61: Student CG8's response to question 3 in the pre-test

This student attempted to answer this question. The student only marked the angles that he thought are equal, but did not provide any reasons. It can be assumed that he knew that $\widehat{D\hat{A}C} = \widehat{A\hat{C}T}$ because of alternate angles and $AD \parallel PT$. He also judged the sketch by its appearance (Van Hiele, 1986; 1999) and compared $\widehat{A\hat{D}C}$ and $\widehat{D\hat{A}C}$ visually (Hoiberg & Sharp, 2001) to be equal, i.e., Van Hiele level 1.

Figure 5.62 shows how Student CG8 approached answering question 3 in the post-test. This shows how he was ordering or analysing the information. He showed in the pre-test that he knew nothing about circle geometry and was on Van Hiele level 1 (Van Hiele, 1986).

The figure shows a worksheet for 'VRAAG 3' (QUESTION 3) with a circle diagram and handwritten student work. The diagram features a circle with center O, points A, B, C, D on the circumference, and a tangent line PCT at point C. A line AD is drawn parallel to PCT. The central angle \widehat{AOC} is labeled as 132° . Handwritten notes in Afrikaans and English provide solutions for parts (a) and (b). Part (a) asks for $\widehat{A_1}$, and the student writes $\widehat{A_1} = 66^\circ$ (tangent PC, chord AC). Part (b) asks for $\widehat{C_1}$, and the student writes $\widehat{C_1} = 66^\circ$ (tangent PC, chord DC). The student also uses the property of opposite angles in a cyclic quadrilateral: $\widehat{A} + \widehat{B} = 180^\circ$, leading to $\widehat{B} = 114^\circ$. A circled '8' is written at the bottom right.

Figure 5.62: Student CG8's response to question 3 in the post-test

The abovementioned student's response shows that he gained knowledge of the theorems and corollaries in circle geometry and that he was able to apply this knowledge after being taught by the designed activities and GeoGebra. He showed that he was on Van Hiele level 3. At this level, the student developed logical reasoning (Murray, 1996; 1997). He was able to follow simple deductions and understood the logical deductions between consecutive steps (Lim, 1992).

Most of the students were on Van Hiele level 1 for circle geometry in the pre-test. The majority of them judged the sketches by their appearance and based their arguments on a statement of belief, and not on logical conclusions (Van Hiele, 1986). The students made their decisions based on their perception and not on reasoning (Mason, 1998). There were also students that could not do circle geometry, but managed to answer the Grade 8 – 10 Euclidean Geometry. Most of these students were either on Van Hiele level 2, because they could see the relationship between angles, but did not mention sufficient reasons (Hoiberg & Sharp, 2001; Van Hiele, 1986). Most of the responses of the students in the post-test show that the students moved to Van Hiele level 3 (Van Hiele, 1986). They also started to

use circle geometry reasons to justify arguments. Some managed to use necessary and sufficient reasons to conclude their answers. They were able to follow simple deductions and understood the logical deductions between consecutive steps (Lim, 1992).

The abovementioned responses are purpose samples of students' responses to their students' learning gains. These samples of students' responses also strengthen the qualitative data presented in section 5.3.2.1 answering the sub-question students' explanations of how GeoGebra afforded them an opportunity to understand mathematical concepts and move to higher levels of abstraction. The responses (see section 5.3.2.1) from the students in the in-depth (Student ID2 and ID7) and focus group (Student FG13 and FG21) interviews show how students referred to the visual affordances and enablements of GeoGebra. These responses also show how students acquired physical and logico-mathematical knowledge through the designed activities with GeoGebra. The students' explanations show how GeoGebra afforded them an opportunity to understand mathematical concepts. The fact that some students (see Student CG1, CG2, CG3, CG4 and CG8) moved in certain questions from Van Hiele 1 to 3 shows the students' higher levels of abstraction. There was one student that moved from Van Hiele level 1 to 2 (see Student CG5) and also two students that moved to Van Hiele level 2 to 3 (see Student CG6 and CG7). The students were able to follow simple deductions and understand logical deductions between consecutive steps (Lim, 1992).

The next sub-section will focus on the pre- and post-tests of all the students in the intervention. The qualitative data analyses were only on a purposefully sample of students in the pre- and post-tests on circle geometry. The next section will investigate if there were significant learning gains for all the students in circle geometry.

5.3.2.3 Quantitative analysis of pre- and post-test in circle geometry

The students wrote the pre-test, were exposed to circle geometry with GeoGebra and designed activities. The students explored on their own on prepared GeoGebra applets. After two days of working with GeoGebra and then working for four days with the designed activities (Appendix L), as well as proving the theorems, they wrote the post-test. The paired t-test was conducted to find out whether there were significant learning gains for all the students in the study.

The quantitative data presented in this section were collected from the pre- and post-tests for circle geometry.

Table 5.4 presents the dependent (paired) samples t-test showed that the circle geometry learning gains between the pre-and post-test were statistically significant at the 5% level.

Table 5.4: Paired sample t-test shows the students' means of the pre- and post-test for circle geometry (n= 48).

	Paired Differences					<i>t</i>	df	Sig. (2-tailed)
	Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference				
				Lower	Upper			
Pre-Post-test	53.188	18.558	2.679	47.799	58.576	19.856	47	.000

The means for the pre-and post-test for circle geometry were 17.31% and 70.50% while the standard deviations were 13.606 and 17.932, respectively. Table 5.4 shows there was a statistically significant difference between the students pre- and post-test scores, $t(47) = 19.856$, $p = 0.000$. Hence this study rejected the null hypothesis in favour of the alternative hypothesis. Cohen's d is calculated as $d = 2.66$ which indicates a very large effect size (Ellis, 2009). Although there was also no comparison group for this particular task, the results show that the students improved their understanding of circle geometry. This study did not investigate if the teaching and learning for circle geometry with GeoGebra is better than an approach without it, but to show that it can help with the understanding of the properties of circle geometry. The t-test result showed that there were learnings gains not only for the purposeful sample of students but also for all the students that were exposed to the teaching intervention with GeoGebra and instructional material.

5.3.3 Students' perceptions before and after the intervention

The first section of the post-questionnaire was the same as the pre-questionnaire. The researcher did this as mentioned in Chapter 4 (see section 4.3.2.1) to see if students changed their perceptions on how they should be taught, how they changed after being taught with GeoGebra, with designed activities, and in an active learning approach mixed with a teacher as a facilitator of learning. This section will therefore describe how the students felt they gained knowledge from designed activities.

The following sections present the Fisher's exact test results regarding the pre- and post-intervention questionnaire that were administered to the experimental group. As mentioned in Chapter 4 (see section 4.3.2.1), a five-point Likert scale was used in both questionnaires. The Fisher's exact test for 2 x 5 table (two questionnaires x five-point Likert scale) was subsequently utilised to test if the relationship between the pre- and post-intervention questionnaires were statistically significant. Forty-eight (100%) students completed the pre-questionnaire and 46 (95.8%) students the post-intervention questionnaire. The following sections sought to test the following null hypothesis:

H₀: There is no difference between the students' perceptions

$$(\mu_{\text{pre-intervention}} = \mu_{\text{post-intervention}})$$

H₁: There is a difference between the students' perceptions

$$(\mu_{\text{pre-intervention}} \neq \mu_{\text{post-intervention}})$$

5.3.3.1 Students' perceptions of how they should learn and be taught mathematics

This section sought to establish if there were any differences between the students' perceptions before the intervention and after, for the construct, how they should be taught and should learn mathematics.

Table 5.5 indicates the Fisher's test and Cramer's *V* results that were administered to test for significant differences in the students' perceptions of how they should be taught or learn mathematics. This test investigates if there was a change in students' perceptions from the first day at SciMathUS to the end of September. The Cronbach alpha reliability coefficient for the eight items in this construct is 0.737. A Cronbach alpha of $\alpha \in [0.7; 0.8)$ is acceptable (George & Mallery, 2003).

Table 5.5 Students' perceptions of how they should be taught or learn mathematics before and after the intervention

Items in construct	Fisher exact test for 2 x 5	<i>df</i>	<i>p</i>	Accept /Reject H_0	Cramer's V	Effect size
I am capable of finding out myself how Mathematics works.	17.43	4	0.0016	Reject	0.4306	Strong
To understand certain concepts in Mathematics, the concepts must first fully be explained to me.	6.43	4	0.16952	Accept	0.2615	Not applicable
I like to do Mathematics with short ways and do not need to do the long way to understand it.	7.529	4	0.11042	Accept	0.283	Not applicable
I prefer to first explore the Mathematics on my own instead of the teacher explaining all the time.	8.442	4	0.07667	Accept	0.2996	Not applicable
I think the teacher must give us notes every time he/she starts a new topic.	2.986	4	0.56011	Accept	0.1783	Not applicable
I can only understand Mathematics well when the teacher is showing me how to do it.	2.198	4	0.6999358	Accept	0.1538	Not applicable
I can only learn by memorising what the teacher had taught me.	6.483	4	0.16588	Accept	0.2626	Not applicable
I learn Mathematics the best if I am given examples and practice similar questions afterwards.	0.421	4	0.93588	Accept	0.0668	Not applicable

Table 5.5 shows that the results of the Fisher exact test pointed to no significant differences in the students' perceptions of how they should learn mathematics and be taught before and after the intervention. The null hypothesis is accepted for seven of the items in this construct. This table shows that students do not easily change and unlearn teaching approaches they are used to. The Fisher exact test shows that the alternative to the null hypothesis for only one item was accepted, i.e. students are capable of exploring how mathematics works. Twenty-seven (56%) of the 48 students took a neutral stance in the pre-questionnaire when asked if they were capable of exploring Mathematics themselves. That number decreased to seven (15%) of 46 students in the post-questionnaire. Twelve (25%) of the students strongly agreed, or agreed, in the pre-questionnaire that they were capable of exploring Mathematics themselves, while 25 (54%) strongly agreed, or agreed, in the post-questionnaire. This shows that many students realised that after a year of being taught with

a hybridisation of the student-centred approach and teacher as facilitator of learning approach, they are capable of exploring Mathematics.

The students felt they benefited from exploring, but also still believed that Mathematics is all about giving an example or two and then it has to be drilled in by means of extra activities. Students still rely on the teacher to give notes with the start of a new topic and also expect the teacher to fully explain concepts. It also shows that the students believe that the transmission model of teaching is the best approach of teaching.

Cramer's V was calculated as $V = 0.436$, which indicates a strong effect size (Botsch, 2011) for the item, i.e. students were capable of exploring how mathematics works. This echoed the objectives and principles of teaching and learning theories such as constructivist approach to teaching and RME.

5.3.3.2 Students' perceptions on the use of GeoGebra in the teaching and learning of mathematics

The students did not know GeoGebra in the beginning and that is why the researcher referred to the computer in the pre- and post-questionnaires. Computer refers to GeoGebra, because that was the only software that they were exposed to during the teaching and learning of mathematics. Forty (83%) of the 48 students indicated in pre-questionnaire that they worked on computers before entering the SciMathUS programme. Fourteen (29%) of the total of students indicated that they worked on educational mathematical software. An average of 20 (42%) students took a neutral stance on the use of computers in the teaching and learning of Mathematics in the pre-questionnaire. This average decreased to 4 (9%) of the students in the post-questionnaire. This section sought to establish if there were any differences between the students' perceptions before the intervention and after for the construct and students' perceptions of the use of computers in the teaching and learning of mathematics.

Table 5.6 shows that there were significant differences in the students' perceptions on the use of computers in the teaching and learning of mathematics before and after the intervention as measured by Fisher's exact test (2 x 5 contingency table). The Cronbach alpha reliability coefficient for the five items in this construct is 0.787. A Cronbach alpha of $\alpha \in [0.7; 0.8)$ is acceptable (George & Mallery, 2003).

Table 5.6: Students' perceptions of the use of computers in the teaching and learning of Mathematics before and after the intervention

Items in constructs	Fisher exact test for 2 x 5	<i>df</i>	<i>p</i>	Accept /Reject H_0	Cramer's <i>V</i>	Effect size
I can learn Mathematics on a computer with Mathematics educational software.	29.168	4	0.000007	Reject	0.5571	Strong
Software on a computer cannot explain Mathematics to me. A teacher has to explain Mathematics to me.	14.757	4	0.00523	Reject	0.3963	Strong
Using computers in Mathematics makes Mathematics more confusing.	25.881	4	0.00003	Reject	0.5247	Strong
Although I am not writing my tests and examination on a computer, it can help me to understand certain concepts better.	31.614	4	0.0000	Reject	0.5799	Strong
Using computers in Mathematics means you will not be able to do Mathematics without them.	25.951	4	0.000032	Reject	0.5254	Strong

The null hypothesis rejected all five of the items in this construct. The values of Cramer's *V* for all five items indicate a strong effect size (Botsch, 2011) for all the items in this construct. Hence students reconsidered their perceptions of teaching and learning with software and technology after being exposed to GeoGebra. There is a significant difference between the students' perceptions, before the intervention and after, on the use of computers in the teaching and learning of mathematics. Although students had earlier indicated, as shown in Table 5.5, that they relied on the teacher to explain, they indicated that GeoGebra could help with understanding mathematical concepts.

Figure 5.63 shows responses from the post-questionnaire how the students felt about exploring certain concepts in Mathematics instead of the teacher-researcher explaining them.

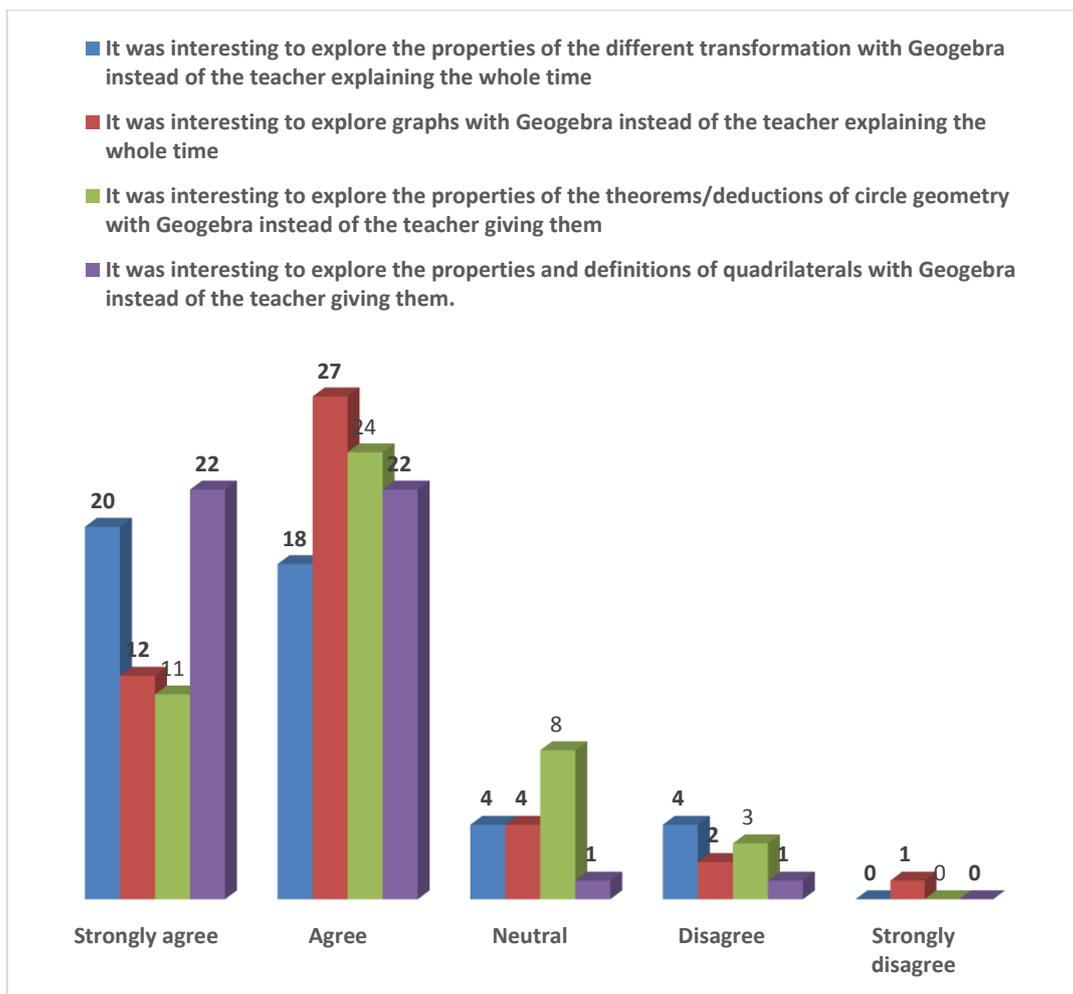


Figure 5.63: Students' responses on exploring

Figure 5.63 shows that most of the students found it interesting to explore topics such as transformation, graphs, circle geometry and quadrilaterals with GeoGebra. An average of 39 (84.7%) students strongly agreed, or agree, it was interesting to explore the abovementioned topics instead of the teacher merely explaining the concepts.

Quantitative data from the post-questionnaire also supported students' views on how GeoGebra helped them to understand concepts in certain topics better. Figure 5.63 shows that students agreed that the worksheets helped them to understand concepts better.

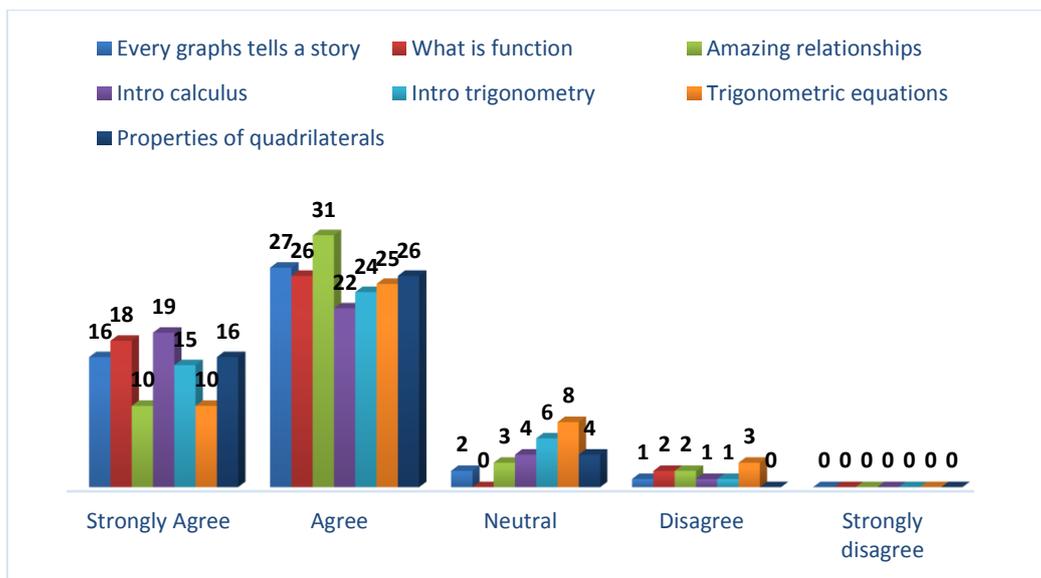


Figure 5.64: Students' agreement that worksheets helped them to understand concepts better (n= 46)

Figure 5.64 shows that students agreed that the worksheets helped them to understand concepts linked to the specific topic better. The Cronbach alpha reliability coefficient for this construct is 0.775. A Cronbach alpha of $\alpha \in [0.7; 0.8]$ is acceptable (George & Mallery, 2003). An average of 41 (88.5%) students agreed that the worksheets helped them with conceptual understanding. In all of the abovementioned worksheets GeoGebra was utilised as a teaching tool by the teacher-researcher. The worksheets *Amazing relationships* (Appendix G); *Every graph tells a story* (Appendix H); and *Introduction Calculus* (examples of activity in Appendix R) consisted of realistic contexts. The worksheets *Introduction to Trigonometric* (see example 9 in Appendix R); *Trigonometric equations* (see example 10 in Appendix R) and *Properties of quadrilaterals* (Appendix K) were of an experiential nature, because students were given the worksheets to complete certain sections and the teacher-researcher did the activities on GeoGebra.

The students were not only shown the algorithms or procedures, but also how they work. When using activities in Appendix G and H, students were not just told that the domain of a parabola is $x \in R$, but the teacher-researcher also demonstrated that the limitation of context, or the formula, determines the domain. Students were thus shown that not any input value can be used within functions based on everyday life situations. It was also proved to them that only certain input values can be used in equations of functions to find the output values, since some of the operations cannot be done. An example is Activity 3 in Appendix H where the formula was $y = \frac{24}{x}$ with $x \neq 0$ and $y \neq 0$.

An average of 42 (91.2%) students felt that the worksheets *Every graph tells a story* and *Amazing relationships* helped them to better understand concepts of function, domain and range. Forty-one (89.1%) of the students indicated that the worksheet *Introduction to Calculus* helped them to better understand the rate of change function concept, average gradient/speed, speed and gradient better. Thirty-nine (84.8%) of the students indicated that the worksheet *Introduction to Trigonometry* helped them to better understand trigonometric ratios. Thirty-five (76.1%) of the students indicated that the worksheet *Trigonometric equations* helped them to better understand the concept of trigonometric equations. Forty-two (91.3%) of the students indicated that the worksheet *Properties of quadrilaterals* helped them to better understand the properties and definitions of quadrilaterals.

The analysis of the post-questionnaire also shows how the students felt GeoGebra helped with different topics. Three and six items respectively from the constructs for transformation geometry and transformations of functions in the post-questionnaire were regrouped to focus on students perceptions on how GeoGebra helped them with transformations. Transformation geometry, as discussed in Chapter 3 (see section 3.9), was part of the HLT for the teaching and learning of transformations of functions. The Cronbach alpha reliability coefficient for nine items in construct for transformations is 0.877. A Cronbach alpha of [0.8;0.9) is good (George & Mallery, 2003).

Figure 5.65 shows the analysis of three of the nine items of how the students felt GeoGebra helped them with the transformation of geometry.

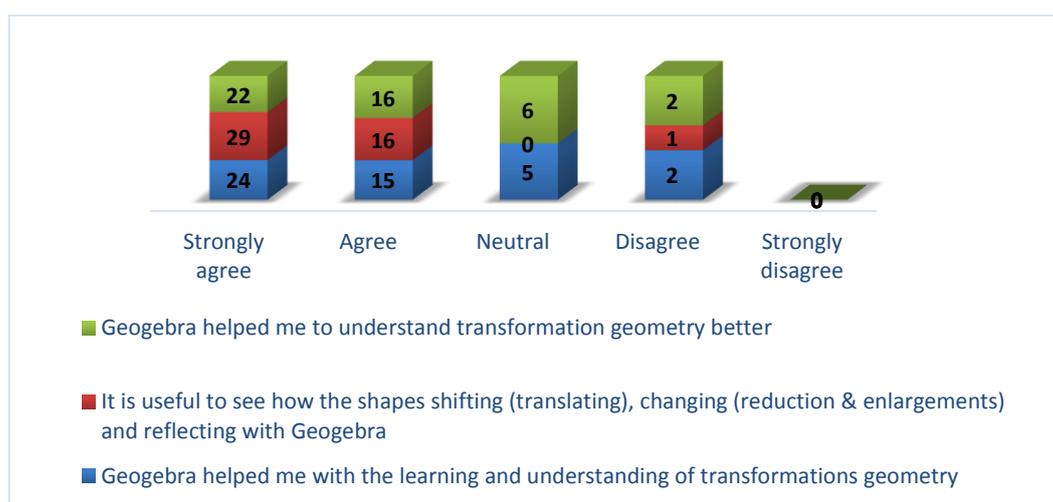


Figure 5.65: Students' perceptions of understanding transformation geometry with GeoGebra (n= 46)

Twenty-four (52%) students strongly agreed and 15 (33%) students agreed that GeoGebra helped them with the learning and understanding of transformations geometry. Twenty-two (48%) students strongly agreed and 16 students agreed that they better understood transformation geometry. Twenty-nine (63%) of the students strongly agreed and 16 (35%) agreed that it was useful to see the shapes shifting, changing, or reflection, in GeoGebra. Students used the designed worksheet (Appendix I), created then in GeoGebra their own polygons, and then explored the properties of transformation. Subsequently more than 80% of the students felt that GeoGebra helped them with learning and understanding of transformation geometry and they better understood transformation geometry because they could see the shapes shifting, changing or reflected in GeoGebra.

Figure 5.66 shows the analysis of six of the nine items of the post-questionnaire of how the students felt GeoGebra helped with the teaching and learning of functions.

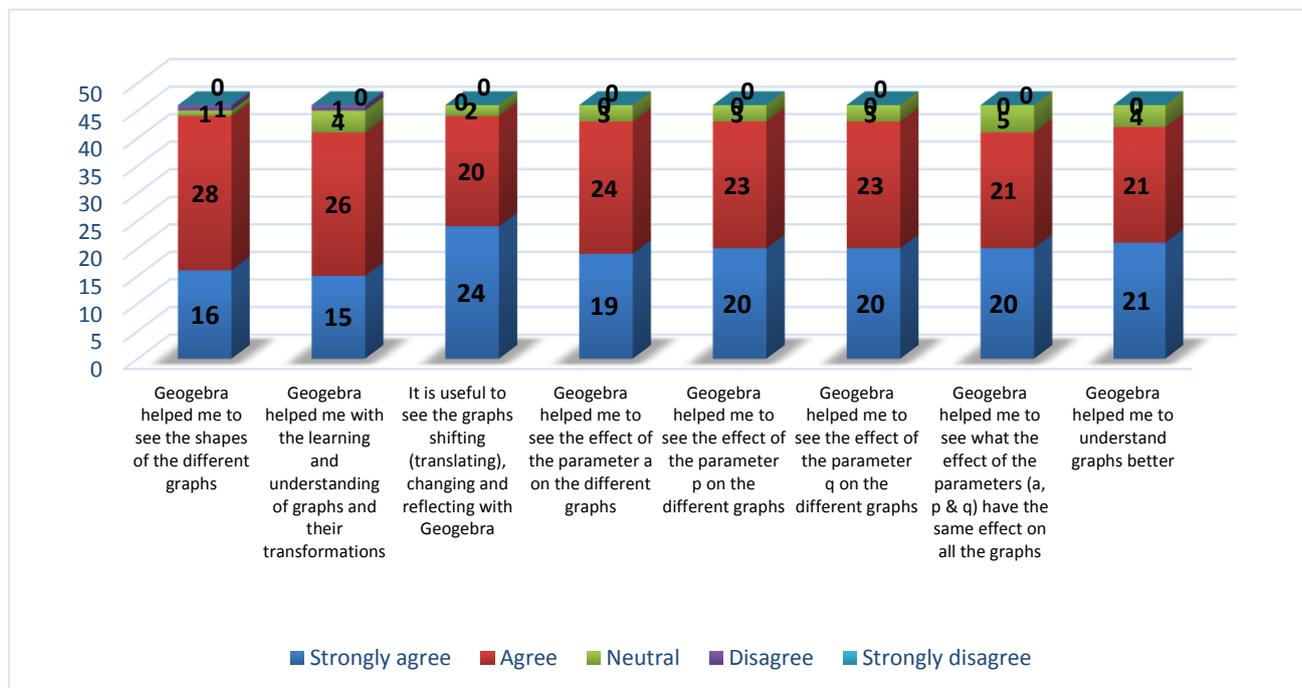


Figure 5.66: Students’ understanding of function transformations with GeoGebra (n= 46)

Figure 5.66 shows an average of almost 43 (90%) students felt that GeoGebra helped them with a better understanding of graphs and functions. The students felt that GeoGebra helped them with the shapes of the graphs, the transformations of functions and they could see the effect of the parameters on the different graphs. They could also see that the effect of the different parameters were the same for all the functions and graphs. The students could see that the parameter a in $y = a(x + p) + q$; $y = a(x + p)^2 + q$; $y = ab^{x+p} + q$;

$y = \frac{a}{x+p} + q$; $y = a \sin b(x+p) + q$; $y = a \cos b(x+p) + q$ and $y = a \tan b(x+p) + q$ has the same effect on the graph, i.e., a vertical stretch or shrinking, etc. The responses of the students in the in-depth and focus group interviews in section 5.3.2.1 on how GeoGebra helped them with transformations of functions, are triangulated with the above quantitative data.

Figure 5.67 shows how students felt GeoGebra helped them with the understanding of the properties of quadrilaterals and circle geometry. The properties of quadrilaterals, as discussed, and indicated in Chapter 4 (see section 4.5.2), was part of the HLT for circle geometry. The Cronbach alpha reliability coefficient for the regrouping of five items in this construct is 0.759. A Cronbach alpha of [0.7;0.9] is acceptable (George & Mallery, 2003).

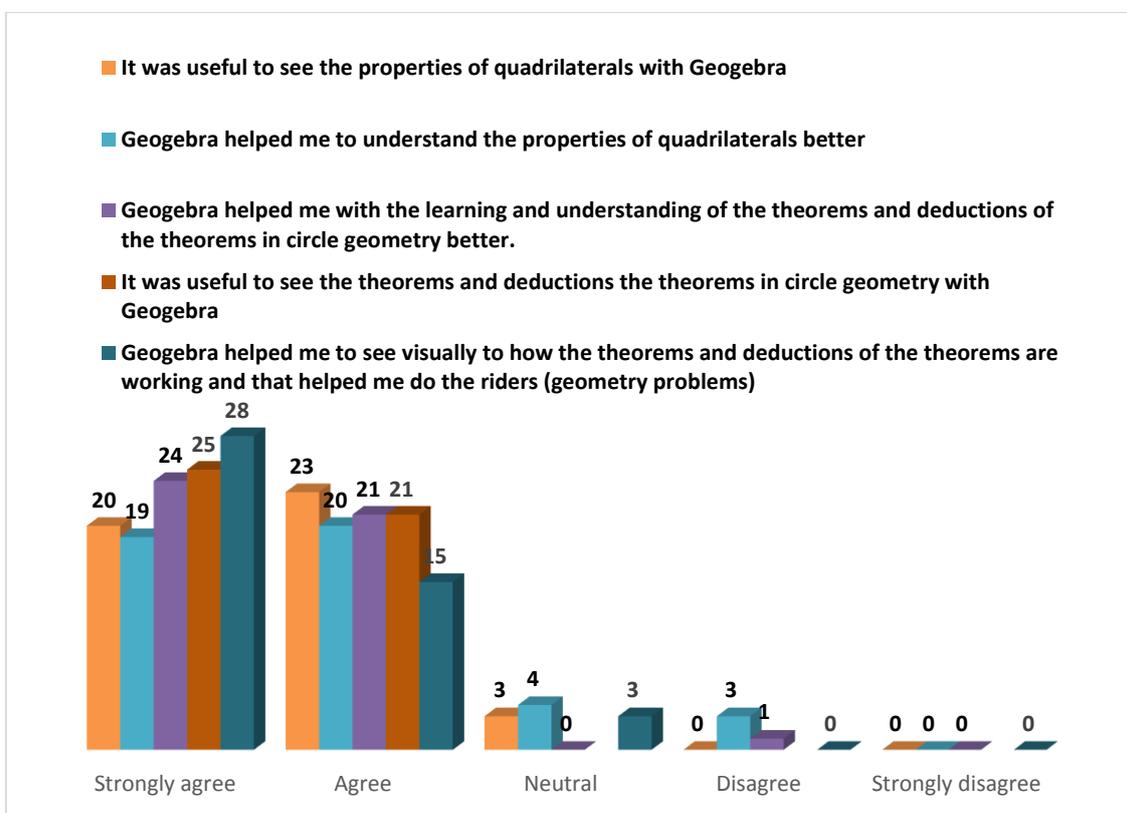


Figure 5.67: Properties of quadrilaterals and circle geometry with GeoGebra

Figure 5.67 shows 20 (44%) students strongly agreed and 23 (50%) students agreed that it was useful to see the properties of quadrilateral with GeoGebra. Nineteen (41%) students strongly agreed and 20 (44%) students agreed that GeoGebra helped them to understand the properties of quadrilaterals. Twenty-four (50%) students strongly agreed and 22 (48%) students agreed that they found GeoGebra had helped them better with the learning of the

theorems and deductions of circle geometry. Twenty-two (48%) students strongly agreed and 24 (50%) students agreed that they found GeoGebra helped them to understand the theorems and deductions of circle geometry. Thus, the majority of the students agreed that GeoGebra helped them to see the properties of quadrilaterals, circle geometry theorems visually, and proofs, as well as related riders. The majority of the students also indicated that GeoGebra helped them to understand the properties of quadrilaterals and circle geometry theorems and their deductions better.

The students' responses in the post-questionnaire on circle geometry refer to activities (see Appendix L) that they did in the computer lab. The students' responses to the properties of the quadrilaterals refer to the interactive lesson with a worksheet (Appendix K), and GeoGebra applets (see Figure 5.68 below) helped them to understand the learning and understanding of the properties of quadrilaterals

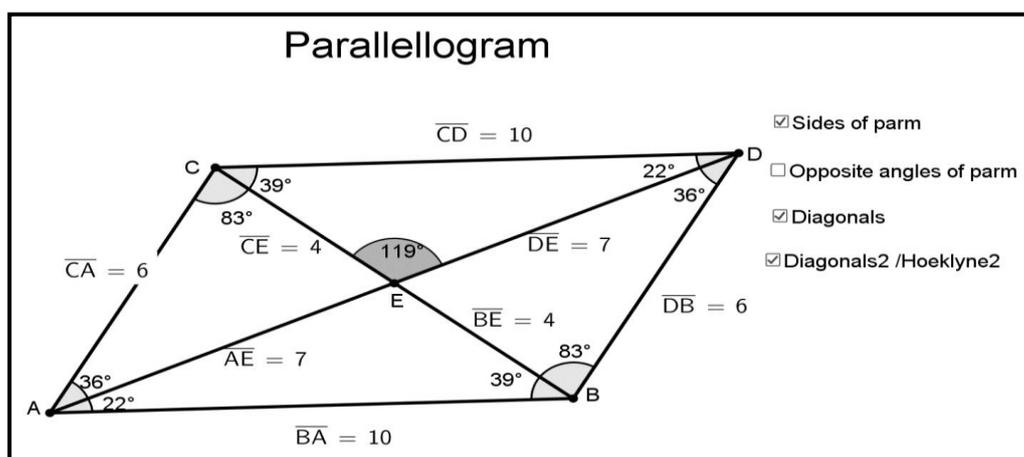


Figure 5.68: GeoGebra applet used with interactive lecturing to teach properties of parallelograms

Figure 5.68 shows the GeoGebra applets used by the teacher-researcher in an interactive lecture on the properties of parallelograms. The teacher-researcher used GeoGebra applets for a rectangle, square, rhombus, kite and trapezium (see example 7 in Appendix R).

All the activities (see Appendixes G – L) mentioned above were done with the students and were guided by questions that were intended to create a cognitive conflict in the students' mind and offer them the opportunity to contend with their own solutions (Waite-Stupiansky, 1997). In this way the teacher-researcher attempted to teach logico-mathematical knowledge (Piaget, 1952; 1954). Even in the interactive lessons with GeoGebra on the properties of quadrilaterals, students were challenged with questions such as:

Can a rectangle be defined as a parallelogram?

Can a rectangle be defined as a square?

Can a square be defined as a rectangle?

In this way the teacher-researcher helped the students understand the relationship, for example, between a square and a rectangle.

The students' responses also show that they felt the interactive lessons with GeoGebra applets (see Figure 5.69 below) helped them with the understanding of concepts, such as trigonometric ratios and general solutions. There were six items in the construct with a Cronbach alpha reliability coefficient of 0.864. A Cronbach alpha of [0.8;0.9) is good (George & Mallery, 2003).

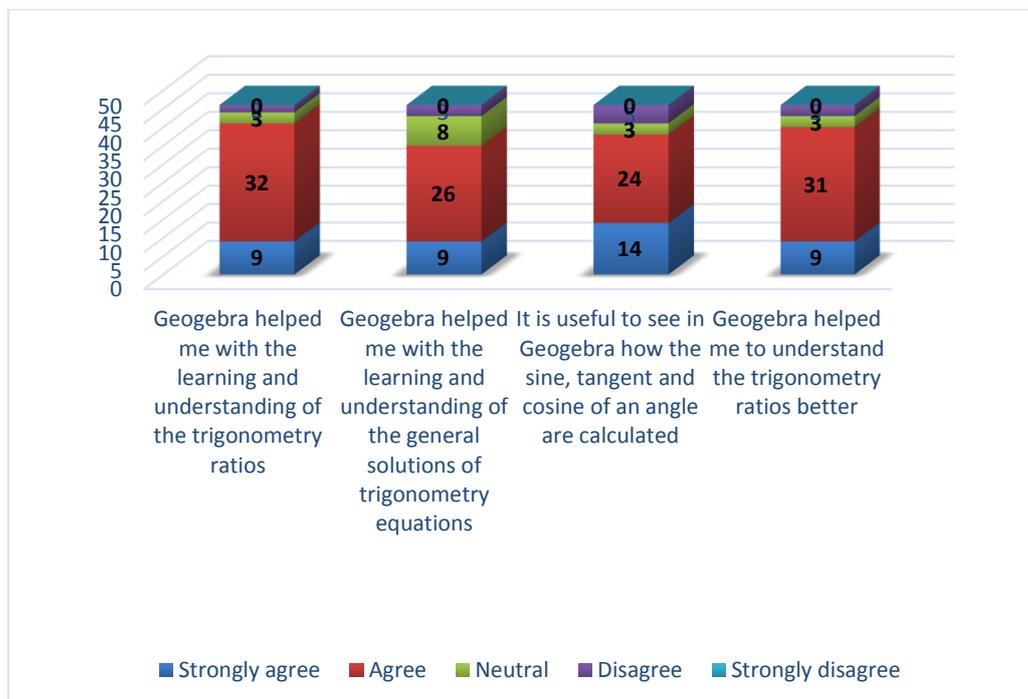


Figure 5.69: Students' perceptions of their learning of trigonometric functions with GeoGebra (n = 46)

Although GeoGebra was utilised in this lesson in a lecturing approach, the teacher-researcher involved the students. The students had to explore (example 9 in Appendix R) the six trigonometric ratios in a right-angled triangle themselves and their answers were then verified by the teacher-researcher with the GeoGebra applet (see Figure 5.70 below). Figure 5.69 shows only four of the six items. Forty-one (89.1%) of the students strongly agreed or agreed that GeoGebra helped them to better understand trigonometry ratios. Thirty-five

(76.1%) students felt that the activity above helped them to understand the concept of general solutions.

Figure 5.70 shows the GeoGebra applet that the teacher-researcher used in the introductory lesson on trigonometric ratios.

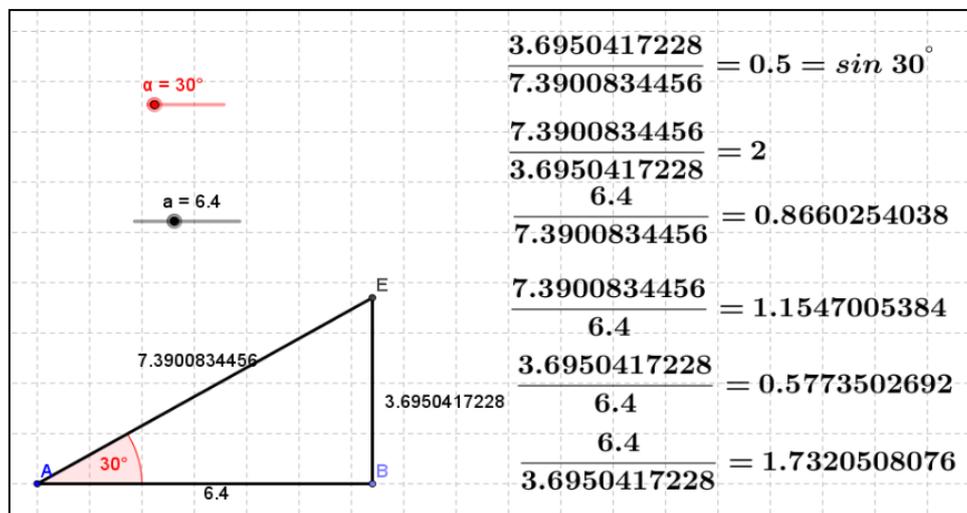


Figure 5.70: Teacher-made applet to explain $\sin 30^\circ$

GeoGebra helped the students to understand the trigonometric ratios, for example “Why is the $\sin 30^\circ = 0.5$?” Students constructed their triangles on graph paper and the teacher-researcher constructed his triangle in GeoGebra (see Figure 5.70). Students could easily see that as long as the angle was 30° the ratio stays 0.5. They were thus exploring with the pen-and-paper activity and compared their answers with each other. They were afforded an opportunity to see that there were six possible ratio that can be computed and that the answer for each ratio remains the same no matter how big or small the triangle. Figure 5.69 shows that forty-one (89.1%) of the students strongly agreed, or agreed, that this interactive lesson helped with understanding of the trigonometry ratios. This activity was also linked to the concept of similarity of triangles. The students made a conjecture on the basis of intuition, numerical investigation and measurement. This refers to quasi-empirical methods of De Villiers (2004). The teacher-researcher validated their answers with GeoGebra.

Figure 5.71 shows the GeoGebra applet the teacher-researcher used in the interactive lecture on trigonometric general solutions.

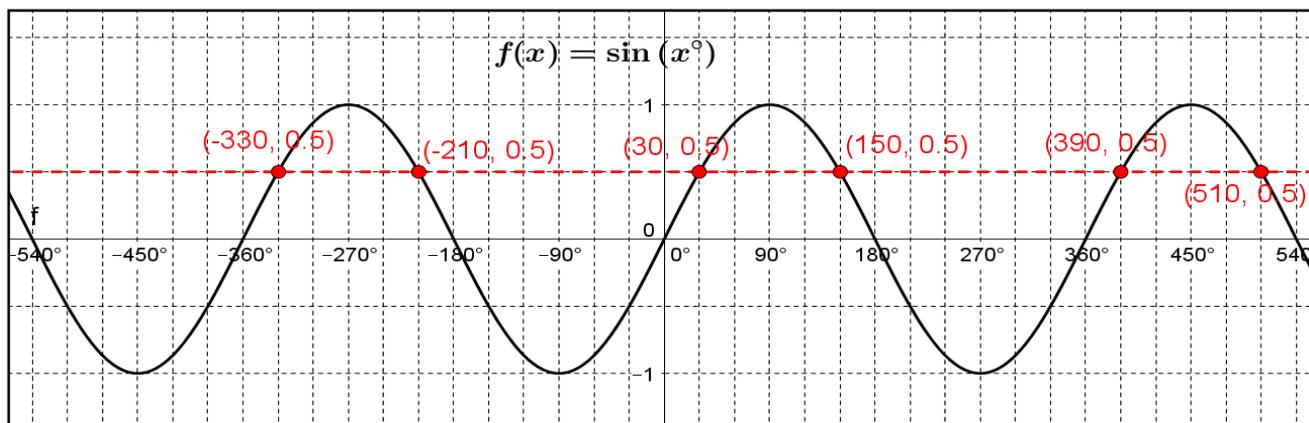


Figure 5.71: GeoGebra applet to explain the concept of general solutions

Although GeoGebra was utilised in a lecturing approach, the teacher-researcher involved the students in this lesson. The students had to complete the activity on Trigonometric equations (example 10 in Appendix R) and then the teacher-researcher used the GeoGebra applet (see Figure 5.71) to show the periodic nature of trigonometric functions and to teach the concept of general solutions. The teacher-researcher therefore involved the students in certain sections of the lesson. The teacher-researcher used the GeoGebra applet to show the students that the value of x in $\sin x = 0.5$ can be computed by adding 360° to 30° , or subtracting from 30° and this results in $360.n$ being added. The students were also shown that the same applies to 150° . The students acquired physical knowledge with GeoGebra when the teacher-researcher showed that this pattern continued by showing them in GeoGebra that the graphs go to positive and negative infinity. The students acquired logico-mathematical knowledge when the numerical investigation was done and the final answer could be written as $30^\circ + 360.n$ or $150^\circ + 360.n$ where $n \in \mathbb{Z}$. The same activity was done for general solutions of the tangent and cosine equations.

One student, after this lesson, stood up and said that she never understood what general solutions were all about. She was just applying the algorithm and never knew that the graph continues on both sides and that the general solutions represent infinite answers to the trigonometric equations. The students' responses to the post-questionnaire show perceptions of learning gains in general solution of trigonometric equations. Other students also responded to the following question in the in-depth and focus group interviews:

Researcher: Do you think GeoGebra helped you to understand general solutions better?

Student ID2: Laughing uhm.

Researcher: When you came to SciMathUS, did you have an understanding of general solutions?

Student ID2: Not from school.

Researcher: What happened when the graphs were viewed to right and left in GeoGebra?

Student ID2: It remained constant.

Researcher: What do you mean by constant?

Student ID2: It goes on and on and on.

Researcher: So do you now understand the concept of general solutions?

Student ID2: Yes, but it doesn't mean I can do it. I understand what it is.

Student ID2 could see by moving the Graphic view in GeoGebra that trigonometric functions are periodic. This student acquired physical knowledge in this way about the trigonometric functions. She also pointed out it does not mean that she could solve a general solution problem but understand the concept of it. Another student pointed out the following:

Student ID6: Ja, ek dink meneer het grafieke met sny punte gehad. Ek dink dit was cos en sin grafiek. Ek het daar gesien hoe om GeoGebra te gebruik, want daar kon ek die sny punte sien en ook aan die kant (sy verwys na Algebra venster in GeoGebra). Ek het in die venstertjie die sin en cos vergelyking gesien en toe wys meneer vir ons dat die sny punte in GeoGebra dieselfde is as ons dit algebraïes uitwerk. (*Yes, I think that sir had graphs with intersection points. I saw there how to use GeoGebra, because I could see the points of intersection on the other side (she refers to the Algebra window in GeoGebra). I could see in the window the sine and cosine equations and sir showed us that the points of intersections in GeoGebra were the same if we work it algebraically.*)

Researcher: Was daar net een, twee of drie punte? (*Were there only one, two or three points?*)

Student ID6: Nee, daar was nog sny punte. (*No, there were more points of intersection.*)

Researcher: Hoe het jy dit in GeoGebra gesien? (*How did you see it in GeoGebra?*)

Student ID6: Ons kon dit sien as ons 360 grade met twee maal (Sy verwys na bv. $360 \cdot k$ waar $k = 2$ is) ... toe het meneer gewys dat nog ander sny punte was. (*We could see if you multiply 360 degrees by two then sir showed that there were more points of intersections.*)

Researcher: Hoe het ek dit gewys? (*How did I show you?*)

Student ID6: As jy sal aangaan op die as (sy verwys na die x -as) het meneer gewys dat daar nog baie sny punte was. (*Sir moved the axis (she referred to x -axis) and showed us that there were more points of intersection.*)

Student ID6 could also see the periodic nature of the trigonometric functions and pointed out that she could see the graphs intersect more than once. The student further explained that she could see the meaning of the notation $360 \cdot k$, where $k \in \mathbb{Z}$. She could see that by substituting k with numerical value that it resulted in another point of intersection. She did not indicate that k has to be an integer. The student referred here to how the teacher-researcher linked the notation with the graphic representation of the answer. She could subsequently link the algebraic and graphic representations. This shows how GeoGebra helped for coherence formation on deeper structural level of the notation $360 \cdot k$, where $k \in \mathbb{Z}$ (Seufert & Brünken, 2004). This also shows how the teacher-researcher's TPACK afforded the students an opportunity to link their graphic observations to the algebraic notation of the answer.

The following student felt that he was shown in school only how to calculate the general solution of trigonometric functions:

Researcher: Do you think GeoGebra helped you to understand general solutions better?

Student ID10: Aah....

Researcher: I am not asking about the solving of it. What was your understanding of general solutions?

Student ID10: Last year I didn't have an understanding what a general solution was. I felt that I was just asked to solve it, but I didn't know conceptually what it actually meant.

Researcher: So why do trigonometric equations have general solutions, but not the quadratic equations?

Student ID10: Trigonometric graphs I think they are continuous (*He meant here it is periodic*). So there has to be a general solution I think because there are many intersection points.

The last response of Student ID10 shows that he now had an understanding of what general solutions conceptually mean. He pointed out that general solutions are applicable to trigonometric equations because of their periodic nature but not to quadratic equations. He further pointed out that trigonometric graphs can intersect each other many times. This showed that he had acquired physical and logico-mathematical knowledge. He could see (physical knowledge) that there is an infinite number of answers that can solve trigonometric equations. The differentiation that the student made between the quadratic and

trigonometric equations referred to the logico-mathematical knowledge he obtained during the interactive lecturing of general solutions of trigonometric equations with GeoGebra.

Students in the focus group interviews also acknowledged that GeoGebra helped them to understand the periodic nature of the trigonometric functions. The following students commented on the solving of $\sin x$ and $\cos x$ equations in this manner:

Student FG8: If you scrolled to the sides you could see where they intersect again.

Student FG9: You could not see it after 360 degrees ... but you could also see it on GeoGebra after 1020 degrees (she meant 1080 degrees)

Student FG11: Ek het laasjaar gedink dat dit gaan net tot daar. (*I thought last year it stopped there.*)

Student FG9 and FG11 thought that the trigonometric functions stopped at 360° . The CAPS (DBE, 2011a) only requires the students to sketch the trigonometric functions between -360° and 360° . That is why they thought that the graphs stopped there. These students in this way acquired physical knowledge about the periodic nature of trigonometric functions and understood, after the interactive lecture, why trigonometric equations have general solutions. The visualization of trigonometric functions in GeoGebra therefore helped the students to make sense of the concept of general solutions. The students' responses in the interviews referred to what they observed during the lesson on general solutions (see Figure 5.70). They could make conjectures from the GeoGebra applet (see Figure 5.70), could do pattern snooping and were seeking to formulate symbolically the given situation (Rasmussen *et al.*, 2005). Treffers (1987) calls this process of negotiating and generalizing informal solution processes, horizontal mathematization. The fact that the students could see that trigonometric functions are oscillating led to their understanding why there is more than one solution to a trigonometric equations and showed the understanding of the symbolic notation $360.k$, where $k \in \mathbb{Z}$. This referred to the students' vertical mathematical reasoning and acquiring logico-mathematical knowledge in this way.

The triangulation of the analysis of the qualitative (QUAL) and quantitative (quan) data in this section show the students' learning gains in transformations of functions and circle geometry. The students' responses during the in-depth and focus group interviews (see section 5.3.1.1) show how GeoGebra afforded them an opportunity to acquire physical and logico-mathematical knowledge of transformations of functions. The students' responses also show how GeoGebra gave them deeper understanding of trigonometric equations.

5.4 STUDENTS' EXPERIENCES AND GESTURES WHEN TAUGHT WITH GEOGEBRA

This section presents both quantitative (quan) and qualitative data (QUAL) that will attempt to answer the following sub-question: *What were students' experiences and gestures when taught with GeoGebra?* This section presents the students' responses on how they experienced working with GeoGebra instructional activities. The challenges that they experienced are covered in section 5.5.

5.4.1 Students' enjoyment, interest and confidence in the different approach to teaching and learning

Results from the pre-questionnaire (see section 5.3.3.1) show that students were not actively involved in the learning process before entering the SciMathUS programme. Only 12 (25%) of 48 students strongly agreed, or agreed, that they were capable of finding out Mathematics on their own. Thirty-five (72.9%) students strongly agreed, or agreed, that they preferred that the teacher should first give notes and explain when (s)he starts a new topic. Twenty-eight (58.3%) students believed that they only understand Mathematics well when the teacher was showing them what to do. This showed that the students were not engaged in the learning process. One of the key foci of the teacher-researcher was to create an environment where students could enjoy Mathematics and to engage them in the learning process. Fallstrom and Walter (2009) point out that enjoyment may boost a student's confidence. Freudenthal (1991) points out that although Mathematics can be abstract, the user must still enjoy it. This intervention with GeoGebra as teaching and learning tool transformations of functions and circle geometry, was based on a constructivist approach, and enjoyment and fun are key aspects to engage students in active learning (Lu, Lin, Lin, & Su, 2007). Tomlinson (2014) observes that when conceptual GeoGebra activities are incorporated, students enjoy calculus more. The teacher-researcher also identified enjoyment as one of the key factors for students to excel in Mathematics. The teacher-researcher is also of the view that if students found Mathematics interesting, they possibly would enjoy it and engage in the learning process. Xie, Alissa and Nima (2008) emphasize that enjoyment and engagement are integral and essential aspects of children's playful learning experiences.

The following responses of students' during the in-depth and focus group interviews show how they enjoyed and found the different approach to teaching. The in-depth interview below was with a student who entered the programme with 63% for NSC Mathematics and then went on to obtain 83% in her SciMathUS' year.

Student ID4: Sometimes it was not boring... but long... it was nice and sir showed us different stuff.

Researcher: How did you experience the activities on amazing relationships? (see Appendix G).

Student ID4: I really didn't know where to go at first, but the next class I could see. But it was fun ... (laughing) at that moment it was nice to do that stuff.

Researcher: Do you think it helped you to understand functions, as well as domain and range better?

Student ID4: I think mainly what functions can be related to. The domain and stuff I understood (*she had a good understanding of domain and range before*), but it was nice to apply it to something. It sticks in your memory.

Researcher: What about the realistic contexts that were used?

Student ID4: It's like something in the back of your mind – so you remember it ... so it refers to something and it triggers something else.

Researcher: Do you think there is a place to teach Mathematics in that way?

Student ID4: Yes I do. It makes it easy to understand.

Although this student scored above 60% when she entered the SciMathUS programme, she enjoyed working with GeoGebra and with realistic contexts. Student ID6 responded to the same question in this way:

Student ID6: You can use different colours [*enablements*]. It is colourful and it *stimulates my brain* and makes it more interesting for me.

Some of the students' focus group interview, found it exciting to work with GeoGebra and responded with regard to how they experienced the teaching of Mathematics during this invention:

Student FG8: A good comparison is to take your mark that you got. Your mark is a *good reflection of how well your brain was entertained* during the lesson. The more entertained your brain was during the lesson, the more it will remember. Your mark is a good reflection ... comparing this year mark and last year's mark.

Student FG10: 'n Mens is eintlik opgewonde om te dink hoe ek dit op GeoGebra gedoen en hoe dit sou gelyk het. Die resepte gaan na die tyd weg. (*You are actually excited when you think how you did it on GeoGebra and how it would have looked like. The recipes go away after a while.*)

Student FG11: GeoGebra het my gehelp om nie goed soos reseppies te memoriseer nie, veral met funksies. (*GeoGebra helped me not to memorise things such as recipes, especially with functions.*)

Student FG11: I told my mother [*potentialities*] about the tool and the apps that we used. When I go home next time, I want to tell my former teachers [*potentialities*] that GeoGebra can help children 100 times better to understand Mathematics instead of memorising it.

Student FG7: Last year ... you ask me not to compare the teachers ... but he was a very good teacher ... he was excellent, but as Student FG11 said it was boring, It was like uhh... it just made Mathematics an enemy. With GeoGebra it rocks (*all of students responded and gave a 'high five' as hand gesture that they all agree.*)

Student FG10 and FG11 felt that GeoGebra excited them and it helped them not just to memorise concepts. This refers to these students' perception of having a deeper understanding and not just memorising. Student ID6 and FG8 responses on how it stimulates the brain refers to the cognitive impact of GeoGebra on the students. According to Sousa (2001; 2006) students' average retention of material can move up to 20% with visual activities.

The same group responded to the question in the following way when the researcher ventured with the statement that same topics can be taught without GeoGebra.

Student FG8: It is the visualising.

Student FG7: It gets stuck in your mind. It doesn't just help you to pass your test, it also gives you greater understanding of Mathematics as a whole.

Student FG9: You also start believing it, because can see it. You also have more faith in your answers. Sometimes you used to draw inverse graphs, you didn't know that if it was correct, but GeoGebra shows you how it looks. When you get a strange equation you actually have more faith because you see it. GeoGebra proves it and it is not just calculations.

Student FG7 and FG9 shared the view that GeoGebra helped them to visualise it and bestowed confidence in Student FG9 that her algebraic calculation made sense. This shows that this student was motivated to use GeoGebra. Moore (2017) asserts that manipulatives

have the ability to help with the cognitive process and also have the advantage of engaging students and increasing both interest in and enjoyment of Mathematics.

The students were asked after the first three days in the computer lab and after the first three months of teaching (AT) to give written and anonymous feedback. The following are sample responses:

Student AT3: It was really *an eye opening experience*, viewing transformation of graphs with dynamic software (referring to GeoGebra). Not only *was it fun*, but I have a much wider, broader and *in-depth understanding* of reflections, translations, enlargements and reductions.

Student AT4: That one is the best, even if you are sleeping at night, it is ringing in your mind. You see all the basic things about the graph.

Student AT5: Well, it is always great to *see the visual stuff* when you are learning. Hence the software is *quite amazing* and it enables us to easily identify the common mistakes we would normally make and learn from them.

Student AT6: I *enjoyed* it. It helped me *to visualise* the graphs.

Student AT7: *Exceptional*. It made concepts easier to grasp.

Student AT8: It was *incredibly* and *indescribably* great and *informative*.

All the students reacted positively to GeoGebra as a teaching and learning tool in the computer lab and the use of the teacher-researcher's interactive instruction. Student AT4 testified that what (s)he learnt kept on ringing in his/her mind even while (s)he slept. This is what the teacher-researcher wished to achieve. The focus was that students should start to talk about Mathematics and that it is not just a subject in the classroom. All the key responses from the students such as *an eye opening experience*, *quite amazing*, *incredibly* and *indescribably*, *informative* and *exceptional* refers to affective aspects of GeoGebra. This accords with Freudenthal (1991) that although Mathematics can be abstract, the user must still enjoy it.

The responses of the abovementioned students show that GeoGebra helped with conceptual understanding because they felt that it helped with visualisation. Students FG1 pointed out that he enjoyed working with GeoGebra, because of his passion for technology.

Student FG1: I really enjoyed it because I like working with computers. Most of the times I am on the phone, using a computer whenever I am doing my work. When you introduced GeoGebra to us, it was actually cool, because it actually

showed us how the graphs looked like. If I had a problem, I could type the function on my phone and was able to see the graph straight away.

Student FG1 downloaded the GeoGebra app on his smart phone. He emphasised that he was one of those students that was always on their phones. He sometimes used the GeoGebra app to draw the graphs if he wished to visualise it. This student thus utilised GeoGebra as an artefact and this emphasises the student's instrumental genesis (Trouche, 2004). The student utilised GeoGebra to shape and it helps him to understand concepts better, i.e. the instrumentalisation process (Trouche, 2004). The instrumentalisation process is how the student used GeoGebra on his own to be a tool, for example to validate their answers and test their conjectures. Oblinger and Oblinger (2005) and Tapscott (1999) allude to the fact that Generation Y is tech savvy and they learn better through discovery than being told.

Student FG16 admitted that although she enjoyed GeoGebra activities she was someone who did things in a certain way. This showed clearly that she was not open to change:

Student FG16: I did not enjoy it that much. I do not know. I am used to doing one thing in a certain way and then I just want to continue in that way. Now when you introduced a new thing, like GeoGebra, I could not put the two things together. I did not really enjoy it, but it was fun.

This student entered the SciMathUS programme with 62% for her NSC Mathematics, but only moved up to 66% in her re-write in her NSC Mathematics in 2014. It can be argued that she was not open to change and did not improve her marks with at least 15 percentage points.

The abovementioned qualitative (QUAL) data will now be triangulated with the quantitative (quan) data gathered from students' in-depth and focus group interviews, as well as their written feedback immediately after their first experience with GeoGebra in the computer lab. The post-questionnaire was also used to assess students' perceptions regarding whether they enjoyed the activities used with GeoGebra for the topics transformation geometry, quadrilaterals, circle geometry and trigonometry. All the items in the different constructs on the transformation geometry, quadrilaterals, circle geometry and trigonometry focused on how the students found it easy and interesting to learn with GeoGebra, and how they enjoyed working with it was regrouped in one construct. The Cronbach alpha reliability coefficient for the eleven items in this regrouped construct for enjoyment and interesting is 0.835. A Cronbach alpha of $\alpha \in [0.8; 0.9]$ is good (George & Mallery, 2003).

Figure 5.72 shows students' responses to four of the eleven items from the post-questionnaire regarding the experiences whilst working with GeoGebra. Figure 5.72 shows four of the eleven items of construct.

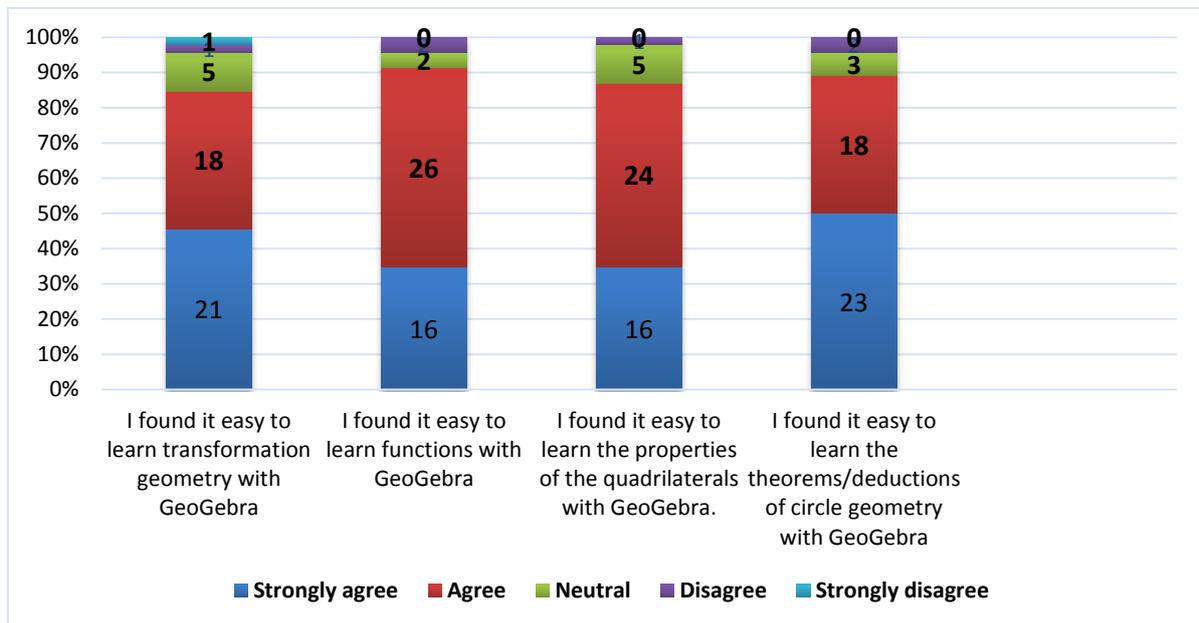


Figure 5.72: Students' responses on whether they found it easy to learn concepts with GeoGebra or not.

Students were asked to explore transformation geometry, transformations of functions and circle geometry in the computer lab. An average of 40 (87%) students indicated that they found it easy to learn using GeoGebra in the computer lab and the teacher using it as a teaching tool. The lesson on the properties of quadrilaterals was with an interactive lesson approach. The students explored circle geometry and transformations of functions with GeoGebra in the computer lab. Most of the students were comfortable to explore GeoGebra and were able to engage in the learning process with it.

Figure 5.73 shows the students' responses to the other seven of the eleven items mentioned in Figure 5.72. Although the students enjoyed the teaching of these topics with GeoGebra, they also found it interesting to explore for themselves instead of the teachers merely giving them rules and algorithms.

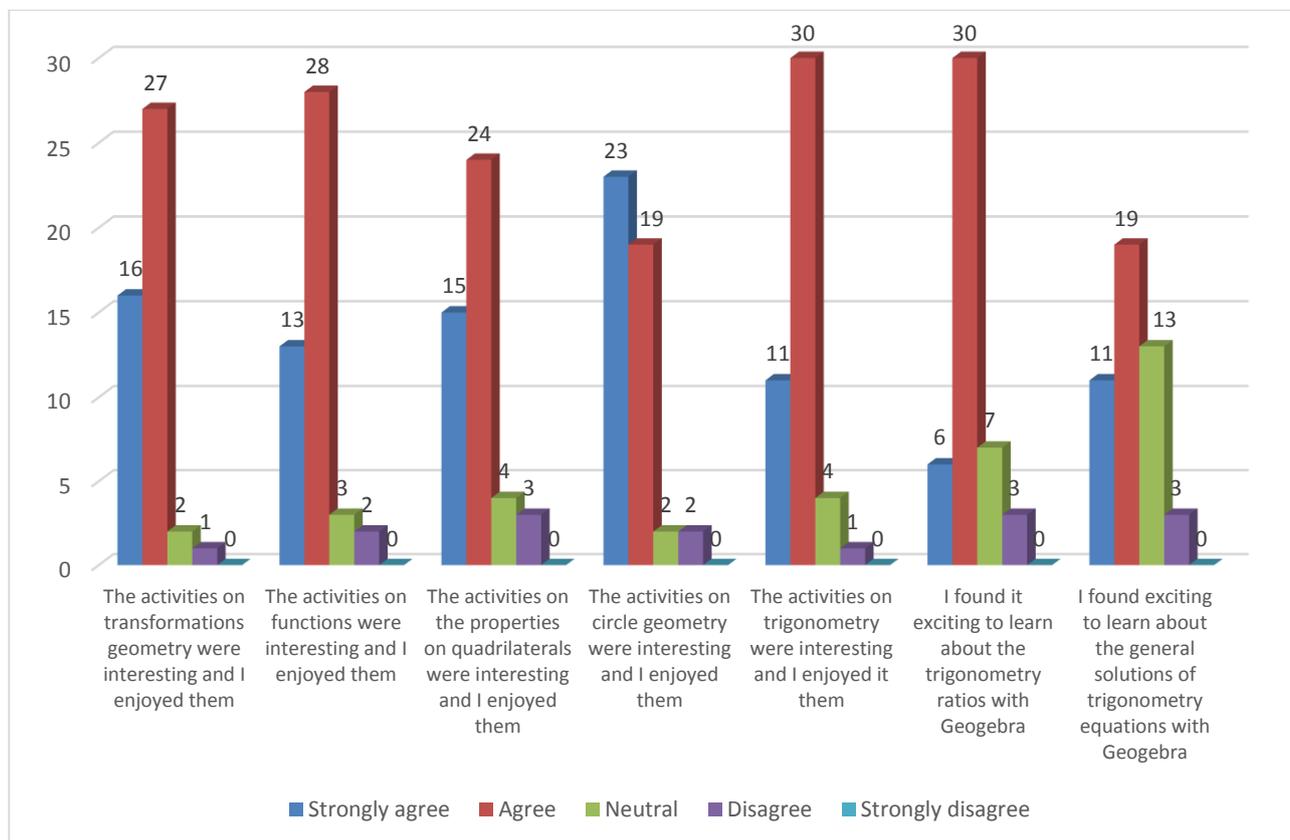


Figure 5.73: Students' enjoyment and interest in GeoGebra activities (n= 46)

Figure 5.73 shows an average of 41 (89%) students strongly agreed or agreed that they enjoyed the GeoGebra activities utilised in the teaching and learning of functions, trigonometry, transformation geometry, quadrilaterals and circle geometry, and trigonometry. An average of 33 (72%) strongly agreed, or agreed, that it was exciting to learn trigonometric ratios and equations with GeoGebra. The teacher-researcher utilised GeoGebra as a teaching tool for trigonometry. Freudenthal (1991) emphasise that students must discover and Mathematics has to be enjoyable so that learning by reinvention, may be motivating. The activities with GeoGebra were designed so that students could learn by guided reinvention. Sutton and Krueger (2002) emphasise that interesting contexts stimulate learning and retention. Students' interest and curiosity can in this way be aroused (Begg, 1999).

The students also gained confidence in using GeoGebra themselves to improve their understanding of Mathematics. Three items (SPEGTTTG6, SPETTTG11 and SPEGTTTEGC5) of the construct for transformation geometry, functions and graphs, processes involved using GeoGebra and circle geometry and, three items (PGTT10, PGTT13 and PGTT15) of the construct teaching or learning different topics with GeoGebra

in the post-questionnaire (see Appendix D) were regrouped together. These items were regrouped to focus on students' confidence. The Cronbach alpha reliability coefficient for the six items in this regrouped construct on students' confidence is 0.700. A Cronbach alpha of $\alpha \in [0.7; 0.8)$ is acceptable (George & Mallery, 2003).

Figure 5.74 presents quantitative responses on how confident they felt about exploring Mathematics with GeoGebra.

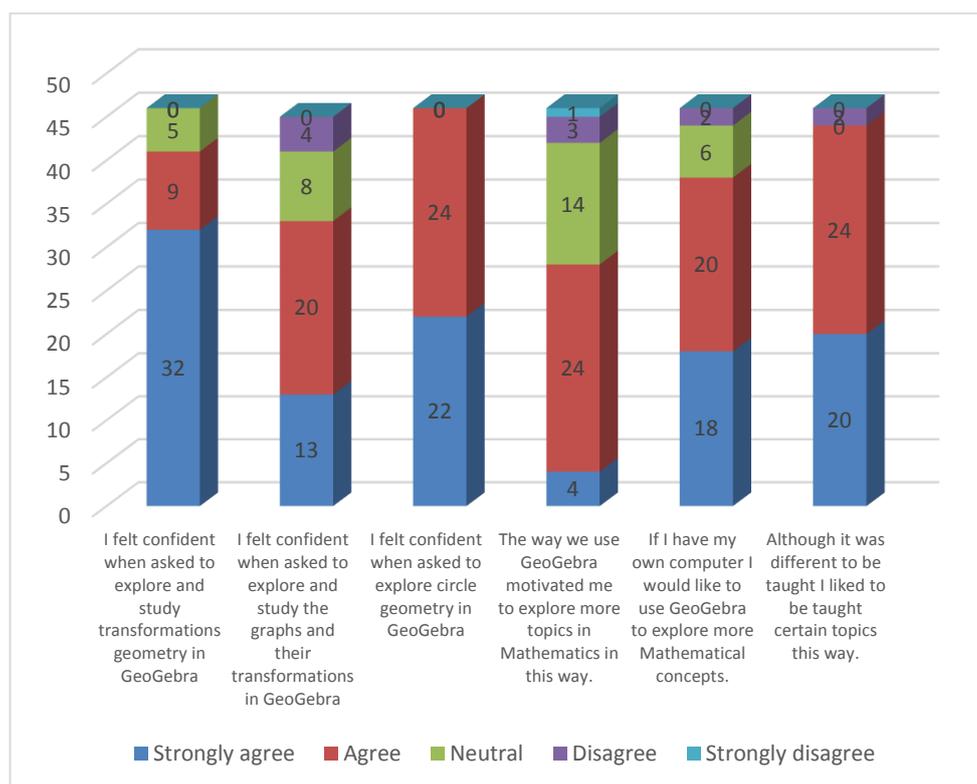


Figure 5.74: Students' confidence to explore with GeoGebra.

Figure 5.74 shows an average of 40 (87%) students strongly agreed, or agreed, that they felt confident when asked to explore transformation geometry, transformations of functions and circle geometry. Thirty-eight (83%) students strongly agreed, or agreed, that the way GeoGebra was used, motivated them to explore more mathematics concepts. Figure 5.74 also shows that 18 (39%) students strongly agreed and 20 (43%) agreed that they would like to explore mathematical concepts if they had their own computer. The majority of the students did not have their own computer and made use of university facilities. Although students indicated that they would like to use it, their responses in the in-depth and focus group interviews did not show that they utilised it frequently on their own. Figure 5.74 further shows 44 (96%) students strongly agreed, or agreed, that they like different approaches of teaching with GeoGebra.

The triangulation of the analysis of the qualitative (QUAL) and quantitative (quan) data show that most of the students enjoyed the different teaching approach and found interesting. The students found it easy and exciting to learn transformation geometry, transformations of functions, circle geometry and general solutions of trigonometric equations with GeoGebra. There were students who found the different teaching approach not enjoyable. This will be discussed under the challenges that the students experienced whilst working with GeoGebra (see section 5.5).

5.4.2 Students' gestures during discussions whilst working with GeoGebra

This section shows how the students utilised hand movements to express their thinking. Hand movements were the dominant gestures observed while the students were interacting and exploring with GeoGebra in the computer lab. The following shows some of the students' hand gestures recorded during the observation while working on GeoGebra (OG) in the computer lab.

In Figure 5.75 one student was showing another student the shifting of the original graph.



Figure 5.75: Student showing that the original graph reflected to the left.

Student OG5 (see Figure 5.75) was pointing at the graph in red (inverse graph of the parabola). She explained to the other student that to get to the red graph (inverse), it appears if the black graph was translated to the left and then rotated. The pointing with her finger is the student's gesture (Scherr, 2004) to show her fellow what she thought happened to parabola when reflected in $y = x$. Her gesture was thus linked to what she did on the GeoGebra (the artefact). She performed the task to drag a point on the parabola, i.e. action scheme (Trouche, 2004), and then found the path of that point reflected in $y = x$.

The two students therefore had an opportunity to validate and make conclusions of the type of transformation on the basis of intuition or experience obtained through GeoGebra. De Villiers (2004) refers to this as quasi-empirical methods.

Figure 5.76 shows the graphs on the computer screen that the student in Figure 5.75 explained.

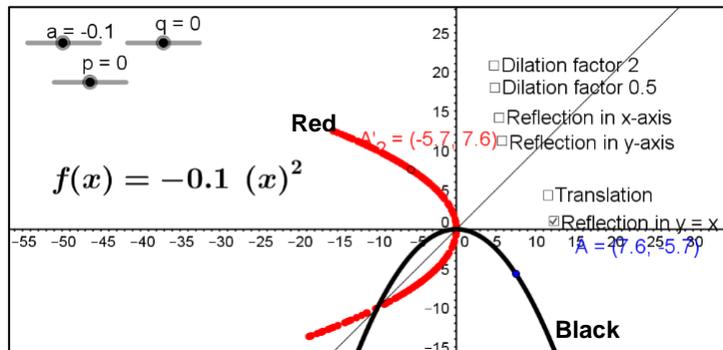


Figure 5.76: Sketches on screen of student in Figure 5.75.

They could not find the equation with the sliders, until the teacher-researcher showed them how to type in the answer that they found algebraically. The student explained what she viewed on the screen, but was actually incorrect because it was a reflection in $y = x$.

Figure 5.77 shows the discussion the teacher-researcher and a student had on the reflection of a function in the line $y = x$.



Figure 5.77: Student OG1 showing hand gestures what happens to x and y -values when reflected in the $y = x$

Student OG1 (see Figure 5.76) showed the teacher-researcher with hand movements, along with speech, that the x - and y -values swapped and were therefore mapped to result in the red graph as presented in Figure 5.78. As discussed in section 5.2.3.1 (see Figure 5.11) the student used hand gestures to explain that the x -intercepts of the parabola (black graph) are mapped to the y -intercepts of the inverse function (red graph) of the parabola. The hand gestures gave visual access to the structure of a reasoning action (Krause, 2015). The hand gestures utilised in Figure 5.75 and 5.77 show a socio-constructivist process of negotiating meaning of reflection in $y = x$.

Figure 5.78 shows how a student also uses hand movements to demonstrate to his classmate the stretching and shrinking of graphs.



Figure 5.78: Student OG6 explains dilation with his hands.

Student OG6 (see Figure 5.78) showed his classmate the meaning of dilation. He said: If the scale factor is two, the graph becomes leaner. He showed with his hands and what he meant was the following graph as shown in Figure 5.79. The hand movements were supplemented by the abovementioned verbal statements (Krause, 2015).

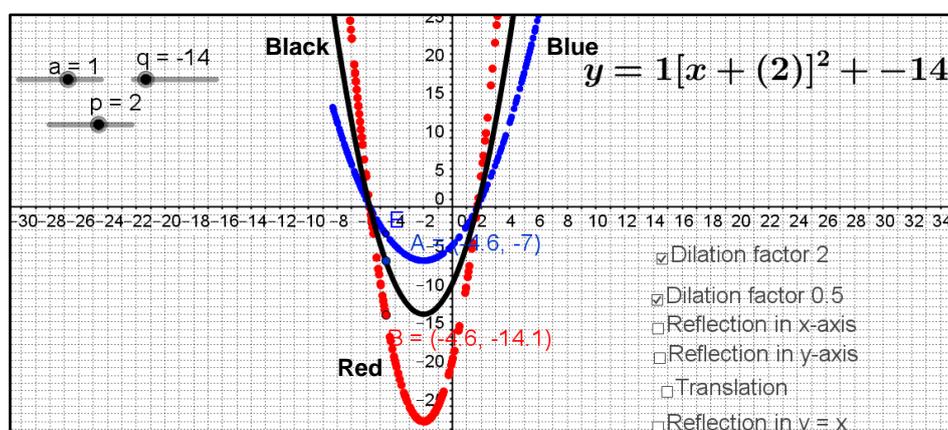


Figure 5.79: Sketches on screen showing dilation (stretching and shrinking)

The using of hands refers to the student's gesture (Scherr, 2004) to show his fellow student (see Figure 5.79) what he thought happened to parabola when he changed the value of slider a (see Figure 5.79) on the GeoGebra (the artefact). He performed the task to drag the slider to transform (in this case dilation) the parabola, i.e. action scheme (Trouche, 2004). Student OG7 put his thoughts into hand gestures (Okrent, 2017) . The hand gesture utilised in Figure 5.78 shows a socio-constructivist process of negotiating meaning of vertical stretching and shrinking. The two students therefore had an opportunity to validate and make conclusions of the type transformation on the basis of intuition or experience obtained through GeoGebra. De Villiers (2004) refers to this as quasi-empirical methods.

Figure 5.80 shows how two students interact and, using hand gestures, explain the type of transformation.

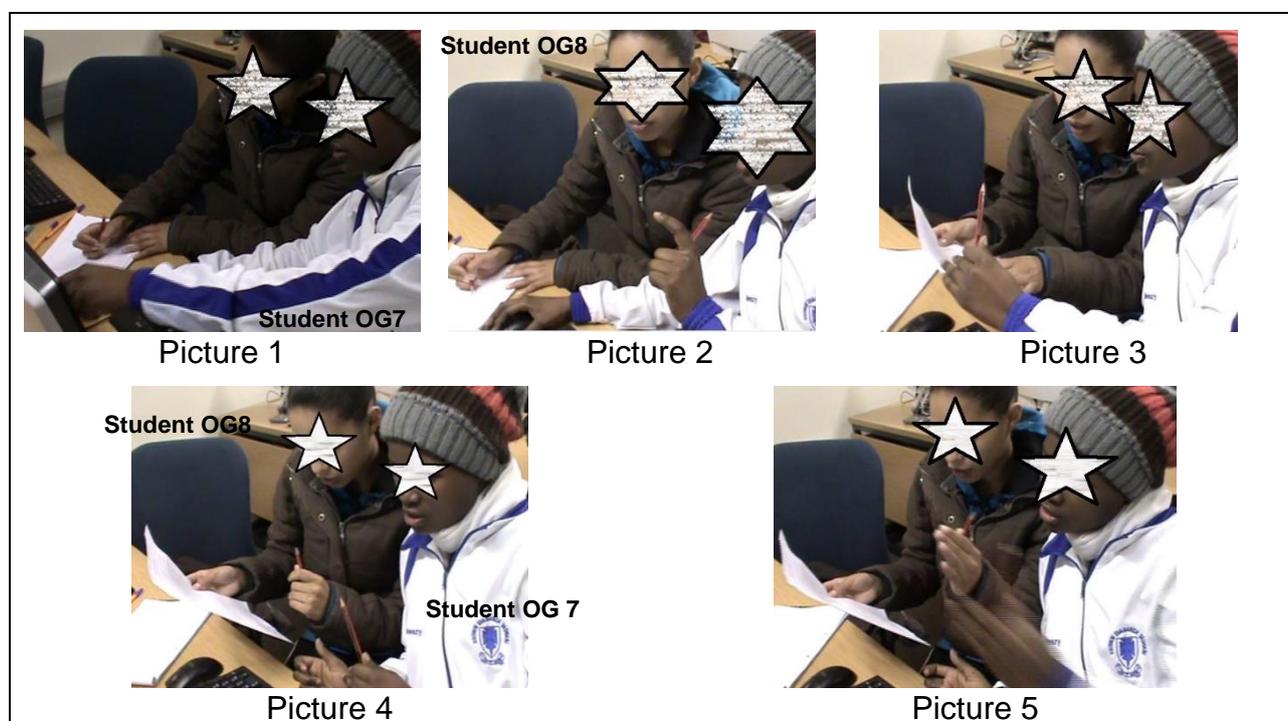


Figure 5.80: Students utilising hand gestures to explain the concept of translation

Student OG7 in Picture 1 wanted to know what happened (in her words) to the original graph (see black graph in Figure 5.81) to result in the transformed graph (see pink graph in Figure 5.81).

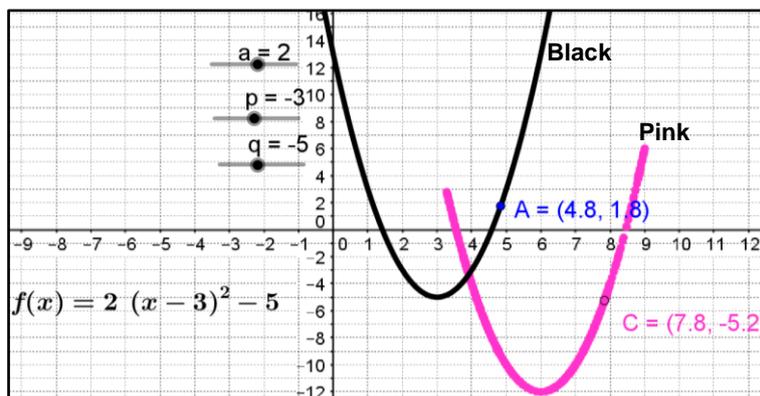


Figure 5.81: Sketch on screen showing the translated graph

Student OG7 suggested and showed with hand gestures that it is an enlargement (see Picture 2), but then she realised (see Picture 3) that the graph was only shifted downwards. She could therefore see that the value of slider a will stay the same. She dropped her hand to show the student on the right. Student OG8 confirmed this (see Picture 4) and also moved her hand downwards. In Picture 5 both students concluded that the graph also moved to the left (see their hand movements to the left). They could therefore see that the transformed graph was a translation of the original graph. The hand gestures in these pictures therefore provided unique and informative evidence regarding the nature of the students' mathematical thinking (Alibali & Nathan, 2012). After this interaction the students continued and used the sliders (see Figure 5.81) to find the equation of the transformed graph. The hand gestures thus helped them to understand the type of transformation and could therefore see that they only had use of sliders p and q . The hand gestures utilised in Figure 5.80 show a socio-constructivist process of negotiating meaning of translation.

Figure 5.82 shows some of the students' gestures recorded during the observation while working on GeoGebra (OG) in the computer lab.



Figure 5.82: Student OG10 using hand movements to explain what he visualized.

The gestures that were recorded were captured during the interaction between the teacher-researcher and students while they were having a discussion.

Student OG11: Sir, we have a discovery here. According to Student OG9 and OG10 ...

Student OG9 interrupted Student OG11.

Student OG9: The angle is double the other angle.

Student OG9 is showing it on the screen. Up to this stage the students were exploring the properties of circle geometry with GeoGebra and they were experimenting pattern snooping and conjecturing (Rasmussen, *et al.*, 2005). They could see that the angle at the centre is double the angle at the circumference, but did not investigate until this stage if it was always the case. They acquired knowledge with horizontal mathematising of their new world of circle geometry and it formed the basis for vertical mathematising. Student OG9 utilised hand movements to articulate the group's thinking about their discovery (Scherr, 2004). The hand movements supplemented the abovementioned verbal statements (Krause, 2015). This shows how the body was involved in thinking and speaking (Alibali & Nathan, 2012).

Teacher: Is it always like that?

They started exploring to confirm. All four students together said "no".

The students' discovery shows how they acquired physical knowledge of circle geometry theorem. Student OG10 responded by showing with his hands (see Figure 5.82) that it is not true at all times. He mentioned that if it passes a certain point (the camera was not on the screen) it is not true anymore. He showed the teacher on the screen. He was referring to the following sketches (Figure 5.76) on GeoGebra to conclude his arguments. Student OG12 was following the discussion.

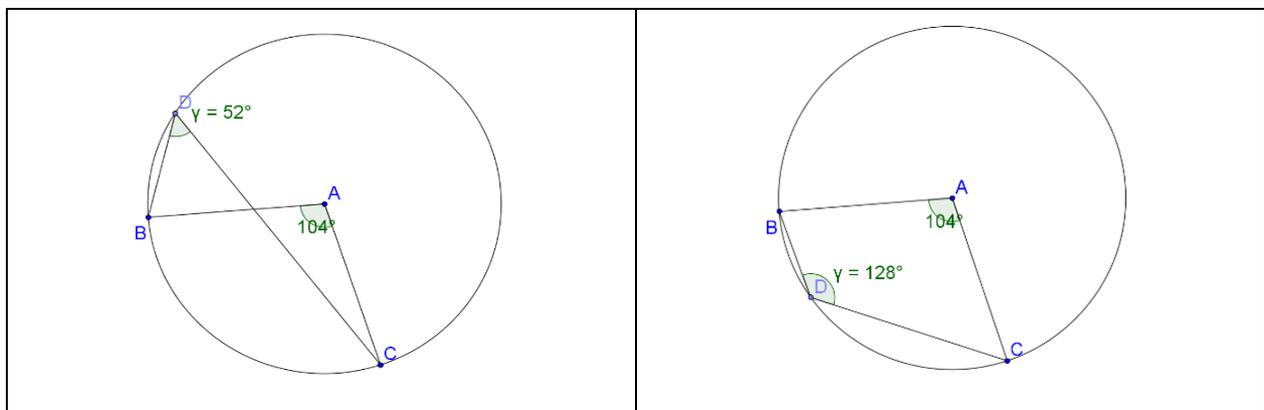


Figure 5.83: Sketches on screen of Student OG12.

Hence the students could conclude that the angle subtended by an arc of a circle at the centre, is double the size of the angle at the circumference, provided both are subtended by the same arc and are in the same segment of the circle. The students did not prove the theorem up to this stage, but started to generalise and formalise. This initial discussions between the students therefore lead to horizontal mathematical reasoning and progress to generalising and formalizing of a theorem, i.e. vertical mathematical reasoning. The hand gesture utilised in Figure 5.82 shows a socio-constructivist process of negotiating meaning of the theorem about the angle, the centre and the circumference. Krause (2015) points out that mathematical knowledge can be acquired with gestures during the social interaction.

Figure 5.84 shows how the teacher-researcher utilised hand gestures to explain vertical stretching and shrinking.

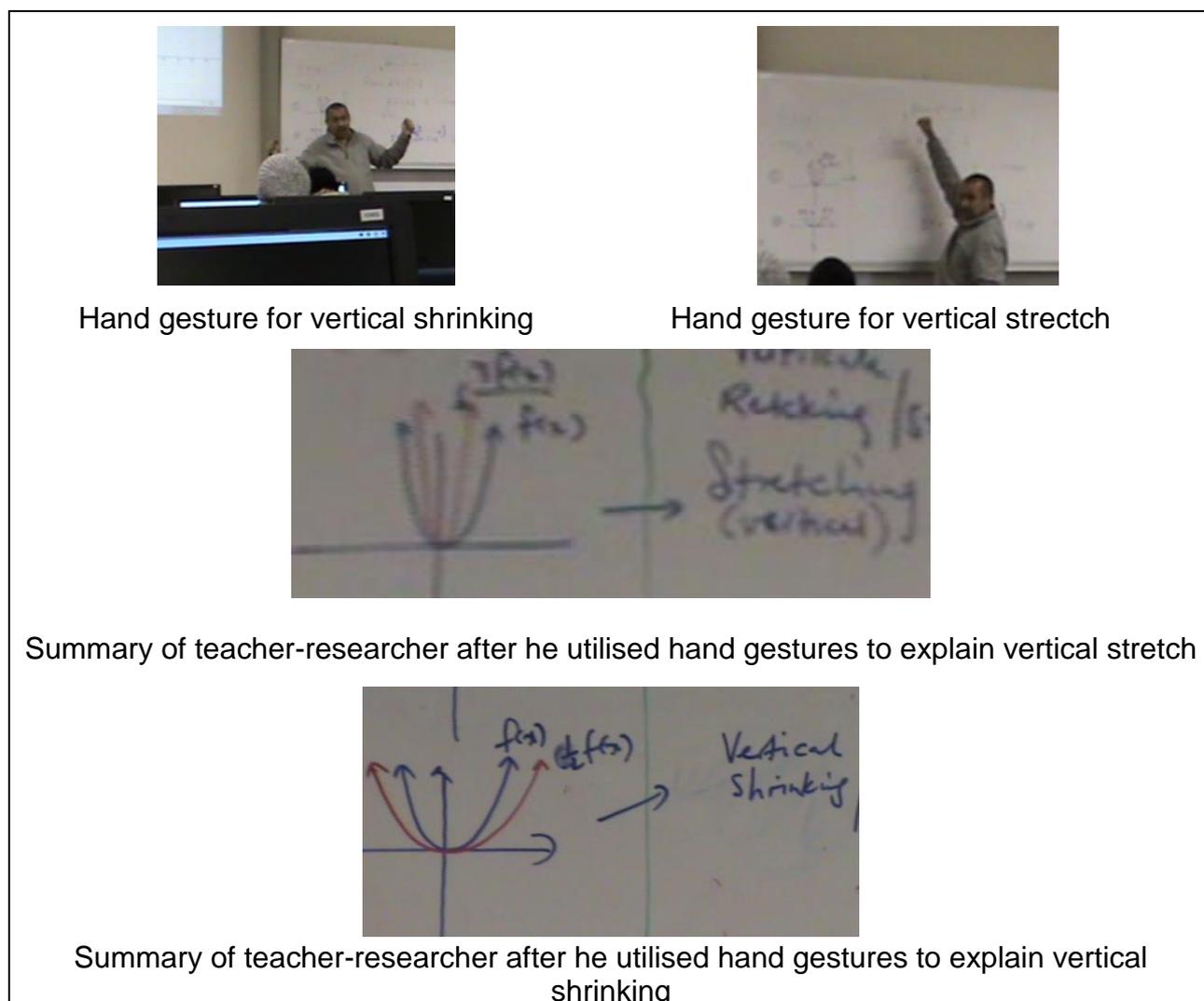


Figure 5.84: Teacher-researcher's hand gestures to explain the concept of vertical stretching and shrinking

Figure 5.84 shows the hand gestures that the teacher-researcher used after the students themselves explored vertical stretch and shrinking of functions. The teacher-researcher used the correct terminology with this presentation. By using terminology such as vertical stretching and shrinking for the dilation of graphs, the students acquired the correct terminology through social knowledge. The hand gestures thus specified aspects of the mathematical object presented in GeoGebra and subsequently enriched the verbal utterance of the teacher-researcher (Krause, 2015). These hand gestures were also showing the students that the transformation is not rigid and therefore the value a in, for example $y = a(x + p)^2 + q$ will have an effect on the shape of the graph.

All the hand movements presented above were supplemented with verbal statements (Krause, 2015). All the hand gestures presented in this section show how GeoGebra helped them to visualise concepts. The hand gestures therefore supplemented the visual affordances of GeoGebra and gave visual access to the structure of the reasoning action (Krause, 2015).

5.5 CHALLENGES THAT STUDENTS EXPERIENCED WHILST WORKING WITH GEOGEBRA

This section presents both qualitative and quantitative data that will attempt to answer the following sub-question: *What are the challenges that students experienced whilst working with GeoGebra?* The tools and commands in GeoGebra (Syntax), the students' computer skills and the different teaching approach were the challenges that the students experienced during the hybridisation of the student-centred approach and teacher as facilitator of learning approach.

5.5.1 Tools and commands in GeoGebra (Syntax)

This section focused on the students' responses during the in-depth and focus group interviews concerning the challenges they experienced whilst working with GeoGebra, the researcher's observations during the lessons in the computer lab and students' responses from the post-intervention questionnaire.

From the students' responses during the in-depth and group interviews, some of them experienced some challenges in the first session of using GeoGebra themselves. The first session in which the students worked on the computer, was on transformation geometry. They also started during this session with transformations of functions. This session was 1

hour 40 minutes long. Some of the challenges that they experienced were to find the icons or the tools. This refers to the student's instrumentation usage schemes to carry out a specific instruction to find the icon or tool (Trouche, 2004).

Some students also indicated in the in-depth interviews that they struggled to draw chords and to zoom in and out. During the observation, the zooming in or out was one of the largest challenges, because a few students, especially in the session on transformations of functions, rolled the wheel on the mouse and could not see where the turning point of the transformed function was (see red graph in Figure 5.85).

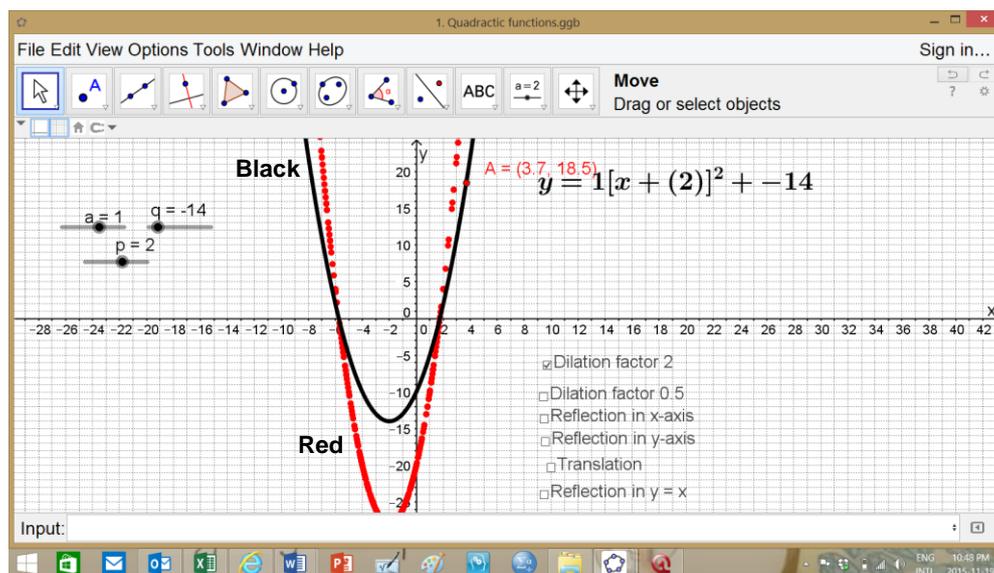


Figure 5.85: Students experiencing with the zoom in or out

Students also mentioned that they struggled mostly in the beginning when they used GeoGebra. However, it became easier as the following comments revealed:

Student ID6: It was tricky at first, because we did not know where all the stuff (icons/tools) was, and we first had to find them. However, once you became used to the program, it became much easier.

Researcher: How long did it take you to get used to it?

Student ID6: It took about half of the first session and more or less the same for the other sessions.

Another general challenge that students experienced during the computer lab sessions, was the measuring of angles. One student summarised this as follows:

Student ID8: It was a bit harder if the teacher did not show us step-by-step what to do. Like with angles you had to measure the angle ... sometimes I got the stuff

wrong, because I clicked on the wrong points and then got the wrong angle. I could see it was the angle on the outside. I tried that on my own, before you showed us. Initially it was a bit hard, but as we worked on it more it became a bit easier.

Student ID8 was referring to the measurement of angles in GeoGebra as shown in Figure 5.86. The student measured \hat{C} reflex and not \hat{C} acute.

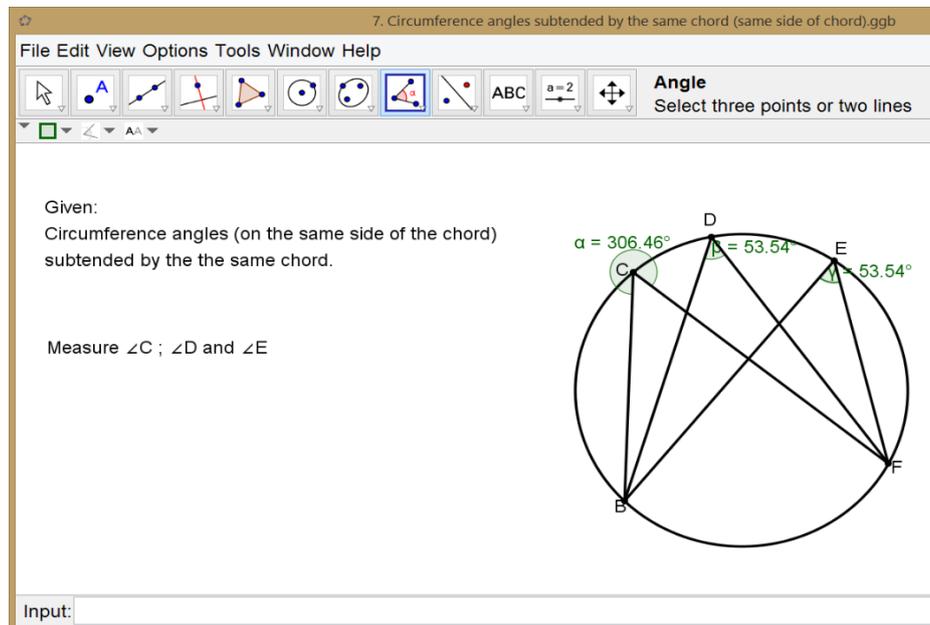


Figure 5.86: Student measuring reflex angle

Figure 5.86 shows the student's instrumentation action schemes to carry out a specific task (Trouche, 2004).

One student summarised why she sometimes found GeoGebra challenging as follows:

Student ID3: It was not really difficult. It was just that you went too fast and sometimes I missed the steps and I didn't know what to do. Then it became difficult because then I did not want to do anything.

This student pointed out that the teacher was sometimes too fast and she did not follow some of the steps properly. The student's computer skills were above average. She was one of the students who did not like the approach of teaching and learning with GeoGebra.

Another example of a challenge with the instructions on the designed activities was Student ID4:

Student ID4: Uhm ... uhm ... most of it was easy but the only thing is that you had to type in the function.

Researcher: In the input bar?

Student ID4: Yes ... the exponent and stuff that were a bit difficult because you had to know what to put in.

This student referred to the typing of exponential function in the input bar in the activity in Appendix L. The command how to type in the exponent was demonstrated in the designed activity. The challenges that this student experienced are referred to as command constraints in the instrumental theory (Trouche, 2004).

One student felt that she experienced challenges at the beginning, because the interface was different from the other computer software or programs:

Student FG20: It was just challenging because the interface was completely different to many other programs I am used to. However, it actually helped in that for graphs it actually showed you what happens and that it was better.

The abovementioned challenges were only sample responses from students' views in the in-depth and focus group interviews. The summary of the different types of instrumental orchestrations in the computer lab in this Chapter (see section 5.2.1), shows that the teacher-researcher spent time on *Technical-demo* orchestrations and supported students with *Walk-and-work by* orchestration with technical issues. Although the teacher-researcher's orchestrations were not a major barrier for the students' GeoGebra experience in the computer lab, the researcher took cognisance of students' challenges in the following year, when he took students to the computer lab.

The quantitative data results from the post-intervention questionnaire show that the majority of students did not experience major problems.

Figure 5.87 shows that most of the students found GeoGebra easy to open and make use of the tools mentioned because they did not experience problems.

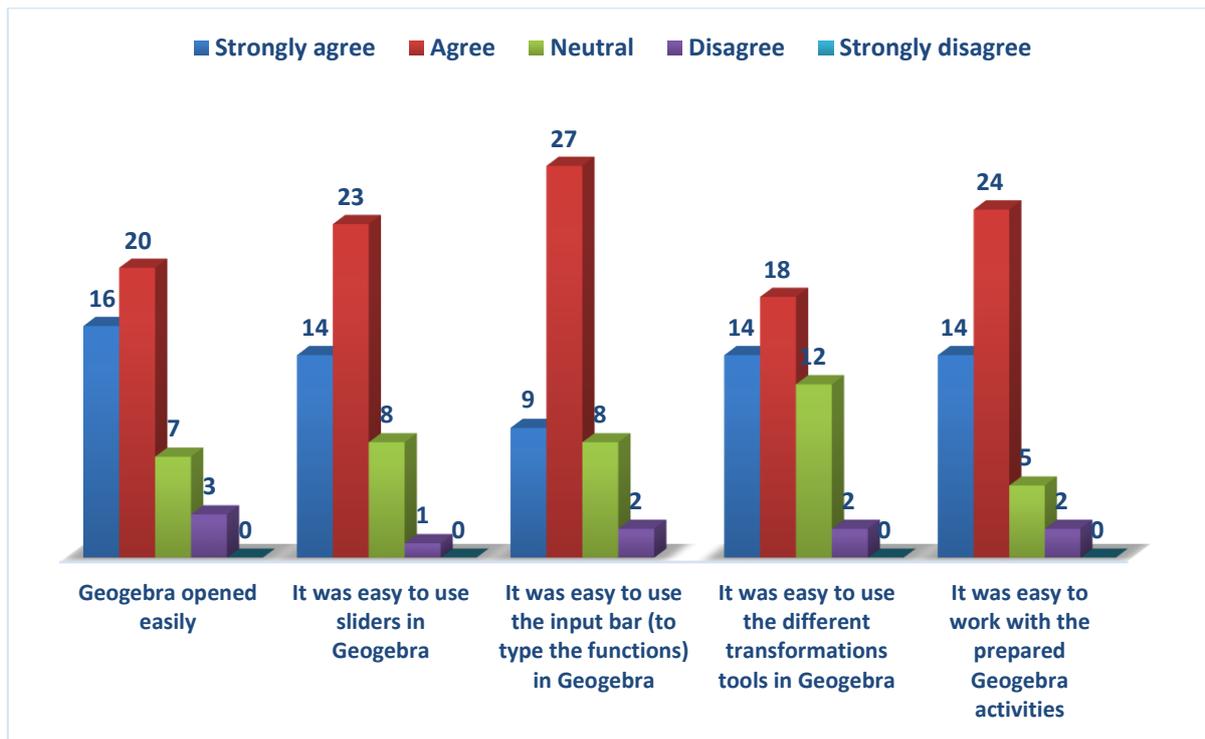


Figure 5.87: Working with tools and commands in GeoGebra

Students were asked about their experiences whilst working on certain tools and commands, such as sliders, transformation tools and the input bar in GeoGebra. Figure 5.87 shows that most of the students found GeoGebra easy to open and make use of the tools mentioned because they did not experience problems. An average of 35 (76.1%) of the students strongly agreed, or agreed, that it was easy to work with the sliders, with the transformation tools and the input bar in GeoGebra. These were amongst the most frequent commands and tools that they were using during the session in the computer lab. Thirty-eight (82.6%) of the students strongly agreed, or agreed, that the designed activities in GeoGebra were easy to work on.

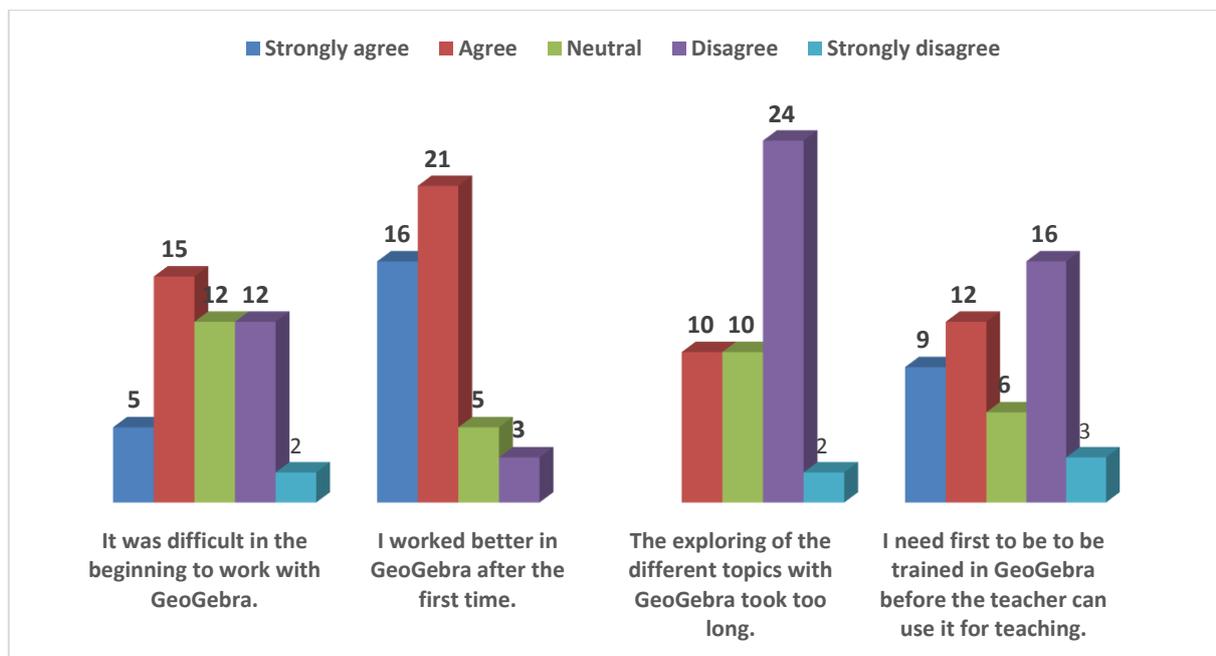


Figure 5.88: Students responses on how they experienced working GeoGebra

Twenty (43%) students strongly agreed, or agreed, that it was difficult to work with GeoGebra in the beginning. Some students indicated during the in-depth and focus group interviews that they experienced some difficulties in the beginning. These challenges are fully discussed in the next section. Thirty-seven (80%) strongly agreed or agreed that they worked better on GeoGebra after the first time. Ten (21.7%) of the students agreed that the teaching and learning of Mathematics with GeoGebra took too long. This shows that there were some students that felt exploring with GeoGebra could be done in less time. Only Student ID4 mentioned during her in-depth interview that the exploring with GeoGebra was too long. Nineteen (41.3%) of the students indicated that they needed training in GeoGebra before the teacher utilised it.

5.5.2 Students' computer skills

The students' computer literacy skills varied from very low to relatively high levels. Students were asked if they had ever worked on computers before (see Appendix C).

Figure 5.88 shows the students' responses from the pre-intervention questionnaire if they had been exposed to working on a computer, before entering SciMathUS.

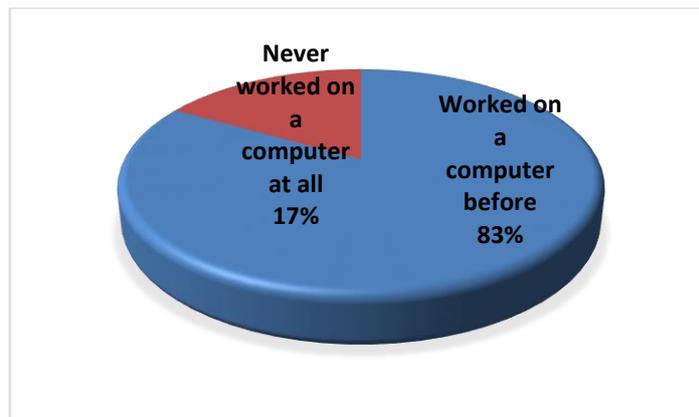


Figure 5.89: Students' prior experience of working on a computer

The students' computer skills also varied widely. Figure 5.89 shows that forty (83%) of the students had worked on a computer before while eight (17%) had never worked on a computer at all. Figure 5.90 shows students' responses to what they were doing on computers before entering SciMathUS.

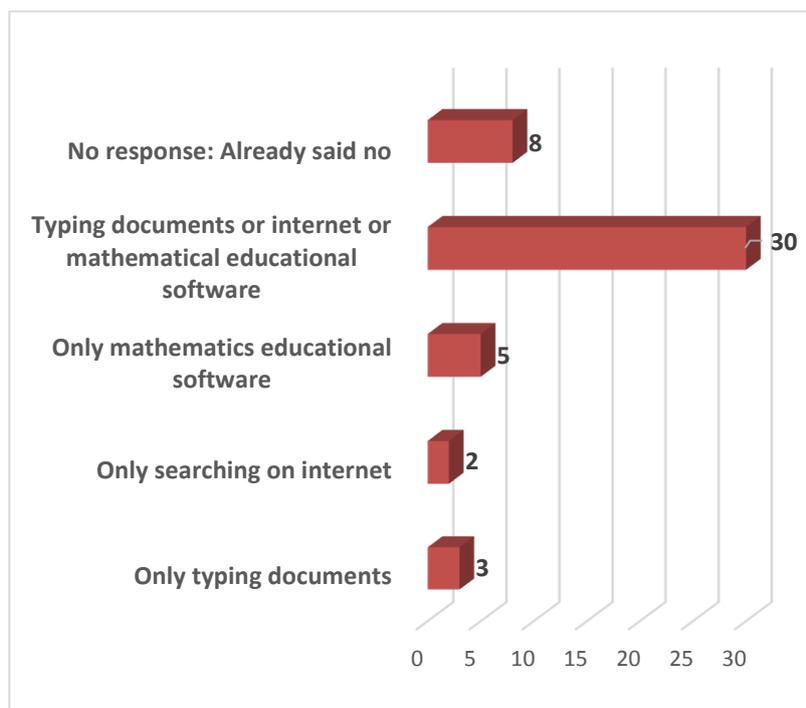


Figure 5.90: Purpose of working on computers (n= 48)

Thirty students (62.5%) indicated that they surfed the internet, or had done mathematics on educational software, or typed documents; five (10.4%) had done only mathematics on a computer; two (4.2%) had only surfed the internet before and three (6.3%) had only done some typing on a computer. Hence figures 5.89 and 5.90 show that the majority of students had been exposed to computers and had some knowledge of working with them. Fourteen

(29.2%) of the students worked on mathematics educational software before entering the SciMathUS programme. Thus most of the students had never before used technology to explore or to do mathematics on a computer. This did not necessarily mean that they were computer literate. Sample responses from the in-depth and focus group interviews show that some students experienced challenges with GeoGebra, because their computer skills were not up to scratch and others felt that their computer skills helped them to experience working on GeoGebra much easier.

The following student mentioned that his computer skills were not so good and he responded in this manner:

Researcher: Do you think your poor computer skills were also a factor?

Student ID9: Yes, yes...

Researcher: Was it difficult to work on GeoGebra.

Student ID9: It was difficult when I worked with it for the first time.

A student who came from a school with three computer labs responded as follows:

Researcher: Was the software the problem or working on a computer?

Student FG18: I think working on a computer. It was working on the computers, because I did not work on computers at school.

Some students indicated that those students, who experienced challenges with GeoGebra, were sometimes the students whose computer skills were not so good.

Student ID4: The people that found it a bit difficult, were those that were not used to computers.

Student FG19: I think it depends on their computer literacy levels, because a person who is more computer literate, would catch it better.

The first session in the computer lab took place six weeks after they started at SciMathUS. All the students were thus exposed to computers for at least six weeks. The researcher acknowledged the students' varied computer literacy levels, but Figure 5.89 shows that the majority of the students had never explored mathematics on a computer. The first session with computers could also pose a problem if students were not used to exploring mathematics. Students changed their perception towards the end of year because at that stage they were capable of exploring mathematics on their own. The Fisher exact test

5.3.1.1 shows that there were significant differences in the students' perceptions of what they were capable of finding out themselves about how mathematics works. The Cramer's V of 0.436 also indicates a strong effect size (Botsch, 2011) after intervention with active learning and interactive instruction with GeoGebra. The Fisher exact test 5.3.1.2 shows that there were significant differences in the students' perceptions on the use of computers, in this case GeoGebra in the teaching and learning of mathematics. The average Cramer's V of 0.52 also indicates a strong effect size (Botsch, 2011) that intervention with active learning and interactive instruction with GeoGebra helped students to explore mathematics.

5.5.3 The different teaching and learning approach

Some students indicated that it was sometimes challenging, because they could not see what the designed activities with GeoGebra attempted to do. The students were not used to exploring on their own and some of them were not open to change. Some students were not familiar with this approach of teaching and that made it challenging.

Student FG13: That lesson you delivered confused me. I could not look at a square, say this is square – it was confusing. Everything I referred to was a parallelogram. I really became confused between the shapes and stuff. I really had to go down and look at this; it is this because of that. Before that, I could look and say that it is a square, but now I must first analyse it before you can say this is square.

Student FG13 assumed that if a quadrilateral looks like a square, it is a square. The lesson with GeoGebra attempted to show students that they cannot assume what type of quadrilateral it is based on how the shape looks like, or insufficient properties. This points to the students' Van Hiele level, in this case Van Hiele level 1. This student showed that there is a need to first analyse the figure, before drawing deductions. This refers to the students' Van Hiele level 2. For example, students assumed that a quadrilateral with equal sides is a square.

The abovementioned students referred to the activity (see Figure 5.90 below) that the teacher-researcher utilised in his interactive lessons with GeoGebra. This student struggled to see that there have to be sufficient properties of the quadrilateral available, before a one can conclude what type of quadrilateral it is.

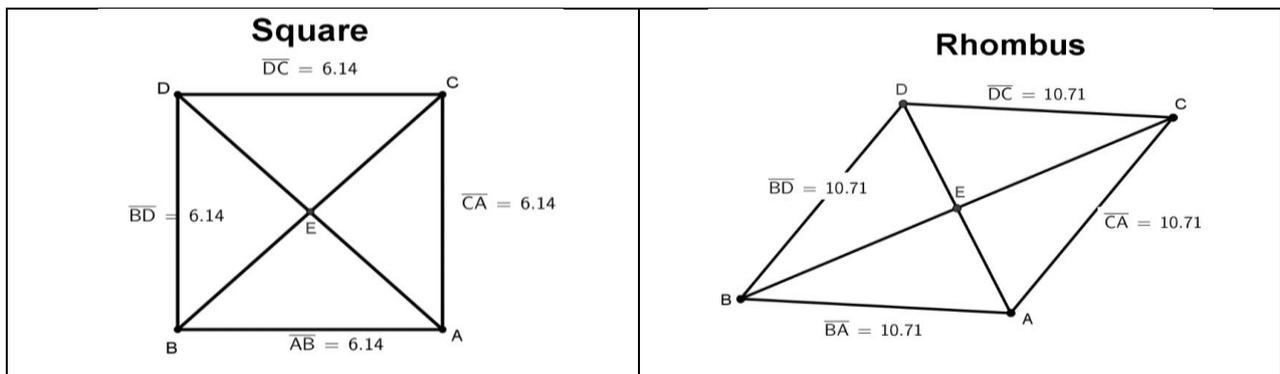


Figure 5.91: Applets utilised by the teacher-researcher in the interactive lesson on quadrilaterals

Another student felt that because he thought harder when working with the activities it made GeoGebra became a challenge. He responded in the following way:

Student FG4: It's quite challenging, but it helps you to think extra, you think extra ... just critical thinking. You have to think out of the box ... you actually force the brain to think extra. In that way you actually create pathways for yourself when you do something ... it's okay when I struggle here. This is how I did it and it actually helps you to obtain an answer, where on the other hand, if you just do it, and leave it, you know sir is going to explain it to me again. Or, you are going to give me the answer now. This way does not help your brain to think.

The following student did not mention that she was experiencing challenges, but then she was not used to the way of teaching:

Student FG15: It was weird at first, but then with your help, we could navigate easily around it. I think it helped. I thought that was nice to do it on a computer.

Researcher: Why are you saying it's weird on computer?

Student FG15: Because it is unusual. It is not something that we did a lot at school.

Some of the abovementioned students were showing resistance and some of them were not prepared to unlearn and continued in the same way. Student FG15 mentioned that she experienced the teaching approach to be challenging, but at a later stage she realised that it was helpful:

Student FG15: Some chapters were difficult, because some of the things that I learnt here compared to some of the things of last year were contradicting. So, I had to choose between which way to go that has meaning. I had to erase some of the things from my mind and replace it with these new concepts, which made it more difficult. Some chapters you were just solving, then coming here, I realised it is not only about solving, it is actually what happens to a certain

equation, which became weird again – surprising in a way. So, as such, I understand some of the chapters much better now.

Figures 5.92 shows Student CG9's responses in her pre-test on circle geometry.

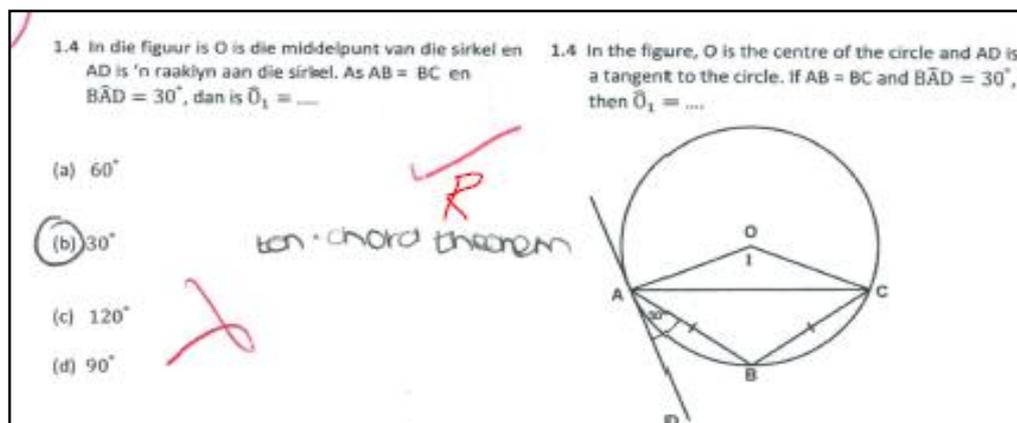


Figure 5.92: Student CG9's response to rider on tangent and chord theorem in the pre-test

The following student did circle geometry in Grade 11. Student O did Euclidean Geometry in Grade 12 and wrote NSC Mathematics Paper 3 in 2013.

Figure 5.93 shows Student O's responses in her post-test on circle geometry.

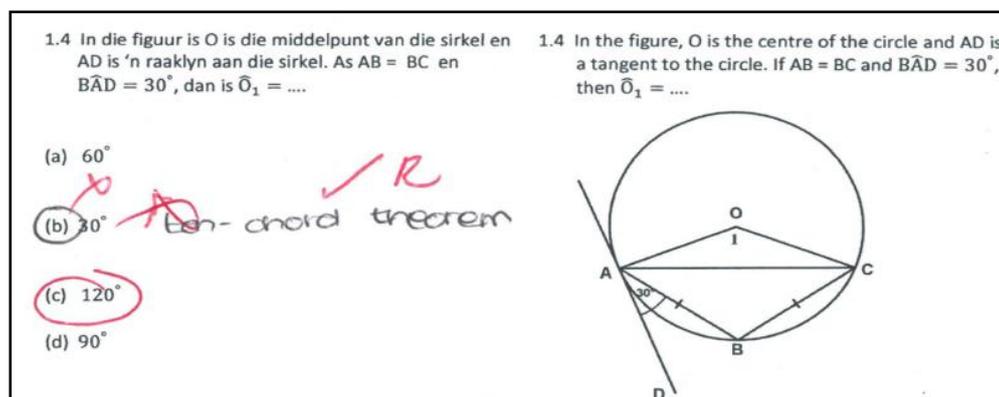


Figure 5.93: Student CG9's response to rider on tangent and chord theorem in the post-test

Student CG9 answered both pre-test (figure 5.90) and post-test (figure 5.91) on the tangent chord theorem. She already knew the theorem in the pre-test, but did not apply it correctly. She was taught with the designed activities and with GeoGebra, but still answered it incorrectly. She already had a misconception about this theorem, but continued with the same mistake after the intervention. This is the challenge that the researcher, as SciMathUS facilitator, has experienced during the past few years. Some students see the programme as a revision programme and for this reason do not give their full cooperation on topics that they think they know well enough. During this computer lab activity the teacher-researcher

emphasised the wording of the theorems. During this session Student CG9 did not cooperate fully in terms of exploring the theorems with GeoGebra. She just continued completing the designed activities based on the knowledge that she had. This student's understanding of Grade 8 – 10 Euclidean Geometry is on the Van Hiele level 2.

The abovementioned section demonstrated the challenges that students experienced whilst working with GeoGebra.

5.6 SUMMARY

The aim of this chapter was to provide an analysis of quantitative and qualitative data in order to answer the primary research question, which is:

What is the role of a GeoGebra-focused learning environment in helping students to develop mathematical knowledge?

This study was guided by the following four secondary research questions:

What were the processes involved in using GeoGebra as a pedagogical tool?

What are students' explanations of how GeoGebra afforded them an opportunity to understand mathematical concepts and move to higher levels of abstraction?

What are students' experiences and gestures when taught with GeoGebra?

What are the challenges that students experience whilst working with GeoGebra?

The quantitative (quan) and qualitative data (QUAL) were triangulated in order to strengthen each other and the secondary research questions. Quantitative data, such as pre- and post-intervention questionnaires, pre- and post-tests for transformations of functions and circle geometry and, qualitative data, such as observations in computer lab, in-depth and focus group interviews and, qualitative analysis of students' responses to pre- and post-tests, were triangulated to answer the abovementioned research questions. In conclusion, all the findings mentioned in this chapter show that GeoGebra creates a learning environment in which the SciMathUS students can construct mathematical knowledge. The next chapter presents a summary, main findings and conclusions, limitations and pedagogical implications, and suggestions, of the study.

CHAPTER 6

DISCUSSION, CONCLUSIONS AND RECOMMENDATIONS

6.1 INTRODUCTION

This chapter presents a summary of the discussions of the chapters, the main findings and conclusions of the study that answered the research questions posed in Chapter 1 and were recapped in Chapters 4 and 5. The chapter then turns to the limitations of the study and makes recommendations for policy, TPACK, HLT's and future research.

6.2 SUMMARY OF CHAPTERS

Chapter 1 gave an introduction and the background of the study, as well as its rationale and significance. The teaching and learning of mathematics in most schools in South Africa are often done through traditional instruction where students are positioned as passive recipients of knowledge (Van der Walt & Maree, 2007). Students also experience difficulties with concepts in the NSC Mathematics examinations, such as transformations of functions and circle geometry that require deeper conceptual understanding (DBE, 2011b; 2012a; 2013; 2014). This study used GeoGebra as an ingredient for teaching and learning of Mathematics to create a learning environment in which students construct mathematical knowledge. Moreover, it afforded them an opportunity to understand mathematical concepts and move to higher levels of abstraction.

Chapter 2 presented a literature review on teaching mathematics effectively, as well as a comprehensive explanation of the constructivist teaching approach with a specific focus on the neo-constructivist theory of Realistic Mathematics Education (RME). The acquiring of social, physical and logico-mathematical knowledge of Piaget (1952; 1954) was also discussed in-depth. Moreover, the Van Hiele theory of geometrical thought development was discussed to support the analyses of the students' levels of thinking in the pre- and post-test in circle geometry. Then the underlying features of a hypothetical learning trajectory (HLT) was discussed. The theoretical framework for the process of coherence

formation for surface and deeper understanding was discussed with a view to analyse the student learning gains in the pre- and post-test on transformations of functions.

Chapter 3 of the study reviewed literature on technology integration in the teaching of mathematics and specifically focused on Technological Pedagogical Content Knowledge (TPACK). Moreover, the chapter discussed the role of GeoGebra as a virtual manipulative and dynamic software. The theory of instrumental genesis (TIG) was discussed for use as a lens to understand students' interaction with technology and the teacher's instrumental orchestration of GeoGebra. The chapter also reviewed literature on bridging, or foundation programmes, and the use of GeoGebra as a teaching or learning tool for mathematics in them, focussing more specifically on the study of transformations of functions and circle geometry. The students explored the properties of circle geometry first and the proofs were done at the start. The quasi-empirical methods of De Villiers (2004) were also discussed to view the approach that was used in this study to teach circle geometry.

Chapter 4 presented the mixed methods approach as the research design of this study with quantitative and qualitative research sub-designs. Research methods and instruments that were used in this study were also discussed and justified. It further described the population and sampling procedures used, the research instruments that were utilised to gather the quantitative and qualitative data, as well as the data analysis procedures. The chapter then proceeded to illustrate the hypothetical learning trajectories for transformations of functions and circle geometry. It concluded with ethical considerations, reliability and validity of data collection procedures and instruments.

Chapter 5 presented the data analysis and interpretation of the various data collection procedures of the research study that emerged from students' responses during and after the implementation of the mixed approach of teaching of teaching, i.e. interactive instruction and active learning with instructional designed activities with GeoGebra.

6.3 SUMMARY OF MAIN FINDINGS

The study sought answers to the following main question:

What is the role of a GeoGebra-focused learning environment in helping students to develop mathematical knowledge?

In order to answer this main question, the following four sub-questions were investigated:

- i. What are the processes involved in using GeoGebra as a pedagogical tool?
- ii. How did GeoGebra afford students an opportunity to understand mathematical concepts and move to higher levels of abstraction?
- iii. What are students' experiences and gestures when taught with GeoGebra?
- iv. What are the challenges that students experience whilst working with GeoGebra?

The summary of findings is also presented in four sections the same way in which both the qualitative and quantitative data presentation and analyses were arranged.

6.3.1 Processes involved when using GeoGebra as a pedagogical tool

The findings of the observations (see section 5.2.1) during the computer lab sessions showed that the teacher-researcher's instrumental orchestrations are important processes to be considered when utilising GeoGebra as a teaching and learning tool for mathematics. *The Guide-and-explain* orchestration was of the type that was most dominant in both the computer lab sessions for transformations of functions and circle geometry. The *Guide-and-explain* orchestration (Drijvers *et al*, 2013a) refers to how the teacher moves around in the classroom while students are working on the computer activities, taking a look, answering questions, providing help or explaining, and continuing walking round in the class. Students work individually or in pairs with computers (see section 5.2.3.1.1). Other orchestration such as *Technical-demo*, *Explain-the-screen*, *Link-the-screen* and *Discuss-the-screen* were also evident during the classroom observation, but the support gradually decreased as the students worked independently. This accords with the view of Amineh and Asl (2015) on how instructors who are facilitators in social constructivism first provide support and help for students, and this support is gradually decreased as students begin to learn independently. These findings also showed the teacher-researcher's exploitation modes.

When integrating the findings of the pre- and post-intervention questionnaire, in-depth and focus group interviews and observations from the computer lab, it showed that the students viewed realistic or experiential context, guidance, exploration and interaction as important aspects of the teaching and learning in mathematics. The quantitative and qualitative results showed that GeoGebra alone is not the only solution of effective teaching. GeoGebra was therefore used with instructional design activities based on some of the principles of RME. The four principles that were investigated in this section with GeoGebra were the guided reinvention, realistic, interaction and activity principles. Both quantitative and qualitative results showed that students felt strongly that designed activities with GeoGebra helped

them to explore and to visualise the mathematics. Students also indicated that it is important for them to see the relevance of mathematics and where it fits into the real life. The students felt that the fact that they could explore, visualise and see the relevance of mathematics helped them to understand concepts better.

The first process involved in using GeoGebra as a pedagogical tool was exploration with guidance. The students were given activities to explore with GeoGebra. These activities were designed along a path that the teacher-researcher thought the students could discover the conceptual understanding of transformations of functions and circle geometry. This accords with the guided reinvention principle of RME, a route to learning along which students are able, in principle, to discover the envisaged mathematics by themselves (Gravemeijer & Terwel, 2000). The designed activities therefore helped with conceptual understanding and this is precisely what the DBE (2014) and researchers such as Freudenthal (1991), Treffers (1987), Gravemeijer (1994), Barnes (2004), etc. promote that teaching must be about conceptual understanding and not about algorithms alone. Most of the students also felt strongly during the in-depth and focus group interviews (see section 5.2.2.1) that GeoGebra helped them with all the topics where it was used, but especially with circle geometry and transformations of functions. The results (see from section 5.2.2.2) of the post-intervention questionnaire showed a Cronbach alpha reliability coefficient of 0.766 for the 12 items in the construct on RME. An average of 44 (96%) students experienced the support and guidance given by the teacher-researcher as positive, or felt that the worksheets helped them to explore concepts, or wish to be actively involved with the learning process. An average of 32 (70%) students indicated that they enjoyed working with questions which requested them to explain their answers, or felt that the worksheets were easy to follow.

Interaction was the second process that was involved in the use of GeoGebra as a pedagogical tool. The results indicated that students viewed interaction as a critical part of the learning process. The responses from in-depth and focus group interviews (see section 5.2.3.1) showed that students felt strongly that the interaction between themselves helped them to build confidence and they started to realise that they could support each other. They also felt the sessions with GeoGebra in the computer lab created an environment where they could talk about what they see on the screen. The results of the construct on RME from the post-questionnaire (see section 5.2.3.2) showed that an average of 43 (93%)

indicated that they enjoyed working with their classmates, or enjoyed the interaction between themselves and the teacher. The activities with GeoGebra have to be designed in such a way that students are able to interact with each other. This confirms the views of social constructivism which allows students to learn from each other and from more capable others (Ertmer & Newby, 2013; Vygotsky, 1978). Sousa (2006) argues that if students can move to a point where they explain to each other their average retention rate after 24 hours can move up to 90%. Interaction according to Freudenthal (1991) can help students to perform better in questions that require higher order level thinking, such as complex and problem solving procedures. The designed activities were tasks that entailed questions requiring completion of tables, making deductions and explaining one's answers, and it promoted interaction. These tasks in the instructional design activities that were used with GeoGebra, were well received by the students. This result coincides with the research by Haciomeroglu *et al.* (2009) that just showing pictures, figures and diagrams are insufficient to facilitate students' learning. Their research shows that GeoGebra worksheets with static drawings have to be converted into dynamic constructions.

The third process is the selection of the type of teaching style the students prefer to use for GeoGebra as a pedagogical tool. The responses from the in-depth and focus group interviews (see section 5.2.4.1) showed that seven of the ten students (70%) in the in-depth interviews (although they had their preferences), while 15 of the 23 students (65%) in the focus group interviews indicated that all three ways of using GeoGebra have a place in the teaching of mathematics. One student (10%) in the in-depth interviews indicated that only the teacher must use it, while 5 students (22%) preferred it in the focus group interviews. Hence the conclusion may be drawn that the majority of the students in the interviews indicated that all three ways of using GeoGebra have a place in the teaching and learning of mathematics. The responses from the in-depth and focus group interviews (see section 5.2.4.1.2) showed that the students preferred to work with prepared GeoGebra applets for circle geometry and for some activities on transformations of functions. The students also preferred (see section 5.2.4.1.2) to themselves do the activities on the properties of transformation and the sessions on typing in functions. The use of GeoGebra by the students refers to students' instrumental genesis (Trouche, 2004). The students discovered the properties of transformation and selected relevant functions from their personalised activities. The students also used GeoGebra (artefact) to sketch their graphs so that they could analyse the data to answer their research questions in the Statistics project. This

process where the student is mastering the artefact is called instrumentalisation (Ndlovu, *et al.*, 2011; Trouche, 2004). The responses of the students from the in-depth and focus group interviews also showed how they acknowledged how they could visualise certain concepts with GeoGebra (see section 5.2.2). Results from the observations (see section 5.2.3.1.1) showed how students interact with each other, with the teacher-researcher and with GeoGebra. The observations also showed how GeoGebra afforded the students an opportunity to acquire physical and logico-mathematical knowledge whilst working with GeoGebra in the computer lab.

The responses from the post-invention questionnaire (see section 5.2.4.2) were triangulated with the findings of the in-depth and focus group interviews. The majority of students indicated in the post-intervention questionnaire that they wanted the teacher to use GeoGebra as a teaching tool, or that they also wished to work with it themselves. This confirmed the need for different styles of instrumental orchestration (Drijvers, *et al.*, 2013a). Students in the experimental group were beginners in the use of GeoGebra. According to Hohenwarter *et al.* (2008) beginner users prefer prepared applets. The results thus confirm the view of De Villiers (2007) that students do not necessarily need to know the software inside out before they can effectively use it to explore, learn, conceptualise, conjecture, etc. This research used ready-made applets for the more complex activities, such as the investigation of the input and output values of the transformed functions by dragging and clicking animation and construction buttons, as suggested by De Villiers (2007). The students could therefore utilise quasi-empirical methods (De Villiers, 2004) to make mathematical conjectures.

Students preferred a certain approach in using GeoGebra, because they could see how they could benefit by it. The findings showed that the students' diverse preferences must be considered when using GeoGebra, although it will not be practicable to consider each one's preferences. These findings are consistent with Hohenwarter *et al.* (2008) namely that GeoGebra can be used in both teacher-centred and student-centred approaches. The results thus suggest that in order to keep students interested and motivated to learn, a variety of orchestration styles (teaching styles) should be used rather than over-reliance on just one.

6.3.2 GeoGebra affords students an opportunity to understand concepts

This section discusses both the quantitative and qualitative results of students' responses in the pre- and post-questionnaire, results of pre-and post-tests in transformation of functions and circle geometry as well as the in-depth and focus group interviews.

The findings from the first 13 questions in the pre- and post-questionnaires revealed how students wished they should learn and be taught mathematics. The results of the Fisher exact test (see section 5.3.3.1) showed that there were no significant differences in the students' perceptions of how they should be taught, or should learn mathematics before and after the GeoGebra intervention. These results showed that students do not easily change and unlearn teaching approaches they are accustomed to. The results from the Fisher exact test (see section 5.3.3.1) showed that the alternative null hypothesis for only one item was adopted, i.e. students are capable of exploring how mathematics works.

The results revealed the Cramer's V as 0.436, which indicates a strong effect size (Botsch, 2011) for the item, i.e. students were capable of exploring how mathematics works. Goodwin and Webb (2014) point out that one of the objectives of constructivism is that students must be active participants in the learning process. Freudenthal (1973, 1991) also suggests that children be given the opportunity to experience a process similar to when a given mathematical topic was invented. These results are thus consistent with the objectives of constructivism and the guided principle of RME. Hence the students viewed a hybridisation of the student-centred approach and teacher as facilitator of learning approach, to be effective for them. The findings show that they wished to explore, but the teacher was still the person who needed to guide them.

The results from pre-and post-intervention questionnaire in section 5.3.3.2 revealed that most of the students made paradigmatic shifts in terms of how they viewed technology, in this study where GeoGebra was the technology, to be an integral part of teaching to help them with their learning of mathematics. The values of Cramer's V for all five items indicate a strong effect size (Botsch, 2011) for all the items in the construct on students' perception on the use of computers in the teaching and learning of mathematics. The results of this study contradict with the results of research done by Green and Robinson (2009). The findings of their research with 200 foundation programme students show some decline between initial expectation and final judgment on GeoGebra as a teaching and learning tool.

The students were exposed to exploring transformations of functions and circle geometry with GeoGebra and could make conjectures and verify their answers. This was consistent with the quasi-empirical methods such as conjectures and verifications (De Villiers, 2004). Students thus reconsidered their perceptions of teaching and learning with software and technology after being exposed to GeoGebra. Students' responses from pre – and post-questionnaire (see section 5.3.3.2) also showed that although they were not writing examinations and tests with GeoGebra, they felt that it helped them to better understand concepts. The findings showed that although expository instruction was still preferred by most of the students, they viewed the addition of technology as beneficial in terms of visualisation and exploration for better conceptual understanding. This confirms the research done by Jaworski and Matthews (2011) in a study on lecturers' expository lecturing with GeoGebra. The aforementioned research shows gains in the attention of the students.

The responses from the students' in-depth and focus group interviews (see section 5.3.3.2) showed how the GeoGebra helped them to understand the concept of general solutions of trigonometric equations (see section 5.3.1.2).

6.3.2.1 Results on transformations of functions

Students' responses from in-depth and focus group interviews (see section 5.3.1.1) also confirmed that most of them better understood the concept of transformation. These responses showed how the instrumentation process of GeoGebra made an impression on the students (Ndlovu, *et al.*, 2013c). *Visual affordances*, *potentialities* and *enablements* were the prominent types of instrumentation processes of GeoGebra that made impressions on the students. The responses also showed how GeoGebra helped the students acquire physical and logico-mathematical knowledge of the meaning of $-f(x)$, $f(-x)$, and inverse graph of the exponential function. It also showed how students progressed from informal to formal reasoning, i.e. horizontal to vertical mathematisation.

Section 5.3.1.2 revealed the qualitative analyses of the pre- and post-tests scripts of transformation of functions that strengthen this claims because students were answering their questions in the post-test using the same approach as they did in GeoGebra. Some of the students also used different ways of answering compared to their attempts in pre-test and these solutions thus showing that they understood the idea of transformations. The results also proved that more students started answering the higher cognitive levels questions.

The way that the students answered the questions on $-f(x)$, $2f(x)$, $-2f(x)$ and $f(x + 30^\circ)$ in the post-test (see section 5.3.1.2) showed how GeoGebra helped for coherence formation on deeper structure level (Seufert & Brünken, 2004). Some students used a technique, or heuristic, to solve this problem, such as drawing sketches to answer a question. Faulkenberry and Faulkenberry (2010) emphasise that teaching and learning transformation of functions are not about memorising formulas and procedures, but we as teachers have to focus on the underlying concepts so that students may know why procedures work, not simply how they work. The intervention on transformations of functions was based on the idea of Faulkenberry and Faulkenberry (2010) to teach transformations of functions with the change of either the input or output values of the function. This study added GeoGebra to explore the change of either the input or output values of the functions. The results of this study confirm the approach of Faulkenberry and Faulkenberry (2010) which gave students conceptual understanding of transformations of functions. This approach also helped students to determine the critical values (turning points, asymptotes, etc.) of the transformed function from the given algebraic notations, and sketching the transformed graph of the cosine graph from the given algebraic notations (see section 5.3.1.2).

The results also confirm research by Bayazit *et al.* (2013) that the actual benefits of GeoGebra can be seen in promoting students' structural conception of the functions. Research by Bayazit *et al.* (2013) shows that GeoGebra helps with regard to exploring the changes in the general behaviour of a graph of a function in accordance with the changes in its algebraic form. The students were using sliders in GeoGebra to see the change of equation and the graph. This also coincides with Sfard's (1992) view of structural conception of a function, i.e. recognising the same concept in many different disguises. GeoGebra also helped, as Isoda (2007) adds, students to visualise the effect of changing parameters – something that students could only imagine in the past.

The pre-and post-test results for transformations of functions also showed that there was strong evidence that the teaching intervention with GeoGebra improved learning gains for the following domains (see section 5.3.1.3): Determining the equation of a transformed function from the given algebraic notations (Domain 1); determining the critical values (turning points, asymptotes, etc.) of the transformed function from the given algebraic notations (Domain 2); explanation of the transformation in words from the given algebraic notations (Domain 3) and sketching the transformed graph of the cosine graph from the

given algebraic notations (Domain 4). The effect sizes for all the domains were either large or very large. This study explored the teaching and the learning of transformations of functions with transformation geometry. The improved learning gains do not show that the teaching and learning of transformations of functions with GeoGebra is better than a teaching approach without it. The results only show that the teaching with GeoGebra can improve students' understanding of transformations of functions. The results are therefore consistent with literature by Usiskin and Coxford (1972) and, Steketee and Scher (2016) who contend that there are many good reasons, both mathematical and pedagogical, to introduce transformations in the teaching of transformations of functions. The results also confirm the research done by Makgakga and Sepeng (2013), namely that the use of transformations to teach and learn exponential functions had played a fundamental role in their study. The approach of teaching transformations of functions with transformation geometry also refers to the intertwine principle of RME (Freudenthal, 1991; Van den Heuvel-Panhuizen & Drijvers, 2014). Ndlovu (2014) also confirms that the major strength of the intertwining principle is that it stimulates a more coherent experience of the mathematics curriculum. The results from section 5.3.1 were also consistent with research done by Anabousy *et al.* (2014) that suggest students working with GeoGebra generally succeeded with function transformations in their algebraic and graphic representations. The effect size is thus very large for both groups. These results also showed that the intervention with material based on active learning with GeoGebra also helped low achievers. Barnes (2005) recommends that the theory of RME be considered in the design and implementation of interventions with low achievers. The Problem-Centred Learning approach by Murray, Olivier and Human (1998) also reinforces the fact that weaker learners in Mathematics are able to construct mathematical concepts if they are allowed to explore Mathematics on their own.

The triangulation of quantitative and qualitative results in this section revealed that the conclusion may thus be drawn that GeoGebra offered students an opportunity to better understand transformations of functions and moved them to higher levels of abstraction. Consequently GeoGebra can be added to teachers' list of teaching and learning tools to address the DBE (2011b; 2012a; 2013; 2014) outcry that resulted from students' experience in the NSC Mathematics examinations difficulties with concepts which required deeper conceptual understanding.

6.3.2.2 Results on circle geometry

The responses from the in-depth and focus group interviews (see section 5.3.2.1) showed the visual affordances of GeoGebra. Qualitative analyses of the pre- and post-tests scripts of circle geometry (see section 5.3.2.2) were analysed to triangulate with quantitative results of the pre- and post-test. The analyses of the purposive sample of students showed that some students moved from Van Hiele's recognition level (level 1) to ordering level (level 3) for circle geometry. The results thus proved that students started answering the higher cognitive level questions. Students started using sufficient reasons to justify and conclude their answers in the post-test. Students also started to use colour to highlight certain information in circle geometry, because they used colour in GeoGebra. This confirmed the *enablement* of GeoGebra. The qualitative analysis of the pre-test (see section 5.3.2.2) for circle geometry results confirmed research done in South Africa (De Villiers & Njisane, 1987) and elsewhere (Senk, 1989; Usiskin, 1982; Shaughnessy & Burger, 1985) that many school students are only on the Van Hiele recognition (level 1) or analysis level (level 2). The results from the pre-test for circle geometry (see section 5.3.2.2) showed that the students were on the visual level for circle geometry, but on Van Hiele's analysis (level 2), or ordering level (level 3) for the Grade 8 – 10 Euclidean Geometry (DBE, 2011a; 2011c). Post-test results showed that the most of the students had moved to levels which require students in the FET phase (Bennie & Smit, 1999; Van Putten, Howie, & Stols, 2010) to be on the Van Hiele ordering level (level 3) for circle geometry, straight-lines, triangles and quadrilaterals.

The results from the qualitative analyses (see section 5.3.2.1) from in-depth and group interviews were triangulated with the qualitative analysis (pre- and post-test) and quantitative results (pre- and post-test) of section 5.3.2.2 and 5.3.2.3, respectively. Students also indicated that with circle geometry – a topic that was new to 83% of them – GeoGebra made the topic easier to understand because of its *visual affordance*. Students' responses coincide with Van Voorst's (1999) findings that technology helps students to better visualise certain mathematical concepts and that it adds a new dimension to the teaching of mathematics.

The paired t –test results of the pre- and post-test reflected (see sections 5.3.2.3) a 95% level of confidence that the teaching intervention with GeoGebra and instructional material had improved their scores in circle geometry. The calculated Cohen's d of 2.66 (see section 5.3.2.3) showed that the effect size was very large. The effect size was thus very large for

both groups. Research done by Smith (1940), Human and Nel (1989) and, Govender and De Villiers (2004) show the effectiveness of the teaching approach by exposing students to first make constructions in order to explore geometric statements. Govender and De Villiers (2004) use ready-made applets in Sketchpad where the students could play step-by-step and observe as the figure was gradually constructed. This study had a similar approach where the students explored the properties of circle geometry with GeoGebra before proving the theorems. The improved learning gains do not show that the teaching and learning of circle geometry with GeoGebra is better than a teaching approach without it. The results only show that teaching with GeoGebra can improve students' understanding of the properties of circle geometry. The results of this study were consistent with research done by Smith's (1940), Human and Nel (1989) and Govender and De Villiers (2004) improvements in students' understanding of the "if-then" statements by letting them first make constructions to evaluate geometric statements. Chimuka and Ogbonnaya's (2015) study also show that GeoGebra software enhances the students' achievement in circle geometry. The intertwinement principle of RME (Freudenthal, 1991; Van den Heuvel-Panhuizen & Drijvers, 2014) was also used to link the Grade 8 – 10 Euclidean Geometry (DBE, 2011a; 2011c) consisting of straight line, triangle and quadrilateral to circle geometry.

The triangulation of quantitative and qualitative results in this section revealed that the conclusion may be drawn that GeoGebra afforded students an opportunity to better understand properties of circle geometry and moved them to higher Van Hiele levels. Hence GeoGebra can be used for the teaching of circle geometry. Moreover, the results showed that the teacher's TPACK are also important. The teacher's understanding of how students move through the Van Hiele levels and how to use technology, such as GeoGebra, are important to teach circle geometry effectively.

6.3.3 Students' experiences and gestures when taught with GeoGebra

Quantitative data from the post-intervention questionnaire in section 5.4.1 showed that students enjoyed working with GeoGebra and also found it interesting to explore. Students' responses from in-depth and focus group interviews (see section 5.4.1) also revealed that they enjoyed working with GeoGebra. Moore (2017) asserts that manipulatives, in this case GeoGebra, have the ability to help with cognitive processes and also have the advantage of engaging students and increasing both interest in and enjoyment of Mathematics. The responses of the students mentioned in sections 5.3.3.2, 5.3.1.1, 5.3.2.1, 5.3.3.2 and

observations in section 5.2.3.1.1 (also summarized in 6.3.2), show the thinking processes that Moore (2017) refers to. They show how students acquired physical and logico-mathematical knowledge, and how students made sense of general solutions of trigonometric functions, transformations of functions and circle geometry.

Results from the post-intervention questionnaire also indicated that students felt confident to explore transformation geometry, transformations of functions and circle geometry on GeoGebra. Research done by Fallstrom and Walter (2009) suggests that small discoveries made by students may bolster self-confidence, even if the result is well known to others, and may go a long way toward the student enjoying mathematics. This result confirms Clements and Sarama's (2009) findings that learning trajectories open up windows for students to learn mathematics in new ways, making teaching more joyous. Freudenthal (1991) also points out that although Mathematics can be abstract, the user must still enjoy it.

The results (see section 5.4.2) from the observation of computer lab sessions revealed students' hand movements were the dominant gestures while they were interacting and exploring with GeoGebra. The students utilised hand movements to explain different transformations of functions, i.e. reflections, vertical shrinking and stretching, and translations. Hand movements were also utilised by some students in the learning of circle geometry. These gestures were part of how students articulated their ideas to their fellow students or teacher-researcher (Scherr, 2004). This indicated that GeoGebra can create a socio-constructivist environment for students to interact and work collaboratively even though some students preferred to work alone. The pedagogical use of GeoGebra also made it possible for students to conjecture and seek patterns (Rasmussen *et al.*, 2005). This conjectures were on the basis of intuition and numerical investigation (De Villiers, 2004).

Some students worked independently and discussed their findings with the facilitator. This coincides with what Van den Heuvel-Panhuizen (2000) refers to as recognising individual learning paths in RME, and therefore views learners as individuals. Analysis of the classroom observations also showed that certain students had informal discussions and moved from this informal discussion to formal mathematics. This is similar to what Gravemeijer (2004) refers to as the RME guided reinvention heuristic where students mathematise their own mathematical activity to move to a higher, or abstract level. The hypothetical learning trajectories for the transformations of functions and circle geometry for this study showed that the students were able to explore the intended Mathematics and

arrive at personal solutions in the same manner as pointed out by Gravemeijer and Terwel (2000). Freudenthal (1968) refers to the level principle of RME and Treffers (1978; 1987) formulates the process of mathematisation. By exploring the transformations of functions and properties of circle geometry with GeoGebra, the students moved from informal (horizontal mathematising) to more formal (vertical mathematising) mathematical reasoning. The students moved from the Van Hiele recognition (informal reasoning) to ordering level (formal reasoning). Van den Heuvel-Panhuizen and Drijvers' (2014) view is that the strength of the level principle lies in guiding the growth in mathematical understanding and that it gives the curriculum a longitudinal coherency. This long-term perspective is a characteristic of RME. Thus, it could help the students to understand in the long term more complex procedures and equip them with problem solving skills. If students move through these levels it can help them to perform better in NSC examination questions that entail complex procedures and problem solving.

The triangulation of the results from the post-intervention questionnaire, in-depth and focus group interviews, and computer lab session observations showed the students' experiences, productions and gestures when stimulated and afforded by GeoGebra. These results also showed increases in student enjoyment, interest and confidence, as a result of the different (new) approach of teaching and learning mathematics.

6.3.4 Challenges that students experience whilst working with GeoGebra

Responses from in-depth and focus group interviews in section 5.5.1 revealed that some of the challenges that students experienced, were to find the certain icons or tools in GeoGebra. The measurement of an angle that is between 0° and 180° was a problem for some of them. This refers to the incorrect menu command constraint of the students (Ndlovu, 2011), because students click the vertices on the diagram to measure the angle incorrectly. A small percentage of students also experienced challenges to type in certain equations in the Input bar. Ndlovu (2011) refers to this as the student's syntax command constraints in TIG. Certain students could not, for example, type the correct equation for exponential function in the input bar.

Sample responses from the in-depth and focus group interviews (see section 5.5.2) showed that some students experienced challenges with GeoGebra, because their computer skills were lacking and others felt that their computer skills helped them to experience working on GeoGebra much easier. The results from the post-intervention questionnaire (see section

5.4.2), in-depth and focus group interviews (see section 5.5.1) showed they experienced difficulties to work with GeoGebra in the beginning, but that they worked better on GeoGebra after the first session. These results therefore confirm the remark by Hohenwarter *et al.* (2008) that the level of computer skills can be a factor when students are exploring or rediscovering mathematical concepts in GeoGebra.

Qualitative response from in-depth and focus group interviews (see section 5.5.3) showed that there were some students that found the different teaching and learning approach challenging at times because they could not see intended mathematics with the designed activities in GeoGebra. Other students were not used to exploring mathematics with or without GeoGebra, or any other software on their own. There were others (see section 5.5.3) that seemed not to be open to change and they merely continued in the same old ways which they were taught before. Students were not used to explaining and justifying their answers and were not used to exploring mathematics concepts on their own. Some students were not used to this approach of teaching and this paradigm shift made learning mathematics for them a challenge and confusing. A few students indicated that they did not enjoy working with GeoGebra and found the student-centred part of the mixed-mode approach of teaching and learning, challenging (see section 5.5.3). One said that this different approach of teaching and learning is challenging because it forces him/her to think. Doyle (2008) suggests that students' resistance can be linked to a number of reasons that include the fact that old habits die hard and students do not like taking learning risks; that student-centred teaching does not resemble what students are familiar with at school; and that many students prefer the path of least resistance in their learning. This may explain why some of the students showed resistance to take responsibility for their own learning.

The abovementioned discussion demonstrates some of the challenges that students experienced whilst working with GeoGebra, but it cannot be said to be permanent since this study exposed the students to GeoGebra over a very short period of time (only eight months).

6.4 LIMITATIONS OF THIS STUDY

As mentioned in section 4.2.1 a control group was unrealistic for the following practical and ethical reasons: The SciMathUS group consisted of 99 students that were divided into four classes; three classes in Stellenbosch and one in Worcester. The researcher was responsible for teaching two of the four classes and had no control over how the other two

facilitators were teaching their students. All the students in the programme were exposed to active learning and problem solving approaches. Although the other two colleagues were not active users of GeoGebra, there was always a possibility that they may use it because it had been installed in the computer lab and they were not prohibited from exposing their classes to GeoGebra. The students' focus in SciMathUS was to gain access to tertiary institutions.

The two classes taught by the researcher could not be divided into a control and an experimental group because of the nature of the timetable. The timetable was so congested it was not feasible to conduct the study during the students' spare time. The two classes that the teacher-researcher taught, had a class together twice a week and 50 of the students on the Stellenbosch campus were accommodated in five houses. As mentioned in section 4.2.1 there was therefore a possibility that the control group could be contaminated.

Some of the students had poor computer skills and this made them work slower. The eight months of exposing the students to different methods of mathematics teaching, (i.e. a hybridisation of the student-centred approach and teacher as facilitator of learning approach) – although their results in their NSC Mathematics in 2014 showed an improvement – was too short to enable students to gain more confidence and to realise that they did not have to rely on the teacher.

During the observation of computer lab sessions more recordings of the students' informal discussions could have been made in order to give more examples of students' feedback on how they moved from horizontal to vertical mathematisation. Due to time constraints, students were also not interviewed on a one-on-one basis to gauge their exact reasons for their responses in pre- and post-tests.

6.5 PEDAGOGICAL IMPLICATIONS AND RECOMMENDATIONS

Future research that embodies a similar study should include participants from a variety of classroom settings. Drawing on the findings, the researcher recommends the integration of GeoGebra with instructional designed activities into teaching and learning of Mathematics. The findings showed that integration of GeoGebra with instructional designed activities enables students to explore mathematics concepts to build their own knowledge and to enhance reasoning on the basis of what they visualised. Hypothetical learning trajectories can help teachers and researchers to design successful teaching and learning paths which

integrate the use of GeoGebra and assist teachers to engage in reflection in action and on action (Schön, 1983) with regard to the integration of ICT in mathematics education.

Lesson materials based on RME characteristics should be used in the teaching and learning of mathematics. Lesson materials such as those highlighted in this study can be integrated into teacher professional learning interventions. Textbooks can be written in the same way as the hypothetical learning trajectories used in this study to teach transformation of functions or circle geometry. Teachers should also be exposed to the understanding of how students acquire knowledge, and the Van Hiele levels, to ensure that both the teacher and the students are talking the same mathematical/geometrical language (De Villiers, 2010). Students must also be exposed to thinking at lower levels before learning formal geometric concepts (Mason, 1998). The Van Hiele geometric thinking levels has to be built into the Curriculum and Assessment Policy Statement (CAPS). According to Van Hiele (1986) the curriculum is sometimes presented by teachers at higher levels. The teachers focus too much and too early on the proofs of the theorems and riders.

The teacher-researcher's TPACK also showed that a teacher who wishes to use technology has to understand how (s)he will integrate appropriate technologies effectively in the teaching and learning of each applicable topic. Students also need to be exposed to teaching and learning to enhance critical thinking so that they can fare better in the higher cognitive level tasks such as complex problem solving procedures. In the past three years SciMathUS received more than 600 applications per year. This shows that there are many students who wish to study but do not meet the minimum requirements to gain access to university. GeoGebra helped this group of students to move to higher levels of abstraction and that is what students need to achieve in order to gain higher marks in National Senior Certificate (NSC) Mathematics.

Further longitudinal research is needed to investigate students' experiences over a long period of being exposed to GeoGebra. The purpose of such research could be to investigate more permanent experiences since this study exposed the students over a very short period of time (only eight months).

The researcher hopes to publish findings for other interested foundation programmes and schools to demonstrate how GeoGebra was implemented at SciMathUS. As a result of this, the researcher hopes that foundation programme mathematics lecturers, or facilitators, and

school mathematics teachers, can be motivated to integrate GeoGebra in their teaching to a much larger extent. Pre- and in-service mathematics student-teachers at universities should have greater exposure to dynamic mathematics technologies such as GeoGebra. There are a few other dynamic software packages, but GeoGebra is free open source material. Stellenbosch University could also establish a GeoGebra institute as many other universities all over the world have done. There are currently only two institutes in South Africa, i.e. Durban's University of KwaZulu Natal (UKZN) and the Nelson Mandela Metropolitan University (NMMU) in Port Elizabeth. These institutes support and train teachers in GeoGebra.

6.6 THE RESEARCH GAP AND POSSIBLE AREAS FOR FURTHER RESEARCH

The researcher could not find much literature on GeoGebra research done in foundation programmes. There is also not much research done with GeoGebra in South Africa. Hence this study is undoubtedly a valuable contribution. There has also been little research done with the theory of Instrumental Genesis in South Africa. This study also adds to that literature with reference to the teaching and learning of transformations of functions and circle geometry. It could be argued that there is a gap in research on teaching and learning with GeoGebra within theoretical frameworks such as Instrumental approach, RME and Van Hiele theory. Future research could also focus on both learner- and teacher as a facilitator of learning approach with GeoGebra with HLTs.

Moreover, future research could be done at secondary school level with teachers and students and also with pre-service students at universities. The Stellenbosch University Centre for Pedagogy (SUNCEP) runs different short courses in Mathematics and will soon be rolling out an Advanced Diploma in Education (ADE). GeoGebra has to be an integral part of the teaching and learning of topics, such as transformations of functions, circle geometry, etc., that lend themselves to technology integration. Research should be done in association with those teachers who are going to be part of this programme.

6.7 A FINAL WORD

The objective for the researcher was to investigate how students experience GeoGebra when it is used as a teaching and learning tool for Mathematics. The study mostly focused on the teaching and learning of transformations of functions and circle geometry, but the researcher is confident that GeoGebra can be used for many other topics as long as the

integration is properly done. What is evident from the research is that an HLT for each topic is necessary for successful integration rather than leaving it to chance. The researcher's main focus was to create a learning environment where students can enjoy Mathematics. When students start enjoying it, they move up to higher levels of abstraction. The following quotations from Freudenthal and from two students who were part of the study:

“A longing for abstract beauty has been a forceful motor and a trustworthy guide in mathematics and in the so-called exact sciences. Yet, whatever beauty may mean for its creator, it is aimless unless there are more people to enjoy it... Therefore, as far as mathematics research is concerned, beauty is not a convincing answer to the question ‘What is the use of it?’”(Freudenthal, 1991, p. 148).

The researcher achieved the objective to get students excited about Mathematics and its use:

“GeoGebra is the best, even if you are sleeping at night it's ringing in your mind.”

“I enjoy my Mathematics class. There was never a dull moment, and for this reason I was excited for Mathematics and to learn more of it. I liked how the teacher connected everything we had done to a new topic he introduced.”

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APPENDIX A

Pre- and Post-test on transformations on functions: Scoring rubric

No calculators must be used.

Consistent accuracy (CA) marking will be applied:

If a candidate makes a mistake in a question and (s)he used the incorrect information in next question or continues to work mathematically correctly, (s)he will be marked positively.

QUESTION 1 (23 marks)

$f(x) = 2(x + 3)^2 + 5$ is given.

1.1 Write down:

(a) the equation of $-f(x)$.

$$\begin{aligned} -f(x) &= -[2(x + 3)^2 + 5] \checkmark \\ &= -2(x + 3)^2 - 5 \checkmark \end{aligned}$$

OR IF ONLY ANSWER IS GIVEN: $-2(x + 3)^2 - 5$ then marks will be allocated for (2)

$$-2(x + 3)^2 \checkmark$$

$$-5 \checkmark$$

$$\text{OR } -f(x) = -2x^2 - 12x - 23 \checkmark\checkmark \quad (2)$$

(b) coordinates of the turning point of $-f(x)$.

$$(-3; -5) \checkmark\checkmark \text{ (mark per coordinate) OR}$$

If student's answer is in $f(x) = ax^2 + bx + c$

$-f(x) = -2x^2 - 12x - 23$ then they used the formula $x = \frac{-b}{2a}$ and substitute the x in $-f(x)$ full marks (2)

(c) in words, the explanation of the transformation $-f(x)$.

Reflection in x -axis : Reflection \checkmark and x -axis \checkmark

1.2 Write down:

(a) the equation of $f(x + 3) - 6$. (2)

$$f(x + 3) - 6 = 2(x + 6)^2 - 1$$

$$2(x + 6)^2 \checkmark$$

$$-1 \checkmark$$

OR $-f(x) = 2x^2 + 24x + 83 \checkmark \checkmark$

(b) coordinates of the new turning point of $f(x + 3) - 6$. (2)

$$(-6; -1) \checkmark \checkmark \text{ (mark per coordinate)}$$

(c) in words, the explanation of the transformation $f(x + 3) - 6$.

Translation 3 units left and 6 units down

Translation \checkmark (3)

3 units left \checkmark

6 units down \checkmark

1.3 Write down:

(a) the equation of $2f(x)$.

$$f(x) = 4(x + 3)^2 + 10$$

$$4(x + 3)^2 \checkmark$$

$$10 \checkmark$$

OR $-f(x) = 4x^2 + 24x + 46 \checkmark \checkmark$ (2)

(b) coordinates of the new turning point of $2f(x)$. (2)

$$(-3; 10) \checkmark \checkmark \text{ (mark per coordinate)}$$

(c) in words, the explanation of the transformation $2f(x)$

Dilation (enlargement) with scale factor of 2

OR Vertical stretch with scale factor 2

Dilation (enlargement) or Vertical stretch \checkmark (2)

Scale factor 2 \checkmark

1.4 Write down:

(a) coordinates of the new turning point of $f^{-1}(x)$.

$$(5; -3) \checkmark \checkmark \text{ (mark per coordinate)}$$

(b) in words the explanation of the transformation $f^{-1}(x)$. (2)

Reflection in $y = x$: (Reflection) \checkmark ($y = x$) \checkmark

QUESTION 2 (9 marks)

Point B(-180 ; -1) lies on the graph of $f(x) = \cos(x)$.

(a) Sketch the graph of $f(x + 30^\circ)$ and also show the new coordinates of B on the sketch.

Name the graph g on the sketch. (4)

See graph:

Shape (S) ✓

Transformation correct (T): Shift ✓ to left 30° ✓

Coordinates of B(-210° ; -1) ✓

(b) Sketch the graph of $-2f(x)$ and also show the new coordinates of B on the sketch.

Name the graph h on the sketch. (4)

See graph:

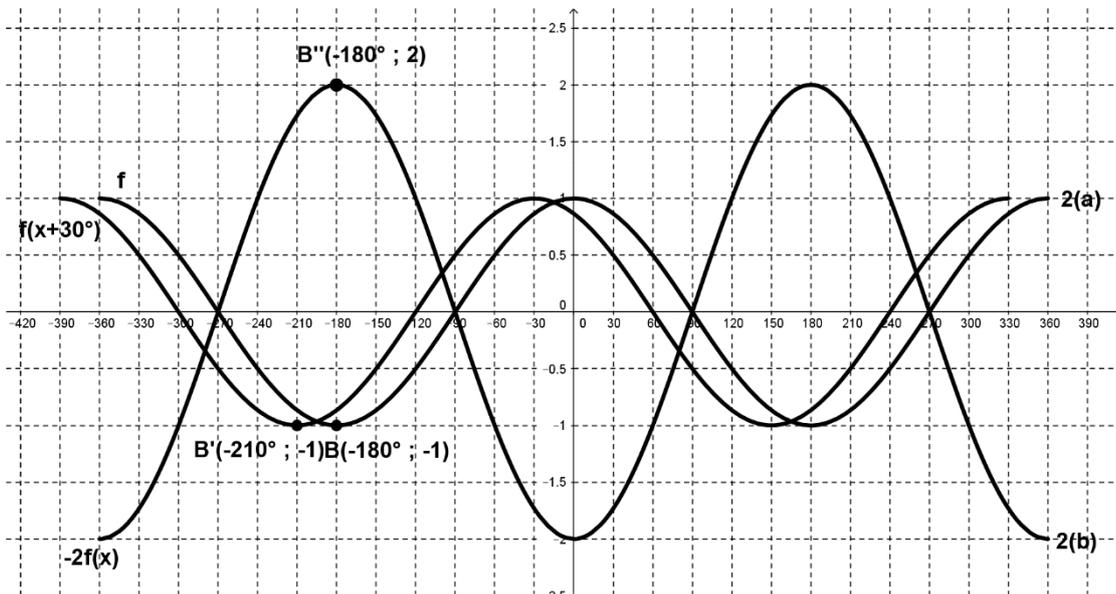
Shape (S) ✓

Transformation correct (T): Stretching ✓ Reflection ✓

Coordinates of B(-180° ; -2) ✓

(c) Write down period of $f\left(\frac{x}{2}\right)$. (1)

Period = 720° ✓



QUESTION 3

Given:

$$f(x) = \frac{8}{x-2} + 3$$

(a) Give the equations of the asymptotes of $f(x)$ reflected in the x -axis. (2)

$x = 2 \quad \checkmark$

$y = -3 \quad \checkmark$

(b) Give the equation of the graph if $f(x)$ is translated 3 units to right and 5 units

$$g(x) = \frac{8}{x-5} - 2$$

$$\frac{8}{x-5} \quad \checkmark$$

$-2 \quad \checkmark$

(c) Describe, in words, the transformation of f to g if $g(x) = \frac{-8}{x+1}$ (3)Reflection in x -axis \checkmark Shift of 3 units to left \checkmark Shift of 3 units upwards \checkmark

OR

Reflection in y -axis \checkmark Shift of 1 unit to left \checkmark Shift of 3 units downwards \checkmark **QUESTION 4**Given: $f(x) = (5)^x$ (a) Give the equation of the asymptote of $f(x)$ if reflected in the y -axis.

$y = 0 \quad \checkmark\checkmark$ (1 mark for y and 1 mark for 0)

$x = 0$ (no marks)

(2)

(b) Give the equation of the new graph if $f(x)$ is translated 3 units to right and 5 units downwards.

$$g(x) = (5)^{x-3} - 5$$

$$(5)^{x-3} \quad \checkmark$$

$-5 \quad \checkmark$

(c) Describe, in words, the transformation of f to g if $g(x) = \log_5 x$ Reflection in $y = x$: (Reflection) \checkmark ($y = x$) \checkmark

(2)

[6]**Total: 45 marks****Time: 1.2 minute per mark – 54 minutes**

APPENDIX B**Pre- and Post-test on circle geometry: Scoring rubric**

Consistent accuracy (CA) marking will be applied:

If a candidate makes a mistake in a question and he/she used the incorrect information in next question it will be marked positively.

Alternative answers will be marked.

QUESTION 1

C: ALL calculations on diagram 1 mark, on the condition are not mentioned in other calculations or reasons mentioned

R: Wrong answer with correct reason (R) ✓

S: Statement (S) ✓

The reasons are linked to students reasoning to find the answer.

Four possible answers have been supplied in questions 1.1. – 1.6. Choose the correct one.

Give reasons for your answer.

1.1 Point A is 8 cm from the circumference of circle O, with radius 5 cm. What is the length of tangent AB? (4)

(a) 13 cm

(b) $\sqrt{194}$ cm

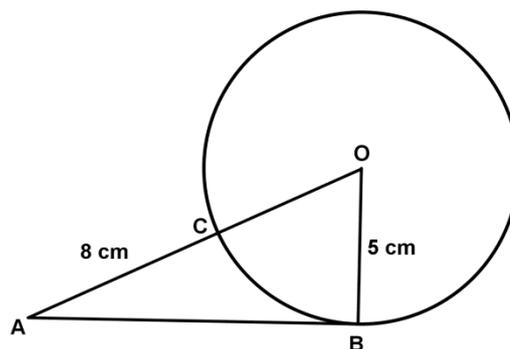
(c) 12 cm ✓

(d) $\sqrt{89}$ cm

Pythagoras ✓

Radii ✓

Rad \perp tangent ✓



If candidate give only answer (1 out 4)

ALL calculations on diagram is 1 mark if candidate did not give all the reasons.

Correct reason but wrong answer (1 mark) e.g $\sqrt{89}$

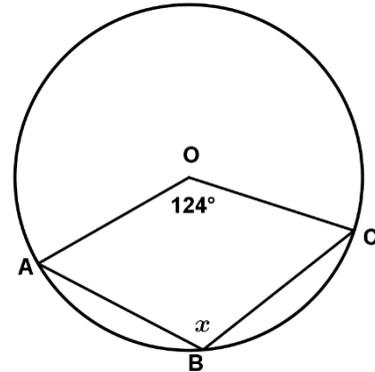
(Pythagoras)

Maximum is 4 marks

1.2 O is the centre of the circle and $\widehat{AOC} = 124^\circ$. The size of $x = \dots$ (3)

- (a) 56°
- (b) 124°
- (c) 118° ✓ S**
- (d) 236°

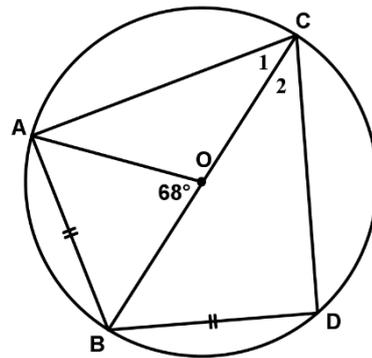
Revolution ✓ R
 \angle at the centre = 2 x Circumf. \angle ✓ R



1.3 In the circle with centre O, $\widehat{AOB} = 68^\circ$ and $AB = BD$. The size of $\widehat{C}_2 = \dots$ (3)

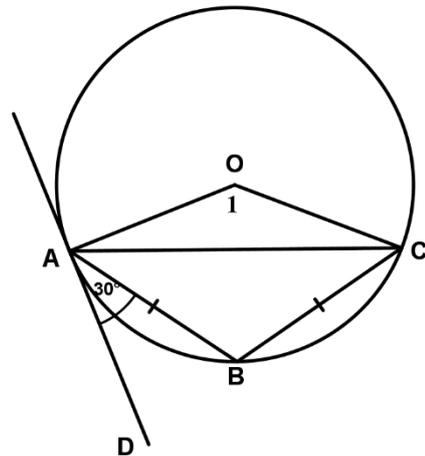
- (a) 68°
- (b) 34° ✓ S**
- (c) 44°
- (d) 56°

\angle at the centre = 2 x Circumf. \angle ✓ R
 Equal chords; equal \angle^s ✓ R
OR ANY OTHER ALTERNATIVE SOLUTIONS
 ✓ for any valid calculation and reason
 Equal chords; equal \angle^s ✓ R



1.4 In the figure, O is the centre of the circle and AD is a tangent to the circle. If $AB = BC$ and $\widehat{BAD} = 30^\circ$, then $\widehat{O_1} = \dots$ (5)

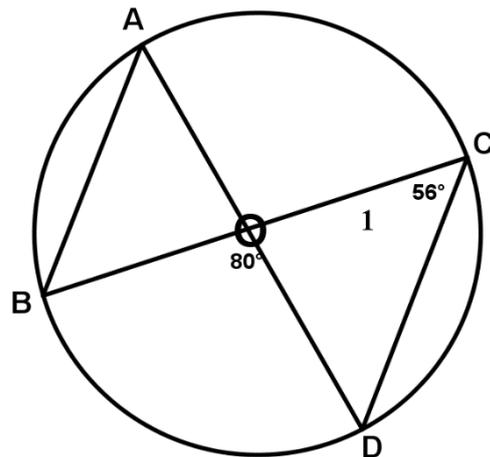
- (a) 60°
- (b) 30°
- (c) 120° ✓ S**
- (d) 90°



Tan chord theorem or **∠ between tan and chord** ✓ R
Isosceles Δ (∠^s opp. = sides) or/and
 ∠^s of Δ ✓ R
Revolution or ∠^s around a point ✓ R
 ∠ at the centre = 2 x Circumf. ∠ ✓ R

1.5 In the circle is A, B, C and D on the circumference of the circle. The size of $\widehat{ABC} = \dots$ (3)

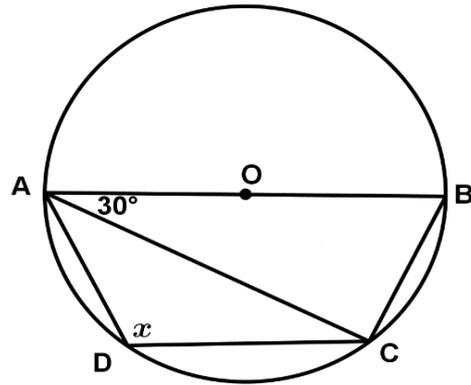
- (a) 40°
- (b) inligting nie genoeg om \widehat{ABC} te bereken /
information not enough to determine \widehat{ABC}
- (c) 28°
- (d) 24° ✓ S**



Straight line ✓ R
Circumf. ∠^s equal or **Circumf. ∠^s equal**
subtended by same chord ✓ R

1.6 AB is a diameter of the circle. $\widehat{BAC} = 30^\circ$. ABCD is a cyclic quadrilateral. The size of $x = \dots$ (4)

- (a) 30°
- (b) 90°
- (c) 120° ✓ S**
- (d) 150°

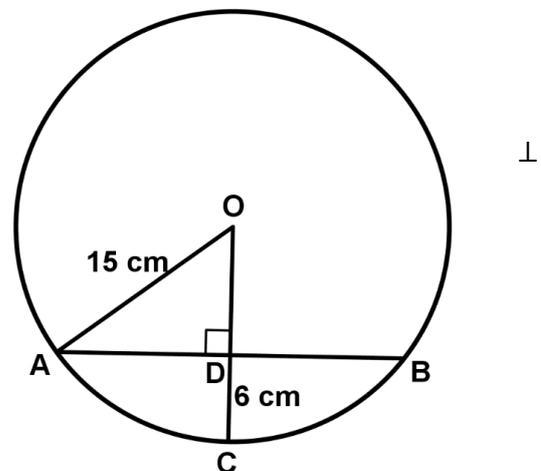


\angle in $\frac{1}{2}$ or
 \angle at the centre = 2 x Circumf. \angle ✓ R
 \angle^s of Δ ✓ R
 Opp \angle^s of cyclic quad ✓ R

QUESTION 2

AB is a chord of the circle with O the centre in the diagram below. OC intersects chord AB at D. $AB \perp OC$. If $OA = 15$ cm and $DC = 6$ cm, calculate with reasons the length of AB. (5)

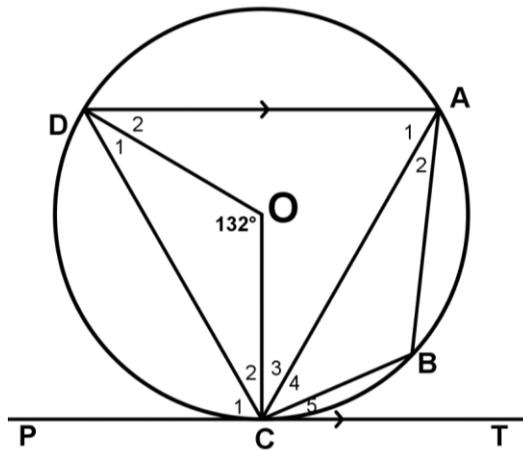
$OD = 9$ cm ✓ S ($AO = OC$; radii) ✓ R
 $AD = 12$ cm S (Pyth) ✓ S / R
 $AB = 24$ cm ✓ S ($OD \perp AB$ or line from centre chord) ✓ R



QUESTION 3

In the diagram below, O is centre of the circle and $\widehat{DOC} = 132^\circ$. PCT is a tangent to the circle and $DA \parallel PCT$. Calculate with reasons the size of:

- a) \widehat{A}_1 (2)
- b) \widehat{C}_1 (2)
- c) \widehat{ADC} (2)
- d) \widehat{B} (2)



- (a) $\widehat{A}_1 = 66^\circ$ ✓ S (\angle at the centre = 2 x Circumf. \angle) ✓ R
- (b) $\widehat{C}_1 = 66^\circ$ ✓ S (Tan chord theorem or \angle between tan and chord) ✓ R
- (c) $\widehat{ADC} = 66^\circ$ ✓ S (Alt. \angle^s ; $AD \parallel PT$) ✓ R
- (d) $\widehat{B} = 114^\circ$ ✓ S (Opp \angle^s of cyclic quad) ✓ R

Total: 35 marks

Time: 1.2 minute per mark : 42 minutes

APPENDIX C**Pre-intervention questionnaire****Instructions**

Please read each question carefully and answer it to the best of your ability. **There are no correct or incorrect responses. I am merely interested in your personal point of view.**

Please remember the following points:

- * Please give a response to every question/statement – do not omit any.
- * Never tick (✓) more than one at a statement.

Gender (M/F) : _____

Year of birth: _____

Please tick (✓) the appropriate box.

1. Students' perception of how they think they should be taught and learn Mathematics		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
LPTLM1	I am capable of finding out myself how Mathematics works.					
LPTLM2	To understand certain concepts in Mathematics, the concepts must first be fully explained to me.					
LPTLM3	I like to do Mathematics with short ways and do not need to do the long way to understand it.					
LPTLM4	I prefer to first explore the Mathematics on my own instead of the teacher explaining the whole time.					
LPTLM5	I think the teacher must give us notes every time he/she starts a new topic.					
LPTLM6	I can only understand Mathematics well when the teacher is showing me how to do it.					
LPTLM7	I can only learn by memorising what the teacher had taught me.					
LPTLM8	I learn Mathematics the best if I am given examples and practice similar questions afterwards.					

2. Students' perception on the use of computers in the teaching of Mathematics		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
LPCTM 1	I can learn Mathematics on a computer with Mathematics educational software.					
LPCTM 2	Software on a computer cannot explain Mathematics to me. A teacher has to explain to me Mathematics.					
LPCTM 3	Using computers in Mathematics makes Mathematics more confusing.					
LPCTM 4	Although I am not writing my tests and examination on a computer it can help me to understand certain concepts better.					
LPCTM 5	Using computers in Mathematics means you won't be able to do Mathematics without them.					

3. Do you like Mathematics?

 Yes

OR

 No

4. Why are you doing Mathematics?

5(a) Have you worked on a computer before?

 Yes

OR

 No

Only answer the following question if you ticked yes in 5(a)

5(b) If yes, what did you do on it?

APPENDIX D**Post-intervention questionnaire****Instructions**

Please read each question carefully and answer it to the best of your ability. There are no correct or incorrect responses; I am merely interested in your personal point of view and your experiences during the sessions with GeoGebra.

Please remember the following points:

- * Be sure to answer all items – do not omit any.
- * Never tick (✓) more than one at a statement.

Gender (M/F) : _____

Year of birth: _____

Student's perspective of how they think they should be taught and to learn Mathematics		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
SPTLM1	I am capable of finding out myself how Mathematics works.					
SPTLM2	To understand certain concepts in Mathematics it must be fully explained to me first.					
SPTLM3	I like to do Mathematics with short ways and do not need to do the long way to understand it.					
SPTLM4	I prefer to first explore the Mathematics on my own instead of the lecturer explaining the whole time.					
SPTLM5	I think the lecturer must give us notes every time he/she starts a new topic.					
SPTLM6	I can only understand Mathematics well when the lecturer is showing me how to do it.					
SPTLM7	I can only learn by memorising what the lecturer had taught me.					
SPTLM8	I learn Mathematics the best if I am explained an example and practice similar questions afterwards.					

Student's perspective on the use of computers in the teaching of Mathematics		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
SPCTM1	I can learn Mathematics on a computer with Mathematics educational software.					
SPCTM2	Software on a computer cannot explain Mathematics to me. The lecturer has to explain to me Mathematics.					
SPCTM3	Using computers in Mathematics makes Mathematics more confusing					
SPCTM4	Although I am not writing my tests and examination on a computer it can help me to understand certain concepts better.					
SPCTM5	Using computers in Mathematics means you won't be able to do Mathematics without them.					

Student's perspective of Realistic Mathematics Education		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
SPRME1	I liked the way the lecturer helped and guided me through the worksheets (activities on paper).					
SPRME2	I liked the worksheets (activities on paper) because they helped me to explore the different concepts.					
SPRME3	I liked the fact that we started some topics with short "real life" situation or "experiential" context and then moved to Mathematics in it.					
SPRME4	I can now see that Mathematics is in our daily life.					
SPRME5	I was actively involved with the learning process (with the paper activities & activities in GeoGebra).					
SPRME6	We (my classmates and I) assist each other.					
SPRME7	I helped my classmate(s) to understand some concepts better.					
SPRME8	I also learned from my classmate.					
SPRME9	I enjoyed the interaction with my classmate(s).					
SPRME10	I enjoyed the interaction between the lecturer and I.					
SPRME11	I enjoyed the questions where I needed to explain why is the answer is like that.					
SPRME12	The worksheets (activities on paper) were easy to follow.					

Student's perspective on the relevance of experiential or "realistic" context to understand certain concepts in Mathematics		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
SPRMC1	The activity on "Every graph tells a story" helped me to understand the concepts like function, domain and range better.					
SPRMC2	The activity on "What is a function" helped me to understand the concepts like function, domain and range better.					
SPRMC3	The activity on "Amazing relationships" helped me to understand the concepts like function, domain and range better.					
SPRMC4	The activity on "Calculus" helped me to understand the concepts like rate of change, average gradient/speed, and gradient and speed better.					
SPRMC5	The activity on "Introduction to Trigonometry" helped me to understand the trigonometric ratios better.					
SPRMC6	The activity on "Trigonometry equations" helped me to understand the concept of general solutions of trigonometric equations better.					
SPRMC7	The activities on "Financial Mathematics" helped me to understand the concepts like compound and simple interest and annuities better before using the formulae.					
SPRMC8	The activity on "Properties of quadrilaterals" helped me to understand the properties and definitions of quadrilaterals better.					

Student's perspective of the effectiveness of GeoGebra as a learning tool for transformation geometry whilst working themselves with GeoGebra in the computer lab		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
SPEGTTG1	I found it easy to learn transformation geometry with GeoGebra.					
SPEGTTG2	It was interesting to explore the properties of the different transformation with GeoGebra instead of the lecturer explaining the whole time.					
SPEGTTG3	GeoGebra helped me with the learning and understanding of transformations geometry.					
SPEGTTG4	It is useful to see how the shapes are shifting (translating), changing (reduction & enlargements) and reflecting with GeoGebra.					
SPEGTTG5	GeoGebra helped me to understand transformation geometry better.					
SPEGTTG6	I felt confident when asked to explore and study transformation geometry in GeoGebra.					

Student's perspective of the effectiveness of GeoGebra as a learning tool for functions and its graphs whilst working themselves with GeoGebra in the computer lab		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
SPEGTTG1	I found it easy to learn functions with GeoGebra.					
SPEGTTG2	It was interesting to explore graphs with GeoGebra instead of the lecturer explaining the whole time.					
SPEGTTG3	GeoGebra helped me to see the shapes of the different graphs.					
SPEGTTG4	GeoGebra helped me with the learning and understanding of graphs and their transformations.					
SPEGTTG5	It is useful to see how the graphs are shifting (translating), changing and reflecting with GeoGebra.					
SPEGTTG6	GeoGebra helped me to see the effect of the parameter a on the different graphs.(moving of slider a)					
SPEGTTG7	GeoGebra helped me to see the effect of the parameter p on the different graphs.(moving of slider p)					
SPEGTTG8	GeoGebra helped me to see the effect of the parameter q on the different graphs.(moving of slider q)					
SPEGTTG9	GeoGebra helped me to see that the effect of the parameters (a , p & q) have the same effect on all the graphs. (moving of sliders a , p & q)					
SPEGTTG10	GeoGebra helped me to understand graphs better.					
SPEGTTG11	I felt confident when asked to explore and study the graphs and their transformations in GeoGebra					

Process involved in using GeoGebra as a learning tool whilst working themselves with GeoGebra		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
PGTT1	GeoGebra opened easily.					
PGTT2	It was easy to use sliders in GeoGebra.					
PGTT3	It was easy to use the input bar (to type the functions) in GeoGebra.					
PGTT4	It was easy to use the different transformations tools in GeoGebra.					
PGTT5	I need first to be to be trained in GeoGebra before the lecturer can use GeoGebra for teaching.					
PGTT6	It was difficult in the beginning to work with GeoGebra.					
PGTT7	I worked better in GeoGebra after the first time.					
PGTT8	I worked at my own pace with GeoGebra different from the other students.					
PGTT9	It was easy to work with the prepared GeoGebra activities					
PGTT10	Although it was different I liked to be taught certain topics in this way.					

PGTT11	The use of GeoGebra in learning Mathematics gave me opportunity to work with a classmate.					
PGTT12	The use of GeoGebra in learning Mathematics gave me opportunity to share my views with my classmate.					
PGTT13	The way we use GeoGebra motivated me to explore more topics in Mathematics in this way.					
PGTT14	The exploring of the different topics with GeoGebra took too long.					
PGTT15	If I have my own computer I would like to use GeoGebra to explore more Mathematical concepts.					
PGTT16	I think my lecturer must continue teaching Mathematics with GeoGebra or any other technology.					

Student's perspective of the effectiveness of GeoGebra as a teaching/learning tool for quadrilaterals		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
SPEGTTEG1	I found it easy to learn the properties of the quadrilaterals with GeoGebra.					
SPEGTTEG2	It was interesting to explore the properties of the different quadrilaterals with GeoGebra instead of the lecturer giving me the properties and definitions of quadrilaterals.					
SPEGTTEG3	GeoGebra helped me to learn and understand the properties of quadrilaterals better.					
SPEGTTEG4	It is useful to see the properties of quadrilaterals with GeoGebra.					
SPEGTTEG5	GeoGebra helped me to understand the properties of quadrilaterals better.					

Student's perspective of the effectiveness of GeoGebra as a teaching/learning tool for circle geometry		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
SPEGTTEGC1	I found it easy to learn the theorems/deductions of circle geometry with GeoGebra.					
SPEGTTEGC2	It was interesting to explore the properties of the theorems/deductions of circle geometry with GeoGebra instead of the lecturer giving me the theorems and deductions.					
SPEGTTEGC3	GeoGebra helped me with the learning and understanding of the theorems and deductions in circle geometry better.					
SPEGTTEGC4	It is useful to see the theorems and deductions in circle geometry with GeoGebra.					
SPEGTTEGC5	I felt confident when asked to explore circle geometry in GeoGebra.					

SPEGTTEGC6	GeoGebra helped me to visualize the theorems and deductions of the theorems and that helped me do the riders (geometry problems).					
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Student's perspective of the effectiveness of GeoGebra as a teaching/learning tool to introduce the trigonometry ratios (sine, cosine & tangent) and general solutions of trigonometric equations		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
SPEGTTTR1	I found it exciting to learn about the trigonometry ratios with GeoGebra.					
SPEGTTTR2	I found it exciting to learn about the general solutions of trigonometry equations with GeoGebra.					
SPEGTTTR3	It was interesting to explore the trigonometry ratios with GeoGebra instead of the lecturer just explaining or giving it.					
SPEGTTTR4	GeoGebra helped me with the learning and understanding of the trigonometry ratios.					
SPEGTTTR5	GeoGebra helped me with the learning and understanding of the general solutions of trigonometry equations.					
SPEGTTTR6	It is useful to see in GeoGebra how the sine, tangent and cosine of an angle are calculated.					
SPEGTTTR7	GeoGebra helped me to understand the trigonometry ratios better.					

Teaching/learning different topics with GeoGebra		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
DTGGB1	The activities on functions were interesting and I enjoyed it.					
DTGGB2	The activities on trigonometry were interesting and I enjoyed it.					
DTGGB3	The activities on transformations geometry were interesting and I enjoyed it.					
DTGGB4	The activities on the properties of quadrilaterals were interesting and I enjoyed it.					
DTGGB5	All the other activities (not mentioned above) in GeoGebra that my lecturer used during this year were interesting and I enjoyed it.					
DTGGB6	The activities on circle geometry were interesting and I enjoyed it.					

Different ways of using GeoGebra as a teaching/learning tool		Strongly agree	Agree	Neutral	Disagree	Strongly disagree
WGTLT1	I liked the way the lecturer used GeoGebra as teaching tool to explain concepts in class.					
WGTLT2	I liked way my lecturer used GeoGebra in the class to help me understand certain topics better.					
WGTLT3	I liked the way I have learnt where I was given prepared activities/applets in GeoGebra and had to explore with it.					
WGTLT4	I liked the way I have learnt where I had to construct my own activities/applets in GeoGebra.					
WGTLT5	I used GeoGebra on my own to help me to understand certain concepts.					
WGTLT6	I used GeoGebra to do the project "What determines academic success?"					

APPENDIX E

Student Engagement and Instrumental orchestration checklist in computer lab

Date of Observation: _____

Number of Students Present During Observation: _____

	The whole time	Not so frequently	Non - Existent
1. Body language:			
Eyes on facilitator to see the instruction on screen and then continue on the computer			
Nonverbal response (nodding, moving)			
Students not looking interested			
Degree of participation:			
Actively busy with activities			
Discussing with classmate(s)			
Following instructions			
2. Degree of student confidence:			
Students working independently			
Student is relying on classmate to work			
3. Fun and Excitement:			
Smiling /Enjoyment			
Responding to humour appropriately			
Verbal Positive Feedback			
4. Orchestration type:			
Technical-demo orchestration			
Explain-the-screen orchestration			
Link-screen-board orchestration			
Discuss-the-screen orchestration			
Sherpa-at-work orchestration			
Guide-and-explain orchestration			

APPENDIX F**Semi-interview schedule for both in-depth and focus group interviews**

- The interviewees were welcomed and encouraged to relax.
- The objectives of the interview were discussed and the topic areas of the interview were shared with the interviewees.
- The structure of the interview was also discussed with the interviewees.
- Interviewees were also encouraged to explain as much as possible and to be honest about their responses.
- The conditions for the institutional permission were also discussed and interviewees signed the consent form.
- Probing was also used.

Questions:

1. Did you do Mathematics on a computer before coming to SciMathUS? If so, what were you doing?
2. How was it doing Mathematics on a computer this year? Explain.
(Depending on participant's responses)
 - Did you enjoy working with GeoGebra? Why?
 - Was it difficult to use GeoGebra? Explain.
 - Was there anything that you did not like whilst working with GeoGebra?
3. Do you think more training is needed before we use GeoGebra as learning tool in the computer lab? Explain.
4. How did you experience lessons taught with GeoGebra and worksheets? Explain. Probe to check for challenges whilst students were using GeoGebra & worksheets. Check how they experience the realistic & experiential context.
 - What lessons do you think help you to understand concepts better?
 - In what way do you think GeoGebra helped you in your understanding of graphs?
 - In what way do you think GeoGebra helped you in your understanding of transformations? (Could you derive the rules from what you saw on the screen)
 - In what way do you think GeoGebra helped you in your understanding of trigonometry ratios?
 - In what way do you think GeoGebra helped you in your understanding of general solutions?
 - In what way do you think GeoGebra helped you in your understanding of circle geometry?

- Although you were not drawing with paper-and-pen, did GeoGebra help you to have a better understanding of certain concepts? What concepts and what can you say how did you experience it?
5. Compare the use of GeoGebra as a teaching and learning tool vs the traditional way of teaching of last year or before.
 6. I used GeoGebra applets in the class; you worked with the prepared applets and also created your own activity in GeoGebra. Which of these activities did you prefer? Why?
 7. How did you experience the interaction between you and your classmate(s)?
 8. How did you feel about exploring Mathematics on your own or with classmate at first and not explaining by me?
 9. Was the support given by the lecturer (me) enough for you to continue working on your own? Explain.
 10. How did you experience working with GeoGebra in the project? Did it help? In which way?

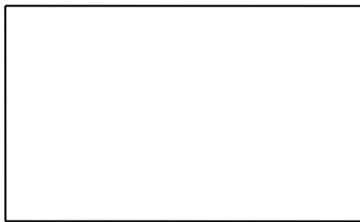
APPENDIX G

Students worked in groups of five

Amazing relations

Activity 1

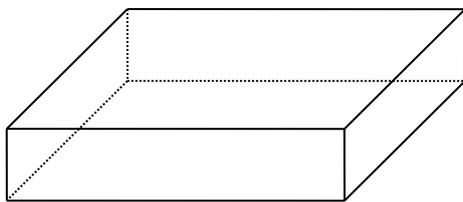
1. Each group must take 5 sheets of paper.



2. Each member of the group must fold a rectangular box (without a lid). The group must have a total of 5 rectangular boxes. The 5 boxes must all be different.



Cut the corners only and fold the paper to form an open rectangular box.



3. Measure the dimensions of each box. Draw a table with the height, length, width.
4. Calculate the total surface area and volume for each box.
5. Draw two graphs:
 - (a) height (x) and total surface area (y) for each box
 - (b) height (x) and volume (y) for each box

Select an appropriate scale for your graphs.

Activity 2: Sunrise for ANY city in the world

Find the time of sunrise for any town/city on the first day of the month for a year.

- Draw a table with the day and the time. Draw a graph.
- Select an appropriate scale for your graph.

Activity 3: Metal weights

Investigate the displacement of water when a weight is added one by one up to a total of 7 weights.

- Draw a table with your number of weights and the volume. Make sure that your measurements are accurate.
- Draw a graph. Select an appropriate scale for your graph.

Activity 4: Saving

Investigate graphically how your money grows yearly.

Invest R10 000 for 10 years at 10% per annum, compounded annually.

- Draw a table for the number of years and the investment after a year.
- Draw a graph. Select an appropriate scale for your graph.

Activity 5: Fixed area for rectangle

The group decides on a fixed area for a rectangle. Please be innovative and creative. Do not select a small area like 10 cm^2 .

- List all the possible measurements (be creative and innovative) for this rectangle in a table.
- Draw a graph to show the relationship between the length and the area of the rectangle. Select an appropriate scale for your graph.

APPENDIX H

Students worked in pairs or individually

Every graph tells a story

Story 1 (Telebel)

Jolene's father wants to buy a cell phone. She considers the options with TeleBel.

TeleBel
basic monthly subscription fee: R25 and then 50¢ per minute

- Set up a table and draw the graph of it.
- What happens (see table) to the monthly cost (y -values) if the minutes (x -values) increase?
- What shape is this graph?
- What possible values for x can we use to draw this graph?
Hint: Can x be negative numbers?
- Write down the equation of the graph that represents the above story.

Story 2 (Sum of two numbers)

The sum of any two numbers is 15. What is the highest product of the two numbers?

- Set up a table and draw the graph of it.
- What happens (see table) to the product (y -value) if the first number (x -value) increases?
- What possible values for x can we use to draw this graph?
Hint: Can x be negative numbers?
- Write down the equation of the graph that represents the above story.

Story 3 (Product of two numbers is always 24)

The product of two numbers is 24. What are the numbers?

- Set up a table and draw the graph of it.
- What happens (see table) to second number (y -value) if the first number (x -value) increases?
- What shape is this graph?
- What possible values for x can we use to draw this graph?
Hint: Can x be negative numbers?
- Write down the equation of the graph that represents the above story.

Story 4 (Growing waterplant)

In various places in Belgium one finds a certain type of pond which they call *vijers*. In one particular pond there is a fast growing water plant. On 1st of January this year the water plant covered 5m^2 of the surface of this pond. The area covered by the plant doubles every year.



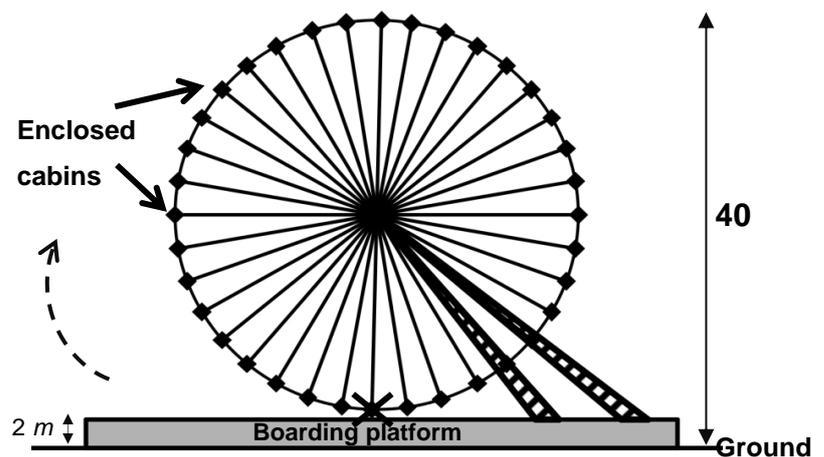
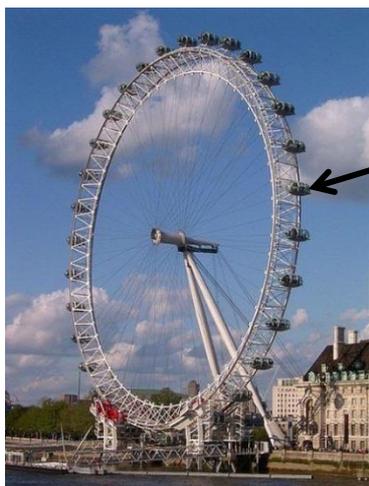
- Set up a table and draw the graph of it.
- What happens (see table) to y-values if the x-values increase?
- What shape is this graph?
- What possible values for x can we use to draw this graph?
Hint: Can x be negative numbers?
- Write down the equation of the graph that represents the above story.

Story 5 (Wheel of Excellence: Ferris Wheel)

The Wheel of Excellence is 40 meters above the ground. The radius of the wheel is 19 meters.

Photo of Wheel of excellence

Diagram of Wheel of excellence



The wheel is moving at a constant speed throughout the trip of 20 minutes long. It takes you 2 minutes to reach the top, 40 m above the ground if you board at the cross (X). The wheel makes one revolution every 4 minutes.

- (a) Set up a table and draw the graph of it.
- (b) What happens (see table) to y -values if the x -values increase?
- (c) What shape is this graph?
- (d) What possible values for x can we use to draw this graph?
Hint: Can x be negative numbers?
- (e) Write down the equation of the graph that represents the above story.

General form of the functions in the stories above:

$$y = ax + q$$

$$y = a(x + p)^2 + q$$

$$y = \frac{a}{x+p} + q$$

$$y = a \cdot b^{x+p} + q$$

$$y = a \sin(bx + p) + q$$

$$y = a \cos(bx + p) + q$$

$$y = a \tan(bx + p) + q$$

Story	Equation from the story	What general form fits the equation of the story?	General shape of graph
Telebel			
The sum of any two numbers is 15			
The product of two numbers is 24			
Growing water plant			
Wheel of Excellence: Ferris Wheel			

APPENDIX I

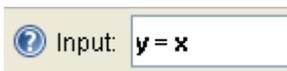
Students exploring the properties of transformation geometry with GeoGebra: They created their own activities from this instruction below.

Rigid Transformations (Movements)

1.1 Reflections (mirror images)

Reflect a polygon on set of axis with respect about a line

- Draw any polygon.
- Reflect your object in the following lines and complete the table as you progress:
 - ✓ y -axis (complete table below)
 - ✓ x -axis (complete table below)
 - ✓ $y = x$. Click in the Input box (bottom left of your screen) and type in $y = x$.

 Input: $y = x$ and press Enter on keyboard

- ✓ $y = -x$. Click in the Input box and type in $y = -x$.

What are the coordinates of the image points of the vertices of this polygon after the different reflections?

Coordinates of your original polygon	Coordinates of image when reflected y -axis	Coordinates of image when reflected in x -axis	Coordinates of image when reflected in y	Coordinates of image when reflected in $y = -x$
A	A'	A'	A'	A'
B	B'	B'	B'	B'
C	C'	C'	C'	C'

- What do you notice about the coordinates of the image when reflected in:
(give rules to explain your answers)
 - ✓ y -axis : $(x ; y) \rightarrow$
 - ✓ x -axis : $(x ; y) \rightarrow$
 - ✓ $y = x$: $(x ; y) \rightarrow$
 - ✓ $y = -x$: $(x ; y) \rightarrow$

Investigate the properties of reflection (discuss it with classmate)

Move the vertices of the original polygon, as well as the line of reflection. Check whether the following is true:

- The mirror image of a figure always lies on the **opposite** side of the mirror line.
- The line that connects the original figure to its mirror image is always perpendicular to the axis of symmetry (mirror line) and is bisected by the axis of symmetry.
- When a figure is reflected, its shape and size do not change. The original figure and its image have the same shape and size. A line segment and its image line segment have equal lengths and an angle and its image have equal sizes. The original and its image are therefore congruent.

1.2 Rotations (Turning)

Rotate a polygon on set of axes about the origin

- Draw any polygon.
- Create a point of rotation at the origin.
- Rotate your polygon 90° anti-clockwise about the origin; Rotate the polygon 90° clockwise about the origin; Rotate the polygon 180° about the origin and complete the table as you progress:

What are the coordinates of the image points of the vertices of this polygon after the different rotations?

Coordinates of your polygon	Coordinates of image when rotated 90° anti-	Coordinates of image when rotated 90° clockwise	Coordinates of image when rotated 180°
A	A'	A'	A'
B	B'	B'	B'
C	C'	C'	C'

- What do you notice about the coordinates of the image when rotated in the following way:(give rules to explain your answers)
 - ✓ 90° clockwise : $(x ; y) \rightarrow$
 - ✓ 90° anti-clockwise : $(x ; y) \rightarrow$
 - ✓ 180° clockwise : $(x ; y) \rightarrow$

Investigate the properties of rotation (discuss it with classmate)

Move the vertices of the original polygon and check whether the following are true when rotating:

- After rotating, the image has the same distance from the centre of rotation as the original figure;
- After a rotation around a centre of rotation, the angle that is formed by any point on the original figure, the centre of rotation and the corresponding image point is equal to the angle of rotation.
- When a figure is rotated, its shape and size do not change. The original figure and its image have the same shape and size. A line segment and its image line segment have equal lengths and an angle and its image have equal sizes. The original and its image are therefore congruent.

1.3 Translation (moving/displacement)

Translate a polygon on set of axes

- Draw any polygon.
- Create a vector with starting point at any point and end point at other point. Translate your polygon and complete the table as you progress:

Coordinates of original polygon	Coordinates of image
A	A'
B	B'
C	C'

- Translate your polygon by changing the coordinates of the vector to: starting point at (-6 ; 6) and end point at (-6 ; 5) and complete the table as you progress:

Coordinates of original polygon	Coordinates of image
A	A'
B	B'
C	C'

- Translate your polygon by changing the coordinates of the vector to: starting point at (-6 ; 6) and end point at (-2 ; 5) and complete the table as you progress:

Coordinates of original polygon	Coordinates of image
A	A'
B	B'
C	C'

- Translate polygon 3 **units** to the left and 2 **units** up and complete the table as you progress:

Coordinates of original polygon	Coordinates of image
A	A'
B	B'
C	C'

- Translation (moving/displacement) changes the position of each point. But there are things that stay the same. Make a summary of what happens when you translate an object. Make use of the information in the above tables.
- What do you notice about the coordinates of the image when translated in the following way:(give rules to explain your answers)
 - ✓ Horizontal translation (p) : $(x ; y) \rightarrow$
 - ✓ Vertical translation (q) : $(x ; y) \rightarrow$

Investigate properties of translation (discuss it with classmate)

Move the vertices of the original polygon and check whether the following are true when translating:

- the line segments that connect any original point in the figure to its image are all equal in length;
- the line segments that connect any original point in the figure to its image are all parallel.
- When a figure is translated, its shape and size do not change. The original figure and its image have the same shape and size. A line segment and its image line segment have equal lengths and an angle and its image have equal sizes. . The original and its image are therefore congruent.

Summary:

Rigid transformations are movements made without changing the shape or size of a two dimensional or three dimensional objects.

A transformation that does not change the size or shape is called a **rigid transformation**, or an **isometry**. The image is always congruent to the original. The following are three types of rigid transformations and they are often referred to as a "*flip*", a "*slide*" or a "*turn*".

Dilation Transformations: Enlargements and reductions

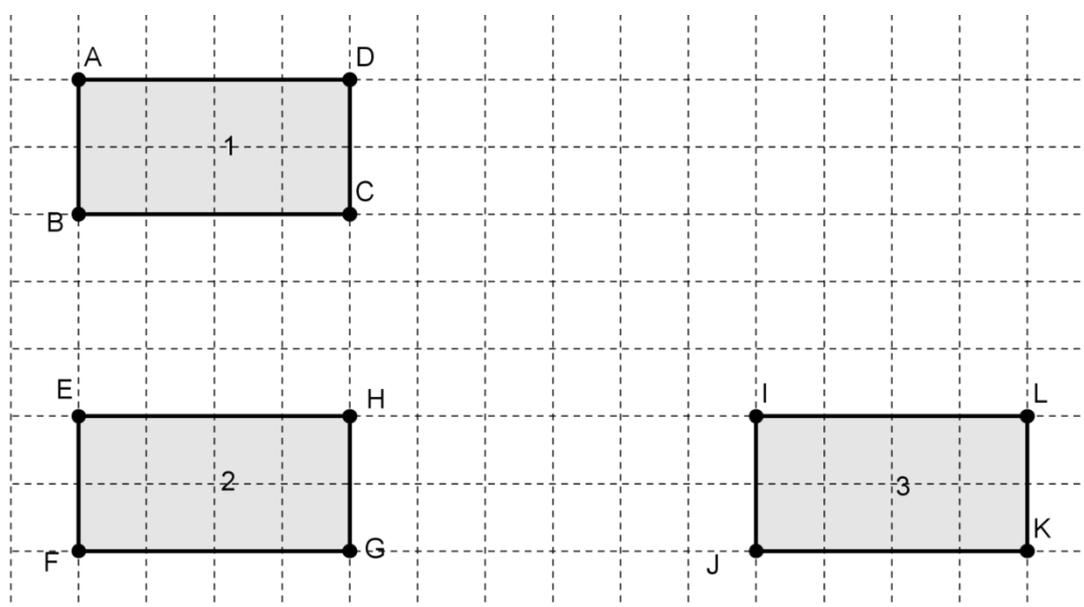
You can also let figures change in size without affecting the shape. This happens for example when you watch slides on a screen. The slide is 5cm by 5cm, but on the screen you can obtain an image of 2m by 2m. An architect makes a scale model of a building, which is a reduction of the actual building. In these cases the **shapes** stay the **same** but the **sizes** are **different**.

Below is a figure which is enlarged a number of times. Notice that an original line segment is parallel to its image but not the same length (i.e. shorter or longer).

We say figures are **similar** when corresponding **angles are equal** and corresponding **line segments are in the same proportion**.

Dilate polygon with a scale factor.

- Hide the **Algebra window** and **axes**. Display **Grid**.
- Draw rectangles with measurement 4 units by 2 units (see below).



- Enlarge  rectangle with a scale factor 0,5.
Click on a rectangle and then on point B and then type in 0,5 in the next dialog box.
- Enlarge  rectangle with a scale factor 2.
Click on a rectangle and then on point F and then type in 2 in the next dialog box.
- Enlarge  rectangle with a scale factor 3.
Click on a rectangle and then on point J and then type in 3 in the next dialog box.

Answer the following questions:

- (i) How many times are the length of sides of the image shorter than the length of the sides of original rectangle if enlarged with a scale factor of 0,5?
- (ii) How many times are the length of sides of the image longer than the length of the sides of original rectangle if enlarged with a scale factor of 2?
- (iii) How many times are the length of sides of the image longer than the length of the sides of original rectangle if enlarged with a scale factor of 3?

Complete the table for the PERIMETERS of the polygon and its image after the following dilations:

Perimeter of original rectangle	Perimeter of the image with enlargement factor of $\frac{1}{2}$	Perimeter of the image with enlargement factor of 2	Perimeter of the image with enlargement factor of 3

- Write down a rule that explains the relationship of the perimeter of the original polygon and the perimeter of its image.

Complete the table for the AREAS of the polygon and its image after the following dilations:

Area of original rectangle	Area of the image with enlargement factor of $\frac{1}{2}$	Area of the image with enlargement factor of 2	Area of the image with enlargement factor of 3

- Write down a rule that explains the relationship of the area of the original polygon and the area of its image.

Dilate a polygon with a scale factor on a set of axes

- Draw triangle A (1 ; 1), B (2 ; 3), C (5 ; 1).
- Create a centre or point of dilation (enlargement/reduction) at the origin
- Dilate (enlarge or reduce) your polygon with factor and complete the table as you progress:
 - ✓ 2 (complete the table below)
 - ✓ 3 (complete the table below)
 - ✓ $\frac{1}{2}$ (complete the table below)

Coordinates of original polygon	Coordinates of image with enlargement of factor of 2	Coordinates of image with enlargement factor of 3	Coordinates of image with reduction of factor of $\frac{1}{2}$
A	A'	A'	A'
B	B'	B'	B'
C	C'	C'	C'

- What do you notice about the coordinates of the image when reduced or enlarged in the following way:(give rules to explain your answers)

✓ If scale factor is a ; then

$(x; y) \rightarrow$

➤ Determine complete tables as you progress:

$\frac{\text{Length of side of image with enlargement of factor of 2}}{\text{Length of side of original polygon}}$
$\frac{A'B'}{AB} =$
$\frac{B'C'}{BC} =$
$\frac{A'C'}{AC} =$

$\frac{\text{Length of side of image with enlargement of factor of 3}}{\text{Length of side of original polygon}}$
$\frac{A'B'}{AB} =$
$\frac{B'C'}{BC} =$
$\frac{A'C'}{AC} =$

$\frac{\text{Length of side of image with enlargement of factor of } \frac{1}{2}}{\text{Length of side of original polygon}}$
$\frac{A'B'}{AB} =$
$\frac{B'C'}{BC} =$
$\frac{A'C'}{AC} =$

Complete the table for the PERIMETERS of the polygon and its image after the following dilations:

Perimeter of ΔABC	Perimeter of $\Delta A'B'C'$ with dilation factor 2	Perimeter of $\Delta A'B'C'$ with dilation factor 3	Perimeter of $\Delta A'B'C'$ with dilation factor $\frac{1}{2}$

- Write down a rule that explain the relationship of the perimeter of the original polygon and the perimeter of its image.
- Complete the table for the AREAS of the polygon and its image after the following dilations:

Area of ΔABC	Area of $\Delta A'B'C'$ with dilation factor 2	Area of $\Delta A'B'C'$ with dilation factor 3	Area of $\Delta A'B'C'$ with dilation factor $\frac{1}{2}$

- Write down a rule that explains the relationship of the area of the original polygon and the area of its image.

Investigate properties of dilation (discuss it with classmate)

Move the vertices of the original polygon and check whether the following are true when dilating (reducing and enlarging):

Similar figures are enlargements or reductions of each other.

If two or more figures are **similar**, then the **corresponding angles** of the figures are **equal**.

In **similar** figures, the **lengths of the corresponding sides are in proportion**.

APPENDIX J

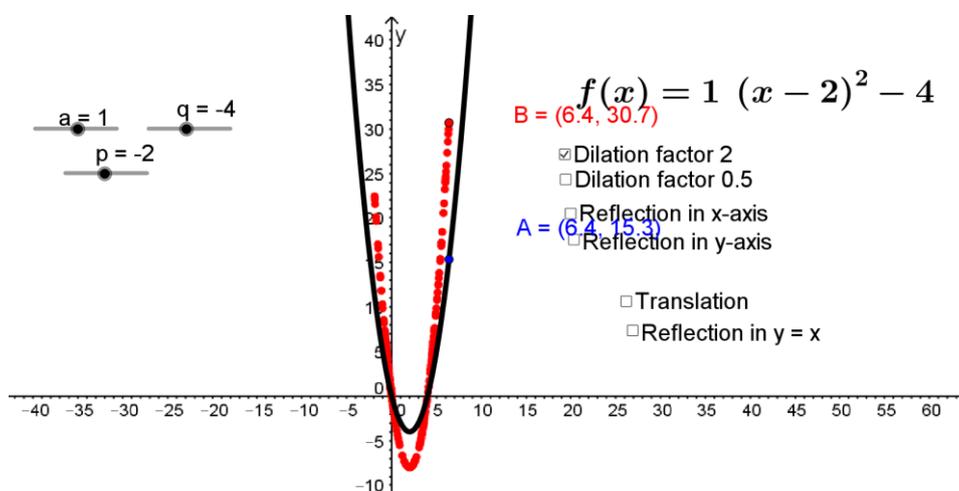
Transformations of functions:

Lesson 1 (Students work on GeoGebra applets)

List of all the GeoGebra applets (some of them has been modified since 2014) that were shared with the students to explore the properties of transformations of functions themselves:

Name	Date modified	Type	Size
1. Quadractic functions.ggb	2015-09-15 02:23 ...	GeoGebra File	19 KB
2. Hyperbola.ggb	2014-03-10 10:23 ...	GeoGebra File	8 KB
3. Exponential function.ggb	2014-03-10 09:53 ...	GeoGebra File	7 KB
4. Sine graph with sliders.ggb	2014-03-10 10:44 ...	GeoGebra File	8 KB
5. Cosine graph with sliders.ggb	2014-03-10 10:48 ...	GeoGebra File	8 KB
6. Tangent graph with sliders.ggb	2013-05-31 10:37 ...	GeoGebra File	11 KB
7. Sin graphs.ggb	2013-05-30 10:34 ...	GeoGebra File	8 KB
8. Cos graphs.ggb	2013-05-30 10:36 ...	GeoGebra File	8 KB
9. tan graphs.ggb	2013-05-30 10:59 ...	GeoGebra File	11 KB

Example of one the Geogebra applet utilised by the students during lesson 1

Move your sliders a , p & q .

- Write down your equation of your graph (simplest form)
- Click in the Transformation (Dilation, Translation, etc.) box. ONLY ONE TRANSFORMATION AT A TIME
- Move point A.
- Move sliders again to get the equation to fit your newly transformed graph.
- Write down the equation of your newly transformed graph.
- What do you notice? Discuss it with the person next to you.

Lesson 2 (students type equation in GeoGebra): They created their own creations

1. $f(x) = 2(x + 4)^2 - 3$.Type in GeoGebra in Input bar $2(x+4)^2 - 3$ and then Enter.
Turning point coordinates :

Type in the following into the input bar. Also write down each time the new equation, then simplify it and describe the transformation and the coordinates of the new turning point.

(a) $-f(x)$	(b) $-3f(x)$	(c) $f(x) - 2$
(d) $f(x) + 3$	(e) $f(x - 3)$	(f) $f(x + 3)$
(g) $f(x + 4) + 2$	(h) $f(x + 1) - 5$	(i) $f^{-1}(x)$ (find it algebraically)

2. $f(x) = x^2 - 2x - 8$ Type in GeoGebra in Input bar $x^2 - 2x - 8$ and then Enter.

For turning point: Type in inputbar: **extremum(f)**

Type in the following into the input bar. Also write down each time the new equation, then simplified it & describe transformation and the coordinates of the new turning point.

(a) $-f(x)$	(b) $-3f(x)$	(c) $f(2x)$
(d) $f\left(\frac{x}{2}\right)$	(e) $f(x) - 2$	(f) $f(x) + 3$
(f) $f(x - 3)$	(g) $f(x + 3)$	(h) $f(x + 4) + 2$
(i) $f(x + 1) - 5$		

3. $f(x) = \frac{4}{x+2} - 3$ type in GeoGebra in Input bar $f(x) = 4/(x+2) - 3$.

Equation of horizontal asymptote:

Equation of vertical asymptote:

Type in the following into the input bar. Also write down each time the new equation, then simplify it & describe the transformation and the equations of the vertical asymptote and horizontal asymptote.

(a) $-f(x)$	(b) $-3f(x)$	(c) $f(2x)$
(d) $f\left(\frac{x}{2}\right)$	(e) $f(x) - 2$	(f) $f(x) + 3$
(f) $f(x - 3)$	(g) $f(x + 3)$	(h) $f(x + 4) + 2$
(i) $f(x + 1) - 5$		

4. $f(x) = 2^{x+1} - 3$ type in GeoGebra in Input bar $f(x) = 2^{(x+1)} - 3$.

Equation of horizontal asymptote:

Type in the following into the input bar. Also write down each time the new equation, then simplify it & describe the transformation and the equation of the horizontal asymptote.

(a) $-f(x)$	(b) $-3f(x)$	(c) $f(x) - 2$
(d) $f(x) + 3$	(e) $f(x - 3)$	(f) $f(x + 3)$
(g) $f(x + 4) + 2$	(h) $f(x + 1) - 5$	(i) $f^{-1}(x)$ (find it algebraically)

5. $f(x) = \sin(x + 30^\circ)$, $-360^\circ \leq x \leq 360^\circ$ (trig graphs prepared applets)

Coordinates where you find the maximum value:

Coordinates where you find the minimum value:

Period:

Amplitude:

Type in the following into the input bar. Write down each time the new equation, then simplify it and describe the transformation. Also write down the period and the amplitude of your transformed graph.

(a) $-f(x)$	(b) $-3f(x)$
(c) $f(x) - 2$	(d) $f(x) + 3$
(e) $f(x - 30^\circ)$	(f) $f(x - 120^\circ)$
(g) $f(x + 30^\circ) + 2$	(h) $f(x + 60^\circ) - 5$
(i) $f(2x)$	(j) $f\left(\frac{x}{3}\right)$

6. $f(x) = \cos(x - 60^\circ)$, $-360^\circ \leq x \leq 360^\circ$

Coordinates where you find the maximum value :

Coordinates at where you find the minimum value :

Period:

Amplitude:

Type in the following into the input bar. Write down each time the new equation, then simplify it and describe the transformation. Also write down the period and the amplitude of your transformed graph.

(a) $-f(x)$	(b) $-3f(x)$
(c) $f(x) - 2$	(d) $f(x) + 3$
(e) $f(x + 30^\circ)$	(f) $f(x - 60^\circ)$
(g) $f(x + 30^\circ) + 2$	(h) $f(x + 60^\circ) - 5$
(i) $f(2x)$	(j) $f\left(\frac{x}{3}\right)$

7. $f(x) = \tan(x)$, $-360^\circ \leq x \leq 360^\circ$.

Equation of horizontal asymptote:

Period:

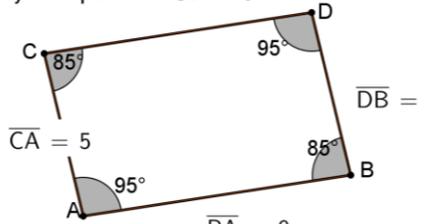
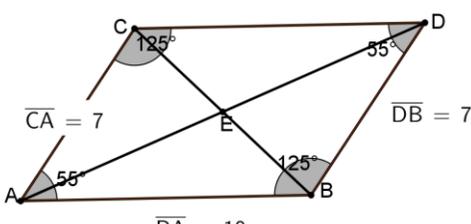
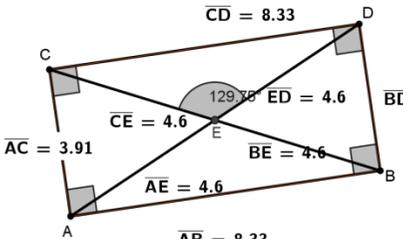
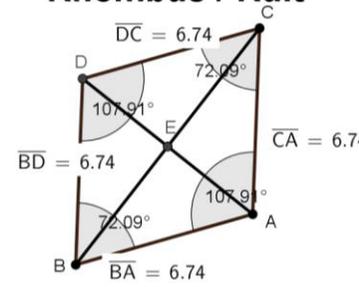
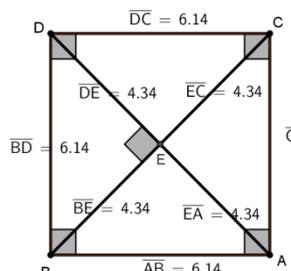
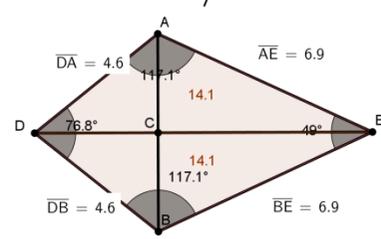
Type in the following into the input bar. Write down each time the new equation, then simplify it and describe the transformation. Also write down the period of your transformed graph.

(a) $-f(x)$	(b) $-3f(x)$
(c) $f(x) - 2$	(d) $f(x) + 3$
(e) $f(x - 30^\circ)$	(f) $f(x - 120^\circ)$
(g) $f(x + 30^\circ) + 2$	(h) $f(x + 60^\circ) - 5$
(i) $f(2x)$	(j) $f\left(\frac{x}{3}\right)$

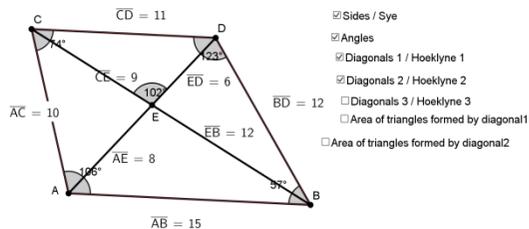
APPENDIX K

Properties of quadrilaterals

The teacher-researcher will show the display the properties of the quadrilaterals with GeoGebra applets. By ticking of check boxes in the GeoGebra applets will display the properties. The students will observe the properties and then complete the table below the diagrams.

<p>Parallelogram (\parallel^m or parm)</p> <ul style="list-style-type: none"> <input checked="" type="checkbox"/> Sides of parm / Sye van parm <input checked="" type="checkbox"/> Opposite angles of parm / Teenoorst. sye van parm $\overline{CD} = 9$ <input type="checkbox"/> Diagonals 1 / Hoeklyne 1 <input type="checkbox"/> Diagonals 2 / Hoeklyne 2 <input type="checkbox"/> Diagonals 3 / Hoeklyne 3 <input type="checkbox"/> Area of triangles in sq units 	<p>Parallelogram (\parallel^m or parm)</p> <ul style="list-style-type: none"> <input checked="" type="checkbox"/> Sides of parm / Sye van parm <input checked="" type="checkbox"/> Opposite angles of parm / Teenoorst. sye van parm $\overline{CD} = 10$ <input checked="" type="checkbox"/> Diagonals 1 / Hoeklyne 1 <input type="checkbox"/> Diagonals 2 / Hoeklyne 2 <input type="checkbox"/> Diagonals 3 / Hoeklyne 3 <input type="checkbox"/> Area of triangles in sq units 
<p>Rectangle / Reghoek</p>  <ul style="list-style-type: none"> <input checked="" type="checkbox"/> Sides of rectangle / Sye van reghoek <input checked="" type="checkbox"/> Angles of rectangle / Hoeke van reghoek <input checked="" type="checkbox"/> Diagonals 1 / Hoeklyne 1 <input checked="" type="checkbox"/> Diagonals 2 / Hoeklyne 2 <input type="checkbox"/> Area of triangles 	<p>Rhombus / Ruit</p>  <ul style="list-style-type: none"> <input checked="" type="checkbox"/> Sides / Sye <input checked="" type="checkbox"/> Angles / Hoeke <input checked="" type="checkbox"/> Diagonals 1 / Hoeklyne 1 <input type="checkbox"/> Diagonals 2 / Hoeklyne 2 <input type="checkbox"/> Diagonals 3 / Hoeklyne 3 <input type="checkbox"/> Area of triangles formed by the diagonals
<p>Square / Vierkant</p>  <ul style="list-style-type: none"> <input checked="" type="checkbox"/> Sides of square <input checked="" type="checkbox"/> Angles of square <input checked="" type="checkbox"/> Diagonals 1 / Hoeklyne 1 <input checked="" type="checkbox"/> Diagonals 2 / Hoeklyne 2 <input type="checkbox"/> Diagonals 3 / Hoeklyne 3 <input type="checkbox"/> Area of triangles formed by the diagonals 	<p>Kite / Vlieër</p>  <ul style="list-style-type: none"> <input checked="" type="checkbox"/> Sides / Sye <input checked="" type="checkbox"/> Angles <input checked="" type="checkbox"/> Diagonals 1 / Hoeklyne 1 <input type="checkbox"/> Diagonals 2 / Hoeklyne 2 <input type="checkbox"/> Diagonals 3 / Hoeklyne 3 <input type="checkbox"/> Area of triangles formed by diagonal 1 <input checked="" type="checkbox"/> Area of triangles formed by diagonal 2

Trapezium / Trapezium



- Sides / Side
- Angles
- Diagonals 1 / Hoeklyne 1
- Diagonals 2 / Hoeklyne 2
- Diagonals 3 / Hoeklyne 3
- Area of triangles formed by diagonal1
- Area of triangles formed by diagonal2

The students had to answer the following questions after complete the table below:

- Is a rectangle a parallelogram?
- Is a parallelogram a rectangle?

- Is a rhombus a parallelogram?
- Is a parallelogram a rhombus?

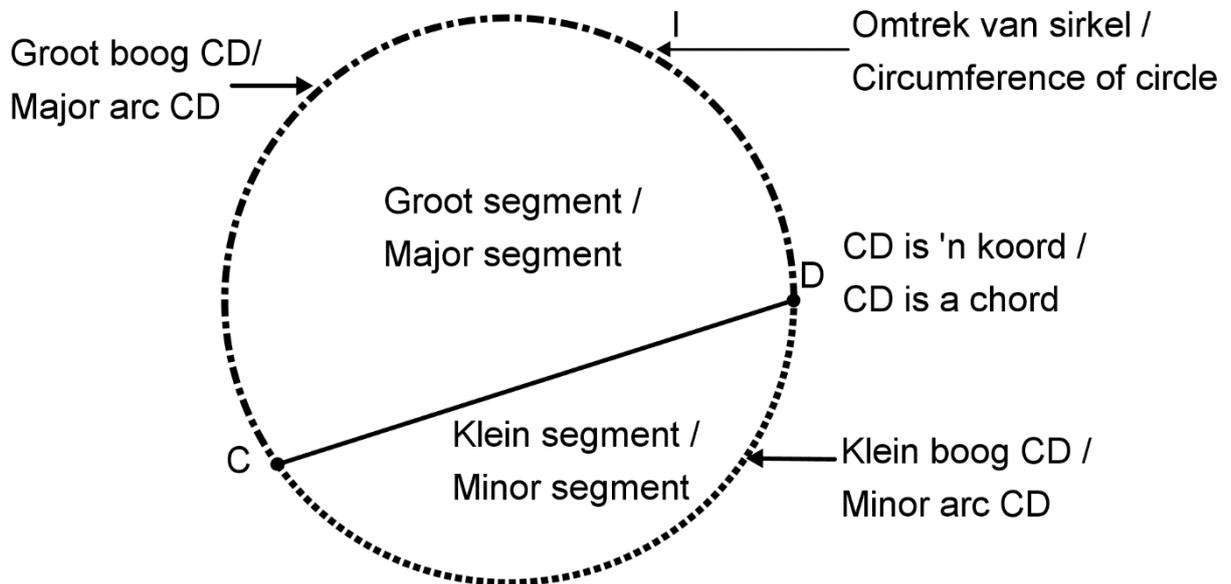
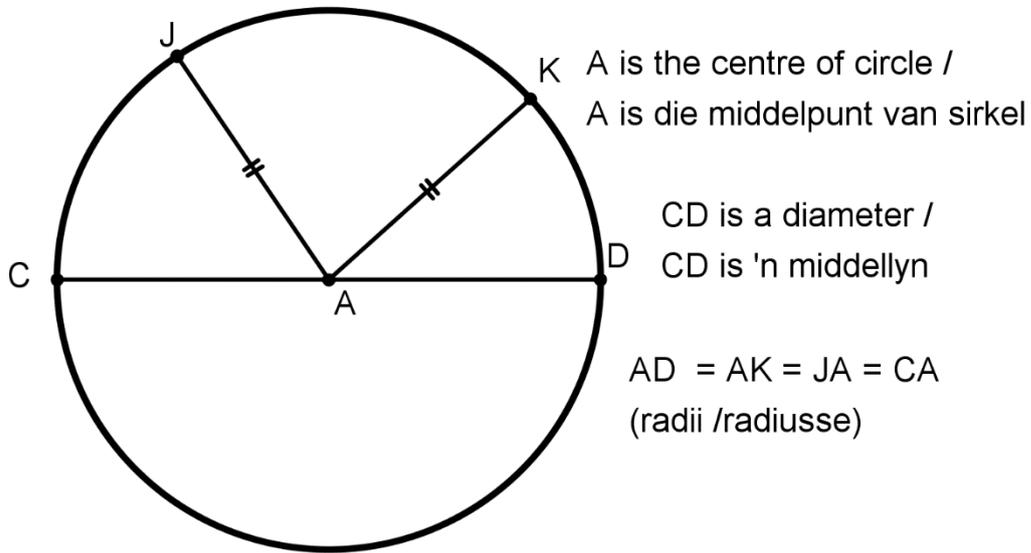
- Is a square a rectangle?
- Is a rectangle a square?

Properties	Parallelogram	Rectangle	Rhombus	Square	Kite	Trapezium
Only 1 pair of sides						
Opposite sides are						
Opposite sides are =						
All sides =						
2 pair adjacent sides are =						
Opposite angles are =						
All angles are =						
Only 1 diagonal bisect the other one						
Diagonals bisect each other						
Diagonals are =						
Diagonals are ⊥						
Only 1 diagonal is bisecting opposite angles						
Both diagonals bisect the angles						
Only 1 diagonal is bisecting area of quadrilateral						
Both diagonals are bisecting area of quadrilateral						

APPENDIX L

STUDENTS EXPLORING CIRCLE GEOMETRY WITH GEOGEBRA APPLETS

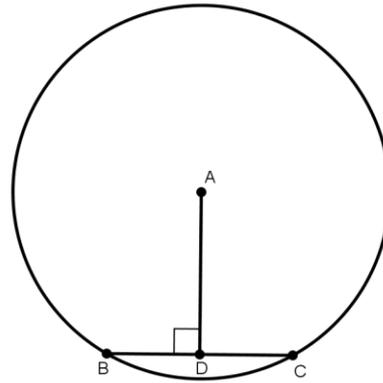
Terminology in circle (Shared as social knowledge)



Example of one of the GeoGebra applet utilised by the students to complete the following theorems

Given:
Line drawn from the centre of circle perpendicular to a chord.

Measure the length of BD and DC

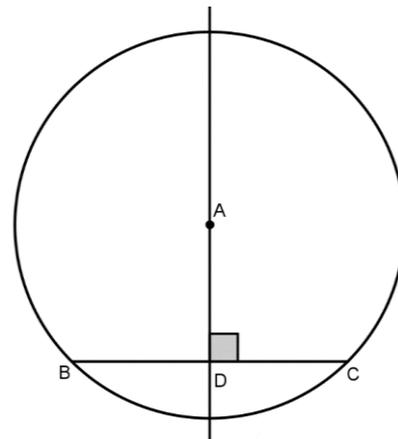


All the GeoGebra applets (some of them has been modified since 2014) that were shared with the students to explore the properties of circle geometry themselves:

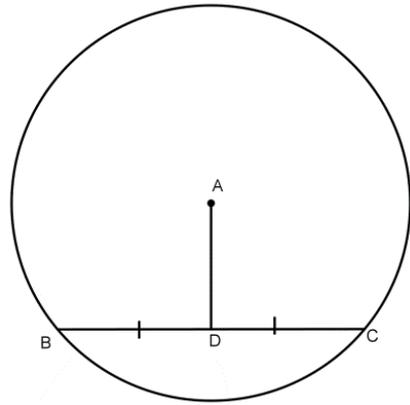
Name	Date modified	Type	Size
1. Perpendicular line drawn from centre of circle.ggb	2014-04-09 12:34 ...	GeoGebra File	5 KB
2. Segment from midpoint of chord to centre of circle.ggb	2014-04-10 02:05 ...	GeoGebra File	5 KB
3. Perpendicular bisector of chord.ggb	2014-04-10 02:07 ...	GeoGebra File	5 KB
4. Angle subtended by an arc at the circle Test the conjecture.ggb	2016-09-15 10:48 ...	GeoGebra File	23 KB
4. Angle subtended by an arc at the circle.ggb	2016-07-17 01:39 ...	GeoGebra File	13 KB
4. No converse for angle at centre.ggb	2016-04-27 06:40 ...	GeoGebra File	10 KB
5. Angle is semicircle.ggb	2014-04-08 09:52 ...	GeoGebra File	5 KB
6. Angle is semicircle (converse).ggb	2014-04-23 08:26 ...	GeoGebra File	7 KB
7. Circumference angles subtended by the same chord (same side of chord).ggb	2015-11-22 04:03 ...	GeoGebra File	9 KB
8. Circumference angles subtended by equal chords (same side of the chords).ggb	2014-04-08 09:54 ...	GeoGebra File	6 KB
9. Opposite angles of cyclic quadrilateral.ggb	2014-04-08 09:55 ...	GeoGebra File	5 KB
10. Exterior angles of cyclic quadrilateral.ggb	2015-05-02 01:31 ...	GeoGebra File	7 KB
11. Tangent to a circle.ggb	2012-04-01 07:09 ...	GeoGebra File	4 KB
12. Two tangents to a circle.ggb	2012-04-01 07:12 ...	GeoGebra File	6 KB
13. Angle between tangent & chord.ggb	2014-04-08 09:56 ...	GeoGebra File	5 KB

Complete the following:

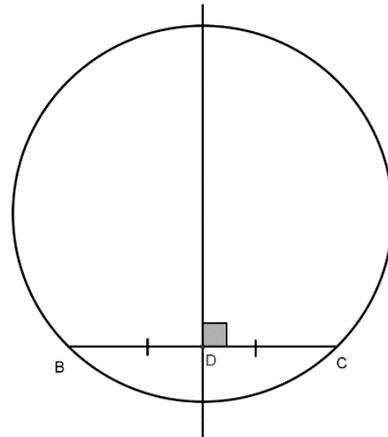
- 1. The line constructed from the centre of a circle perpendicular to a chord _____.**



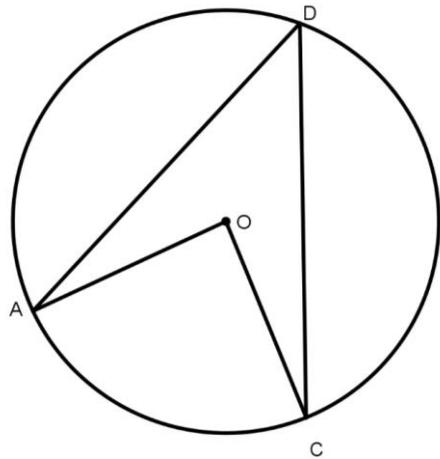
2. The line segment from the centre of a circle to the midpoint of a chord _____.



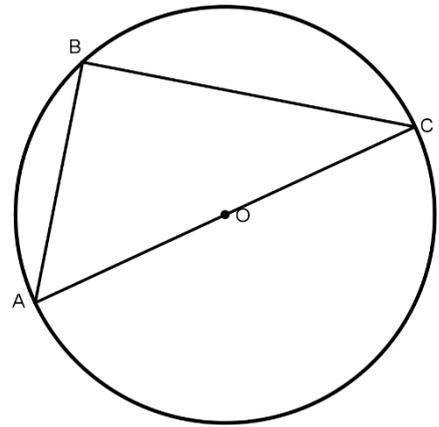
3. The perpendicular bisector of a chord _____.



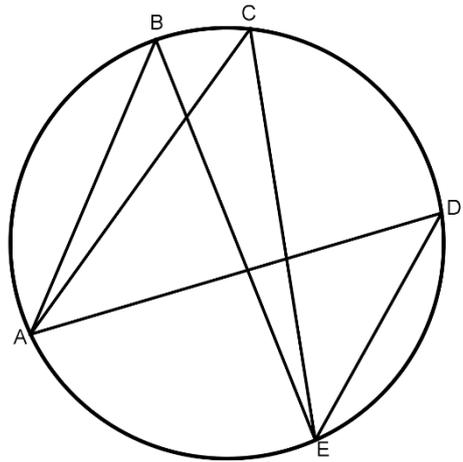
4. The angle subtended by an arc of a circle at the centre _____



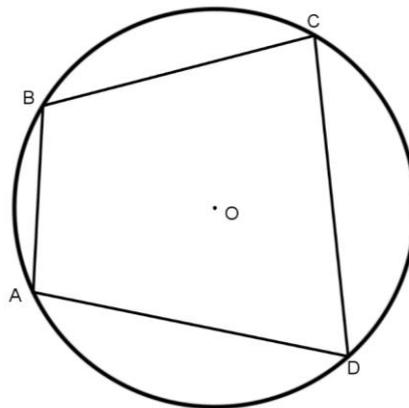
5. The angle at the circumference of a circle subtended by a diameter _____.



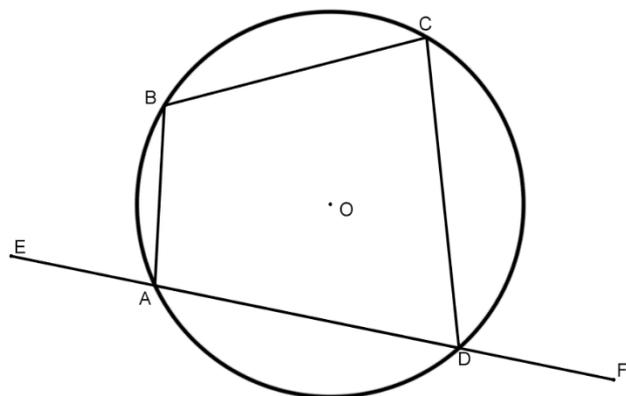
6. Angles subtended by a chord of the circle, on the same side of the chord _____.



7. The opposite angles of a cyclic quadrilateral _____.

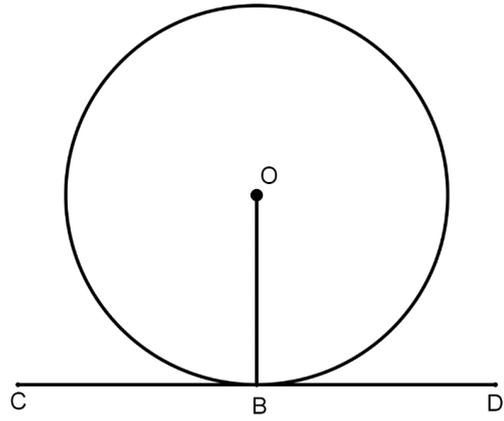


8. The exterior angle of a cyclic quadrilateral _____.



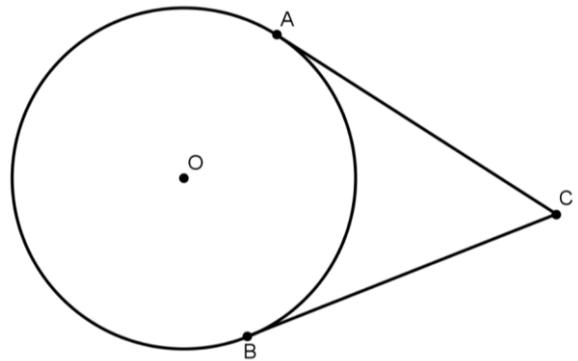
9. A tangent to a circle

_____.



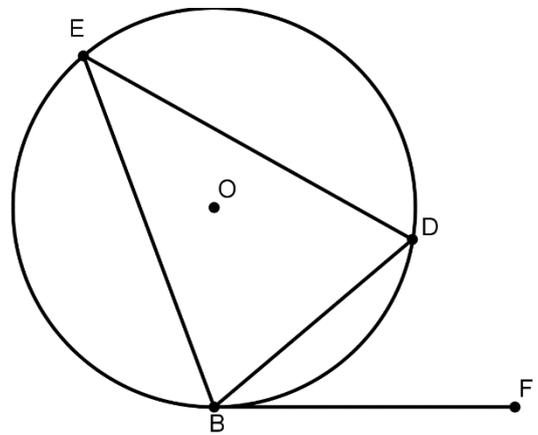
10. Two tangents constructed to a circle from the same point outside the circle

_____.



11. The angle between the tangent to a circle and the chord construct from the point of contact

_____.



APPENDIX M

Students exploring circle geometry with riders and proving circle geometry theorems after exploring with GeoGebra

PROOF OF LINE FROM CENTRE OF CIRCLE TO CHORD THEOREM AFTER EXPLORING IT ON GEOGEBRA

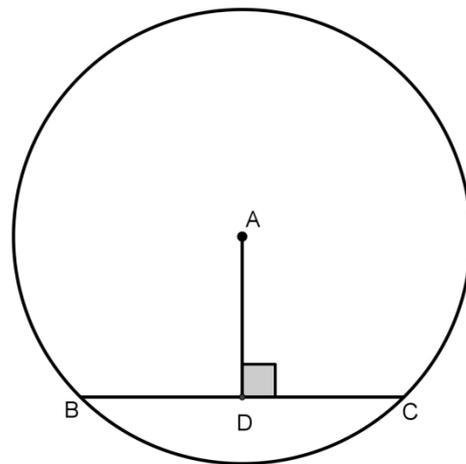
The line drawn from the centre of a circle perpendicular to a chord bisects the chord.
(line from centre \perp to chord)

Given: $AD \perp BC$

To prove: $BD = DC$

Construction:

Proof:



RIDERS TO EXPLORE THE ANGLE AT THE CENTRE AND CIRCUMFERENCE ANGLES OF CIRCLE AFTER EXPLORING IT ON GEOGEBRA

You **cannot** use the theorem that states that: **angle at the centre is = 2 x circumference angle.**

O is the centre of circle and A, C and D are on the circumference of circle.

Determine the size of:

(a) $\hat{A}CD$

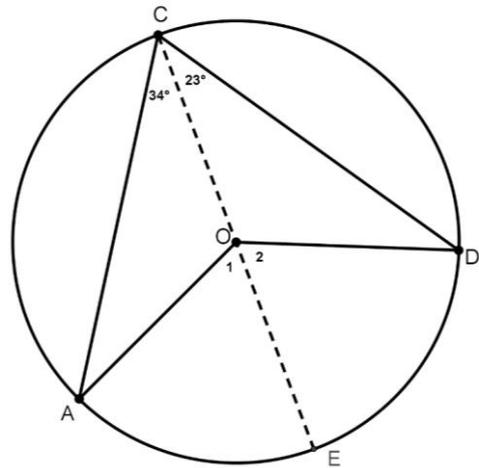
(b) \hat{A}

(d) \hat{O}_1

(e) \hat{D}

(f) \hat{O}_2

(g) \hat{AOD}



(a) $\hat{A}CD$

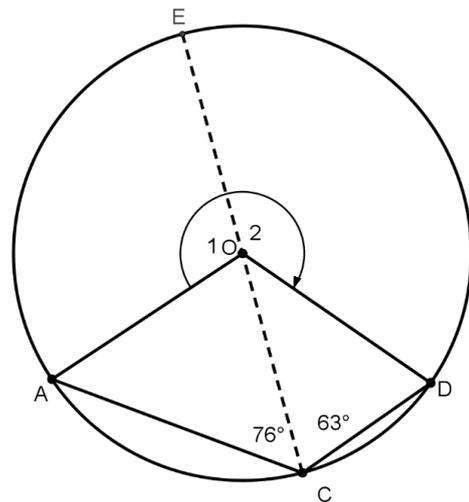
(b) \hat{A}

(d) \hat{O}_1

(e) \hat{D}

(f) \hat{O}_2

(g) \hat{AOD} (reflex)



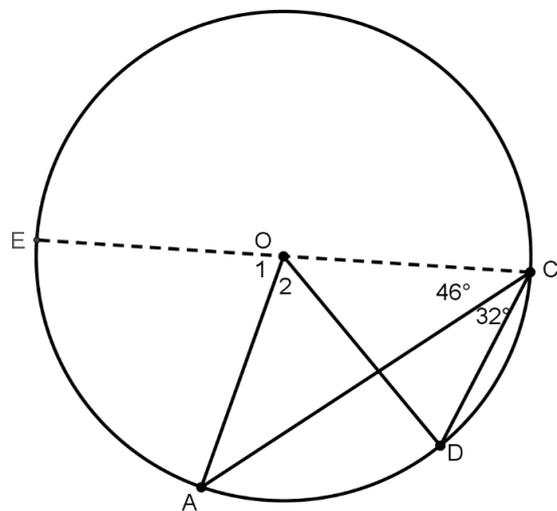
(a) \hat{A}

(b) \hat{O}_1

(c) \hat{D}

(d) \hat{EOD}

(e) \hat{O}_2



**PROOF OF THE ANGLE AT THE CENTRE AND CIRCUMFERENCE ANGLE OF THE CIRCLE THEOREM
AFTER EXPLORING IT WITH RIDERS AND ON GEOGEBRA**

The angle subtended by an arc of a circle at the centre is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre).

(\angle at centre = $2 \times \angle$ at circumference)

Given: A, C and D are on the circumference of the circle with centre at O.

To prove: $\widehat{AOC} = 2\widehat{ADC}$

Construction:

Proof:

Diagram 1

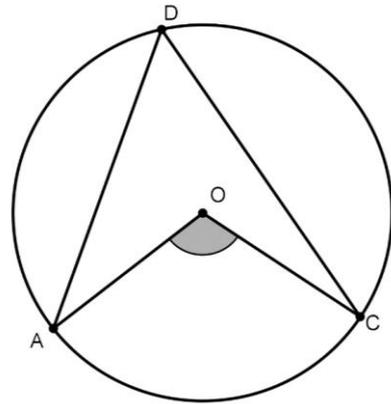


Diagram 2

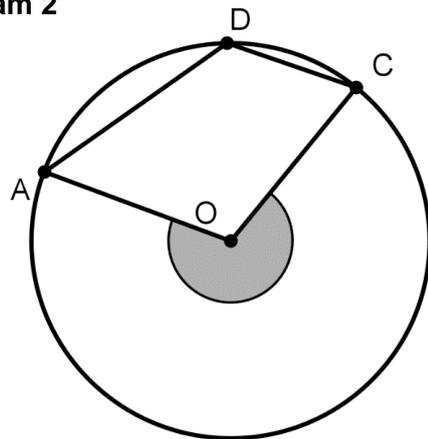
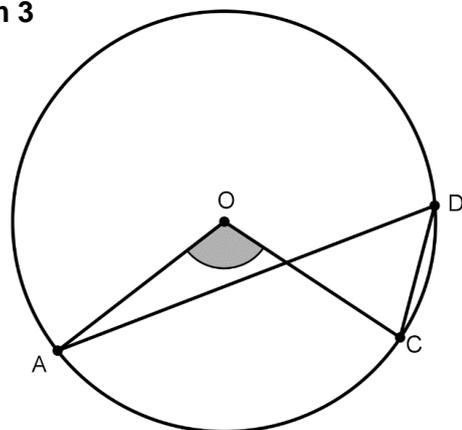


Diagram 3



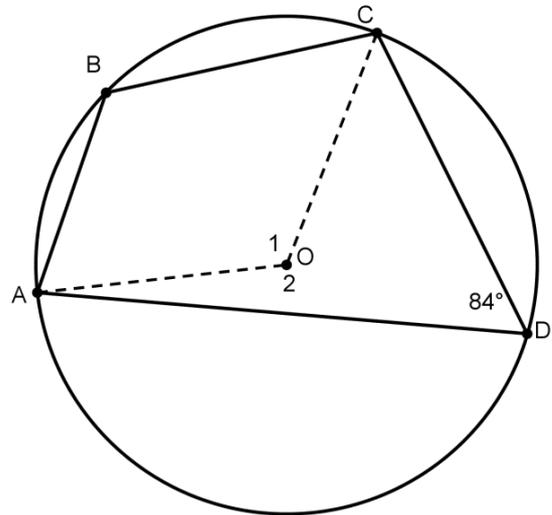
RIDERS TO EXPLORE THE SUM OF THE OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL AFTER EXPLORING IT ON GEOGEBRA

You **cannot** use the theorem that states that:

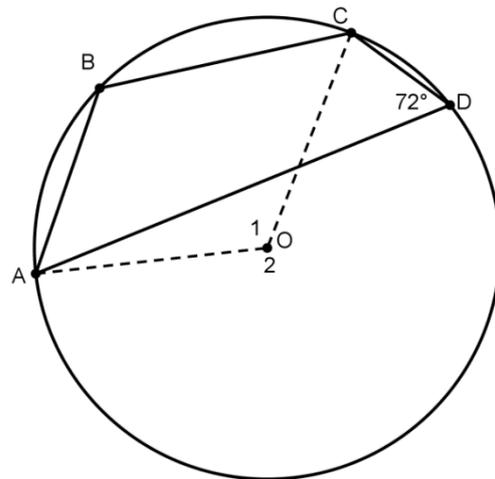
Sum of the opposite angles of quadrilateral is 180°

O is the centre of circle and ABCD is a cyclic quadrilateral. Determine the size of:

- (a) \widehat{O}_1
- (b) \widehat{O}_2
- (c) \widehat{B}
- (d) $\widehat{B} + \widehat{D}$
- (e) $\widehat{A} + \widehat{C}$



- (a) \widehat{O}_1
- (b) \widehat{O}_2
- (c) \widehat{B}
- (d) $\widehat{B} + \widehat{D}$
- (e) $\widehat{A} + \widehat{C}$



PROOF OF THE SUM OF OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL THEOREM AFTER EXPLORING IT WITH RIDERS AND ON GEOGEBRA

The opposite angles of a cyclic quadrilateral are supplementary (180°).
(opp \angle s of cyclic quad)

Given: Cyclic quadrilateral ACDE with O at the centre .

To prove: $\hat{A} + \hat{D} = 180^\circ$ &
 $\hat{C} + \hat{E} = 180^\circ$

Construction:

Proof:

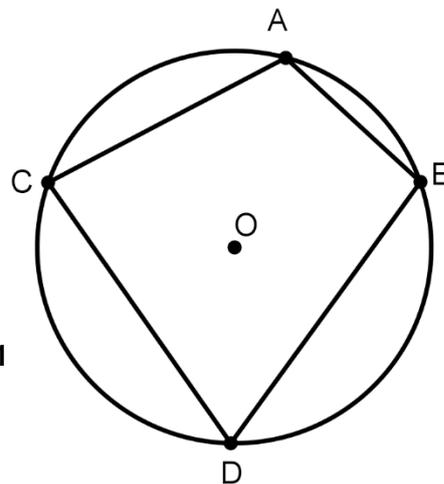


Diagram 1

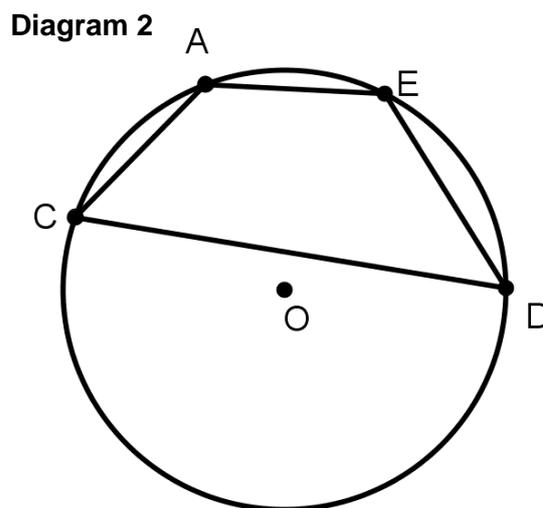


Diagram 2

RIDERS TO EXPLORE THE CHORD & TANGENT OF CIRCLE THEOREM AFTER EXPLORING IT ON GEOGEBRA

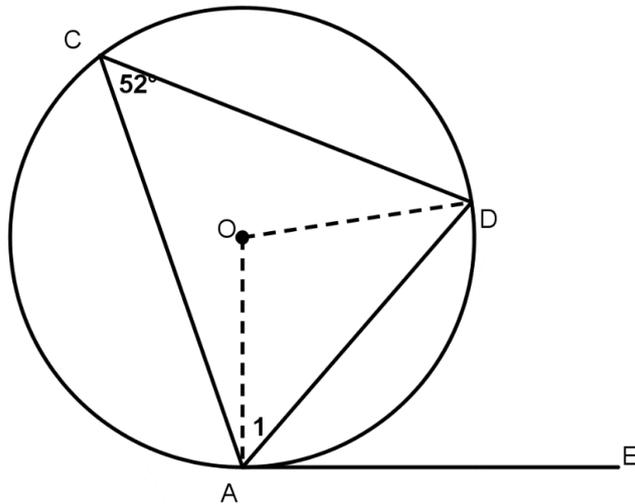
You **cannot** use the theorem that states that:

Angle between tangent & chord (chord & tangent theorem)

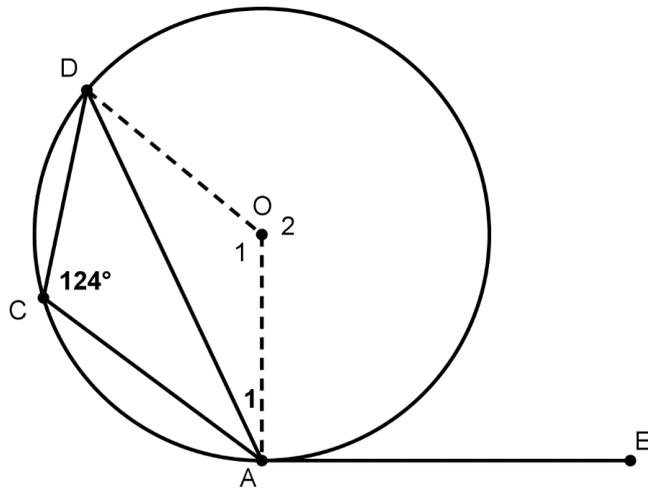
O is the centre of circle and A, C and D are on the circumference of circle.

Determine the size of:

- (a) \widehat{AOD}
- (b) \widehat{A}_1
- (c) \widehat{OAE}
- (d) \widehat{DAE}



- (a) \widehat{O}_2
- (b) \widehat{O}_1
- (c) \widehat{A}_1
- (d) \widehat{DAE}



PROOF OF THE TANGENT AND CHORD OF A CIRCLE THEOREM AFTER EXPLORING IT WITH RIDERS AND ON GEOGEBRA

The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment. (\angle between tangent & chord / tangent chord theorem)

Given: Circle O with tangent DB and A, D and C on the circumference of the circle.

To prove: $\hat{A}DB = \hat{A}CD$

Construction:

Proof:

Diagram 1

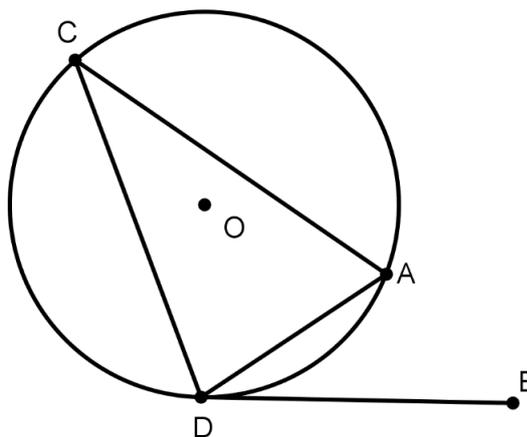
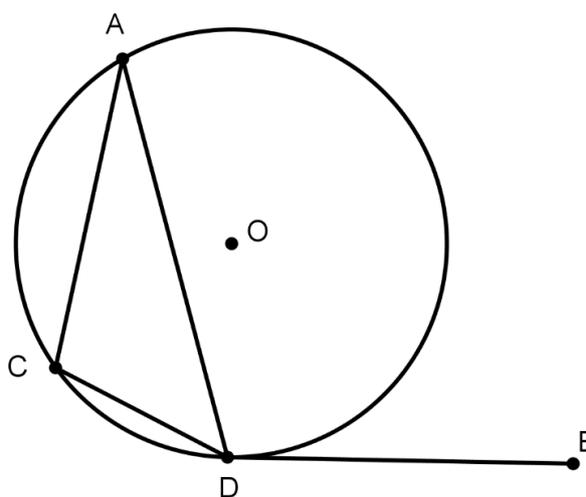


Diagram 2



APPENDIX N

WCED institutional permission for research in two schools



WESTERN CAPE
Education Department
Provincial Government of the Western Cape

RES
EAR
CH

Audrey.wyngaard2@pgwc.gov.za
tel: +27 021 476 9272
Fax: 0865902282
Private Bag x9114, Cape Town, 8000
wced.wcape.gov.za

REFERENCE: 20111114-0046
ENQUIRIES: Dr A T Wyngaard

Mr Cerenus Pfeiffer
17 Second Street
Strand
7140

Dear Mr Cerenus Pfeiffer

RESEARCH PROPOSAL: THE ROLE OF GEOGEBRA IN CREATING A LEARNING ENVIRONMENT IN WHICH LEARNERS CAN CONSTRUCT MATHEMATICAL KNOWLEDGE

Your application to conduct the above-mentioned research in schools in the Western Cape has been approved subject to the following conditions:

1. Principals, educators and learners are under no obligation to assist you in your investigation.
2. Principals, educators, learners and schools should not be identifiable in any way from the results of the investigation.
3. You make all the arrangements concerning your investigation.
4. Educators' programmes are not to be interrupted.
5. The Study is to be conducted from **01 March 2012 till 31 May 2012**
6. No research can be conducted during the fourth term as schools are preparing and finalizing syllabi for examinations (October to December).
7. Should you wish to extend the period of your survey, please contact Dr A.T Wyngaard at the contact numbers above quoting the reference number.
8. A photocopy of this letter is submitted to the principal where the intended research is to be conducted.
9. Your research will be limited to the list of schools as forwarded to the Western Cape Education Department.
10. A brief summary of the content, findings and recommendations is provided to the Director: Research Services.
11. The Department receives a copy of the completed report/dissertation/thesis addressed to:

The Director: Research Services
Western Cape Education Department
Private Bag X9114
CAPE TOWN
8000

We wish you success in your research.

Kind regards.

Signed: Audrey T Wyngaard
for: HEAD: EDUCATION
DATE: 14 November 2011

MELD ASSEBLIEF VERWYSINGSNOMMERS IN ALLE KORRESPONDENSIE / PLEASE QUOTE REFERENCE NUMBERS IN ALL CORRESPONDENCE /
NCEDA UBHALE INOMBOLO ZESALATHISO KUVO YONKE IMBALELWANO

GRAND CENTRAL TOWERS, LAER-PARLEMENTSTRAAT, PRIVAATSAK X9114, KAAPSTAD 8000
GRAND CENTRAL TOWERS, LOWER PARLIAMENT STREET, PRIVATE BAG X9114, CAPE TOWN 8000

WEB: <http://wced.wcape.gov.za>

INBELSENTRUM /CALL CENTRE

INDIENSNEMING- EN SALARISNAVRAE/EMPLOYMENT AND SALARY QUERIES ☎0861 92 33 22
VEILIGE SKOLE/SAFE SCHOOLS ☎ 0800 45 46 47

APPENDIX O



UNIVERSITEIT•STELLENBOSCH•UNIVERSITY
jou kennisvennoot • your knowledge partner

STELLENBOSCH UNIVERSITY CONSENT TO PARTICIPATE IN RESEARCH

The role of *GeoGebra* in creating a learning environment in which students construct mathematical knowledge.

You are asked to participate in a research study conducted by Cerenus Roderick Pfeiffer, B.A., H.D.E, B.Ed.(Hons) and M.Ed., from the Department of Curriculum Studies at Stellenbosch University. The quantitative and qualitative data that will be generated from this research will contribute towards a dissertation presented in fulfillment of the requirements for the degree of Doctoral of Philosophy. You were selected as possible participants in this study because you are part of the bridging programme of SciMathUS and were exposed to active learning and the learning and teaching with *GeoGebra*.

1. PURPOSE OF THE STUDY

The focus of the research is to determine the effectiveness of GeoGebra teaching/learning in the teaching of Mathematics. The study will be mixed method approach and will be used to describe what impact GeoGebra has on students' attitude towards Mathematics. Students will be expose to a learning environment in which they construct with active learning and guided reinvention mathematical knowledge.

2. PROCEDURES

If you volunteer to participate in this study, I would ask you to do the following things:

- The researcher will be using GeoGebra as teaching/learning tool throughout the year. As part of his teaching he will use in his classrooms lessons and you will also use GeoGebra in the computer laboratory.
- Your involvement whilst using GeoGebra as a learning tool in computer lab will be observed and I will videotape the certain lessons.
- Completing questionnaire
- A sample will be selected from table of random numbers to take part in semi-structured interviews.
- Your results in Mathematics will only be used in this study.

3. POTENTIAL RISKS AND DISCOMFORTS

To avoid any inconvenience the participant will be requested to suggest a suitable time and location for the interview. There were no physical or psychological risks associated with this study.

4. POTENTIAL BENEFITS TO SUBJECTS AND/OR TO SOCIETY

The findings of this study may:

- have implications for the teachers to realise that technology can be effective in the teaching of Mathematics.
- also help to show that if designed activities that are provided to teachers, it will help with the proficiency and fluency of use technology.
- change the "chalk-and-talk" way of teaching to teaching where students are constructing their own knowledge or learning with reinventing.
- students will be expose to the virtualization of certain Mathematical concepts. They will explore and visualise geometrical properties and transforming figures in ways beyond the scope of traditional paper-and-pencil activities.

5. PAYMENT FOR PARTICIPATION

Participants will not receive any payment to take part in the study.

6. CONFIDENTIALITY

Any information that is obtained in connection with this study and that can be identified with you will remain confidential and will be disclosed only with your permission or as required by law. Confidentiality will be maintained by means of safe guarding of the data with the researcher and anonymity of the participant. The name of the institution, SciMathUS will be used in the study. The researcher will be the only person having access to the information.

7. PARTICIPATION AND WITHDRAWAL

The researcher will be using GeoGebra as teaching /learning throughout the year. You can choose whether you want to complete the questionnaires, to take part in the semi-structured interviews or not. If you volunteer to be in this study, you may withdraw at any time without consequences of any kind. You may also refuse to answer any questions you don't want to answer and still remain in the study. The investigator may withdraw you from this research if circumstances arise which warrant doing so.

8. IDENTIFICATION OF INVESTIGATORS

If you have any questions or concerns about the research, please feel free to contact Cerenus Pfeiffer, (021) 808 2904, 083 5333189, crp2@sun.ac.za

9. RIGHTS OF RESEARCH SUBJECTS

You may withdraw your consent at any time and discontinue participation without penalty. You are not waiving any legal claims, rights or remedies because of your participation in this research study. If you have questions regarding your rights as a research subject, contact Ms Maléne Fouché [mfouche@sun.ac.za; 021 808 4622] at the Division for Research Development.

SIGNATURE OF RESEARCH SUBJECT OR LEGAL REPRESENTATIVE

The information above was described to me by Cerenus Pfeiffer in Afrikaans/ English and I, am in command of this language. I was given the opportunity to ask questions and these questions were answered to my satisfaction.

I hereby consent voluntarily to participate in this study. I have been given a copy of this form.

Name of Participant

N/A

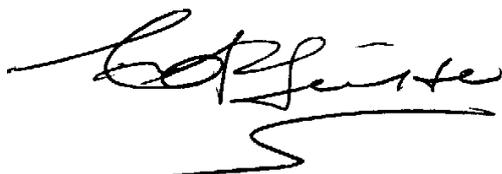
Name of Legal Representative (if applicable)

Signature of Participant

Date

SIGNATURE OF INVESTIGATOR

I declare that I explained the information given in this document to _____. He/ she was encouraged and given ample time to ask me any questions. This conversation was conducted in Afrikaans/ English and no translator was used.



Signature of Investigator

06.02.2014

Date

APPENDIX P

Stellenbosch University Institutional permission



UNIVERSITEIT • STELLENBOSCH • UNIVERSITY
jou kennisvenoot • your knowledge partner

22 April 2013

Mr Cerenus Pfeiffer
Institute for Mathematics
and Science Teaching
Stellenbosch University

Dear Mr Pfeiffer

Re: *The role of Geogebra in creating a learning environment in which students can construct mathematical knowledge*

The researcher has institutional permission to solicit the participation of 55 SciMathUS students for this research project as stipulated in the research proposal. The researcher has institutional permission to use the mathematics results of this group of students for the purpose of this research project. Institutional permission is granted on the following conditions:

- the researcher must obtain ethical clearance from the SU Research Ethics Committee,
- the researcher must obtain the participants' full informed consent for all the facets of their participation,
- participation is voluntary,
- students who choose not to participate may not be penalized as a result of non-participation,
- participants may withdraw their participation at any time, and without consequence,
- data must be collected in a way that ensures the anonymity of all participants,
- individuals may not be identified in the results of the study,
- data that is collected may only be used for the purpose of this study,
- data that is collected must be destroyed on completion of this study,
- the privacy of individuals must be respected and protected.

The researcher must act in accordance with SU's principles of research ethics and scientific integrity as stipulated in the *Framework Policy for the Assurance and Promotion of Ethically Accountable Research at Stellenbosch University*.

Best wishes,



Jan Botha
Senior Director Institutional Research and Planning Division



Afdeling Institusionele Navorsing en Beplanning • Institutional Research and Planning Division
Privaatsak/Private Bag X1 • Stellenbosch • 7602 • Suid-Afrika/South Africa
Tel. +27 21 808 3967 • Faks/Fax +27 21 808 4533

APPENDIX Q

Stellenbosch University Institutional permission: Approval Notice-New Application



UNIVERSITEIT-STELLENBOSCH-UNIVERSITY
jou kennisvenoot • your knowledge partner

Approval Notice New Application

20-May-2013
Pfeiffer, Cerenus CR

Proposal #: DESC_Pfeiffer2013

Title: The role of Geogebra in creating a learning environment in which students can construct mathematical knowledge

Dear Mr. Cerenus Pfeiffer,

Your DESC approved New Application received on 09-May-2013, was reviewed by members of the Research Ethics Committee: Human Research (Humanities) via Expedited review procedures on 17-May-2013 and was approved.

Please note the following information about your approved research proposal:

Proposal Approval Period: 17-May-2013 -16-May-2014

Please take note of the general Investigator Responsibilities attached to this letter. You may commence with your research after complying fully with these guidelines.

Please remember to use your proposal number (DESC_Pfeiffer2013) on any documents or correspondence with the REC concerning your research proposal.

Please note that the REC has the prerogative and authority to ask further questions, seek additional information, require further modifications, or monitor the conduct of your research and the consent process.

Also note that a progress report should be submitted to the Committee before the approval period has expired if a continuation is required. The Committee will then consider the continuation of the project for a further year (if necessary).

This committee abides by the ethical norms and principles for research, established by the Declaration of Helsinki and the Guidelines for Ethical Research: Principles Structures and Processes 2004 (Department of Health). Annually a number of projects may be selected randomly for an external audit.

National Health Research Ethics Committee (NHREC) registration number REC-050411-032.

We wish you the best as you conduct your research.

If you have any questions or need further help, please contact the REC office at 0218839027.

Included Documents:

Research proposal
Informed consent
Permission letter
DESC form
Questionnaire

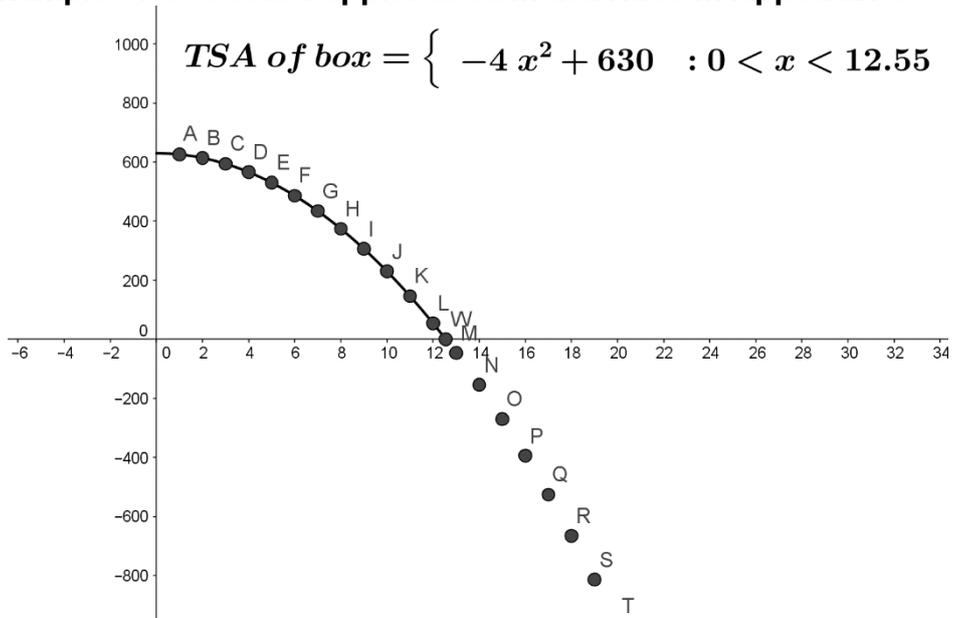
Sincerely,

Susara Oberholzer
REC Coordinator
Research Ethics Committee: Human Research (Humanities)

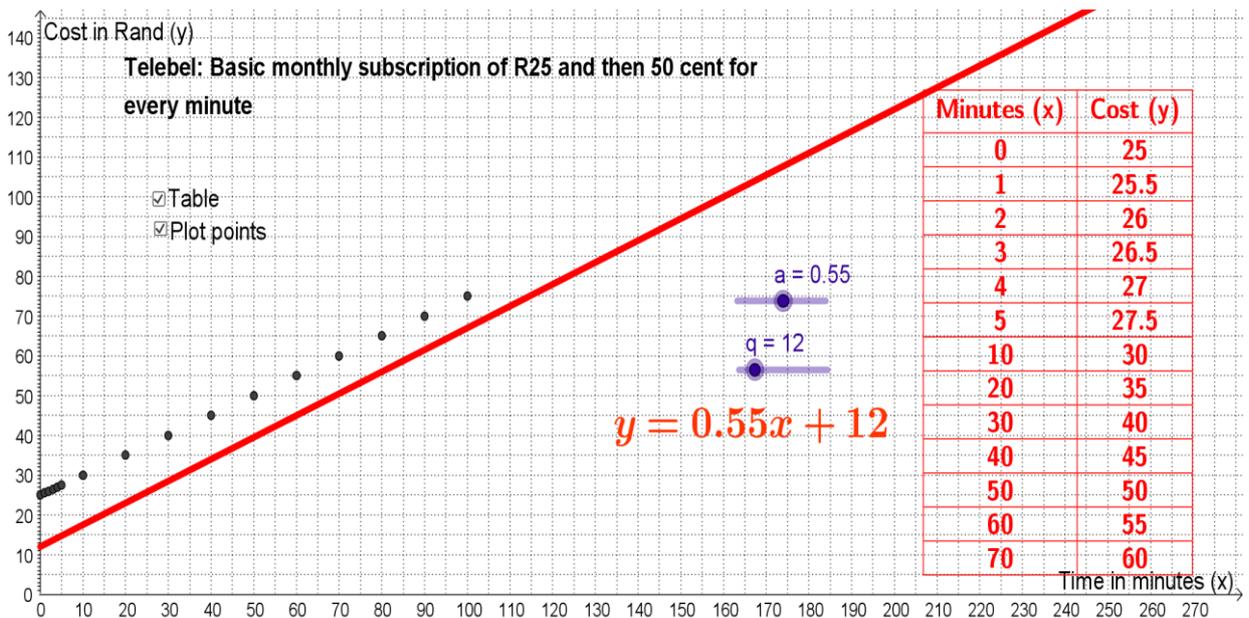
APPENDIX R

Examples of GeoGebra applets utilised by teacher-researcher with interactive lecturing

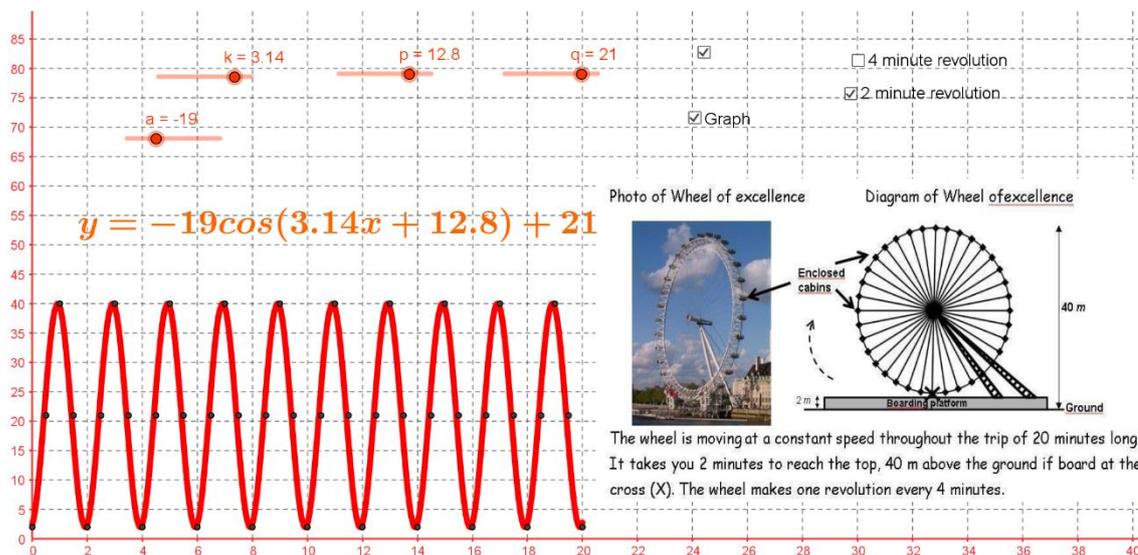
1. An example of GeoGebra applet used in activities in Appendix G



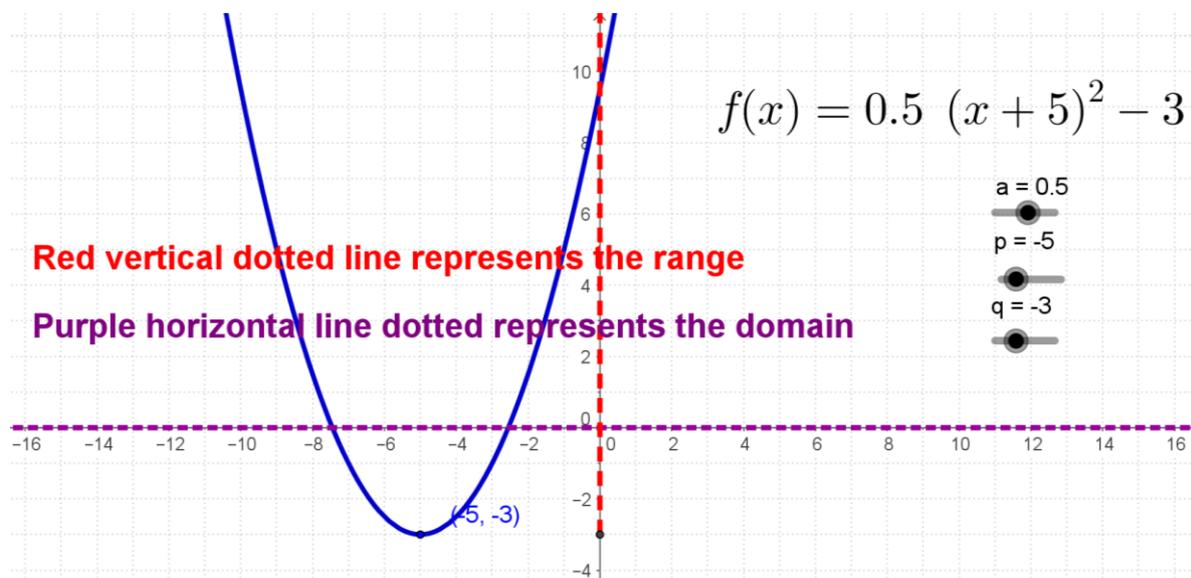
2. An example of GeoGebra applet used in activities in Appendix H



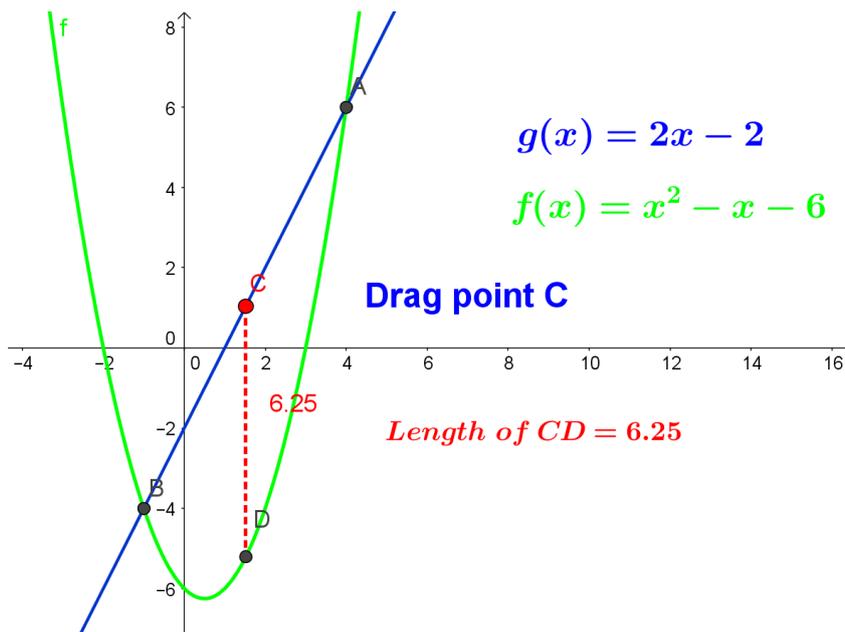
3. An example of GeoGebra applet used in connecting mathematics with real-life: Wheel of excellence in Cape Town



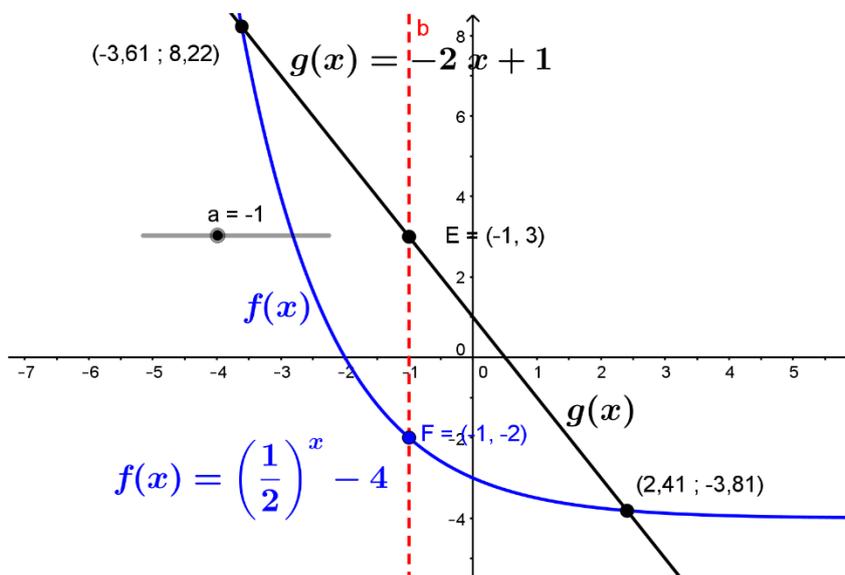
4. An example of GeoGebra applet used in the teaching of domain and range



5. An example of GeoGebra applet used in the teaching of functions

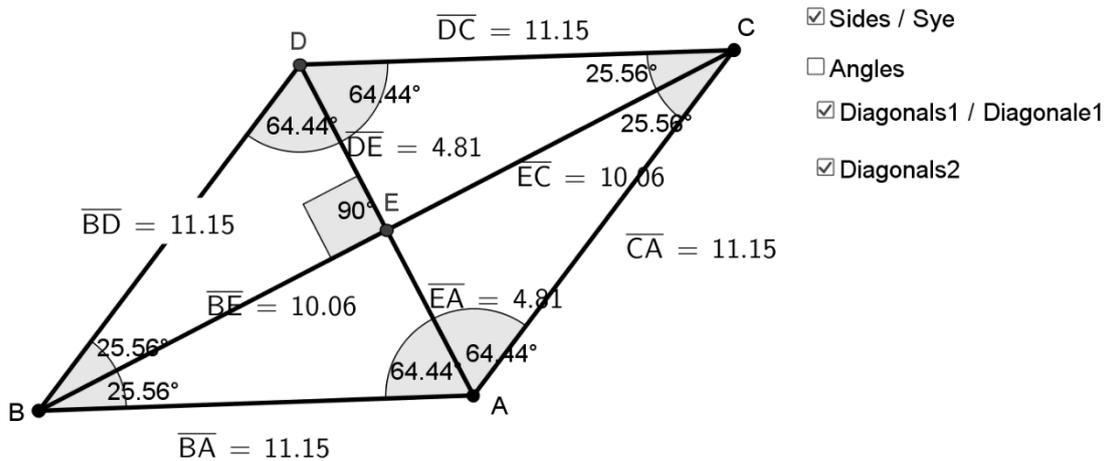


6. An example of GeoGebra applet used in the teaching of $f(x) \geq g(x)$, etc.

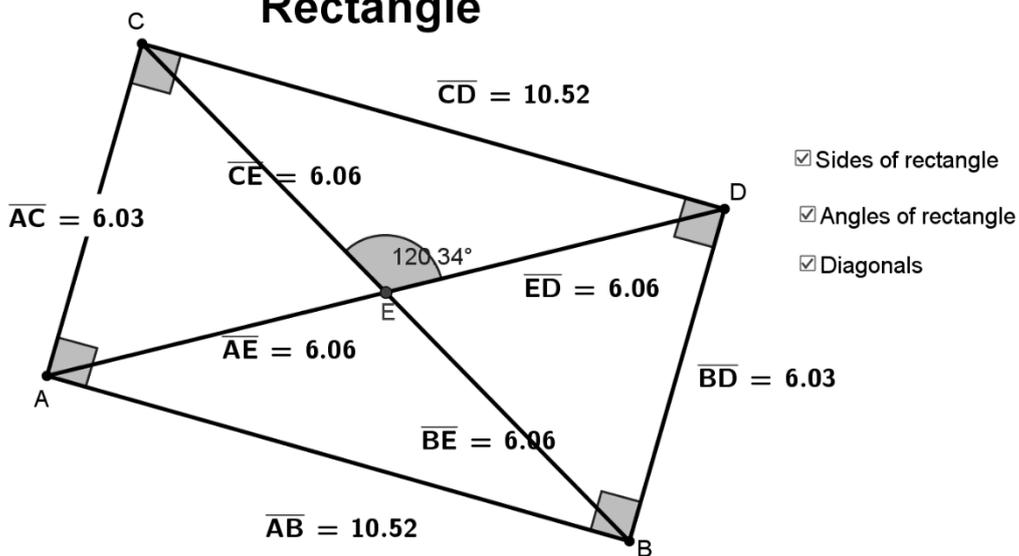


7. Examples of GeoGebra applets used in quadrilaterals activity (in Appendix K)

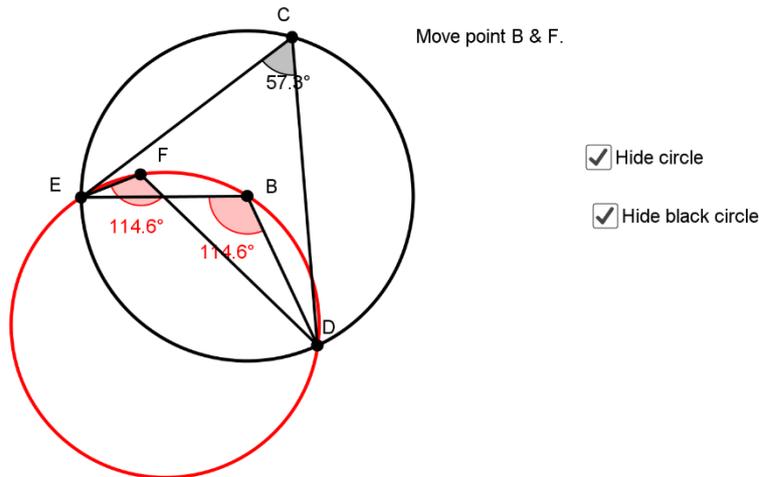
Rhombus /Ruit



Rectangle



8. An example of GeoGebra applet used in teaching why the angle at the centre theorem does not have a converse

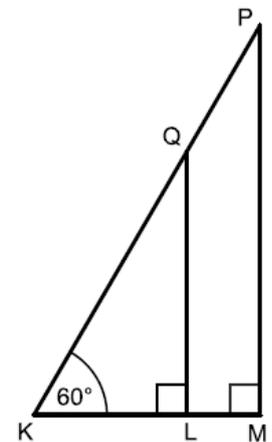


9. Introduction to trigonometry

Construct your own diagram as shown on graphpaper and then complete the following tables

(a) Measure the lengths of the sides in the triangles:

In Δ KLQ	KL =	QL =	KQ =
In Δ KMP	KM =	PM =	KP =



(b) Complete the table (round off to 2 decimals)

$\frac{QL}{KL} =$	$\frac{KL}{QL} =$	$\frac{KL}{KQ} =$	$\frac{KQ}{KL} =$	$\frac{QL}{KQ} =$	$\frac{KQ}{QL} =$
$\frac{PM}{KM} =$	$\frac{KM}{PM} =$	$\frac{KM}{KP} =$	$\frac{KP}{KM} =$	$\frac{PM}{KP} =$	$\frac{KP}{PM} =$

(c) Determine the following with a calculator to 2 decimals:

$\sin 60^\circ =$

$\cos 60^\circ =$

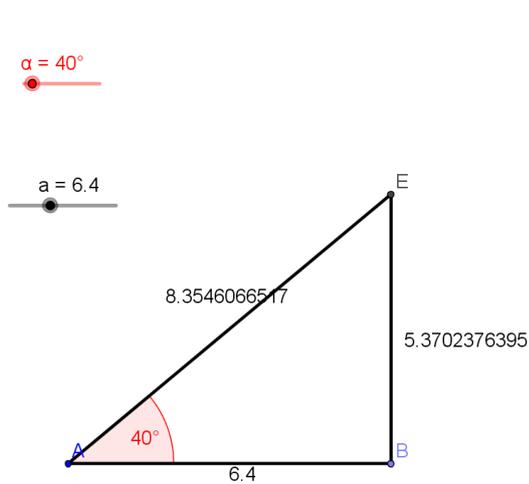
$\tan 60^\circ =$

(d) Complete (i) $\sin 60^\circ = \frac{QL}{KP} = \frac{\quad}{\quad} \approx 0,87$

(ii) $\cos 60^\circ =$

(iii) $\tan 60^\circ =$

GeoGebra applets utilised by teacher-researcher to verify students' answers



$$\frac{BE}{AE} = \frac{5.3702376395}{8.3546066517} = 0.6427876097$$

$$\frac{AE}{BE} = \frac{8.3546066517}{5.3702376395} = 1.5557238269$$

$$\frac{AB}{AE} = \frac{6.4}{8.3546066517} = 0.7660444431$$

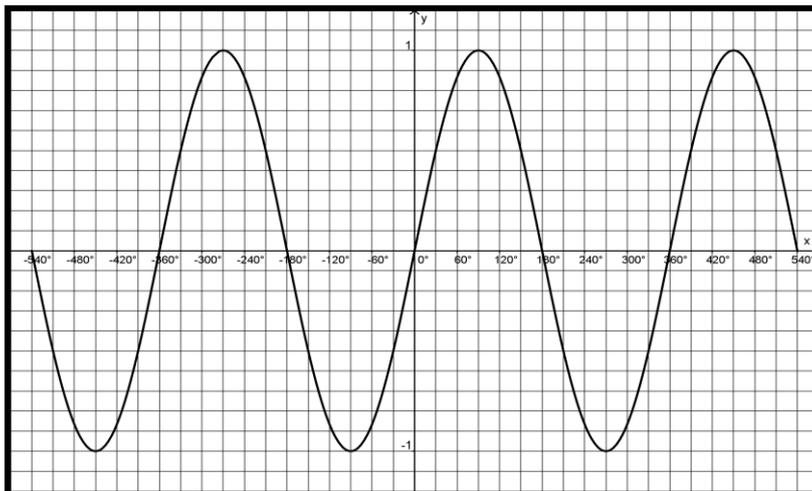
$$\frac{AE}{AB} = \frac{8.3546066517}{6.4} = 1.3054072893$$

$$\frac{BE}{AB} = \frac{5.3702376395}{6.4} = 0.8390996312$$

$$\frac{AB}{BE} = \frac{6.4}{5.3702376395} = 1.1917535926$$

10. Activity used to teach concept of general solutions of trigonometric equations

1. The graph represents $h(x) = \sin x$ for $x \in [-540^\circ ; 540^\circ]$.



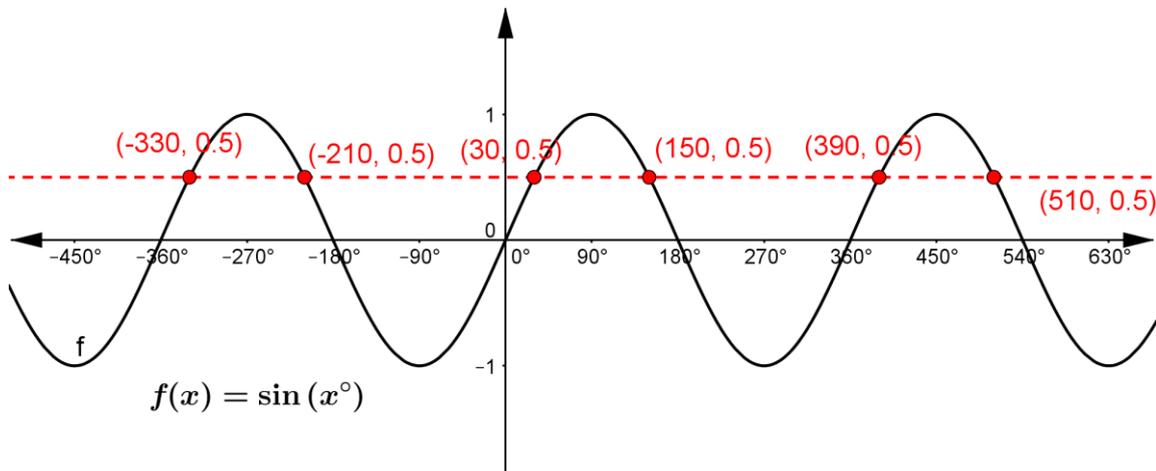
1.1 Read from the graph:

- (a) $\sin 30^\circ$ (b) $\sin (-90^\circ)$ (c) $\sin 150^\circ$ (d) $\sin (270^\circ)$

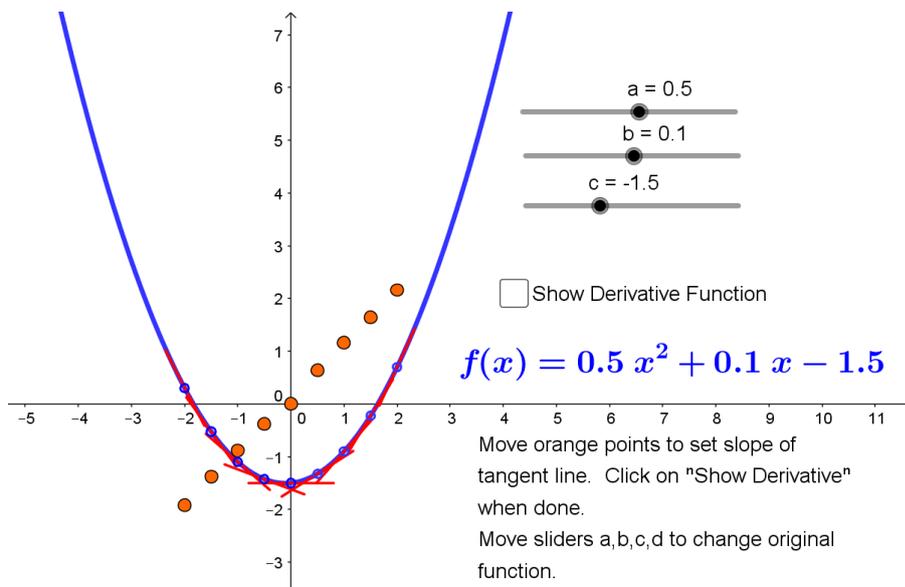
1.2 At which x -value(s) can you see on the graph:

- (a) $\sin x = 0,5$ (b) $\sin x = 1$ (c) $\sin x = 0$ (d) $\sin x = -1$
 (e) $\sin x = 2$ (f) $\sin x = -3$

An example of the GeoGebra applet used by the teacher-researcher to show the periodic nature of trigonometric functions and to teach the concept general solutions



11. An example of GeoGebra applet used to find the derivative of a parabola with tangents



12. An example of real-life activity utilised with introduction to Calculus

Activity 1: The growth of a young boy

The growth of a young boy is shown in the following table:

Age (in years)	Height (in cm)
0	45
3	95
9	130
12	140
18	180

To determine during which period the boy grew most rapidly we must compare the **average speed of growth** or the **increase in height per year**. Calculate the average speed of growth for each period with the aid of the table below.

Period	Speed of growth (in cm per year) $\left(\frac{\text{end height} - \text{initial height}}{\text{number of years}} \right)$
from 0 to 3 years	$\frac{95 - 45}{3} = 16,67 \text{ cm / year}$
from 3 to 9 years	
from 9 to 12 years	
from 12 to 18 years	

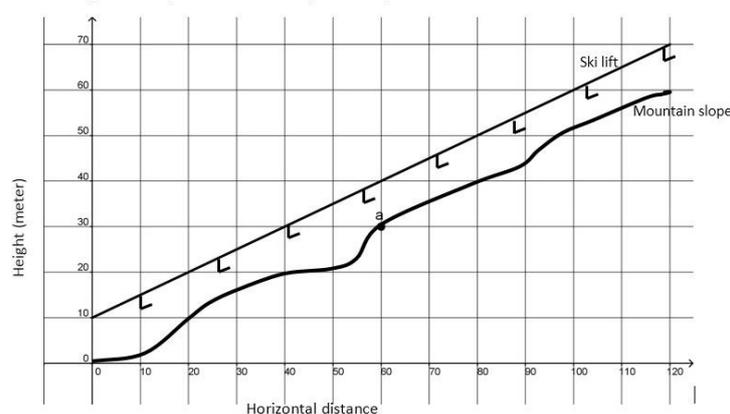
During which period did the boy grow most rapidly? Explain your answer

To study the **rate at which something changes**, we must always compare the average differences. In connection with this you must bear the following in mind:

- The change in height alone tells us nothing about the steepness of the slope, but rather it is the change in height per horizontal distance.
- The change in height tells us nothing about the speed at which the boy is growing. What is actually necessary is the change in height per year (increase in height per year).

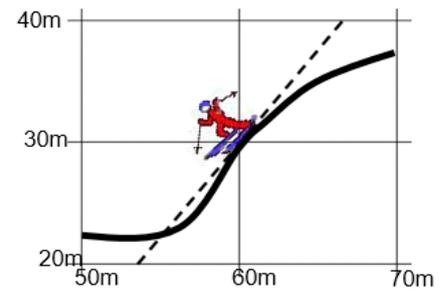
Activity 2: The gradient of a curved line (or curve)

The diagram shows part of a mountain slope and also part of a ski lift.



- The ski lift ascends steeply. What is the slope (or gradient) of the ski lift?
- The profile (outline) of the mountain shows that the slope of the mountain is not constant, but that the slope changes. Thus it is difficult to express the slope of the mountain with a single number. We can certainly talk about an "average slope". What is the average slope of the mountain?
- Where is the slope the steepest?

We wish to express the slope of the mountain profile (outline) at a as a number. To help us numerically express the slope of the mountain, we look at the slope of the skis in the adjacent sketch. This sketch is an enlargement of the sketch of the mountain slope near the point a . The skis form a straight line and at each point they follow the slope of the downhill ski run. The slope of the skis is thus a good indication of the slope of the mountain at a given moment.



- (d) What is the gradient (or slope) of the mountain at the point where the skier is? (To calculate this calculate the gradient of the skis)

We can find the gradient or slope of a curve at any point by using an appropriate straight line passing through the point. Such a straight line, which at a given point has the same gradient as the curve, is called a **tangent** to the curve at that point.