# Geometry of Complex Polynomials: On Sendov's Conjecture

by

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# Abstract


### Geometry of Complex Polynomials: On Sendov's Conjecture



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Sendov's conjecture states that if all the zeroes of a complex polynomial $P(z)$ of degree at least two lie in the unit disk, then within a unit distance of each zero lies a critical point of $P(z)$. In a paper that appeared in 2014, Dégot proved that, for each $a \in (0,1)$, there is an integer $N$ such that for any polynomial $P(z)$ with degree greater than $N$, $P(a) = 0$ and all zeroes inside the unit disk, the disk $|z - a| \leq 1$ contains a critical point of $P(z)$. Basing on this result, we derive an explicit formula $N(a)$ for each $a \in (0,1)$ and, furthermore, obtain a uniform bound $N$ for all $a \in [\alpha, \beta]$ where $0 < \alpha < \beta < 1$. This addresses the questions posed in Dégot's paper.

# Uittreksel

## Meetkunde van Komplekse Polinome en die Vermoeding van Sendov

*("Geometry of Complex Polynomials: On Sendov's Conjecture")*

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Die vermoede van Sendov lui dat, as alle nulpunte van 'n komplekse polinoom $P(z)$ van graad minstens twee binne die eenheidssirkel lê, dan is daar 'n kritieke punt van $P(z)$ binne 'n afstand van een van elke nulpunt. In die artikel wat 2014 verskyn het, het Dégot bewys dat daar vir elke $a \in (0, 1)$ 'n heelgetal $N$ bestaan sodat, vir elke polinoom $P(z)$ van graad groter as $N$ met $P(a) = 0$ en met alle nulpunte binne die eenheidskyf, die skyf $|z - a| \leq 1$ 'n kritieke punt van $P(z)$ bevat. Gebaseer op hierdie werk bepaal ons 'n formule $N(a)$ vir elke $a \in (0, 1)$, en verder bepaal ons 'n uniforme bogrens $N$ vir alle $a \in [\alpha, \beta]$ waar $0 < \alpha < \beta < 1$. Dit spreek die vrae aan wat in Dégot se artikel gestel is.

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# Dedications

*To F, who set the wheel in motion,*

*To G, who keeps the wheel turning.*

# Contents

# Chapter 1

# Introduction

The conjecture of Sendov simply states that, if a complex polynomial $P(z)$ of degree at least two has all of its zeros in the unit disk, then within a unit distance of each zero, there is a critical point of $P(z)$. Since its inception in 1958, up to present day, the problem remains open despite more than eighty research articles dedicated to it. However, over the years, many special cases have been verified, and possible attack strategies refined.

Chapter 1 is a general introduction to Sendov's conjecture. Herein, we give a brief but panoramic review of the literature on the conjecture. This is illustrated mainly by sampling through various special cases and their proof strategies, in a somewhat chronological order. We end the chapter with a new and elementary proof of a result of Rubinstein, which is one of the well known special cases.

Let us illustrate a common method which often succeeds in establishing certain special cases of the conjecture. One begins with a polynomial $P(z)$, and assumes that it is a counterexample to Sendov's conjecture at some zero, say $z_k$. With additional assumptions about $P(z)$, such as either on the geometrical configuration of the other zeros, or the size of its degree, one then shows that the hypothetical polynomial $P(z)$ violates a result known to be true in the general theory of *Geometry of Complex Polynomials*, and hence, one would have established the conjecture for polynomials $P(z)$ satisfying the said assumptions.

Indeed, one of the recent breakthroughs on the conjecture uses this approach. In his paper that appeared in 2014, Jérôme Dégot [4] established the conjecture for polynomials with "high enough" degree $n$. The degree $n$ is required to be greater than some integer bound $N(a)$, which depends on the choice of the root $a$ for which the conjecture is being investigated. The above mentioned paper is the focus of Chapter 3, our main chapter.

Dégot starts by fixing a polynomial $P(z)$ with a zero at $a \in (0,1)$ and degree $n$, assumed to contradict Sendov's conjecture at $a$ (that is, all the critical points of $P(z)$ are more than a unit distance away from $a$). By studying closely the geometry of $P(z)$, he obtains lower and upper bounds on the quantity $|P(c)|$ for some $c \in (0,a)$. He then proceeds to show that if $n$ is greater than the bound $N(a)$, a contradiction on the size of $|P(c)|$ ensues, and hence the disk $|z-a| \leq 1$ must have a zero of $P'(z)$.

Worthy of note is that aside from an existence proof, there was no explicit formula given for calculating $N(a)$ for any given $a \in (0,1)$. In fact, upon closer inspection, one notices that the method used to obtain it depended on additional parameters associated with the polynomial $P(z)$. More precisely, a crucial technical inequality that $N(a)$ has to satisfy depended on the quantity $m$, defined as the real part of the mean of the zeroes of $P(z)$. Dégot does indicate afterwards that this dependence can, in principle, be removed by using a certain estimate on the size of $m$. Through a heuristic method, Dégot then calculates a few values of $N(a)$ at the end of his paper.

Carefully following the treatment in Dégot's paper, we extract information from and modify his **Theorems 5, 6** and **7**. Each of these theorems introduced conditions which for a given $a \in (0,1)$, an integer bound $N_1$ ($N_2$ and $N_3$ respectively), has to satisfy in order to draw the requisite conditions on the size of $|P(c)|$. By studying closely these conditions, we systematically remove the extra dependencies on other parameters, and obtain explicit and continuous analogues of the bounds $N_1, N_2$ and $N_3$. We shall refer to these new formulas as $\mathcal{N}_1(a), \mathcal{N}_2(a)$ and $\mathcal{N}_3(a)$ respectively.

This allows us to obtain the conclusions of each of Dégot's main theorems and hence, ultimately his main result, but now with explicit constants which depend continuously on $a$. So we can then obtain, as a by-product of the continuity of our analogous functions, a uniform bound $N$ independent of $a \in [\alpha, \beta]$ for any $0 < \alpha < \beta < 1$. This addresses the two questions that Dégot posed in the conclusion of his paper.

# Chapter 2

# A general introduction to Sendov's conjecture

## 2.1 Introduction

The main part of our thesis is focused on improving a special case of Sendov's conjecture, hence, some remarks on what is known about the conjecture in general are in order. The goal of this section therefore, is to familiarize the reader with what is known in the literature. We begin by motivating the conjecture via the Gauss-Lucas theorem. After this, we briefly survey some of the other special cases in the literature, with a historical flavour. We end the section with a new and direct proof of a known special case proven first by Rubinstein in 1968.

We begin with the statement of the Gauss-Lucas theorem:

**Theorem 2.1.1.** *(Gauss-Lucas, [14], p. 25): Let $P(z) \in \mathbb{C}[z]$ be a polynomial of degree $n$ with zeroes $z_1, \ldots, z_n$. Then the critical points of $P(z)$ lie in the convex hull of the set $\{z_1, \ldots, z_n\}$.*

This theorem was first proven by Gauss with a view towards physical interpretations. The case he considered was that of relating the location of point charges on the plane to that of the resultant neutral zones. It was Lucas who later formalized the result to the general format in which it is stated above. A full discussion of the result, including the physical interpretations and historical aspects, can be found in the book [9].

There are many generalizations and further sharpenings of the Gauss-Lucas theorem. To mention just a few, the theorem has been generalized to entire functions as well as rational functions in [9]. In the 2004 paper of Curgus and Mascioni [3], the authors showed that in the case where $P(z)$ only has simple zeroes, the critical points lie in a region strictly within the boundary of the convex hull of the zeroes. This had been known for a long time, albeit qualitatively, but it was in [3] that this result was quantified. Hence it is a further sharpening of the Gauss-Lucas theorem.

We now give the statement of Sendov's conjecture:

**Conjecture 2.1.2.** *(Sendov, [5], p. 25): Let $P(z) = \prod_{j=1}^{n}(z - z_j)$, be a polynomial of degree $n \geq 2$ such that $|z_j| \leq 1$ for all $j = 1, \ldots, n$. Then each of the disks $|z - z_j| \leq 1$, $j = 1, \ldots, n$ contains a critical point of $P(z)$.*

**Remark 2.1.3.** *This conjecture can be viewed as a further attempt towards sharpening the Gauss-Lucas theorem. This is because, immediately from the Gauss-Lucas theorem, one obtains the following corollary:*

**Corollary 2.1.4.** *Suppose all the zeroes of $P(z)$ lie in the disk $|z - a| \leq r$, for some $a \in \mathbb{C}$, $r > 0$. Let $z_1$ be one of the zeroes. Then the disk $|z - z_1| \leq 2r$ contains all the critical points of $P(z)$.*

*Proof.* This is an immediate consequence of Theorem 2.1.1. For if $z_1 \in D = \{z \in \mathbb{C} : |z - a| \leq r\}$, then $\max_{z \in D}\{|z - z_1|\} \leq 2r$. Hence the disk $|z - z_1| \leq 2r$ in particular contains $D$, and consequently *all* the critical points of $P(z)$. $\square$

As Sendov himself mentions in [13], in 1958 he was intrigued by this quantity "$2r$". He reckoned that if $2r$ were to be replaced by $r$, then $|z - z_j| \leq r$ should have (at least) one critical point of $P(z)$. This is how the conjecture was conceived.

**Remark 2.1.5.** *Without loss of generality, we can take $r = 1$ and $a = 0$. This is because the act of scaling the radius $r$ of the disk containing all the zeroes of $P(z)$ to $r = 1$ and then translating it to the unit disk is an affine transformation. Hence the geometrical configuration of the zeroes and critical points of $P(z)$ is preserved.*

It is worth noting that from this perspective the conjecture is sharp, in the sense that the polynomial $P(z) = z^n - e^{i\theta}$ has all its zeroes on the unit circle, and only one critical point, which is located at the origin.

With that being said, we now take a brief look at some of the known special cases.

## 2.2 Some known results on Sendov's conjecture

Over the years since its inception, more than 80 research articles have been published investigating Sendov's conjecture. As such, many special cases, often contributed independently by several authors, were proven. The few we mention here are far from an exhaustive list. Thus, for a more extensive survey of what is known, we refer the curious reader to the excellent treatment by Rahman and Schmeisser in the book [11]. For a survey with a view towards recent literature, one may also consult [14]. The latter reference ends the section on Sendov's conjecture with several enlightening pointers towards possible previously unexplored approaches towards the conjecture. Interestingly enough, in early articles, the conjecture is often referred to as Illief's conjecture. This is due to a misattribution in [5], where the problem first appeared in print.

In his 1968 paper, Rubinstein in [12] verified the conjecture for polynomials with degree $n \leq 4$. Shortly thereafter, Meir and Sharma proved the case when $n = 5$. The chronological order of all these results, and authors leading up to the case of $n = 6$ is given fully in [11]. All these cases are then recovered in the 1999 paper of Brown and Xiang [2], who proved the conjecture for all polynomials with degree $n \leq 8$, as well as for polynomials with arbitrary degree but at most 8 distinct zeroes.

In [1], Bojanov et al showed that for $P(z) = \prod_{j=1}^{n}(z - z_j)$, $n \geq 2$, with $|z_j| \leq 1$ for all $j = 1, \ldots, n$, the disk

$$D(z_j; (1 + |z_1 \cdots z_n|)^{\frac{1}{n}})$$

contains a critical point of $P(z)$.
As an immediate corollary of their theorem, they obtained the following result obtained first independently by Schmeisser in 1969:

**Corollary 2.2.1.** *(Schmeisser, 1969): Sendov's conjecture is true for polynomials with a zero at the origin.*

*Proof.* If $P(z) = z\prod_{j=2}^{n}(z - z_j)$, then the product of all the zeroes, $\prod_{j=1}^{n} z_j = 0$. Hence by the above result, $D(z_j; 1)$ contains a critical point of $P(z)$ for all $z_j$, $j = 1, \ldots, n$. □

Another known result in the literature is that Sendov's conjecture is true at zeroes with modulus one, that is:

**Theorem 2.2.2.** *Let $P(z) = \prod_{j=1}^{n}(z - z_j)$, with $|z_1| = 1$ and $|z_j| \leq 1$ for $j = 2, \ldots, n$. Then the disk $|z - z_1| \leq 1$ contains a critical point of $P(z)$.*

The first verification of this result was by Rubinstein in [12].

**Remark 2.2.3.** *Subsequent to the above result, Phelps and Rodriguez [10], showed that the conjecture is true for $P(z)$ if the vertices of the convex hull of the zeroes lie on the unit circle.*

Two particular cases from our brief survey are of interest. First, the result that the conjecture is true for $P(z)$ if $P(0) = 0$, as well as at a zero $z_j$ if $|z_j| = 1$. Furthermore, we note that the transformation $z \mapsto e^{i\theta}z$, a rotation of $P(z)$ by an angle $\theta$, does not affect the relative configurations of the zeroes and critical points. Combining these three points, we get the following common reformulation of Sendov's conjecture:

**Conjecture 2.2.4.** *(Sendov): Let*

$$P(z) = (z - a)\prod_{j=1}^{n-1}(z - z_j), \text{ with } a \in (0, 1),\ |z_j| \leq 1 \text{ for } j = 1, \ldots, n - 1.$$

*Then the disk $|z - a| \leq 1$ contains a critical point of $P(z)$.*

This is the format in which the conjecture appears in most modern treatments, and it is the convention we have adopted as well in the thesis.

It is evident that most of the special cases proved are of an "*ad hoc*" nature. One begins with some geometrical condition imposed either on the zeroes or the critical points of $P(z)$, and then proceeds to show that Sendov's conjecture is true for the family of polynomials satisfying the *a priori* conditions. We illustrate this approach by deducing such a result. In preparation for the result we will need the next two lemmas:

**Lemma 2.2.5.** *(Bisector Theorem, [14], p. 200): Let $P(z)$ be a polynomial, $a$ and $b$ be distinct points such that $P(a) = P(b)$. Then there exists a critical point $w$ of $P(z)$ such that:*

$$|w - a| \leq |w - b|.$$

This result, attributed to G Szego in [14], implies that the polynomial $P(z)$ has a critical point on both sides of (or on) the perpendicular bisector of the line segment joining two zeroes $z_1$ and $z_2$ of $P(z)$.

**Proposition 2.2.6.** *Let $P(z) = (z - a)\prod_{j=1}^{n-1}(z - z_j)$, $a \in (0, 1)$, $|z_j| \leq 1$ for all $j = 1, \ldots, n-1$. Let $w_1, \ldots, w_{n-1}$ be the critical points of $P(z)$. Suppose $P(z)$ contradicts Sendov's conjecture at $a$, that is, $|w_j - a| > 1$ for all $j = 1, \ldots, n-1$. Then $\Re(w_j) < \frac{a}{2}$ for all $j = 1, \ldots, n-1$.*

*Proof.* The real part of the intersection of the circles $|z| = 1$ and $|z - a| = 1$ is $\frac{a}{2}$. Hence if $|w_j - a| > 1$, then $\Re(w_j) < \frac{a}{2}$. □

We are now ready to state the result:

**Proposition 2.2.7.** *Let*

$$P(z) = (z - z_1)(z - z_2)\prod_{j=3}^{n}(z - z_j), \text{ with } |z_j| \leq 1 \text{ for } j = 1, \ldots, n.$$

*Suppose that $\arg(z_1) = \arg(z_2)$ and $|z_1| > |z_2|$. Let $w_1, \ldots, w_{n-1}$ denote the critical points of $P(z)$. Then the disk $|z - z_1| \leq 1$ contains a critical point of $P(z)$.*

*Proof.* Suppose to the contrary that the claim is false. We apply the transformation $z \mapsto e^{-i\arg(z_1)}z$, so that $z_1 \mapsto a$, and $z_2 \mapsto b$, with $a, b \in [0, 1]$.

By Proposition 2.2.6, the supposition that $|w_j - a| > 1$ for all $j = 1, \ldots, n-1$ implies that

$$x_j = \Re(w_j) < \frac{a}{2} \text{ for all } j = 1, \ldots, n-1. \tag{2.1}$$

On the other hand, Lemma 2.2.5 tells us that there exists a $w_j$ such that

$$|w_j - a| \leq |w_j - b|.$$

This amounts to saying that $x_j = \Re(w_j) \geq \frac{a+b}{2} \geq \frac{a}{2}$, contradicting Equation

2.1 above.

Hence there exists at least one $j \in \{1, \ldots, n-1\}$ such that $|w_j - z_1| \leq 1$. $\square$

**Remark 2.2.8.** *Almost trivially, one notices that the result implies that* $|w_j - z_2| \leq 1$ *also. The disk* $|z - z_2| \leq 1$ *contains the intersection of* $|z| \leq 1$ *and* $|z - z_1| \leq 1$*, and thus also contains* $w_j$*.*

The following diagram illustrates the geometry behind the above proposition.

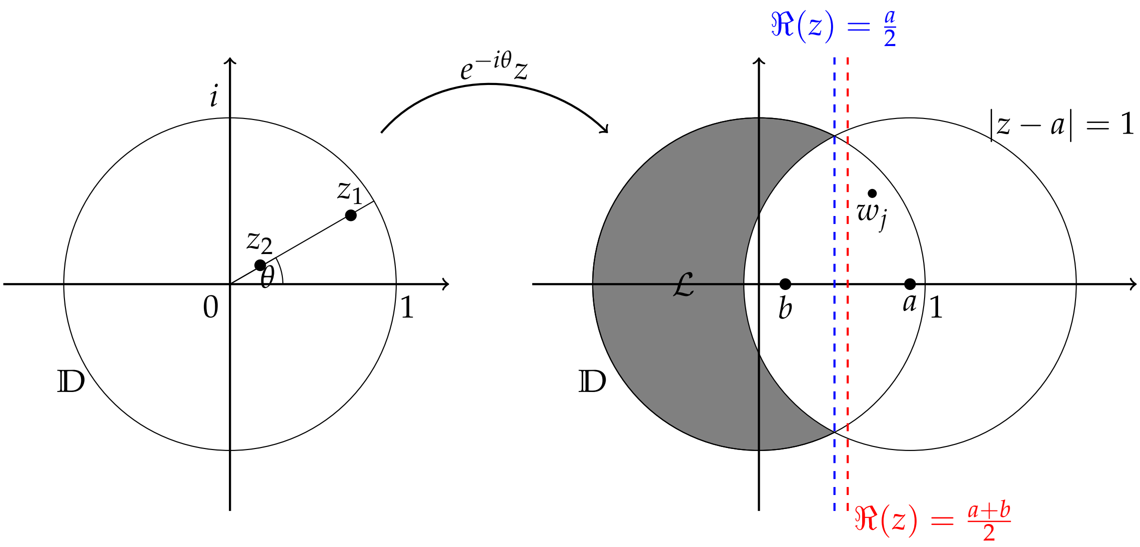



Figure 2.1: If $P(z)$ contradicts Sendov's conjecture at $a \in (0, 1)$, then all the zeros of $P'(z)$ lie in the shaded region $\mathcal{L}$, contradicting Lemma 2.2.5.

The above result can be found in [14], Section 6.4.3, p. 216, wherein it is proven for the case where $z_1$ and $z_2$ are in $[0, 1]$. As an immediate corollary, we obtain yet another proof of Corollary 2.2.1, Schmeisser's result.

**Corollary 2.2.9.** *Sendov's conjecture is true for all polynomials* $P(z)$ *such that* $P(0) = 0$*.*

*Proof.* In Proposition 2.2.7, let $z_2 = 0$. Then any other zero $z_j$ is collinear with $z_2$, and hence has a critical point within a unit distance of itself. $\square$

We end this chapter with a new elementary proof of an early result of Rubinstein, Theorem 3 in [12].

**Theorem 2.2.10.** *(Rubinstein, 1968): Let* $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$*,* $n \geq 2$*. If* $P(z_1) = 0$ *and* $|P'(z_1)| < n$*, then* $P'(z)$ *has a zero in the disk* $|z - z_1| < 1$*.*

*Proof.* Since $P(z)$ is monic, we write it as

$$P(z) = (z-a)\prod_{j=2}^{n}(z - z_j),$$

with $a$ taking the role of $z_1$ in the theorem statement. Consequently,

$$P'(z) = n\prod_{j=1}^{n-1}(z - w_j), \tag{2.2}$$

where $w_1, \ldots, w_{n-1}$ are the critical points of $P(z)$. From the theorem statement, $|P'(a)| < n$, and from Equation 2.2

$$|P'(a)| = n\prod_{j=1}^{n-1}|a - w_j|.$$

Combining the assumption that $|P'(a)| < n$ and Equation 2.2, we obtain:

$$n\prod_{j=1}^{n-1}|a - w_j| < n,$$

and from this, we deduce that:

$$\prod_{j=1}^{n-1}|a - w_j| < 1.$$

This implies that there is at least one $j \in \{1, \ldots, n-1\}$ such that $|a - w_j| < 1$. □

This brings us to the end of the introductory section. We now proceed to the main part of the thesis.

# Chapter 3

# On the paper of Dégot

**Introductory Remarks**

In this chapter we study the 2014 paper of Dégot, [4]. As well as extending the result obtained therein, we endeavour to present a self contained exposition of the paper. Below is a brief (and non-technical) rendition of how Dégot arrived at his result, as well as how we address the questions raised at the end of the paper.

Dégot's *modus operandi* can be roughly divided into a three part strategy. The first part consists of a series of lemmas (technical inequalities) that would then later support the key results in the second part.

The lemmas are a meticulous study of the geometry of a polynomial that is assumed to contradict Sendov's conjecture at some particular zero $a \in (0,1)$. The characterisation of such a polynomial is given in terms of the possible location of the other zeroes, or in terms of the lower and upper bounds of $|P(z)|$ in the unit disk. For instance, Dégot's **Theorem 3** provides what he calls the "exclusion domain" for the zeroes of such a polynomial. This turns out to be a geometrical area within the unit disk which should be devoid of any zeroes of a polynomial assumed to contradict Sendov's conjecture.

After the first part, Dégot then proceeds to what we consider to be the main part of his paper. This is the estimation of the size of $|P(c)|$ for some $c \in (0,a)$, where $P(a) = 0$, assuming that $P(z)$ contradicts Sendov's conjecture at $a$. Using some of the established lemmas, an upper bound of the form

$|P(c)| \leq 1 + a$ is obtained, with an assumption on the size of $n$, the degree of $P(z)$. Thereafter, efforts are directed towards finding a lower bound for $|P(c)|$. It is then ultimately established that $|P(c)| \geq CK^n$, for some specifically defined positive constants $C$ and $K$, and the degree $n$ of the polynomial being greater than some established integer bound.

In the final part, Dégot then combines the two results obtained on the size of $|P(c)|$. Upon a suitable assumption on the degree of $P(z)$, it is shown that the two results yield a contradiction. The section is then ended with two follow-up questions regarding the possibility of extending the results obtained in the paper.

We closely follow the same approach as outlined above, presenting more detailed versions of his proofs and, where we provide only a sketch, we shall highlight the key ideas involved. To some of Dégot's results we have made improvements, in a bid to address the issues raised at the end of his paper. More precisely, as mentioned in the introduction, there was no explicit formula $N(a)$ for each $a \in (0,1)$. Furthermore, the technical inequality through which $N_1$ was defined further depended on the mean of the zeroes of $P(z)$. Thus, the bound $N_1$ as defined still depended on $P(z)$. We address these issues and come up with an explicit formula $\mathcal{N}(a)$. Where such a result is encountered, we shall briefly explain or remark on his original version, followed by a complete treatment of our contribution.

In an effort to aid continuity, we shall not always state Dégot's supporting lemmas (and proofs) in the same order in which they appear in [4]. As opposed to having a section dedicated to preliminaries, results shall be quoted as and when they are needed. We may now proceed to study the paper.

## 3.1 Dégot's preparatory lemmas

We begin this section with a series of technical lemmas from the Geometry of Polynomials. The first is a result from [11].

**Lemma 3.1.1.** *([11], p. 100): Let $P(z)$ be a polynomial of degree $n \geq 2$. Suppose $w \in \mathbb{C}$ such that $P(w) \neq 0$ and $P'(w) \neq 0$. Then every circle $\mathcal{C}$ that passes* *through $w$ and $w - n\frac{P(w)}{P'(w)}$ separates at least two zeroes of $P(z)$, unless all the zeroes are contained in the circle.*

**Remark 3.1.2.** *From the above lemma (retaining the notation therein), we have that in particular, the (unique) closed disk $\mathcal{D}$ for which the points $w$ and $w - n\frac{P(w)}{P'(w)}$ are antipodal contains (at least) one zero of $P(z)$. The diameter of $\mathcal{D}$ in this case is*

$$\left| w - (w - n\frac{P(w)}{P'(w)}) \right|,$$

*which simplifies to $n\left|\frac{P(w)}{P'(w)}\right|$.*

From Lemma 3.1.1, we obtain the following corollary.

**Corollary 3.1.3.** *Let $P(z) = (z-a)\prod_{j=1}^{n-1}(z-z_j)$, $a \in (0,1)$ and $P'(a) \neq 0$. Let $c \in \mathbb{D}$ such that $P'(c) \neq 0$. Then there exists a $\gamma$ in the closed disk centered at $a$ with radius $R = n\left|\frac{P(c)}{P'(a)}\right|$ such that $P(\gamma) = P(c)$.*

*Proof.* We apply Lemma 3.1.1 and Remark 3.1.2 to the polynomial $Q(z) = P(z) - P(c)$. We note that, $Q(a) = P(a) - P(c) = -P(c)$ and $Q'(a) = P'(a) \neq 0$. If $-P(c) = 0$, then (using the notation from the hypothesis of the corollary) formally $R = 0$ and therefore $\gamma = a$, so that trivially we get that $P(\gamma) = P(a) = 0 = P(c)$ as required. Henceforth we can assume that $-P(c) \neq 0$ and proceed to apply Lemma 3.1.1 to $Q(z)$.

Proceeding, let $\mathcal{D}_1$ be the closed disk on which the two points $a$ and $a + n\frac{P(c)}{P'(a)}$ are antipodal. It follows that $Q(z)$ will have a zero (call it $\gamma$) in $\mathcal{D}_1$. Note that the diameter of $\mathcal{D}_1$ will be $D = n\left|\frac{P(c)}{P'(a)}\right|$. We thus have that $Q(\gamma) = 0 = P(\gamma) - P(c)$, hence $P(c) = P(\gamma)$. Noting that $\mathcal{D}_1$ is wholly contained in the closed disk $\mathcal{D}_2$ centered at $a$ and radius $D = n\left|\frac{P(c)}{P'(a)}\right|$, this completes the proof.

□

The next result one encounters in [4] is one we have already encountered before in the previous chapter, in the form of Lemma 2.2.5, the "bisector theorem".

**Remark 3.1.4.** *Another way of interpreting Lemma 2.2.5 is that it tells us that the perpendicular bisector of the line segment $[a, b]$ intersects the convex hull of the critical points of $P(z)$.*

We shall soon encounter the above lemmas in action in the proofs of results we will be considering shortly. More specifically, Lemma 2.2.5 features in the very next result we address, and both lemmas will be used in the proof of Dégot's **Theorem 4**, which we encounter later on.

In a bid to characterize the geometry of a polynomial $P(z)$ initially assumed to contradict Sendov's conjecture at some zero $a \in (0,1)$, the following result of Dégot describes a region in the plane which should be devoid of any other zero of the polynomial.

**Lemma 3.1.5.** *([4], Theorem 3): Let $P(z) = (z-a)\prod_{j=1}^{n-1}(z-z_j)$ with $a \in (0,1)$ and $|z_j| \leq 1$ for $j = 1, \ldots, n-1$. Suppose $P(z)$ contradicts Sendov's conjecture at $a$ (that is, all critical points of $P(z)$ are more than a unit distance from $a$). Then for any $c \in (0,a)$, $P(z)$ cannot have a zero in the disk of center $c$ and radius $1-\sqrt{1-c(a-c)}$.*

*Proof.* Suppose on the contrary that there is a zero, say $\gamma$, in the disk described in the hypothesis. Then this means that $|c-\gamma| \leq 1-\sqrt{1-c(a-c)}$. We can assume that $\Im(\gamma) \leq 0$ and let $\xi$ lie in the intersection of the circles $|z| = 1$ and $|z-a| = 1$ with $\Im(\xi) > 0$. We have not lost generality in the above assumption since by symmetry, the geometrical argument we are pursuing would still work for the reverse geometrical configuration. Proceeding, we note that therefore $\xi$ satisfies

$$|\xi - a| = |\xi| = 1.$$

We note that $\Re(\xi) = \frac{a}{2}$, and hence $\Im(\xi) = y = \sqrt{1-\frac{a^2}{4}}$. This implies that

$$|\xi - c| = \sqrt{\left(\frac{a}{2}-c\right)^2 + \left(1-\frac{a^2}{4}\right)} = \sqrt{1+c^2-ac}.$$

Recall that by assumption, $|c-\gamma| \leq 1-\sqrt{1-c(a-c)}$, combining this with the triangle inequality, we get that:

$$|\xi-\gamma| \leq |\xi-c|+|c-\gamma| \leq \left(\sqrt{1+c^2-ac}\right)+\left(1-\sqrt{1-ac+c^2}\right) = 1 = |\xi-a|.$$

Consider the triangle $\Delta\xi\gamma a$, (see Figure 3.1 below). Since $\Im(\gamma) \leq 0$, and having shown that $|\xi-\gamma| \leq |\xi-a|$, we deduce that the perpendicular bisector of the side $[\gamma, a]$ passes "on the right hand side" of the vertex $\xi$ (and hence certainly still on the right hand side of $\bar{\xi}$). In particular, the perpendicular bisector does not intersect the crescent shaped region $\mathcal{C}$ defined by

$$\mathcal{C} = \{z \in \mathbb{C} : |z| \leq 1 \text{ and } |z - a| \geq 1\}.$$

Since by the Gauss-Lucas theorem all critical points of $P(z)$ lie in the unit disk, and by assumption none of them lies within a unit distance of $a$, the region $\mathcal{C}$ is exactly the feasible region in which the critical points of $P(z)$ lie. On the other hand, $\gamma$ and $a$ are zeroes of $P(z)$ and therefore $P(\gamma) = P(a) = 0$. Hence by Lemma **??**, the perpendicular bisector of the segment $[\gamma, a]$ should intersect $\mathcal{C}$. This leads to a contradiction and hence the theorem is proven.

□

**Remark 3.1.6.** *The geometrical argument used in the above lemma is illustrated in Figure 3.1 below.*

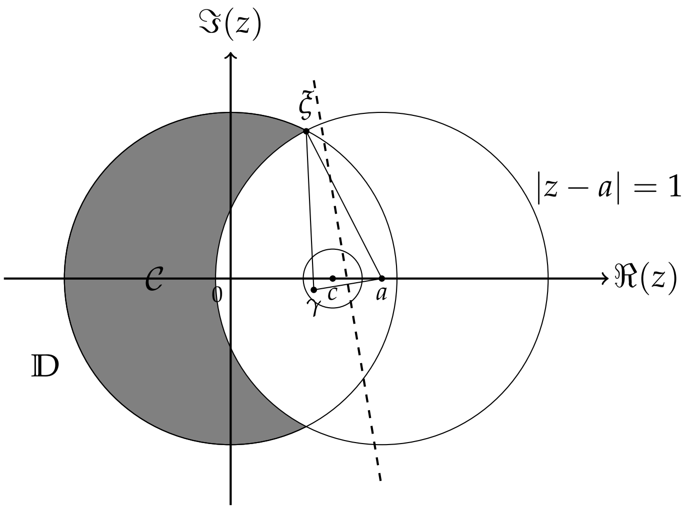



Figure 3.1: Illustration of the proof of Lemma 3.1.5. (*Not to scale*)

After establishing the above results, the next section in [4] was dedicated to a series of lemmas (essentially technical inequalities) which built up towards the eventual establishment of the upper bound of $|P(c)|$. We study the lemmas below.

As a prelude towards the first of Dégot's lemmas we give the following definition:

**Definition 3.1.7.** *By the real part of the mean of the roots (which also coincides with that of the critical points) of a polynomial $P(z)$ of degree $n$, we are referring to the quantity*

$$m = \frac{1}{n}\Re\left(\sum_{j=1}^{n} z_j\right).$$

**Remark 3.1.8.** *For the sake of completeness, let us prove the claim in Definition 3.1.7 above.*

**Proposition 3.1.9.** *The mean of the zeroes of a polynomial $P(z)$ is invariant under differentiation.*

*Proof.* We express $P(z)$ as

$$P(z) = \sum_{j=1}^{n} a_j z^j = a_n \prod_{j=1}^{n}(z - z_j).$$

Expanding the multiplicative expression of $P(z)$ yields:

$$\begin{aligned} P(z) &= a_n[(z - z_1)\cdots(z - z_n)] \\ &= a_n[z^n - (z_1 + \cdots + z_n)z^{n-1} + \cdots + (-1)^n z_1 \cdots z_n] \\ &= a_n z^n - a_n(z_1 + \cdots + z_n)z^{n-1} + \cdots + (-1)^n a_n z_1 \cdots z_n. \end{aligned} \tag{3.1}$$

On the other hand, $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$.

Comparing the coefficients of $z^{n-1}$ from the above equation and Equation 3.1 yields the identity

$$\frac{z_1 + \cdots + z_n}{n} = -\frac{a_{n-1}}{na_n}. \tag{3.2}$$

Similarly,

$$P'(z) = \sum_{j=0}^{n-1}(j+1)a_{j+1}z^j = na_n \prod_{j=1}^{n-1}(z - w_j).$$

Using the Identity 3.2, we deduce that the mean of the roots of $P'(z)$ is

$$\frac{w_1 + \cdots + w_{n-1}}{n-1} = -\frac{1}{n-1}\left(\frac{(n-1)a_{n-1}}{na_n}\right) = -\frac{a_{n-1}}{na_n} = \frac{z_1 + \cdots + z_n}{n}.$$

This establishes the proposition.

□

Before proceeding, let us recall the definition of a convex function.

**Definition 3.1.10.** *A real-valued function $f$ is convex on the interval $[a, b]$ if for all $x_1, x_2 \in [a, b]$, and any $\lambda \in [0, 1]$, we have that*

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2).$$

**Remark 3.1.11.** *If $f(x)$ has a second derivative, then an equivalent characterization of convexity is that $f''(x) \geq 0$ for all $x \in [a, b]$. We shall switch freely between these two definitions, depending on context. If the inequality in Definition 3.1.10 is reversed, then $f(x)$ is said to be concave.*

We will also require the following well known special case of a result by Jensen:

**Lemma 3.1.12.** *(Jensen's Inequality, [6], p. 118): Let $\varphi$ be a real continuous function which is convex. Suppose $x_1, \ldots, x_n$ are real numbers in the domain of $\varphi$. Then:*

$$\varphi\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \leq \frac{\sum_{i=1}^{n} \varphi(x_i)}{n}. \tag{3.3}$$

**Remark 3.1.13.** *For a concave function $\psi$, the inequality in Equation 3.3 is reversed.*

We are now ready to state the result. The geometrical insight behind the lemma is that given a polynomial $P(z)$ of degree $n$, we can bound the product of the distances from a fixed zero of $P(z)$ (or indeed, from some other fixed point of our choice) to all the other zeroes in terms of the average location of the zeroes and the degree $n$ of the polynomial. The statement follows:

**Lemma 3.1.14.** *([4], Lemma 1): Let $P(z) = \prod_{j=1}^{n}(z - z_j)$, with $|z_j| \leq 1$ for $j = 1, \ldots, n$. Let $m = \frac{1}{n}\Re(\sum_{j=1}^{n} z_j)$ be the real part of the mean of the zeroes of $P(z)$. Suppose $\alpha \in (0, 1)$, then:*

$$\prod_{j=1}^{n} |\alpha - z_j| \leq \left(\sqrt{1 + \alpha^2 - 2\alpha m}\right)^n.$$

*Proof.* We note that, given $\Re(z_j)$ for each $z_j$, $j = 1, \ldots, n$, the quantity $|\alpha - z_j|$ is maximized when $|z_j| = 1$. We can henceforth assume that $|z_j| = 1$ for all $j = 1, \ldots, n$. We thus write each $z_j$ as:

$$z_j = e^{i\theta_j} = \cos\theta_j + i\sin\theta_j.$$

Consider the mapping $\psi(x) = \frac{1}{2}\log(1 + \alpha^2 - 2\alpha x)$. We note that,

$$\psi''(x) = \frac{-2\alpha^2}{(1 + \alpha^2 - 2\alpha x)^2} < 0. \tag{3.4}$$

Hence $\psi(x)$ is concave. We can thus establish that:

$$\begin{aligned}
\log\left(\left|\prod_{j=1}^{n}(\alpha - z_j)\right|^{\frac{1}{n}}\right) &= \frac{1}{n}\sum_{j=1}^{n}\log|\alpha - z_j| \\
&= \frac{1}{n}\sum_{j=1}^{n}\log|\alpha - \cos\theta_j - i\sin\theta_j| \\
&= \frac{1}{n}\sum_{j=1}^{n}\frac{1}{2}\log[(\alpha - \cos\theta_j)^2 + (-\sin\theta_j)^2] \\
&= \frac{1}{n}\sum_{j=1}^{n}\frac{1}{2}\log[\alpha^2 - 2\alpha\cos\theta_j + \sin^2\theta_j + \cos^2\theta_j] \\
&= \frac{1}{n}\sum_{j=1}^{n}\psi(\cos\theta_j) \\
&\leq \psi\left(\frac{1}{n}\sum_{j=1}^{n}\cos\theta_j\right) \text{ by Remark 3.1.13,} \\
&= \frac{1}{2}\log\left(1 + \alpha^2 - 2\alpha\frac{1}{n}\sum_{j=1}^{n}\Re(z_j)\right) \\
&= \log\left(\sqrt{1 + \alpha^2 - 2\alpha m}\right).
\end{aligned}$$

Taking exponential on both sides and then raising everything to $n$ establishes the lemma.

□

The next lemma that we encounter considers the geometry of $P'(z)$ when $P(z)$ is assumed to contradict Sendov's conjecture at some zero $a \in (0, 1)$. It gives an upper bound on the quantity $\left|\frac{P'(c)}{P'(a)}\right|$ where $c \in (0, a)$.

**Lemma 3.1.15.** *([4], Lemma 2): Suppose the polynomial $P(z)$ contradicts Sendov's conjecture at $a \in (0,1)$ and let $w_1, \ldots, w_{n-1}$ be its critical points. Let $c \in (0,a)$ and $q = \frac{\frac{a}{2}-m}{\frac{a}{2}+1}$, where $m$ is as defined in Definition 3.1.7. Then:*

$$\left|\frac{P'(c)}{P'(a)}\right| = \prod_{j=1}^{n-1}\left|\frac{c-w_j}{a-w_j}\right| \leq \left\{\left(\frac{1+c}{1+a}\right)^q \left(\sqrt{1+c^2-ac}\right)^{1-q}\right\}^{n-1}.$$

*Proof.* The proof is somewhat similar in flavour to the proof of the previous lemma. We start by first verifying the following claim:

**Claim**: Given the $\Re(w_j)$ for each $j = 1, \ldots, n-1$, the quantity $\left|\frac{c-w_j}{a-w_j}\right|$ is maximized when $|w_j| = 1$.

*Proof.* It suffices to show that given $u$ and $v$ such that $|u| \leq 1$, $|v| \leq 1$, $\Re(u) = \Re(v)$, $|\Im(u)| < |\Im(v)|$ and $|u-a| > 1$, $|v-a| > 1$ where $a \in (0,1)$, then $\left|\frac{c-u}{a-u}\right| < \left|\frac{c-v}{a-v}\right|$. We can write $u = x + iy$ and $v = x + iky$ for $k > 1$. Proving the claim then amounts to showing that

$$|c-v||a-u| - |c-u||a-v| > 0.$$

Substituting the rectangular form of $u$ and $v$ into the above equation, we see that we have to show that

$$\left[(c-x)^2 + k^2y^2\right]^{\frac{1}{2}}\left[(a-x)^2 + y^2\right]^{\frac{1}{2}} - \left[(c-x)^2 + y^2\right]^{\frac{1}{2}}\left[(a-x)^2 + k^2y^2\right]^{\frac{1}{2}} > 0.$$

Since all of the quantities in the square brackets in the previous inequality are positive, we can prove the claim by showing that

$$\left[(c-x)^2 + k^2y^2\right]\left[(a-x)^2 + y^2\right] - \left[(c-x)^2 + y^2\right]\left[(a-x)^2 + k^2y^2\right] > 0.$$

After expanding and cancelling terms out, this amounts to showing that

$$y^2[(c-x)^2 - (a-x)^2] + k^2y^2[(a-x)^2 - (c-x)^2] > 0.$$

The above inequality is true since it simplifies to

$$y^2[k^2((a-x)^2 - (c-x)^2) - ((a-x)^2 - (c-x)^2)] > 0.$$

This is true since $k > 1$. Since $|w_j| \leq 1$ for $j = 1, \ldots, n-1$, given $\Re(w_j)$, the quantity $\left|\frac{c-w_j}{a-w_j}\right|$ is maximized when $|w_j| = 1$.

$\square$

Henceforth we can assume that $|w_j| = 1$ for all $j$. Consider the mapping $\psi(x)$ defined on $[-1, \frac{a}{2}]$ by:

$$\psi(x) = \frac{1}{2}\log\left(\frac{1+c^2-2cx}{1+a^2-2ax}\right).$$

Since $\psi(x)$ can be written as

$$\psi(x) = \frac{1}{2}\log(1+c^2-2cx) - \frac{1}{2}\log(1+a^2-2ax).$$

From Equation 3.4, we can deduce that the second derivative of $\psi$ is given by

$$\psi''(x) = \frac{2a^2}{(1+a^2-2ax)^2} - \frac{2c^2}{(1+c^2-2cx)^2}.$$

Writing the right hand side of the above equation as a single term, one obtains that

$$\psi''(x) = 2\left[\frac{a^2(1+c^2-2cx)^2 - c^2(1+a^2-2ax)^2}{(1+a^2-2ax)^2(1+c^2-2cx)^2}\right].$$

Expanding and simplifying the numerator of the above equation, one ends up with

$$\psi''(x) = \frac{2(a-c)(1-ac)((a+c)(1+ac)-4acx)}{(1+c^2-2cx)^2(1+a^2-2ax)^2}.$$

We note that:

$$\begin{aligned}
(a+c)(1+ac) - 4acx &= a + c + ac(a + c - 4x) \\
&\geq a + c + ac(a + c - 2a) \\
&= a + c + ac(c - a) \\
&\geq a + c + (c - a) = 2c > 0
\end{aligned}$$

since $c \in (0, a)$ and $x \in [-1, \frac{a}{2}]$.

This implies that $\psi''(x) > 0$ and hence $\psi(x)$ is convex on its domain. Before proceeding, let us take note of the following remark which will help justify an inequality we encounter shortly:

**Remark 3.1.16.** *Consider the interval $[\alpha, \beta]$ and a set of points $S = \{x_1, x_2, \ldots, x_{n-1}\} \subset [\alpha, \beta]$. Let $\bar{x} = \frac{1}{n-1}\sum_{j=1}^{n-1} x_j$ be the arithmetic mean of the points in $S$. Also, we know that, for each $x_j \in S$, there exists $\mu_j \in [0, 1]$ such that*

$$x_j = \mu_j \alpha + (1 - \mu_j)\beta$$

*We then have that*

$$\sum_{j=1}^{n-1} x_j = \alpha \sum_{j=1}^{n-1} \mu_j + \beta \sum_{j=1}^{n-1} (1 - \mu_j)$$

*Hence,*

$$\begin{aligned} \bar{x} &= \alpha \left( \frac{1}{n-1} \sum_{j=1}^{n-1} \mu_j \right) + \beta \left( \frac{1}{n-1} \sum_{j=1}^{n-1} (1 - \mu_j) \right) \\ &= \mu\alpha + (1 - \mu)\beta, \end{aligned}$$

*where $\mu = \frac{1}{n-1}\sum_{j=1}^{n-1} \mu_j$.*

*If $\phi$ is a convex function (defined over $[\alpha, \beta]$), then*

$$\phi(x_j) = \phi(\mu_j \alpha + (1 - \mu_j)\beta) \leq \mu_j \phi(\alpha) + (1 - \mu_j)\phi(\beta).$$

*Hence,*

$$\begin{aligned} \sum_{j=1}^{n-1} \phi(x_j) &\leq \phi(\alpha) \sum_{j=1}^{n-1} \mu_j + \phi(\beta) \sum_{j=1}^{n-1} (1 - \mu_j) \\ &= (n-1) \left[ \mu\phi(\alpha) + (1 - \mu)\phi(\beta) \right]. \end{aligned}$$

We can now proceed to the rest of the proof. Writing each $w_j$ as $w_j = e^{i\theta_j} = \cos\theta_j + i \sin\theta_j$, we note that:

$$\begin{aligned}
\log\left(\prod_{j=1}^{n-1}\left|\frac{c-w_j}{a-w_j}\right|\right) &= \sum_{j=1}^{n-1}\log\left|\frac{c-w_j}{a-w_j}\right| \\
&= \sum_{j=1}^{n-1}\log\left|\frac{c-\cos\theta_j-i\sin\theta_j}{a-\cos\theta_j-i\sin\theta_j}\right| \\
&= \sum_{j=1}^{n-1}\frac{1}{2}\log\left(\frac{1+c^2-2c\cos\theta_j}{1+a^2-2a\cos\theta_j}\right) \\
&= \sum_{j=1}^{n-1}\psi(\cos\theta_j) \\
&\leq (n-1)\left[q\psi(-1)+(1-q)\psi\left(\frac{a}{2}\right)\right],
\end{aligned}$$

where the inequality is true since $q \in [0,1]$, and the mean $m$ can be expressed as

$$m = q(-1)+(1-q)\frac{a}{2}. \tag{3.5}$$

Proceeding we note that:

$$\psi(-1) = \frac{1}{2}\log\left(\frac{1+c^2+2c}{1+a^2+2a}\right) = \log\left(\frac{1+c}{1+a}\right)$$

and,

$$\psi\left(\frac{a}{2}\right) = \frac{1}{2}\log\left(\frac{1+c^2-ac}{1+a^2-a^2}\right) = \log\left(\sqrt{1+c^2-ac}\right).$$

Therefore,

$$\log\left(\prod_{j=1}^{n-1}\left|\frac{c-w_j}{a-w_j}\right|\right) \leq (n-1)\left[q\log\left(\frac{1+c}{1+a}\right)+(1-q)\log\left(\sqrt{1+c^2-ac}\right)\right].$$

Taking the exponential both sides yields the result.

□

After stating the above lemmas, Dégot went on to prove two more lemmas, his **Lemma 3** and **Lemma 4**. However, since we will not be using these lemmas in the next section, we shall not address them here. We thus relegate them to a later section in which they will play a role.

We are now ready to start working towards the result that will give us an upper bound on the quantity $|P(c)|$. We carry this out in the next section.

## 3.2 Towards $\mathcal{N}_0$

This section corresponds to **Section 4** in [4], which Dégot called the "Upper estimation of $|P(c)|$". Indeed, the section was dedicated towards establishing an upper bound of the form $|P(c)| \leq 1 + a$. Pertaining to our approach, we subdivide this section into three components.

The first subsection takes a closer look at the behaviour of the mean $m$ of a polynomial $P(z)$ assumed to contradict Sendov's conjecture when the degree of the polynomial is very large. We shall then take advantage of this result and extract information about the reverse direction. That is, given the location of $m$, how much can we say about the size of the degree of $P(z)$?

In the second subsection, we study two theorems of Dégot that give conditions on the size of the degree of the polynomial in order to be able to say something about the size of $|P(c)|$. We then dedicate the establishment of the bounds on $|P(c)|$ to a separate section.

### 3.2.1 The mean of a polynomial assumed to contradict Sendov's conjecture

As previously mentioned, in this subsection we study a result that gives us a handle on the arithmetic mean (and hence the geometrical configuration) of the zeroes of a polynomial that would contradict Sendov's conjecture at a zero $a \in (0, 1)$. We begin with Dégot's **Theorem 4**, followed by an extensive treatment of a particularly useful corollary.

From the outset, let:

$$P(z) = (z - a) \prod_{j=1}^{n-1} (z - z_j), \ a \in (0, 1), \text{ and } |z_j| \leq 1 \text{ for all } j = 1, \ldots, n-1.$$

**Lemma 3.2.1.** *([4], Theorem 4): Assume $P(z)$ contradicts Sendov's conjecture at $a \in (0,1)$. Suppose $\delta \in (0,a)$. Then:*

$$\left|\frac{P(\delta)}{P'(a)}\right| \geq \frac{1-\sqrt{1+\delta^2-a\delta}}{n}.$$

**Remark 3.2.2.** *We would like to clarify that the $\delta \in (0,a)$ considered here is not the same as the $c \in (0,a)$ which we have encountered before. The reader will notice that the explicit formulas for $\delta$ and $c$ in terms of $a$ that we are going to formulate are different. We proceed to the proof of the above lemma.*

*Proof.* Suppose not, and define the quantity $R = n\left|\frac{P(\delta)}{P'(a)}\right|$. By the supposition, we have that $R < 1-\sqrt{1+\delta^2-a\delta}$. Let $\xi$ be the complex number satisfying $|\xi - a| = |\xi| = 1$, with $\Im(\xi) > 0$. That is, $\xi$ is at one of the points of intersection of the two circles of unit radius and centers 0 and $a$. This implies that

$$\Re(\xi) = \frac{a}{2} \text{ and } \Im(\xi) = \sqrt{1-\frac{a^2}{4}}.$$

By Corollary 3.1.3, we can find $\gamma$ in the disk centered at $a$ with radius $R = n\left|\frac{P(\delta)}{P'(a)}\right|$ such that $P(\gamma) = P(\delta)$. Of course by definition, we have that $|a-\gamma| \leq R$.

By symmetry, there is no loss in generality upon assuming that $\Im(\gamma) \geq 0$. First, we would like to deduce that $|\xi - \delta| \geq |\xi - \gamma|$. If $\delta = \gamma$, the inequality is formally true, henceforth we can assume that $\delta \neq \gamma$. Recall that $\delta \in (0,a)$ and consider the quantity $\delta - 1 + \sqrt{1+\delta^2-a\delta}$.

**Claim**: $\delta - 1 + \sqrt{1+\delta^2-a\delta} > 0$.

*Proof.* This is equivalent to showing that $\sqrt{1+\delta^2-a\delta} > 1-\delta$, which is true if and only if $1+\delta^2-a\delta > 1-2\delta+\delta^2$, if and only if $2\delta > a\delta$, which is true. □

From the above claim we can thus deduce that $\frac{a+\delta-1+\sqrt{1+\delta^2-a\delta}}{2} > \frac{a}{2}$.

By assumption, we have that $R < 1-\sqrt{1+\delta^2-a\delta}$. Combining this with the above inequality, we get that

$$\frac{\delta + a - R}{2} > \frac{a}{2}.$$

We can thus deduce that the midpoint of the line joining $\delta$ and $\gamma$ has real part greater than $\frac{a}{2}$. Since $P(\gamma) = P(\delta)$, Lemma 2.2.5 tells us that the perpendicular bisector of the segment $[\delta, \gamma]$ intersects the convex hull of the critical points of $P(z)$. This implies that the triangle $\Delta\xi\delta\gamma$ has base $\delta\gamma$ and shorter side $\xi\gamma$ (since otherwise the perpendicular bisector of $[\delta, \gamma]$ would pass on the right hand side of $\xi$). We thus have that $|\xi - \delta| \geq |\xi - \gamma|$. The argument is illustrated in Figure 3.2 below.

Since $R \geq |a - \gamma|$, we have that

$$R + |\xi - \gamma| \geq |a - \gamma| + |\gamma - \xi| \geq |a - \xi| = 1.$$

Therefore,

$$n \left| \frac{P(\delta)}{P'(a)} \right| = R \geq 1 - |\xi - \gamma| \geq 1 - |\xi - \delta|, \tag{3.6}$$

the last inequality following from the fact that $|\xi - \delta| \geq |\xi - \gamma|$.

We note that, $|\xi - \delta|$ evaluates to

$$|\xi - \delta| = \left[ \left( \frac{a}{2} - \delta \right)^2 + 1 - \frac{a^2}{4} \right]^{\frac{1}{2}} = \sqrt{1 + \delta^2 - a\delta}.$$

Combining the preceding discussion with Equation 3.6, we get that

$$n \left| \frac{P(\delta)}{P'(a)} \right| = R \geq 1 - \sqrt{1 + \delta^2 - a\delta},$$

which implies that

$$\left| \frac{P(\delta)}{P'(a)} \right| \geq \frac{1 - \sqrt{1 + \delta^2 - a\delta}}{n}.$$

This contradicts the supposition and hence the lemma is true.

□

We are now ready to consider the corollary:

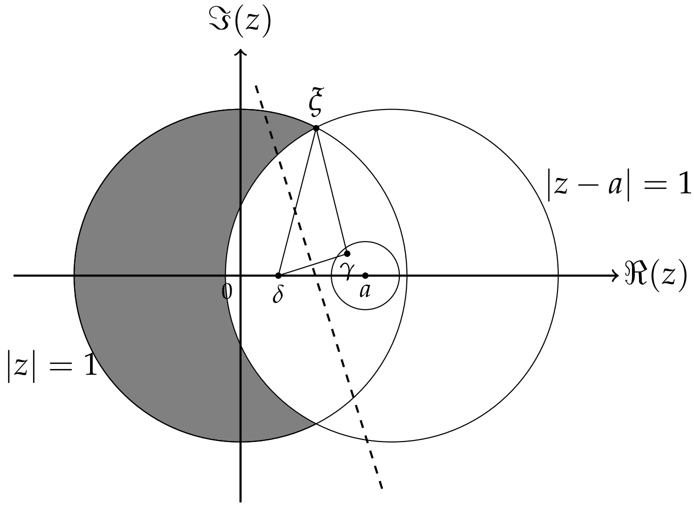



Figure 3.2: Illustration of the proof of Lemma 3.2.1. (*Not to scale*)

**Lemma 3.2.3.** *([4], Corollary 1): Let $P(z)$ be a polynomial of degree $n$ assumed to contradict Sendov's conjecture at $a \in (0,1)$. Then:*

$$m \leq \inf_{\delta\in(0,a)} \left( \frac{\delta}{2} - \frac{1}{\delta n} \log(1 - \sqrt{1+\delta^2 - \delta a}) \right).$$

*Proof.* If $\delta \in (0,a)$, then by Lemma 3.2.1 above,

$$|P(\delta)| \geq |P'(a)| \frac{1-\sqrt{1+\delta^2 - a\delta}}{n}.$$

On the other hand,

$$|P'(a)| = n \prod_{j=1}^{n-1} |a-w_j| \geq n \cdot 1 \text{ (since } |a-w_j| \geq 1 \text{ for all } j = 1, \ldots, n-1 \text{ by assumption).}$$

This implies that

$$|P(\delta)| \geq 1 - \sqrt{1+\delta^2 - a\delta}.$$

Lemma 3.1.14 gives us the inequality:

$$|P(\delta)| \leq \left(\sqrt{1+\delta^2 - 2\delta m}\right)^n.$$

Combining the above two inequalities yields:

$$1 - \sqrt{1+\delta^2 - a\delta} \leq \left(\sqrt{1+\delta^2 - 2\delta m}\right)^n.$$

Taking log both sides, we get:

$$\frac{1}{n}\log\left(1-\sqrt{1+\delta^2-a\delta}\right) \leq \frac{1}{2}\log(1+\delta^2-2\delta m) \leq \frac{\delta^2-2\delta m}{2},$$

where the last inequality follows from the fact that since $m < \frac{a}{2}$, then $\delta^2 - 2\delta m > \delta^2 - 2\delta\left(\frac{a}{2}\right) = \delta(\delta - a) > -1$, and $\log(1+x) \leq x$ for $x > -1$.

Hence,

$$m \leq \left(\frac{\delta}{2} - \frac{1}{\delta n}\log(1-\sqrt{1+\delta^2-\delta a})\right). \tag{3.7}$$

And thus:

$$m \leq \inf_{\delta\in(0,a)}\left(\frac{\delta}{2} - \frac{1}{\delta n}\log(1-\sqrt{1+\delta^2-\delta a})\right).$$

as claimed.

□

The above lemma is of crucial importance. This is because it allows us to obtain for any $\beta \in (0,1)$ an explicit formula $\mathcal{M}_\beta(a)$, such that for the polynomial $P(z)$ with a zero at $a$, if the degree of $P(z)$ is greater than $\mathcal{M}_\beta(a)$, then $m \leq \beta \cdot a$.

Upon gaining this control over the size of $m$, we can then remove the dependence on $m$ from other future parameters without loss of generality. This shall become clear when we call upon this new quantity later. In the meantime, let us extract the formula.

We obtain the formula for the particular case where we require $m \leq \frac{a}{4}$. That is, in our notation, we work out the formula for $\mathcal{M}_{0.25}(a)$:

From Equation 3.7, suppose we want

$$m \leq \left(\frac{\delta}{2} - \frac{1}{\delta n}\log(1-\sqrt{1+\delta^2-\delta a})\right) \leq \frac{a}{4},$$

this in particular holds when:

$$n \geq \frac{-4\log(1-\sqrt{1+\delta^2-\delta a})}{a\delta - 2\delta^2}.$$

Hence, we simply let:

$$\mathcal{M}_{0.25}(a,\delta) = \frac{-4\log(1-\sqrt{1+\delta^2-\delta a})}{a\delta - 2\delta^2},\ \delta \in (0, \frac{a}{2}).$$

This implies that for any polynomial $P$ assumed to contradict Sendov's conjecture at $a \in (0,1)$, if $\deg(P) = n \geq \mathcal{M}_{0.25}(a,\delta)$, for some $\delta \in (0, \frac{a}{2})$, then $m \leq \frac{a}{4}$. This is in particular true for, say, $\delta = \frac{a}{4}$. With this in mind, we obtain the following function:

$$\mathcal{N}_0(a) = \mathcal{M}_{0.25}(a, \frac{a}{4}) = \frac{-32\log\left(1 - \frac{\sqrt{16-3a^2}}{4}\right)}{a^2}. \tag{3.8}$$

## 3.3 Obtaining explicit analogues of Dégot's bounds

In this section we study **Theorems 5** and **6** from Dégot's paper. From **Theorem 5**, we study closely the quantities that go into the definition of $N_1$. We will see that the bound $N_1$ as originally defined depends on the real part of the mean of the zeroes of the polynomial $P(z)$. After exposing Dégot's method, we work towards circumventing this dependence. The procedure is then repeated for **Theorem 6**.

The end result is that we come up with the formulas $\mathcal{N}_1(a)$ and $\mathcal{N}_2(a)$, the explicit and continuous analogues of Dégot's $N_1$ and $N_2$ respectively. These new quantities have the following advantages over Dégot's:

- they are explicitly given,
- they are continuous in $a$,
- they depend only on $a$.

We begin with **Theorem 5** from [4].

### 3.3.1 Towards $\mathcal{N}_1(a)$

**Lemma 3.3.1.** *([4], Theorem 5): Suppose $P(z)$ contradicts Sendov's conjecture at $a$. Let $q = \frac{\frac{a}{2}-m}{1+\frac{a}{2}}$ and let $N_1$ be the smallest integer such that*

$$\left(\frac{1+\frac{a}{2}}{1+a}\right)^q \leq \left(\frac{1-\sqrt{1-\frac{a^2}{4}}}{an}\right)^{\frac{1}{n-1}} \text{ for all } n \geq N_1. \tag{3.9}$$

*Then, if $n \geq N_1$,*

$$|P'(a)| \leq \frac{16n}{a^2} \text{ and } |P(0)| \geq \frac{a^2}{16}.$$

Before providing the proof of the theorem, we quickly verify that indeed Dégot's $N_1$ exists.

**Proposition 3.3.2.** *There exists $N_1 \in \mathbb{N}$ such that Inequality 3.9 is true.*

*Proof.* First we note that the left hand side of Inequality 3.9 is

$$\left(\frac{1+\frac{a}{2}}{1+a}\right)^q < 1.$$

On the other hand, since $a \in (0,1)$, we have that:

$$\left(\frac{1-\sqrt{1-\frac{a^2}{4}}}{n}\right)^{\frac{1}{n-1}} \leq \left(\frac{1-\sqrt{1-\frac{a^2}{4}}}{an}\right)^{\frac{1}{n-1}} \leq \left(\frac{1}{an}\right)^{\frac{1}{n-1}}.$$

Furthermore,

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1-\sqrt{1-\frac{a^2}{4}}}{n}\right)^{\frac{1}{n-1}} &= \exp\left\{\lim_{n\to\infty}\frac{\log\left(1-\sqrt{1-\frac{a^2}{4}}\right)-\log(n)}{n-1}\right\} \\ &= e^0 = 1.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{an}\right)^{\frac{1}{n-1}} &= \exp\left\{-\lim_{n\to\infty}\left(\frac{\log(a)}{n-1}+\frac{\log(n)}{n-1}\right)\right\} \\ &= e^0 = 1.\end{aligned}$$

Hence:

$$\lim_{n\to\infty}\left(\frac{1-\sqrt{1-\frac{a^2}{4}}}{an}\right)^{\frac{1}{n-1}}=1$$

and thus, there is an $N_1 \in \mathbb{N}$ such that Inequality 3.9 holds.

□

Recall that $q = \frac{a/2-m}{1+a/2}$ and $m < \frac{a}{2}$. This implies that

$$1-q=\frac{1+a/2}{1+a/2}-\frac{a/2-m}{1+a/2}=\frac{1+m}{1+a/2}>0.$$

We may now proceed with the proof of Lemma 3.3.1.

*Proof.* Suppose that $n \geq N_1$. Then by Lemma 3.1.15, we have that for all $c \in (0, \frac{a}{2})$,

$$\begin{aligned}\left|\frac{P'(c)}{P'(a)}\right| &\leq \left[\left(\frac{1+c}{1+a}\right)^q\left(\sqrt{1+c^2-ac}\right)^{1-q}\right]^{n-1}\\ &\leq \left(\frac{1+\frac{a}{2}}{1+a}\right)^{q(n-1)}\\ &\leq \left(\frac{1-\sqrt{1-\frac{a^2}{4}}}{an}\right),\end{aligned} \tag{3.10}$$

where the last inequality follows from the hypothesis of the lemma. On the other hand, we also have that:

$$\frac{1}{|P'(a)|}\cdot|P(0)-P(a/2)|=\frac{1}{|P'(a)|}\left|\int_0^{\frac{a}{2}}P'(z)\mathrm{d}z\right|\leq\frac{a}{2}\cdot\sup_{c\in[0,\frac{a}{2}]}\left|\frac{P'(c)}{P'(a)}\right|.$$

This, combined with Inequality 3.10 above, tells us that:

$$\left|\frac{P(0)}{P'(a)}-\frac{P(a/2)}{P'(a)}\right|\leq\frac{1-\sqrt{1-\frac{a^2}{4}}}{2n}.$$

Whence, we can of course deduce that

$$\left|\frac{P(a/2)}{P'(a)}\right|\leq\left|\frac{P(0)}{P'(a)}\right|+\frac{1-\sqrt{1-\frac{a^2}{4}}}{2n}. \tag{3.11}$$

But from Lemma 3.2.1, we also get that

$$\left|\frac{P(a/2)}{P'(a)}\right| \geq \frac{1-\sqrt{1-\frac{a^2}{4}}}{n}. \tag{3.12}$$

Combining Equations 3.11 and 3.12 above, we deduce that

$$\left|\frac{P(0)}{P'(a)}\right| \geq \frac{1-\sqrt{1-\frac{a^2}{4}}}{2n}. \tag{3.13}$$

Before proceeding with the proof, let us quickly verify the following technical inequality:

**Claim**:

$$1-\sqrt{1-a^2/4} \geq \tfrac{a^2}{8}.$$

*Proof.* This is true if and only if

$$1-\frac{a^2}{8} \geq \sqrt{1-a^2/4},$$

if and only if

$$\frac{a^4}{64} \geq 0.$$

This is true since $a \in (0,1)$. □

Combining the above claim with Inequality 3.13, we deduce that:

$$\left|\frac{P(0)}{P'(a)}\right| \geq \frac{a^2}{16n}. \tag{3.14}$$

Finally, we note that:

$$|P(0)| = a\prod_{j=1}^{n-1}|z_j| \leq a \leq 1,$$

and

$$|P'(a)| = n\prod_{j=1}^{n-1}|a-w_j| \geq n.$$

So, from Equation 3.14, we get that:

$$|P(0)| \geq \frac{a^2}{16n}|P'(a)| \geq \frac{a^2}{16}$$

and

$$|P'(a)| \leq \frac{16n}{a^2}|P(0)| \leq \frac{16n}{a^2}.$$

This completes the proof.

□

**Remark 3.3.3.** *This is one of the pivotal results in [4], however, aside from the dependency of $N_1$ on $P(z)$, there was no explicit formula $N_1(a)$ such that whenever $n \geq N_1(a)$, the Inequality 3.9 holds. Our goal here is to obtain such a function.*

Building on the previous remark, we would like to mention that we are adopting a slight but noticeable change in our discourse. Hitherto, our approach has mostly focused on exposing the results of Dégot that we have discussed. Henceforth, we shall continue our discussion with a view towards our final goal. We still maintain the exposition, however for the most part, rather than beginning with a particular result of Dégot, we shall first give our analogue or version, and explain beforehand or afterwards how the corresponding original version was stated or proved. This will gradually lead us towards the main result. We proceed.

Upon closer inspection of Inequality 3.9, we note that the bound $N_1$ depends on $q = q(a, m)$. But $m$ is the real part of the mean of the zeroes of $P(z)$, hence this implies that $N_1$ as defined depends on the polynomial $P(z)$. This is an unnecessary restriction on the generality of the result. We would ultimately like to obtain a bound which only depends (at most) on $a \in (0, 1)$ and nothing else. The first task towards this is to redefine $q$ such that the dependence on $m$ is removed.

With the help of the quantity $\mathcal{N}_0(a)$ obtained in the previous section, we do this in the following steps:

- First, we note that by definition, $\mathcal{N}_0(a)$ gives us a high enough degree bound such that any polynomial with degree $n \geq \mathcal{N}_0(a)$ has $m \leq 0.25a$.

- The quantity $\left(\frac{1+\frac{a}{2}}{1+a}\right) \in (0,1)$, hence for any $0 < q_1 < q_2$:

$$\left(\frac{1+\frac{a}{2}}{1+a}\right)^{q_1} \geq \left(\frac{1+\frac{a}{2}}{1+a}\right)^{q_2}.$$

- Therefore this implies that whenever $m \leq \frac{a}{4}$, then

$$\left(\frac{1+\frac{a}{2}}{1+a}\right)^{q} \leq \left(\frac{1+\frac{a}{2}}{1+a}\right)^{\frac{\frac{a}{4}}{1+\frac{a}{2}}}.$$

  Thus, from Inequality 3.9, if we have that

$$\left(\frac{1+\frac{a}{2}}{1+a}\right)^{\frac{\frac{a}{4}}{1+\frac{a}{2}}} \leq \left(\frac{1-\sqrt{1-\frac{a^2}{4}}}{an}\right)^{\frac{1}{n-1}},$$

  it would then follow that:

$$\left(\frac{1+\frac{a}{2}}{1+a}\right)^{q} \leq \left(\frac{1+\frac{a}{2}}{1+a}\right)^{\frac{\frac{a}{4}}{1+\frac{a}{2}}} \leq \left(\frac{1-\sqrt{1-\frac{a^2}{4}}}{an}\right)^{\frac{1}{n-1}}$$

  for all $m \leq \frac{a}{4}$.

Hence, the $N_1(a)$ obtained by replacing $m$ with $\frac{a}{4}$ (and hence $q = \frac{\frac{a}{4}}{1+\frac{a}{2}}$) works for all $m \leq \frac{a}{4}$. With this in mind, we replace the quantity $q(a,m)$ with the new quantity $q'(a) := \frac{\frac{a}{4}}{1+\frac{a}{2}}$ which only depends on $a \in (0,1)$. Since the final degree bound $\mathcal{N}$ will be large enough such that $m \leq 0.25a$, at each occurrence of $q$ in Dégot's results we need to verify that the version where $q$ is replaced with $q'$ still holds, taking into account that $m \leq 0.25a$, and hence $q \geq q'$. The preceding discussion is an instance of such an argument. We may now proceed.

**Proposition 3.3.4.** *There is an explicit and continuous function $\mathcal{N}_1(a)$ such that for all $n \geq \mathcal{N}_1(a)$, Inequality 3.9 (with $q'$ in the place of $q$) holds.*

*Proof.* We construct such a function: Replacing $q$ by $q'$ in Inequality 3.9 and then taking log on both sides, we get that the new inequality holds if and only if:

$$q' \log\left(\frac{1+\frac{a}{2}}{1+a}\right) \leq \frac{1}{n-1}\log\left(\frac{1-\sqrt{1-\frac{a^2}{4}}}{an}\right)$$

But this is true if and only if:

$$n-1 \geq \frac{1}{q'\log\left(\frac{1+\frac{a}{2}}{1+a}\right)} \cdot \left(\log\left(1-\sqrt{1-\frac{a^2}{4}}\right) - \log(a) - \log(n)\right).$$

This is true if and only if:

$$n \geq 1 + \frac{\log\left(1-\sqrt{1-\frac{a^2}{4}}\right) - \log(a)}{q'\log\left(\frac{1+\frac{a}{2}}{1+a}\right)} - \frac{\log(n)}{q'\log\left(\frac{1+\frac{a}{2}}{1+a}\right)}. \tag{3.15}$$

More succinctly, we write:

$$n \geq 1 + n_1(a) + n_2(a)\cdot\log(n), \tag{3.16}$$

where:

- $n_1(a) = \frac{\log\left(1-\sqrt{1-\frac{a^2}{4}}\right)-\log(a)}{q'\log\left(\frac{1+\frac{a}{2}}{1+a}\right)}$, and
- $n_2(a) = \frac{-1}{q'\log\left(\frac{1+\frac{a}{2}}{1+a}\right)}$.

For ease of notation, in calculations we shall simply refer to these quantities as $n_1$ and $n_2$.

Recall that $\log(n) \leq n$ for any $n > 0$, hence $\log(\sqrt{n}) \leq \sqrt{n}$, from which we get that $\log(n) \leq 2\sqrt{n}$.
We note that, Inequality 3.9 will still hold if:

$$n \geq 1 + n_1 + 2\sqrt{n}n_2 \geq 1 + n_1 + n_2\log(n).$$

From the first inequality above, after a little algebra, "solving for $n$" yields that Inequality 3.9 will hold when:

$$n \geq \left[n_2 + (1 + n_1 + n_2^2)^{\frac{1}{2}}\right]^2$$

We then let:

$$N_1(a) = \left[n_2 + (1 + n_1 + n_2^2)^{\frac{1}{2}}\right]^2. \tag{3.17}$$

And then,

$$\mathcal{N}_1(a) := \max\{N_1(a), \mathcal{N}_0(a)\}.$$

□

**A closer look at the constituents of $N_1(a)$**

From the previous subsection, we defined $N_1(a)$ to be:

$$N_1(a) = \left[n_2(a) + (1 + n_1(a) + n_2(a)^2)^{\frac{1}{2}}\right]^2.$$

The quantity $\log\left(\frac{1+\frac{a}{2}}{1+a}\right)$ is defined, continuous and negative on $(0,1)$. Of course this implies that $n_2(a) > 0$ on $(0,1)$.

Finally, the quantity $1 - \sqrt{1 - \frac{a^2}{4}} \in (0, 1 - \frac{\sqrt{3}}{2})$. Hence $\log\left(1 - \sqrt{1 - \frac{a^2}{4}}\right)$ is defined on $(0,1)$.

**Proposition 3.3.5.** $n_1(a) > 0$.

*Proof.* Since the denominator of $n_1(a)$ is negative, the claim amounts to saying that $\log\left(1 - \sqrt{1 - \frac{a^2}{4}}\right) - \log(a) < 0$ for all $a \in (0,1)$. But this is true since:

$$\begin{aligned}
\log\left(1 - \sqrt{1 - \frac{a^2}{4}}\right) - \log(a) < 0 &\iff \log\left(\frac{a}{1 - \sqrt{1 - \frac{a^2}{4}}}\right) > 0 \\
&\iff \frac{a}{1 - \sqrt{1 - \frac{a^2}{4}}} > 1 \\
&\iff 1 - \frac{a^2}{4} > 1 - 2a + a^2 \\
&\iff 5a^2 - 8a < 0 \\
&\iff 0 < a < 1.6
\end{aligned}$$

Since $a \in (0,1)$, the claim is true. □

From the above observations, we can thus conclude that $N_1(a)$, and therefore $\mathcal{N}_1(a)$ is continuous on $(0,1)$.

## Towards $\mathcal{N}_2(a)$

Having obtained the explicit formula $\mathcal{N}_1(a)$, we now turn our attention to Dégot's **Theorem 6**, wherein conditions to be satisfied by the second bound $N_2$ were stipulated. The statement given below stipulates such a condition. We state our version, the only difference from his being that we replaced the appearance of $q$ with $q'$.

**Definition 3.3.6.** *Let $c \in (0,a)$. For $x \in (0,1)$ set:*

$$D(x) := \max\left\{\left(\frac{1}{1+a}\right)^x; \left(\frac{1+c}{1+a}\right)^x \left(\sqrt{1+c^2-ac}\right)^{1-x}\right\}.$$

It is easy to see that $D(x) < 1$ for all $x \in (0,1)$.

**Proposition 3.3.7.** *$D$ is a decreasing function with respect to $x$.*

*Proof.* Writing $D(x)$ as $D(x) = \max\{D_1(a,x); D_2(a,x,c)\}$, it is clear that if $0 < x_1 < x_2$, then $D_1(a,x_1) > D_1(a,x_2)$. So we only need to show that the function $D_2(a,x,c) = \left(\frac{1+c}{1+a}\right)^x (\sqrt{1+c^2-ac})^{1-x}$ is a decreasing function of $x$, $x \in (0,1)$. We proceed as follows:

Consider the function $f(x) = a^x b^{1-x}$. Then, $f'(x) = \log\left(\frac{a}{b}\right) a^x b^{1-x}$.

By the same reasoning, we then have that:

$$D_2{}'(x) = \log\left(\frac{(1+c)}{(1+a)(1+c^2-ac)^{\frac{1}{2}}}\right) \cdot \left(\frac{1+c}{1+a}\right)^x (\sqrt{1+c^2-ac})^{1-x}.$$

Now, $\frac{1+c}{1+a} > 0$ and $\sqrt{1+c^2-ac} > 0$. Thus, it remains to show that

$$\log\left(\frac{(1+c)}{(1+a)(1+c^2-ac)^{\frac{1}{2}}}\right) < 0,$$

which is equivalent to showing that:

$$\frac{(1+c)}{(1+a)(1+c^2-ac)^{\frac{1}{2}}} < 1.$$

Now this holds:

$$\begin{aligned}
&\iff (1+a)(1+c^2-ac)^{\frac{1}{2}} > 1+c\\
&\iff (1+2a+a^2)(1+c^2-ac) > 1+2c+c^2\\
&\iff 1+c^2-ac+2a+2ac^2-2a^2c+a^2+a^2c^2-a^3c > 1+2c+c^2\\
&\iff 2(a-c)+(a^2c+2ac)(c-a)+a(a-c) > 0\\
&\iff (2+a)(a-c)+(a^2c+2ac)(c-a) > 0\\
&\iff (2+a)-(a^2c+2ac) > 0\\
&\iff 2 > 2ac+a^2c-a = a(2c+ac-1)
\end{aligned}$$

But

$$a(2c+ac-1) < a(2+ac-1) = a(1+ac) < 2.$$

This proves the claim.

□

Proceeding, define $N_2$ to be the smallest integer such that

$$D(q')^{n-1} \le \frac{a}{16n} \text{ for all } n \ge N_2. \tag{3.18}$$

We note that, if $Q_1$ and $Q_2$ are quantities such that $0 < Q_1 < Q_2 < 1$ and ${Q_2}^{N_2-1} \le \frac{a}{16N_2}$ for some positive integer $N_2$, then ${Q_1}^{N_2-1} \le {Q_2}^{N_2-1} \le \frac{a}{16N_2}$. That is, if an $N_2$ is obtained for our quantity $q'(a)$, then it also works for $q(a,m)$ for all polynomials with $m \le \frac{a}{4}$. We may now proceed.

As in the previous subsection, we are interested in an explicit formula $N_2(a)$ which would guarantee that the Inequality 3.18 holds for all $n \ge N_2(a)$. The method used to extract this explicit formula from 3.18 is very much similar to what we did to obtain $\mathcal{N}_1(a)$ in the previous subsection. However, we include the full analysis here as it may have its own merit.

**Proposition 3.3.8.** *There is an explicit formula $\mathcal{N}_2(a,c)$ which guarantees that for all $n \ge \mathcal{N}_2(a,c)$, Inequality 3.18 holds.*

*Proof.* We construct such a formula: Let $D = D(q')$.

If $D^{n-1} \leq \frac{a}{16n}$, this is equivalent to

$$\log(n) + (n-1)\log(D) \leq \log(\frac{a}{16}).$$

This is true if and only if:

$$n \geq 1 + \frac{\log(\frac{a}{16})}{\log(D)} - \frac{\log(n)}{\log(D)}.$$

For ease of notation, we write the above inequality as:

$$n \geq 1 + n_1(a,c) + \log(n) \cdot n_2(a,c), \tag{3.19}$$

where, again for ease of notation:

$$n_1 = \frac{\log(\frac{a}{16})}{\log(D)} > 0, \text{ and } n_2 = \frac{-1}{\log(D)} > 0, \text{ (since } D < 1).$$

But then,

$$1 + n_1 + 2\sqrt{n} \cdot n_2 \geq 1 + n_1 + n_2 \cdot \log(n).$$

Hence, $n$ would satisfy the Inequality 3.18 if:

$$n \geq 1 + n_1 + 2\sqrt{n} \cdot n_2,$$

and this is true if

$$n \geq \left[n_2 + (1 + n_1 + n_2^2)^{\frac{1}{2}}\right]^2.$$

Hence, let

$$N_2(a,c) = \left[n_2 + (1 + n_1 + n_2^2)^{\frac{1}{2}}\right]^2,$$

then, $D^{n-1} \leq \frac{a}{16n}$ for all $n \geq N_2(a,c)$
We then define $\mathcal{N}_2(a,c)$ as:

$$\mathcal{N}_2(a,c) = \max\{N_2(a,c), \mathcal{N}_0(a)\}.$$

□

**A closer look at the constituents of $N_2(a,c)$**

We now need to verify that $N_2(a,c)$, and hence $\mathcal{N}_2(a,c)$ as defined is continuous. As in the discussion of $\mathcal{N}_1(a)$, we proceed term by term:
The quantity $\left(\frac{1}{1+a}\right)^{q'}$ is a continuous function of $a$, hence if $D = \left(\frac{1}{1+a}\right)^{q'}$, there is nothing to check.
So suppose

$$\begin{aligned} D &= \left(\frac{1+c}{1+a}\right)^{q'} \left(\sqrt{1+c^2-ac}\right)^{1-q'} \\ &= g(a,c)^{q'} \cdot h(a,c)^{\frac{1-q'}{2}} \end{aligned}$$

We note:

- $g(a,c) = \frac{1+c}{1+a}$ is the ratio of two continuous functions, each non-zero on their respective domains, hence $g(a,c)$ is continuous.
- $h(a,c) = 1+c^2-ac > 0$ for all $a,c$ such that $a \in (0,1)$ and $c \in (0,a)$
- $h(a,c)$ is continuous on $\mathbb{R}^2$, and in particular positive on the subdomain $(0,1)\times(0,a)$ for any $a \in (0,1)$. Hence, $h(a,c)^{\frac{1-q'}{2}}$ is continuous.

We can thus conclude that $\mathcal{N}_2(a,c)$ is continuous on $\{(x,y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } 0 < y < x\}$.

## 3.4 Bounds on the size of $|P(c)|$

### 3.4.1 Towards the upper bound of $|P(c)|$

In this section, we now put to use the bounds $\mathcal{N}_1(a)$ (and hence the conclusion of Lemma 3.3.1), and $\mathcal{N}_2(a,c)$ to obtain an upper bound on the size of $|P(c)|$. The following lemma is our version of Lemma 3.1.15. Herein we have replaced Dégot's $q(a,m)$ with $q'(a) = \frac{a/4}{1+a/2}$.

**Lemma 3.4.1.** *([4], Lemma 2): Suppose the polynomial $P(z)$ contradicts Sendov's conjecture at $a \in (0,1)$ with $m \leq \frac{a}{4}$ and let $w_1,\ldots,w_{n-1}$ be its critical points. Let $\delta \in (0,a)$ and $q'$ be as previously defined. Then:*

$$\left|\frac{P'(\delta)}{P'(a)}\right| = \prod_{j=1}^{n-1}\left|\frac{\delta - w_j}{a - w_j}\right| \leq \left\{\left(\frac{1+\delta}{1+a}\right)^{q'}\left(\sqrt{1+\delta^2 - a\delta}\right)^{1-q'}\right\}^{n-1}. \tag{3.20}$$

*Proof.* The only crucial observation we have to make here is that the right hand side of the Inequality 3.20 is $D_2(a, q', \delta)^{n-1}$. We have already verified that $D_2(a, x, c)$ is a decreasing function with respect to $x$. Since $q(a, m) \geq q'(a)$ (because $m \leq \frac{a}{4}$), applying Lemma 3.1.15 completes the proof. □

We can now state the result that gives us the upper bound:

**Theorem 3.4.2.** *Suppose $P(z)$ contradicts Sendov's conjecture at $a \in (0, 1)$ and let $c \in (0, a)$. If $\deg(P) = n \geq \max\{\mathcal{N}_1(a), \mathcal{N}_2(a, c)\}$, then,*

$$|P(c)| \leq 1 + a.$$

*Proof.* Recall that $D = D(q')$, we have that:

$$D^{n-1} = \max\left\{\left(\frac{1}{1+a}\right)^{q'(n-1)}; \left[\left(\frac{1+c}{1+a}\right)^{q'}\left(\sqrt{1+c^2-ac}\right)^{1-q'}\right]^{n-1}\right\}.$$

For ease of reference, we write the above equation as

$$D^{n-1} = \max\{[h_0]^{n-1}; [h_1(c)]^{n-1}\},$$

where the function $h_1 : [0, c] \longrightarrow \mathbb{R}$ defined by

$$h_1(x) = \left(\frac{1+x}{1+a}\right)^{q'}\left(1 + x^2 - ax\right)^{\frac{1-q'}{2}}.$$

By definition, $D^{n-1} \geq [h_1(c)]^{n-1}$. We also note that, $h_1(0) = h_0$. We now turn our attention to the behaviour of $h_1(x)$ for $x \in (0, c]$. For this, let $f(x) = \log(h_1(x))$, so that

$$\begin{aligned} f(x) &= \log\left[\left(\frac{1+x}{1+a}\right)^{q'}\left(1+x^2-ax\right)^{\frac{1-q'}{2}}\right], \\ &= q'\log\left(\frac{1+x}{1+a}\right) + \frac{1-q'}{2}\log(1+x^2-ax). \end{aligned}$$

We have that,

$$
\begin{aligned}
f'(x) &= q'\left(\frac{1}{1+x}\right) + \frac{1-q'}{2}\left(\frac{2x-a}{1+x^2-ax}\right),\\
&= \frac{2(1+x^2-ax)q' + (1-q')(1+x)(2x-a)}{2(1+x)(1+x^2-ax)},\\
&= \frac{(1+x^2-ax)q' + (1-q')(x^2+(1-a/2)x-a/2)}{(1+x)(1+x^2-ax)}.
\end{aligned}
$$

We note that, the denominator of $f'(x)$ is positive since $x \in (0,c]$, so that, $f'(x)$ shares the sign with the numerator. Substituting $q'(a) = \frac{a/4}{1+a/2}$ into the numerator and simplifying, we get that $f'(x)$ shares the sign with the quadratic function

$$
g(x) = x^2 + \left(\frac{8-3a^2-2a}{2(4+2a)}\right)x - \left(\frac{2a+a^2}{2(4+2a)}\right). \tag{3.21}
$$

We note that $g(0) < 0$. The $x$-coordinate of the turning point of $g$ is given by $x = \frac{2a+3a^2-8}{4(4+2a)}$, and this is negative. We can thus deduce that $g$ is increasing on $[0,c]$. This gives two cases, depending on whether $g$ cuts the $x-$axis on the left or right hand side of $c$:

- (i) Either there is a $\delta \in [0,c]$ such that $h_1(x)$ is decreasing on $[0,\delta]$ and increasing on $[\delta,c]$. It can thus be possible that for $x \geq \delta$, $h_1(x) \geq h_0$. Either way, we still have by definition that $D^{n-1} = \max\{[h_0]^{n-1}; [h_1(c)]^{n-1}\}$.

- $h_1(x)$ could be decreasing on $[0,c]$, so that $h_0 \geq h_1(x)$ for all $x \in [0,c]$.

In either case, we can conclude that

$$
D^{n-1} \geq \sup_{\delta\in[0,c]}\left[\left(\frac{1+\delta}{1+a}\right)^{q'}\left(\sqrt{1+\delta^2-a\delta}\right)^{1-q'}\right]^{n-1}
$$

By Lemma 3.4.1, and then by the assumption on the degree of $P(z)$, the fact that $D^{n-1} \leq \frac{a}{16n}$ for all $n \geq \mathcal{N}_2(a,c)$, we can deduce that:

$$
\sup_{\delta\in[0,c]}\left|\frac{P'(\delta)}{P'(a)}\right| \leq \frac{a}{16n}.
$$

From Lemma 3.3.1, we have that $|P'(a)| \leq \frac{16n}{a^2}$. From this, together with the inequality above we can deduce that:

$$|P(0)-P(c)| = \left|\int_0^c P'(z)\mathrm{d}z\right| \leq c \cdot \sup_{\delta\in[0,c]} \left|\frac{P'(\delta)}{P'(a)}\right| \cdot |P'(a)| \leq \frac{c}{a} \leq 1.$$

Whence

$$|P(c)| \leq 1 + |P(0)| \leq 1 + a.$$

□

### 3.4.2 Towards the lower bound of $|P(c)|$

In this section we look at Dégot's **Theorem 7**. The goal is to obtain constants $C > 0$ and $K > 1$ such that for large enough degree $n$, the value of $P(c)$ satisfies $|P(c)| \geq C \cdot K^n$.

#### 3.4.2.1 Technical inequalities towards the lower bound of $|P(c)|$

As in the previous section, before arriving at the main result, we go through a series of lemmas which establish some technical inequalities which will subsequently be used in the proof of the main result of the section. The first of such lemmas that we consider is Dégot's **Lemma 3**. In preparation for the proof of the lemma, we briefly recall the main statement of the theory of Lagrange multipliers.

Although this can be easily stated for any general multivariate problem, for our specific purpose we only discuss the result for a three variable problem subject to two constraints.

**Lagrange multipliers theory**
When preparing the following discussion, we referred heavily to the 2002 reprint of Lasdon's book on Optimization [7].

Let $g(x_1, x_2, x_3), f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be functions with continuous partial derivatives. Consider the following optimization problem:

**maximize** $g(x_1, x_2, x_3)$ subject to $f_1(x_1, x_2, x_3) = k_1$ and $f_2(x_1, x_2, x_3) = k_2$ where $k_1, k_2 \in \mathbb{R}$ are constants.

The Lagrangian function $\mathcal{L}(x_1, x_2, x_3, \lambda_1, \lambda_2)$ is then defined as:

$$\mathcal{L}(x_1, x_2, x_3, \lambda_1, \lambda_2) = g(x_1, x_2, x_3) - \lambda_1 f_1(x_1, x_2, x_3) - \lambda_2 f_2(x_1, x_2, x_3).$$

The theory then asserts that if $g(x_0, y_0, z_0)$ is a maximum of $g(x, y, z)$ for the original constrained problem, and the Jacobian matrix $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}' (x_0, y_0, z_0)$ has full row rank (in this case rank 2), then there exist $\lambda_1$ and $\lambda_2$ such that the gradient of the Lagrangian function vanishes at $(x_0, y_0, z_0, \lambda_1, \lambda_2)$. That is:

$$\nabla \mathcal{L}(x_1, x_2, x_3, \lambda_1, \lambda_2)\Big|_{(x_0, y_0, z_0, \lambda_1, \lambda_2)} = 0$$

where $\nabla$ is the "grad operator":

$$\nabla \mathcal{L} = \left( \frac{\partial \mathcal{L}}{\partial x_1}, \frac{\partial \mathcal{L}}{\partial x_2}, \frac{\partial \mathcal{L}}{\partial x_3} \right).$$

We may now proceed with the statement and proof of the lemma.

**Lemma 3.4.3.** *([4], Lemma 3): Let $a, b$ be real numbers such that $0 < a < 1 < b$ and let $w_1, \ldots, w_{n-1}$ denote complex numbers such that $|w_j| \leq 1$ and $|w_j - a| \geq 1$, for all $j = 1, \ldots, n-1$. Let $m = \frac{1}{n-1} \sum_{j=1}^{n-1} \Re(w_j) < \frac{a}{2}$. Furthermore, define the following quantities:*

$$p = \frac{a/2 - m}{1 - a/2}, \; q = \frac{a/2 - m}{1 + a/2},$$

*and*

$$B_1 = (1 + b - a)^p (\sqrt{1 + b^2 - ba})^{1-p}, \; B_2 = (1 + b)^q (\sqrt{1 + b^2 - ba})^{1-q}.$$

*Then,*

$$\prod_{j=1}^{n-1} |b - w_j| \geq \min\{B_1, B_2\}^{n-1}.$$

*Proof.* We begin by first verifying the following claim.

**Claim**: For $j \in \{1, \dots, n-1\}$, let $\Re w_j$ be given. Then $|b - w_j|$ is minimized when $|a - w_j| = 1$, or if $w_j$ lies on the real line and $-1 \leq w_j \leq -1 + a$.

*Proof.* We start by showing that if $u$ and $v$ satisfy the constraints imposed on the $w_j$'s in the statement of the lemma, with $\Re(u) = \Re(v)$ and $|\Im(u)| < |\Im(v)|$, hence $|u| < |v|$, then $|b - u| < |b - v|$. Thus let $u = x + iy$ and $v = x + iky$, where $k > 1$. Then

$$|b - u| = \sqrt{(b-x)^2 + y^2} < \sqrt{(b-x)^2 + k^2y^2} = |b - v|.$$

Since $|a - u| \geq 1$, the distance $|b - u|$ is minimum when $|a - u| = 1$.

If $\Im(u) = 0$, then by definition $u \in [-1, a-1]$, and in this case $|b - u|$ is minimum when $u \in [-1, a-1]$. This verifies the claim

□

Henceforth, we can assume that $|a - w_j| = 1$ or $w_j \in [-1, a-1]$.

Consider the mapping $\Phi : [-1, \frac{a}{2}] \longrightarrow \mathbb{R}$ defined by

$$\Phi(x) = \begin{cases} \frac{1}{2}\log(1 + (b-a)(a+b-2x)) = \Phi_1(x) & \text{if } x \geq a-1 \\ \log(b-x) = \Phi_2(x) & \text{if } x \in [-1, a-1] \end{cases}$$

**Remark 3.4.4.** *The motivation for the definition of the function $\Phi$ will become apparent in the proof of the lemma. However, roughly speaking, each factor $|b - w_j|$ of the product $|P'(b)|$ corresponds to* $\exp(\Phi(\Re(w_j)))$*, we can study $|P'(c)|$ both as a product and as a sum (by taking* $\log$*). The function $\Phi$ then gives a lower approximation of the corresponding factor, and thus we can get a lower bound on $|P'(c)|$, which is what we set out to obtain.*

We note that at $x = a - 1$,

$$\begin{aligned}\Phi_1(a-1) &= \frac{1}{2}\log(1+(b-a)(a+b-2a+2)) \\ &= \frac{1}{2}\log(1+(b-a)(b-a+2)) \\ &= \frac{1}{2}\log(1+b^2-2ab+a^2+2b-2a) \\ &= \frac{1}{2}\log((b+1-a)^2) = \Phi_2(a-1),\end{aligned}$$

and thus $\Phi$ is well defined.

**Claim**: Let $g : I \longrightarrow \mathbb{R}$ be a concave function positive on the interval $I$. Then the function $f(x) = \log(g(x))$ is also concave on $I$.

*Proof.* By definition, we have that $g(\lambda x + (1-\lambda)y) \geq \lambda g(x) + (1-\lambda)g(y)$ for $\lambda \in [0,1]$ and $x, y \in I$. Then

$$\begin{aligned}f(\lambda x + (1-\lambda)y) &= \log(g(\lambda x + (1-\lambda)y)) \\ &\geq \log(\lambda g(x) + (1-\lambda)g(y)) \\ &\geq \lambda \log(g(x)) + (1-\lambda)\log(g(y)) \\ &= \lambda f(x) + (1-\lambda)f(y).\end{aligned}$$

□

We note that $\Phi$, (i.e $\Phi_1$ and $\Phi_2$) is defined as the composition of log and a linear function, so that by the preceding argument, each of the pieces $\Phi_1$ and $\Phi_2$ is concave on its respective domain.

Recall that if $|a - w_j| = 1$, writing $w_j = x_j + iy_j$, then we get that $y_j^2 = 1 - (a-x_j)^2$. Also, by concavity, if we have $\alpha, \beta, \gamma \in (0,1)$ such that $\alpha + \beta + \gamma = 1$, then $\Phi_k(\alpha x + \beta y + \gamma z) \geq \alpha\Phi_k(x) + \beta\Phi_k(y) + \gamma\Phi_k(z)$, where $x, y, z$ are in the respective domain of each $\Phi_k$, $k = 1, 2$. Without loss of generality we can assume that the $w_j$'s are numbered such that for $j = 1, \ldots, l$, $|a - w_j| = 1$ and for $j = l+1, \ldots, n-1$, $w_j \in [-1, a-1]$. With this in mind, we can then deduce that:

$$\begin{aligned}\log\left(\prod_{j=1}^{n-1}|b-w_j|\right) &= \sum_{j=1}^{l}\log|b-w_j| + \sum_{j=l+1}^{n-1}\log|b-w_j| \\ &= \sum_{j=1}^{l}\Phi_1(\Re(w_j)) + \sum_{j=l+1}^{n-1}\Phi_2(\Re(w_j)).\end{aligned}$$

For $j = 1, \ldots, l$ we write $\Re(w_j) = \alpha_j(-1) + \beta_j(a-1) + \gamma_j(\frac{a}{2})$, where $\gamma_j = 0$ and $\alpha_j + \beta_j + \gamma_j = 1$. By Remark 3.1.16 (the only changes being that each of $\Phi_1$ and $\Phi_2$ are concave), we can get that:

$$\sum_{j=1}^{l}\Phi(\Re(w_j)) \geq \sum_{j=1}^{l}\alpha_j\Phi(-1) + \sum_{j=1}^{l}\beta_j\Phi(a-1). \tag{3.22}$$

Similarly for $j = l+1, \ldots, n-1$ we write $\Re(w_j) = \alpha_j(-1) + \beta_j(a-1) + \gamma_j(\frac{a}{2})$ where $\alpha_j = 0$ and $\alpha_j + \beta_j + \gamma_j = 1$. We similarly get that

$$\sum_{j=l+1}^{n-1}\Phi(\Re(w_j)) \geq \sum_{j=l+1}^{n-1}\beta_j\Phi(a-1) + \sum_{j=l+1}^{n-1}\gamma_j\Phi(\frac{a}{2}). \tag{3.23}$$

Adding Equations 3.22 and 3.23, we get

$$\begin{aligned}\sum_{j=1}^{n-1}\Phi(\Re(w_j)) &\geq \sum_{j=1}^{n-1}\alpha_j\Phi(-1) + \sum_{j=1}^{n-1}\beta_j\Phi(a-1) + \sum_{j=1}^{n-1}\gamma_j\Phi(\frac{a}{2}), \\ &= (n-1)\left[\alpha\Phi(-1) + \beta\Phi(a-1)\gamma\Phi(\frac{a}{2})\right],\end{aligned}$$

where $\alpha = \frac{1}{n-1}\sum_{j=1}^{n-1}\alpha_j$, $\beta = \frac{1}{n-1}\sum_{j=1}^{n-1}\beta_j$, and $\gamma = \frac{1}{n-1}\sum_{j=1}^{n-1}\gamma_j$. It follows that

$$\log\left(\prod_{j=1}^{n-1}|b-w_j|\right) \geq (n-1)\min\{\alpha\Phi(-1) + \beta\Phi(-1+a) + \gamma\Phi(\frac{a}{2})\}. \tag{3.24}$$

where the minimum is taken over the set of all triples $(\alpha, \beta, \gamma) \in \mathbb{R}^3_+$ such that

$$\begin{cases} \alpha + \beta + \gamma = 1, \\ -\alpha + (a-1)\beta + \frac{a}{2}\gamma = m. \end{cases}$$

Consider the mappings $g, f_1, f_2 : \mathbb{R}^3_+ \longrightarrow \mathbb{R}_+$ defined by:

$$\begin{aligned} g(\alpha,\beta,\gamma) &= \alpha\log(b+1) + \beta\log(b+1-a) + \frac{\gamma}{2}\log(1+b^2-ab), \\ f_1(\alpha,\beta,\gamma) &= \alpha+\beta+\gamma, \\ f_2(\alpha,\beta,\gamma) &= -\alpha + (-1+a)\beta + \frac{a}{2}\gamma. \end{aligned}$$

From Equation 3.24, we can thus deduce that

$$\frac{1}{n-1}\log\left(\prod_{j=1}^{n-1}|b-w_j|\right) \geq \min_{(\alpha,\beta,\gamma)\in\mathbb{R}^3_+}\{g(\alpha,\beta,\gamma); f_1(\alpha,\beta,\gamma)=1, f_2(\alpha,\beta,\gamma)=m\}. \tag{3.25}$$

Suppose the minimum in Equation 3.25 is reached at $(\alpha_0,\beta_0,\gamma_0) \in (\mathbb{R}^*_+)^3$, then from our discussion of the Lagrange multipliers theory, there exist multipliers $\lambda_1$ and $\lambda_2$ such that:

$$\nabla g = \lambda_1 f_1 + \lambda_2 f_2.$$

That is:

$$\begin{pmatrix} \log(b+1) \\ \log(b+1-a) \\ \frac{1}{2}log(1+b^2-ab) \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ -1+a \\ \frac{a}{2} \end{pmatrix}. \tag{3.26}$$

This implies that:

$$\begin{aligned} \log(1+b) &= \lambda_1 - \lambda_2, \\ a\lambda_2 + \log(b+1-a) &= \lambda_1 - \lambda_2, \\ \log(1+b^2-ab) &= 2\lambda_1 + a\lambda_2. \end{aligned}$$

The first two equations imply that:

$$\lambda_2 = \frac{1}{a}\log\left(\frac{1+b-a}{1+b}\right),$$

and,

$$\lambda_1 = \log(1+b) + \frac{1}{a}\log\left(\frac{1+b-a}{1+b}\right).$$

But substituting the above values of $\lambda_1$ and $\lambda_2$ into the right hand side of the third equation does not equal $\log(1+b^2-ab)$, the left hand side. Hence, if $\alpha$, $\beta$ and $\gamma$ are all non-zero, then Equation 3.26 above does not have a solution (i.e there are no critical points).

We thus deduce that if the minimum in Equation 3.25 is reached at $(\alpha, \beta, \gamma)$, then either $\alpha = 0$, or $\beta = 0$ or $\gamma = 0$. Let us study the consequences of these possibilities by systematically substituting them into $g(\alpha, \beta, \gamma)$, bearing in mind that $-\alpha + (a-1)\beta + \frac{a}{2}\gamma = m$ and $\alpha + \beta + \gamma = 1$.

- if $\alpha = 0$, then $\beta = \frac{a/2-m}{1-a/2}$ and, from Equation 3.25, we have that:

$$\log\left(\prod_{j=1}^{n-1}|b-w_j|\right) \geq (n-1)g(0,\beta,\gamma)$$
$$= (n-1)\left[\beta\log(b+1-a) + \frac{1-\beta}{2}\log(1+b^2-ab)\right]$$

  Taking the exponential both sides yields:

$$\prod_{j=1}^{n-1}|b-w_j| \geq \left((b+1-a)^{\beta}(\sqrt{1+b^2-ab})^{1-\beta}\right)^{n-1} = B_1^{n-1}.$$

- Similarly, if $\beta = 0$, then $\alpha = \frac{a/2-m}{1+a/2}$, and

$$\log\left(\prod_{j=1}^{n-1}|b-w_j|\right) \geq (n-1)g(\alpha,0,\gamma)$$
$$= (n-1)\left[\alpha\log(b+1) + \frac{1-\alpha}{2}\log(1+b^2-ab)\right]$$

  Taking the exponential both sides yields:

$$\prod_{j=1}^{n-1}|b-w_j| \geq \left((b+1)^{\alpha}(\sqrt{1+b^2-ab})^{1-\alpha}\right)^{n-1} = B_2^{n-1}.$$

- Finally, if $\gamma = 0$, then

$$\prod_{j=1}^{n-1}|b-w_j| \geq (1+b-a)^{n-1} \geq B_1^{n-1}.$$

This completes the proof.

□

We now proceed to study Dégot's **Lemma 4**.

**Lemma 3.4.5.** *([4], Lemma 4): Let $c$ and $r$ be real numbers such that $0 < c < 1$ and $0 < r < 1 - c$. Suppose $z_1, \ldots, z_n$ are complex numbers satisfying $0 < |z_j| \leq 1$ and $\left|\frac{c-z_j}{1-cz_j}\right| \geq r$, for $j = 1, \ldots, n$. Then*

$$\prod_{j=1}^{n} \left| \frac{c - z_j}{1 - cz_j} \right| \geq r^{\beta}, \quad where \quad \beta = \frac{\sum_{j=1}^{n} \log |z_j|}{\log \left( \frac{c+r}{1+cr} \right)}.$$

*Proof.* Let us begin by first verifying the following claim:
**Claim**:
Given $z_j$ such that $0 < |z_j| \leq 1$ and $\left|\frac{c-z_j}{1-cz_j}\right| \geq r$, we can always find $z \in \mathbb{R}$ such that $\left|\frac{c-z}{1-cz}\right| = \left|\frac{c-z_j}{1-cz_j}\right|$ and $\frac{c+r}{1+cr} \leq z \leq 1$.

*Proof.* Let $k = \left|\frac{c-z_j}{1-cz_j}\right|$ and consider the equation $k = \left|\frac{c-z}{1-cz}\right|$. This implies that $|c - z| = k|1 - cz|$ and hence $|c - \bar{z}| = k|1 - c\bar{z}|$. We deduce that the equation $k = \left|\frac{c-z}{1-cz}\right|$ defines a circle $\mathcal{C}$ which is symmetric about the real axis since it is preserved by conjugation. In particular, $\mathcal{C}$ has two real points.

The mapping $z \mapsto \frac{c-z}{1-cz}$ gives a bijection between $\mathcal{C}$ and the circle $\mathcal{C}_0$ centred at the origin with radius $k$, and is equal to its inverse. Since $-k$ lies on $\mathcal{C}_0$, we have that $\frac{c+k}{1+ck}$ lies on $\mathcal{C}$. We thus let $z = \frac{c+k}{1+ck}$. The derivative of the map $\frac{x+c}{cx+1}$ is $\frac{1-c^2}{(cx+1)^2}$ and is positive (where $x \in \mathbb{R}$). This implies that $\frac{c+k}{1+ck} = z \geq \frac{c+r}{1+cr}$ since $k \geq r$.

Write $z_j = x + iy$ and recall that $k = \left|\frac{c-z_j}{1-cz_j}\right|$. We note that, $|1 - cz_j| \geq |c - z_j|$ if and only if $|(1 - cx) - ciy| \geq |(c - x) - iy|$, if and only if $(1 - cx)^2 + c^2y^2 \geq (c - x)^2 + y^2$, if and only if $(1 - c^2) \geq (x^2 + y^2)(1 - c^2)$, if and only if $1 \geq x^2 + y^2 = |z_j|^2$, which is true. This verifies that $k = \left|\frac{c-z_j}{1-cz_j}\right| \leq 1$. We therefore have that $1 + ck - (c + k) = (1 - c)(1 - k) \geq 0$ since $k \leq 1$. This implies that $z \leq 1$. This completes the proof.

□

Henceforth, we can always assume that $\frac{c+r}{1+cr} \leq z_j \leq 1$.

Proceeding, for each $z_j$ let $\alpha_j = \log(z_j)$ and consider the mapping $\Phi : \left[\log\left(\frac{c+r}{1+cr}\right), 0\right] \longrightarrow \mathbb{R}$ defined by

$$\Phi(\alpha) = \log\left(\frac{e^\alpha - c}{1 - ce^\alpha}\right).$$

We note that, since $(1 - ce^\alpha) - (e^\alpha - c) = (1+c)(1-e^\alpha) \geq 0$, then

$$\Phi''(\alpha) = \frac{ce^\alpha}{(1-ce^\alpha)^2} - \frac{ce^\alpha}{(e^\alpha - c)^2} \leq 0.$$

This implies that $\Phi$ is concave, hence:

$$\begin{aligned}
\log\left(\prod_{j=1}^{n}\left|\frac{c - z_j}{1 - cz_j}\right|\right) &= \sum_{j=1}^{n} \log\left|\frac{c - z_j}{1 - cz_j}\right| \\
&= \sum_{j=1}^{n} \log\left|\frac{c - e^{\alpha_j}}{1 - ce^{\alpha_j}}\right| \\
&= \sum_{j=1}^{n} \Phi(\alpha_j) \\
&= \sum_{j=1}^{n} \Phi(\beta_j \log\left(\frac{c+r}{1+cr}\right) + (1-\beta_j)\cdot 0),
\end{aligned}$$

where $\beta_j = \frac{\log(z_j)}{\log\left(\frac{c+r}{1+cr}\right)}$. Hence, by concavity, we get that:

$$\begin{aligned}
\log\left(\prod_{j=1}^{n}\left|\frac{c - z_j}{1 - cz_j}\right|\right) &\geq \beta\Phi(\log\left(\frac{c+r}{1+cr}\right)) + (n-\beta)\Phi(0) \\
&= \beta\Phi(\log\left(\frac{c+r}{1+cr}\right)),
\end{aligned}$$

where $\beta = \frac{\log(\prod_{j=1}^{n}|z_j|)}{\log\left(\frac{c+r}{1+cr}\right)}$, and also noting that $\Phi(0) = \log\left(\frac{1-c}{1-c}\right) = 0$.

Taking the exponential on both sides, the lemma is obtained.

□

The next technical inequality we encounter is Dégot's **Lemma 5**. Below is the statement.

**Lemma 3.4.6.** *([4], Lemma 5): Let $h$ and $c$ be positive real numbers such that $0 < c < 1-h$. Then, for all $z \in \mathbb{C}$, if $|z| \geq 1-h$, then*

$$|c-z| \geq \frac{c}{1-h}\left|\frac{(1-h)^2}{c} - z\right|.$$

*Proof.* The proof is obtained by establishing the equivalence of the following inequalities:

Given

$$|c-z| \geq \frac{c}{1-h}\left|\frac{(1-h)^2}{c} - z\right|, \tag{3.27}$$

we divide both sides of Inequality 3.27 by $1-h$ and then square both sides to establish that 3.27 holds if and only if:

$$\Leftrightarrow \left|\frac{z}{1-h} - \frac{c}{1-h}\right|^2 \geq \left|1 - \frac{c}{1-h}\frac{z}{1-h}\right|^2. \tag{3.28}$$

On the other hand, Inequality 3.28 is true if and only if:

$$\left|\frac{z}{1-h}\right|^2 + \left(\frac{c}{1-h}\right)^2 \geq 1 + \left|\frac{cz}{(1-h)^2}\right|^2. \tag{3.29}$$

We note that, the above inequality can be expressed in the form $a^2 + b^2 - 1 - a^2b^2 \geq 0$. The latter is true if and only if $(a^2-1)(1-b^2) \geq 0$. Hence Inequality 3.29 holds if and only if

$$\left(\left|\frac{z}{1-h}\right|^2 - 1\right)\left(1 - \left(\frac{c}{1-h}\right)^2\right) \geq 0.$$

But the above inequality is true since from the assumption of the lemma, we have that $|z| \geq 1-h$ and $0 < c < 1-h$, hence each of the factors in the above inequality is non-negative. The lemma is thus established.

□

The next technical inequality we prove is Dégot's **Lemma 6**.

**Lemma 3.4.7.** *([4], Lemma 6): Let $P(z) = (z-a)\prod_{j=1}^{n-1}(z-z_j)$, where $a \in (0,1)$ and $|z_j| \leq 1$ for $j = 1, \ldots, n-1$. Let $w_1, \ldots, w_{n-1}$ be the critical points of $P(z)$. Then, for all $b > 1$, we have*

$$(b-a)\prod_{j=1}^{n-1}|b-z_j| \geq (b-1)\prod_{j=1}^{n-1}|b-w_j|.$$

*Proof.* Taking the logarithm of $P(z)$ yields

$$\log P(z) = \log(z-a) + \sum_{j=1}^{n-1}\log(z-z_j).$$

Taking the derivative on both sides, we obtain:

$$\frac{P'(z)}{P(z)} = \frac{1}{z-a} + \sum_{j=1}^{n-1}\frac{1}{z-z_j}.$$

Whence:

$$\left|\frac{P'(b)}{P(b)}\right| = \left|\frac{1}{b-a} + \sum_{j=1}^{n-1}\frac{1}{b-z_j}\right|.$$

By the triangle inequality,

$$\begin{aligned}\left|\frac{P'(b)}{P(b)}\right| &\leq \frac{1}{b-a} + \sum_{j=1}^{n-1}\frac{1}{|b-z_j|}\\ &\leq \frac{n}{b-1}, \quad \text{since } b>1 \text{ and } |z_j|\leq 1. \end{aligned} \tag{3.30}$$

Since $P'(z)$ can be written as $P'(z) = n\prod_{j=1}^{n-1}(z-w_j)$, it follows that $|P'(b)| = n\prod_{j=1}^{n-1}|b-w_j|$. Combining this with Inequality 3.30, we get that:

$$(b-a)\prod_{j=1}^{n-1}|b-z_j| = |P(b)| \geq \frac{b-1}{n}|P'(b)| = (b-1)\prod_{j=1}^{n-1}|b-w_j|.$$

This is what we set out to prove.

□

The last technical inequality we consider establishes yet another exclusion domain for the zeroes of a polynomial assumed to contradict Sendov's conjecture. This is Dégot's **Lemma 7**.

**Lemma 3.4.8.** *([4], Lemma 7): Suppose $P(z)$ contradicts Sendov's conjecture at $a \in (0,1)$. Let $c$ and $h$ be real numbers such that $0 < h < c < a < 1-h$. Then the disk $\mathcal{D}$ defined by*

$$\mathcal{D} = \left\{ z \in \mathbb{C} : \left| \frac{(c-z)}{(1-h)^2 - cz} \right| \leq \frac{c(a-c)}{2((1-h)^2 - c^2)} \right\}$$

*is devoid of any zeroes of* $P(z)$.

*Proof.* Setting $k = \frac{c(a-c)}{2((1-h)^2-c^2)}$, we note that $\mathcal{D}$ is then defined as the set of all $z = x + iy \in \mathbb{C}$ such that:

$$\frac{|c-z|}{|(1-h)^2 - cz|} \leq k.$$

This translates to $|(c-x) - iy| \leq k|(1-h)^2 - cx - icy|$, and holds

$$\begin{aligned}
&\iff (c-x)^2 + y^2 \leq k^2[((1-h)^2 - cx)^2 + c^2y^2] \\
&\iff (1-k^2c^2)x^2 - 2[c(1+k^2(1-h)^2)]x + (1-k^2c^2)y^2 \leq k^2(1-h)^4 - c^2 \\
&\iff \left[ x - \left( c\frac{1-k^2(1-h)^2}{1-k^2c^2} \right) \right]^2 + y^2 \leq c^2\frac{(1-k^2(1-h)^2)^2}{(1-k^2c^2)^2} + \frac{k^2(1-h)^4 - c^2}{1-k^2c^2} \\
&\iff \left[ x - \left( c\frac{1-k^2(1-h)^2}{1-k^2c^2} \right) \right]^2 + y^2 \leq \frac{k^2(1-h)^4 - 2c^2k^2(1-h)^2 + c^4k^2}{(1-k^2c^2)^2} \\
&\iff \left[ x - \left( c\frac{1-k^2(1-h)^2}{1-k^2c^2} \right) \right]^2 + y^2 \leq k^2\left[ \frac{(1-h)^2 - c^2}{1-k^2c^2} \right]^2.
\end{aligned}$$

The above calculation simply tells us that, with that definition of $k$, the disk $\mathcal{D}$ has center $\omega$ and radius $R$ given by

$$\omega = c\frac{1-k^2(1-h)^2}{1-k^2c^2} \quad \text{and} \quad R = k\frac{(1-h)^2 - c^2}{1-k^2c^2}.$$

**Remark 3.4.9.** *We note that, for* $R$ *in the above equation to make sense, the denominator* $1 - k^2c^2$ *must be positive, or equivalently,* $kc < 1$*. This is true as we will later show that* $k \leq \frac{1}{2}$*, hence we can proceed.*

Recall that Lemma 3.1.5 asserts that, if $P(z)$ contradicts Sendov's conjecture at $a \in (0,1)$, and $c \in (0,a)$, then $P(z)$ cannot have a zero in the disk centered at $c$ with radius $1 - \sqrt{1 + c^2 - ac}$.

Thus, to prove the current lemma, we need only show that

$$R \leq 1 - \sqrt{1 + \omega^2 - \omega a} \quad \text{and} \quad \omega \in (0,a).$$

This is equivalent to showing that

$$R \leq 1 \text{ and } \omega^2 - R^2 \leq \omega a - 2R, \;\; \omega \in (0, a). \tag{3.31}$$

Towards this, in order to apply Lemma 3.1.5, we first need to verify that $\omega \in (0, a)$.

**Claim**: $\omega \in (0, a)$

*Proof.* We already know that $c \in (0, a)$. So, it remains to show that $\frac{1-k^2(1-h)^2}{1-k^2c^2} \in (0, 1)$. But from the assumption of the lemma, we have that $0 < c < 1 - h$ and $h \in (0, 1)$. Hence, $1 - k^2c^2 > 1 - k^2(1 - h)^2$. We note that, since $1 - h < 1$, if $k \in (0, 1)$, then $k(1 - h) < 1$ and hence $1 - k^2(1 - h)^2 > 0$. This is true since we will show that $k \leq \frac{1}{2}$. □

To show that indeed $R \leq 1$, we note that, from the definition of $k$, $R$ can be written as:

$$R = k\frac{(1-h)^2 - c^2}{1 - k^2c^2} = \frac{c(a-c)}{2((1-h)^2 - c^2)} \cdot \frac{(1-h)^2 - c^2}{1 - k^2c^2} = \frac{c(a-c)}{2(1 - k^2c^2)}.$$

Hence,

$$R \leq 1 \iff \frac{ca - c^2}{2} \leq 1 - k^2c^2.$$

We express $k$ as

$$k = \frac{1}{2} \cdot \frac{c(a-c)}{(1-h)^2 - c^2}$$

and note that:

$$(1-h)^2 - c^2 - (c(a-c)) = (1-h)(1-h) - ac \geq 0 \text{ since } 1 - h > a > c.$$

This tells us that $k \leq \frac{1}{2}$ and hence indeed

$$\frac{ca - c^2}{2} \leq \frac{2 - c^2}{2} \leq 1 - k^2c^2. \tag{3.32}$$

This confirms that $R \leq 1$.

We note that,

$$\begin{aligned}
\omega^2 - R^2 &= \frac{c^2[1-k^2(1-h)^2]^2 - k^2[(1-h)^2-c^2]^2}{(1-k^2c^2)^2}, \\
&= \frac{c^2 - 2c^2k^2(1-h)^2 + c^2k^4(1-h)^4 - k^2(1-h)^4 + 2k^2c^2(1-h)^2 - k^2c^4}{(1-k^2c^2)^2}, \\
&= \frac{c^2 + c^2k^4(1-h)^4 - k^2(1-h)^4 - k^2c^4}{(1-k^2c^2)^2}, \\
&= \frac{c^2(1-k^2c^2) - k^2(1-h)^4[1-c^2k^2]}{(1-k^2c^2)^2}, \\
&= \frac{c^2 - k^2(1-h)^4}{1-k^2c^2}.
\end{aligned}$$

On the other hand,

$$a\omega - 2R = \frac{ac[1-k^2(1-h)^2] - 2k[(1-h)^2-c^2]}{1-k^2c^2}.$$

So that $\omega^2 - R^2 \leq a\omega - 2R$ if and only if:

$$\frac{c^2 - k^2(1-h)^4}{1-k^2c^2} \leq \frac{ac[1-k^2(1-h)^2] - 2k[(1-h)^2-c^2]}{1-k^2c^2}. \tag{3.33}$$

We note that $1-k^2c^2 > 0$ by Inequality 3.32 since $0 < c < a$ and hence $ac - c^2 = c(a-c) > 0$. Therefore, Inequality 3.33 is equivalent to:

$$c^2 - k^2(1-h)^4 \leq ac(1-k^2(1-h)^2) + 2k(c^2-(1-h)^2).$$

The above inequality is in turn equivalent to:

$$c^2 - 2ack(c^2-(1-h)^2) \leq k^2(1-h)^4 - ack^2(1-h)^2. \tag{3.34}$$

Recall that $k = \frac{c(a-c)}{2((1-h)^2-c^2)}$, so that the left hand side of Inequality 3.34 reduces to $c(c-a) + c(a-c)$, which equals zero. Therefore, $\omega^2 - R^2 \leq a\omega - 2R$ is equivalent to showing that:

$$k^2(1-h)^2((1-h)^2 - ac) \geq 0.$$

But in proving that $R \leq 1$, we verified that $(1-h)^2 - ac \geq 0$, hence indeed

$$k^2(1-h)^2((1-h)^2 - ac) \geq 0.$$

This establishes the lemma.

□

#### 3.4.2.2 The lower bound of $|P(c)|$

Having established all the requisite inequalities, we now proceed to the main result of this subsection, the lower bound of $|P(c)|$.

Still under the usual assumption that $P(z)$ contradicts Sendov's conjecture at $a \in (0,1)$, recall that $c \in (0,a)$, $q = \frac{a/2-m}{1+a/2}$, where $m$ and $N_1$ are defined in Definition 3.1.7 and Lemma 3.3.1 respectively. Dégot introduced the following new parameters:

$$p = \frac{\frac{a}{2}-m}{1-\frac{a}{2}},\; r = \frac{c(a-c)}{2(1-c^2)},\; \alpha = \frac{\log(\frac{a}{16})}{\log\left(\frac{c+r}{1+cr}\right)}$$

and

$$K = \min\left\{(1+c-ac)^p\sqrt{1+c^2-ac}^{\,1-p}; (1+c)^q\sqrt{1+c^2-ac}^{\,1-q}\right\}. \tag{3.35}$$

**Lemma 3.4.10.** *([4], Theorem 7): For the previously defined parameters, if the degree $n$ of $P(z)$ is such that $n \geq N_1$, then:*

$$|P(c)| \geq \frac{(1-c)(a-c)}{1-ac} r^\alpha K^{n-1}.$$

Although the proof of the above lemma is quite technical, the majority of the work has already been done in proving the technical inequalities in the previous subsection, so that the proof is a straight forward application of the lemmas. We consider it below:

*Proof.* Let $h \in (0, 1-\sqrt{a})$ and index the zeroes of $P(z)$ such that for all $j \geq n_0$, we have that $|z_j| \geq 1-h$. Let $b_h = \frac{(1-h)^2}{c}$. We note, $|P(c)|$ can be expressed as:

$$|P(c)| = |c-a| \prod_{j=1}^{n_0-1} |c-z_j| \prod_{j=n_0}^{n-1} |c-z_j|.$$

Applying Lemma 3.4.6 to each of the $n-n_0$ factors of the product $\prod_{j=n_0}^{n-1} |c-z_j|$, we get that:

$$|P(c)| \geq (a-c) \prod_{j=1}^{n_0-1} |c-z_j| \left(\frac{c}{1-h}\right)^{n-n_0} \prod_{j=n_0}^{n-1} |b_h - z_j|.$$

We would like to express the right hand side of the above inequality according to different factors in such a way that Lemmas 3.4.7 and 3.4.3 can be directly applied to it. We do this by multiplying the right hand side of the inequality by $\left|\frac{b_h-a}{b_h-a}\right| \left|\frac{b_h-z_j}{b_h-z_j}\right|^{n_0-1}$ and then express the product as:

$$\left(\frac{a-c}{b_h-a}\right) \prod_{j=1}^{n_0-1} \left|\frac{c-z_j}{b_h-z_j}\right| \left(\frac{c}{1-h}\right)^{n-n_0} |b_h-a| \prod_{j=1}^{n-1} |b_h-z_j|.$$

Note that since $c \in (0,a)$ and $h \in (0, 1-\sqrt{a})$, we have that $1-h \in (\sqrt{a}, 1)$, hence $c < 1-h$. More importantly we still have that $c < (1-h)^2$ since $(1-h)^2 \in (a,1)$. This implies that $1 < \frac{(1-h)^2}{c} = b_h$. We can thus apply Lemma 3.4.7 to the product $|b_h - a| \prod_{j=1}^{n-1} |b_h - z_j|$. Denoting by $w_1, \ldots, w_{n-1}$ the critical points of $P(z)$, we deduce that:

$$\begin{aligned} |P(c)| &\geq \left(\frac{a-c}{b_h-a}\right) \prod_{j=1}^{n_0-1} \left|\frac{c-z_j}{b_h-z_j}\right| \left(\frac{c}{1-h}\right)^{n-n_0} (b_h-1) \prod_{j=1}^{n-1} |b_h-w_j| \\ &\geq \left(\frac{(b_h-1)(a-c)}{(b_h-a)}\right) \prod_{j=1}^{n_0-1} \left|\frac{(c-z_j)(1-h)}{(b_h-z_j)c}\right| \left(\frac{c}{1-h}\right)^{n-1} \prod_{j=1}^{n-1} |b_h-w_j|, \end{aligned}$$

where the last inequality holds since it differs from the first by the factor $\left(\frac{c}{1-h}\right)^{n-n_0}$, which is less than one.

We apply Lemma 3.4.3 to the factor $\left(\frac{c}{1-h}\right)^{n-1} \prod_{j=1}^{n-1} |b_h - w_j|$ and subsequently conclude that:

$$|P(c)| \geq \left(\frac{(b_h-1)(a-c)}{(b_h-a)}\right) \prod_{j=1}^{n_0-1} \left|\frac{(c-z_j)(1-h)}{(b_h-z_j)c}\right| (K_h)^{n-1}, \tag{3.36}$$

where $K_h$ is defined by:

$$\frac{c}{1-h} \min\left\{(1+b_h-a)^p \sqrt{1+b_h^2-ab_h}^{\,1-p}; (1+b_h)^q \sqrt{1+b_h^2-ab_h}^{\,1-q}\right\}.$$

Let $c' = \frac{c}{1-h}$, $z_j' = \frac{z_j}{1-h}$ and $r_h = \frac{c(a-c)(1-h)}{2((1-h)^2-c^2)}$. By Lemma 3.4.8, we deduce that

$$\left|\frac{(c-z_j)(1-h)}{(b_h-z_j)c}\right| = \left|\frac{(c-z_j)(1-h)}{(\frac{(1-h)^2}{c}-z_j)c}\right| = \left|\frac{(c-z_j)(1-h)}{(1-h)^2-cz_j}\right| \geq r_h.$$

On the other hand,

$$\left|\frac{(c-z_j)(1-h)}{(1-h)^2-cz_j}\right| = \left|\frac{\frac{c}{1-h}-\frac{z_j}{1-h}}{1-\frac{c}{1-h}\cdot\frac{z_j}{1-h}}\right| = \left|\frac{c'-z_j'}{1-c'z_j'}\right|.$$

**Claim**: $0 < r_h < 1$.

*Proof.* We note that, $c(a-c)(1-h) > 0$ and since $c < (1-h)^2$, we deduce that $(1-h)^2 - c^2 > 0$. So $r_h > 0$. On the other hand,

$$r_h = \frac{c(a-c)(1-h)}{2((1-h)^2-c^2)} = \frac{c(a-c)}{2((1-h)^2-c^2)}(1-h) = k(1-h) \leq \frac{1-h}{2} < 1.$$

□

Applying Lemma 3.4.5, we deduce that

$$\prod_{j=1}^{n_0-1}\left|\frac{c'-z_j'}{1-c'z_j'}\right| \geq r_h^{\beta_h},$$

where $\beta_h$ is given by

$$\beta_h = \frac{\log\left(\prod_{j=1}^{n_0-1}|z_j'|\right)}{\log\left(\frac{c'+r_h}{1+c'r_h}\right)}.$$

We note that

$$|z_j'| = |\frac{z_j}{1-h}| \geq |z_j| \text{ since } h \in (0,1).$$

Combining this with the definition of $n_0$, we deduce that

$$\prod_{j=1}^{n_0-1}|z_j'| \geq \prod_{j=1}^{n_0-1}|z_j| \geq \prod_{j=1}^{n-1}|z_j| = \frac{1}{a}|P(0)| \geq \frac{a}{16},$$

where the last inequality follows from Lemma 3.3.1. We thus have that $\log\left(\prod_{j=1}^{n_0-1}|z_j'|\right) \geq \log\left(\frac{a}{16}\right)$. On the other hand, since $c' + r_h - (1 + c'r_h) = (r_h - 1)(1 - c') < 0$ (because $r_h < 1$), we have that $0 < \frac{c'+r_h}{1+c'r_h} < 1$, so that $\log\left(\frac{c'+r_h}{1+c'r_h}\right) < 0$. Therefore,

$$\beta_h \leq \frac{\log\left(\frac{a}{16}\right)}{\log\left(\frac{c'+r_h}{1+c'r_h}\right)} = \alpha_h.$$

Since $0 < r_h < 1$, then ${r_h}^{\beta_h} \geq {r_h}^{\alpha_h}$ and thus

$$\prod_{j=1}^{n_0-1} \left| \frac{c' - z'_j}{1 - c' z'_j} \right| \geq r_h^{\alpha_h}. \tag{3.37}$$

Combining Inequalities 3.37 and 3.36, we deduce that

$$|P(c)| \geq \left( \frac{(b_h - 1)(a - c)}{(b_h - a)} \right) r_h^{\alpha_h} (K_h)^{n-1}.$$

Letting $h \to 0$, we note that

$$b_h \to \frac{1}{c}, \quad r_h \to \frac{c(a-c)}{2(1-c^2)}, \quad \alpha_h \to \frac{\log\left(\frac{a}{16}\right)}{\log\left(\frac{c+r}{1+cr}\right)},$$

and finally,

$$K_h \to \min\left\{ (1 + c - ac)^p \sqrt{1 + c^2 - ac}^{\,1-p}; (1+c)^q \sqrt{1 + c^2 - ac}^{\,1-q} \right\}.$$

This completes the proof.

$\square$

Before proceeding, we would like to bring the reader's attention to two observations:

**Observation 1**: For $K$ as defined above, one can always find $c$ sufficiently close to $a$ such that $K > 1$. That is:

As $c \to a$,

$$(1 + c - ac)^p \sqrt{1 + c^2 - ac}^{\,1-p} \to (1 + a(1-a))^p > 1$$

and similarly

$$(1+c)^q \sqrt{1 + c^2 - ac}^{\,1-q} \to (1+a)^q > 1.$$

This observation was enough for Dégot's results, however we have to bear in mind that we want a bound that depends only on $a$. Thus we would like to obtain an explicit formula $c = c(a)$ which will always yield a $c$ (in terms of $a$) close enough to $a$ such that $K > 1$. We also introduce the quantity

$p'(a) = \frac{\frac{a}{4}}{1-\frac{a}{2}} = p'$ to take the place of $p$ in order to avoid the dependence on $m$. Hence our version of $K$ is:

$$K' = \min\left\{(1+c-ac)^{p'}\sqrt{1+c^2-ac}^{\,1-p'}; (1+c)^{q'}\sqrt{1+c^2-ac}^{\,1-q'}\right\}.$$

For ease of notation, from Equation 3.35, we let $K = \min\{K_1(a,c,p); K_2(a,c,q)\}$. We point out to the reader that for $p_1 \geq p_2 > 0$ and $q_1 \geq q_2 > 0$, we have the following:

**Proposition 3.4.11.** *$K_1(a,c,p_1) \geq K_1(a,c,p_2)$ and $K_2(a,c,q_1) \geq K_2(a,c,q_2)$.*

**Remark 3.4.12.** *The proof of the above proposition is very much similar to that of Proposition 3.3.7, where we studied the derivative of a function of the form $f(x) = a^x b^{1-x}$. We therefore omit it here.*

With the above in mind, we see that the conclusion of Lemma 3.4.10 still holds with $K'$ in place of $K$ whenever $n \geq \mathcal{N}_1(a)$. This will become more clear in the discussion leading towards our Theorem 3.4.19, which, *mutatis mutandis*, is a restatement of Lemma 3.4.10.

We may now proceed and study how one can obtain an explicit lower bound for $K'$. In preparation for the result, we need to first recall the following logarithmic inequalities, one of which we have already used before. The first one follows from the fact that the graph of $1+x$ dominates that of $e^{\frac{x}{2}}$ for all $x \in [0,1]$. The second inequality can be found in [8].

**Lemma 3.4.13.** *(Useful* log *inequalities):*

- $\log(1+x) \geq \frac{x}{2}$ *for* $x \in [0,1]$,
- $\frac{x}{x+1} \leq \log(1+x) \leq x$ *for* $x > -1$.

We proceed to define the quantity $\mu_2(a)$ as follows:

$$\mu_2(a) = \left[\left(\frac{1}{2a}\left(\frac{2}{q'}-2\right)-\frac{1}{2}\right)^2 - \left(\frac{1}{a^2}-\frac{1}{a}\left(\frac{2}{q'}-2\right)\right)\right]^{\frac{1}{2}} + \left[\frac{1}{2}-\frac{1}{2a}\left(\frac{2}{q'}-2\right)\right],$$

and note that this expresses the positive root of the quadratic equation:

$$x^2 + \left[\left(\frac{1}{a}\left(\frac{2}{q'} - 2\right) - 1\right)\right] x + \left(\frac{1}{a^2} - \frac{1}{a}\left(\frac{2}{q'} - 2\right)\right) = 0. \tag{3.38}$$

**Claim**: $0 < \mu_2(a) < 1$.

*Proof.* Recalling that $q'(a) = \frac{\frac{a}{4}}{1+\frac{a}{2}} = \frac{a}{4+2a}$, the quadratic Equation 3.38 can be written as:

$$f(x) = x^2 + \beta(a)x + \rho(a),$$

where:

$$\beta(a) = \frac{8 + 2a - 2a^2}{a^2} \text{ and } \rho(a) = \frac{-7 - 2a}{a^2} < 0.$$

Recall that if $x_1$ and $x_2$ are the roots of the equation $ax^2 + bx + c$, then:

$$x_1 + x_2 = -\frac{b}{a} \text{ and } x_1 x_2 = \frac{c}{a}.$$

For $f(x)$ as defined, $a = 1$, $b = \beta(a)$ and $c = \rho(a) = y$ intercept.

Hence if $x_1$ and $x_2$ are the zeroes of $f(x)$, we have that:

$$x_1 + x_2 = -\frac{\beta(a)}{1} = \frac{2a^2 - 2a - 8}{a^2}, \tag{3.39}$$

and

$$x_1 x_2 = \frac{\rho(a)}{1} = \frac{-7 - 2a}{a^2} < 0. \tag{3.40}$$

Without loss of generality we can assume that $x_1 < 0$ and $x_2 > 0$. Hence, our claim is equivalent to showing that $0 < x_2 < 1$. We argue as follows:

Suppose, for the sake of contradiction that $x_2 \geq 1$. Then from Equation 3.40:

$$x_1 \cdot 1 \geq \frac{-7 - 2a}{a^2} = x_1 x_2, \tag{3.41}$$

from Equation 3.39,

$$x_1 + 1 \leq x_1 + x_2 = \frac{2a^2 - 2a - 8}{a^2},$$

hence:

$$x_1 \leq \frac{a^2 - 2a - 8}{a^2}. \tag{3.42}$$

Combining Equations 3.41 and 3.42 above yields:

$$\frac{-7 - 2a}{a^2} \leq x_1 \leq \frac{a^2 - 2a - 8}{a^2}.$$

In particular, this implies that

$$-7 - 2a \leq a^2 - 2a - 8,$$

or, $a^2 - 1 \geq 0$. But $a \in (0,1)$, hence this is not true. Hence, $\mu_2(a) := x_2 \in (0,1)$ as claimed.

□

We may now proceed to state and prove the proposition.

**Proposition 3.4.14.** *For $a \in (0,1)$ we can find a function $\gamma$ such that $\gamma(a) \in (0,1)$ and $K_2(a, a\gamma(a), q') > 1$.*

*Proof.* For $a \in (0,1)$ let:

$$1 > \gamma > \left[\left(\frac{1}{2a}\left(\frac{2}{q'} - 2\right) - \frac{1}{2}\right)^2 - \left(\frac{1}{a^2} - \frac{1}{a}\left(\frac{2}{q'} - 2\right)\right)\right]^{\frac{1}{2}} + \left[\frac{1}{2} - \frac{1}{2a}\left(\frac{2}{q'} - 2\right)\right] > 0$$

Focusing on the middle inequality, bearing in mind that $\mu_2(a)$ is a root of the Equation 3.38, reversing the "completion of the square" with respect to $\gamma$ , yields:

$$\left[\gamma + \left(\frac{1}{2a}\left(\frac{2}{q'} - 2\right) - \frac{1}{2}\right)\right]^2 > \left(\frac{1}{2a}\left(\frac{2}{q'} - 2\right) - \frac{1}{2}\right)^2 - \left(\frac{1}{a^2} - \frac{1}{a}\left(\frac{2}{q'} - 2\right)\right).$$

Continuing to simplify, we eventually arrive at:

$$q'(1 + \gamma^2 a^2 - \gamma a^2) + 2(1 - q')(a\gamma - a) > 0.$$

And ultimately:

$$q' + (1 - q')\left(\frac{a\gamma - a}{1 + a^2\gamma^2 - a^2\gamma}\right) > \frac{q'}{2}$$

Multiplying the above inequality with the quantity $\frac{a\gamma}{2}$, we obtain:

$$q'\frac{(a\gamma)}{2} + \frac{(1-q')}{2}\left(\frac{\gamma^2 a^2 - \gamma a^2}{1+\gamma^2 a^2 - \gamma a^2}\right) > \frac{aq'\gamma}{4}. \tag{3.43}$$

Invoking Lemma 3.4.13, we note:

- $\log(1+a\gamma) \geq \frac{a\gamma}{2}$, and
- $\log(1+\gamma^2 a^2 - \gamma a^2) \geq \frac{\gamma^2 a^2 - \gamma a^2}{1+\gamma^2 a^2 - \gamma a^2}$

This implies therefore that:

$$\log(K_2(a, a\gamma, q')) = q'\log(1+a\gamma) + \left(\frac{1-q'}{2}\right)\log(1+\gamma^2 a^2 - \gamma a^2) > \frac{aq'\gamma}{4}.$$

Hence,

$$K_2(a, a\gamma, q') = (1+a\gamma)^{q'}(1+\gamma^2 a^2 - \gamma a^2)^{\frac{1-q'}{2}} > e^{\frac{aq'\gamma}{4}} \geq 1 + \frac{aq'\gamma}{4} > 1.$$

□

One may have already noticed that the above proof exposes slightly more than what we set out to prove. More specifically, we have shown that not only is $K_2 > 1$, but it can also be explicitly bounded below by a function of $a$. The same can be done for $K_1$. Since the quantity $\log(K')$ appears in the denominator of the quantity $\mathcal{N}_3(a)$ which we shall encounter later, knowing the explicit lower bound of $K'$ gives us a handle on the growth of $\mathcal{N}_3(a)$. For the most part though, knowing that $K' > 1$ suffices.

We proceed to define the quantity $\mu_1(a)$ as follows:

$$\mu_1(a) = \frac{a^2 p'(1-a) - a(1-p')}{2a^2(p'-ap')} + \left[\left(\frac{a^2 p'(1-a) - a(1-p')}{2a^2(p'-ap')}\right)^2 + \frac{a-p'}{a^2(p'-ap')}\right]^{\frac{1}{2}}.$$

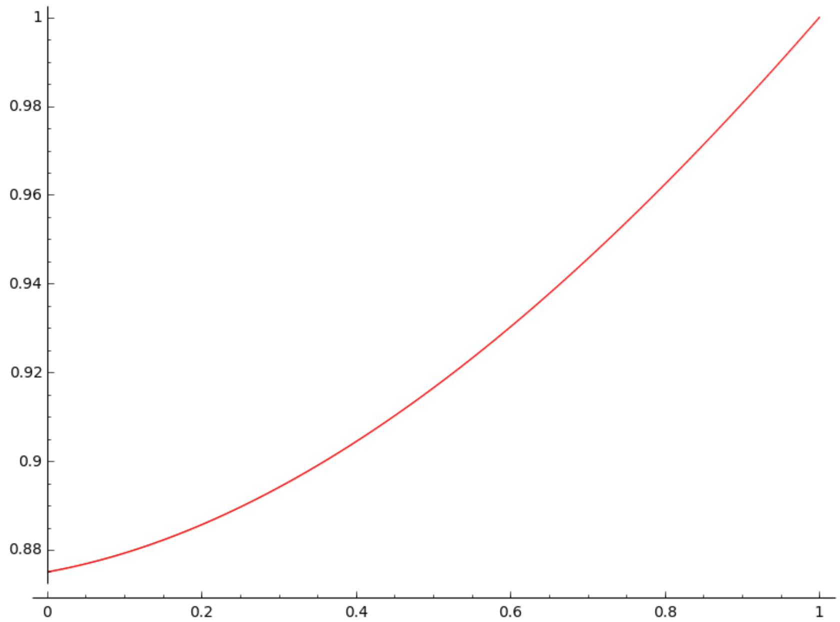
Just as in the previous analysis, we note that $\mu_1(a)$ is simply the positive root of the following quadratic equation:

$$x^2 + \left[\frac{a(1-p') - a^2 p'(1-a)}{a^2(p'-ap')}\right]x - \frac{a-p'}{a^2(p'-ap')} = 0. \tag{3.44}$$

Indeed, repeating the same procedure as before, that is, studying the zeroes of the quadratic equation 3.44, it can be shown analytically that $\mu_1(a) < 1$. However, we would like to illustrate this geometrically:

**Remark 3.4.15.** *Figure 3.4 below shows the plot of $\mu_1(a)$ for $a \in (0,1)$. The important point to note is that $\mu_1(a) \in (0,1)$.*

Figure 3.3: A plot of $\mu_1(a)$.




We can now state and prove the proposition.

**Proposition 3.4.16.** *For $a \in (0,1)$ we can find a function $\eta$ such that $\eta(a) \in (0,1)$ and $K_1(a, a\eta(a), p') > 1$.*

*Proof.* Let $a \in (0,1)$ and suppose that

$$1 > \eta > \frac{a^2p'(1-a) - a(1-p')}{2a^2(p'-ap')} + \left[\left(\frac{a^2p'(1-a) - a(1-p')}{2a^2(p'-ap')}\right)^2 + \frac{a-p'}{a^2(p'-ap')}\right]^{\frac{1}{2}} > 0.$$

Focusing on the middle inequality, bearing in mind that $\mu_1(a)$ is a root of the equation 3.44, reversing the "completion of the square" with respect to $\eta$ , yields:

$$\eta^2 + \left[\frac{a(1-p') - a^2p'(1-a)}{a^2(p'-ap')}\right]\eta > \frac{a-p'}{a^2(p'-ap')}$$

Continuing to simplify, we get that:

$$(1 + a^2\eta^2 - a^2\eta)(p' - ap') + (1-p')(a\eta - a) > 0.$$

Hence,

$$p'(1-a) + (1-p')\left(\frac{a\eta - a}{1 + a^2\eta^2 - a^2\eta}\right) > 0.$$

Multiplying throughout by $\frac{a\eta}{2}$, we obtain:

$$\frac{p'}{2}(a\eta - a^2\eta) + \frac{1-p'}{2}\left(\frac{a^2\eta^2 - a^2\eta}{1 + a^2\eta^2 - a^2\eta}\right) > 0.$$

But then, looking closely at the quantities in the above equation, and invoking Lemma 3.4.13, we note:

- $\log(1 + a\eta - a^2\eta) \geq \frac{a\eta - a^2\eta}{2}$, and
- $\log(1 + a^2\eta^2 - a^2\eta) \geq \frac{a^2\eta^2 - a^2\eta}{1 + a^2\eta^2 - a^2\eta}$

Therefore this implies that:

$$\log(K_1(a, \eta a, p') = p' \log(1 + a\eta - a^2\eta) + \left(\frac{1-p'}{2}\right) \log(1 + a^2\eta^2 - a^2\eta) > 0.$$

Hence,

$$K_1(a, \eta a, p') = (1 + a\eta - a^2\eta)^{p'}(1 + a^2\eta^2 - a^2\eta)^{\frac{1-p'}{2}} > e^0 = 1.$$

□

**Remark 3.4.17.** *From the definition of $K_1(a, c, p')$, we note that at $a = 1$, $K_1(1, c, p') = (1 + c^2 - c)^{\frac{1-p'}{2}} < 1$ for all $c \in (0, 1)$. Hence $K_1(a, c, p') > 1$ only if $a \in (0, 1)$.*

**Remark 3.4.18.** *Define the function $\rho$ as $\rho(a) = \max\{\frac{1+\gamma(a)}{2}; \frac{1+\eta(a)}{2}\}$.*

**Observation 2**: We would like to notify the reader that the lower bound of $n$ required to obtain the conclusion of Dégot's **Theorem 7** is the previously defined $N_1$ from his **Theorem 5** (in our case Lemma 3.3.1). We have already obtained the explicit analogue of this bound in the form of $\mathcal{N}_1(a)$. Hence, as it stands, we have all the necessary ingredients to obtain the conclusion of Dégot's **Theorem 7**.

However, since our ultimate goal is to obtain an explicit $\mathcal{N}(a)$ independent of all the other implicit parameters, it is worthwhile to remark on the new parameters that were introduced in preparation for Lemma 3.4.10.

- $p$ is defined as $p = \frac{\frac{a}{2}-m}{1-\frac{a}{2}} = p(a,m)$. The dependence on $m$ is avoided by the same argument that led to the introduction of $q'(a)$. We simply define the alternative quantity $p' = \frac{\frac{a}{4}}{1-\frac{a}{2}} = p'(a)$ and invoke the quantity $\mathcal{N}_0(a)$ to ensure a high enough degree bound such that the results work.

- The parameter $r$ is defined in terms of $a \in (0,1)$ and $c \in (0,a)$ as $r = \frac{c(a-c)}{2(1-c^2)}$. This poses no problem as we have already demonstrated that $c$ can be ultimately be chosen in terms of $a$, thus obtaining $r = r(a)$.

- Similar reasoning as above applies to the quantity $\alpha = \frac{\log(\frac{a}{16})}{\log(\frac{c+r}{1+cr})}$.

That being said, we arrive at our version of Dégot's **Theorem 7** which depends only on $a \in (0,1)$. We restate the conclusion here for the sake of continuity:

**Theorem 3.4.19.** *Suppose $P(z)$ contradict Sendov's conjecture at $a \in (0,1)$. Let $c = a\rho(a)$. If $\deg(P(z)) = n \geq \mathcal{N}_1(a)$, then:*

$$|P(c)| \geq \frac{(1-c)(a-c)}{1-ac} r^{\alpha} K'^{n-1}.$$

Before proceeding, let us take yet another closer look at these parameters. This analysis will prove useful and simplify notation in the result that follows thereafter.

- the quantity $r$ is defined as $r = \frac{c(a-c)}{2(1-c^2)}$. Clearly $r > 0$. Furthermore:

$$\begin{aligned} c(a-c) - 2(1-c^2) &= ac - c^2 - 2 + 2c^2 \\ &= c^2 + ac - 2 \\ &< 2a^2 - 2 \\ &= 2(a^2-1) < 0 \end{aligned}$$

In other words, $c(a-c) < 2(1-c^2)$, hence $0 < r < 1$ and consequently, $\log(r) < 0$.

- $\log\left(\frac{1+a}{a-c}\right) > 0$ and always defined since $0 < c < a < 1$.
- $\log\left(\frac{1-ac}{1-c}\right) > 0$ and always defined since $ac < c$ for $0 < c < a < 1$.
- The quantity $\frac{c+r}{1+cr} > 0$, but then:

$$c + r - 1 - cr = (r-1)(1-c) < 0, \text{ since } r < 1.$$

Hence $0 < \frac{c+r}{1+cr} < 1$. This implies that:

$$\alpha = \frac{\log(\frac{a}{16})}{\log\left(\frac{c+r}{1+cr}\right)} > 0$$

- Finally, we have shown that we can express $c$ explicitly in terms of $a$. Furthermore, this $c$ is sufficiently close to $a$ such that $K' > 1$. Hence $\log(K') > 0$.

All the above analysis culminates in the following definition of the final degree bound, which we denote by $N_3(a,c)$ as follows:

$$N_3(a,c) = \frac{\log\left(\frac{1+a}{a-c}\right) + \log\left(\frac{1-ac}{1-c}\right) - \alpha \cdot \log(r)}{\log(K')} + 1.$$

We then define $\mathcal{N}_3(a,c)$ to be:

$$\mathcal{N}_3(a,c) = \max\{\mathcal{N}_0(a), N_3(a,c)\}.$$

## 3.5 Main result (Improvement of Dégot's Theorem 8)

In the preceeding sections we reached the pinnacle of all the technical considerations in the thesis. In this section, we tie everything together into one result. This culminates in the proof of Sendov's conjecture for polynomials with large enough degree, but now with an explicit bound $\mathcal{N}(a)$. First, a preamble.

Let $K'(a,c)$, $\mathcal{N}_1(a)$, $\mathcal{N}_2(a,c)$ and $\mathcal{N}_3(a,c)$ be as defined in the previous sections.

Recall that in the previous section, through Propositions 3.4.14 and 3.4.16, we showed that given the function $K'(a,c)$, one can always find $c$ sufficiently close to $a$ such that $K'(a,c) > 1$. Furthermore, this $c$ can be expressed purely in terms of $a$ as $c = a\rho(a)$. The quantity $\rho(a)$ was defined in Remark 3.4.18.

This means that all the quantities that depended on $c$ (in particular $K'(a,c)$, $\mathcal{N}_2(a,c)$ and $\mathcal{N}_3(a,c)$) can now be written as functions of $a$ only. Henceforth, we simply let $c = a\rho(a)$, which we already know lies in $(0,a)$.

**Theorem 3.5.1.** *Let $P(z) = (z-a)\prod_{j=1}^{n-1}(z-z_j)$, with $a \in (0,1)$, $|z_j| \le 1$ for all $j = 1,\dots,n-1$, where $n \ge 2$. If:*

$$\deg P(z) = n > \mathcal{N}(a) := \max\{\mathcal{N}_1(a), \mathcal{N}_2(a, a\rho(a)), \mathcal{N}_3(a, a\rho(a))\},$$

*then $P'(z)$ has a zero in the disk $|z-a| \le 1$.*

*Proof.* We follow Dégot's approach:

Let $c = a\rho(a)$ and suppose to the contrary, that $P'(w) \neq 0$ for all $w \in |z-a| \le 1$. Then, by Theorem 3.4.2 we have that:

$$1 + a \ge |P(c)|.$$

Theorem 3.4.19 tells us that:

$$|P(c)| \geq \frac{(1-c)(a-c)}{1-ac} r^{\alpha} K'^{n-1}.$$

We combine the above two inequalities to get:

$$1+a \geq |P(c)| \geq \frac{(1-c)(a-c)}{1-ac} r^{\alpha} K'^{n-1}.$$

This implies that:

$$\frac{(1-c)(a-c)}{(1-ac)(1+a)} r^{\alpha} K'^{n-1} \leq 1.$$

Taking log on both sides yields:

$$(n-1)\log(K') + \log\left(\frac{1-c}{1-ac}\right) + \log\left(\frac{a-c}{1+a}\right) + \alpha \log(r) \leq 0.$$

Equivalently,

$$(n-1)\log(K') \leq \log\left(\frac{1-ac}{1-c}\right) + \log\left(\frac{1+a}{a-c}\right) - \alpha \log(r).$$

Hence,

$$n \leq \frac{\log\left(\frac{1+a}{a-c}\right) + \log\left(\frac{1-ac}{1-c}\right) - \alpha \cdot \log(r)}{\log(K')} + 1 = \mathcal{N}_3(a).$$

This contradicts the assumption on the degree of $P(z)$.

Hence $P'(w) = 0$ for some $w \in |z-a| \leq 1$.

$\square$

**Some remarks on the behaviour of $\mathcal{N}(a)$ for extremal values of $a$**

Here we take a brief look at the function $\mathcal{N}(a)$ as $a$ approaches 0 and as $a$ approaches 1.

We note that by definition, $\mathcal{N}(a)$ could be one of $\mathcal{N}_1(a)$, $\mathcal{N}_2(a)$ or $\mathcal{N}_3(a)$. Each of the functions $\mathcal{N}_1(a)$, $\mathcal{N}_2(a)$, and $\mathcal{N}_3(a)$ is unbounded as $a$ approaches 0. Hence $\mathcal{N}(a)$ goes to infinity as $a$ goes to 0.

On the other hand, the functions $\mathcal{N}_1(a)$ and $\mathcal{N}_2(a)$ can be evaluated at $a = 1$. They are thus continuous on $(0, 1]$. Let us now turn our attention to $\mathcal{N}_3(a)$.

From its definition, $\mathcal{N}_3(a)$ has $\log(K')$ in the denominator. $K'$ itself is the minimum of two functions (we referred to them as $K_1(a,c), K_2(a,c)$ in previous discussions). From Remark 3.4.17 we note that $K' < 1$ at $a = 1$. Therefore this is why, in general, we restrict the function $\mathcal{N}(a)$ to the open interval $(0,1)$. Of course, in the case where it can be shown that $\max\{\mathcal{N}_1(a), \mathcal{N}_2(a), \mathcal{N}_3(a)\} \neq \mathcal{N}_3(a)$, then it makes sense to evaluate $\mathcal{N}(a)$ as $a$ approaches 1.

## Towards uniformity

From our previous discussions, the functions $\mathcal{N}_0(a), \mathcal{N}_1(a), \mathcal{N}_2(a)$ and $\mathcal{N}_3(a)$ are all continuous functions of $a$ on $(0,1)$. This implies that the function $\mathcal{N} : (0,1) \longmapsto \mathbb{R}^+$ defined in the hypothesis of Theorem 3.5.1 is continuous on $(0,1)$.

In [4], Dégot concludes by asking first for a degree bound $N \in \mathbb{N}$ which is independent of $a \in (0,1)$, or at least an explicit formula $N(a)$.
We note that, the formula $\mathcal{N}(a)$ defined above suffices for the latter request.

However, recall that the *Weierstrass Extreme Value theorem* says that a continuous function on a compact interval attains its extrema on the interval.

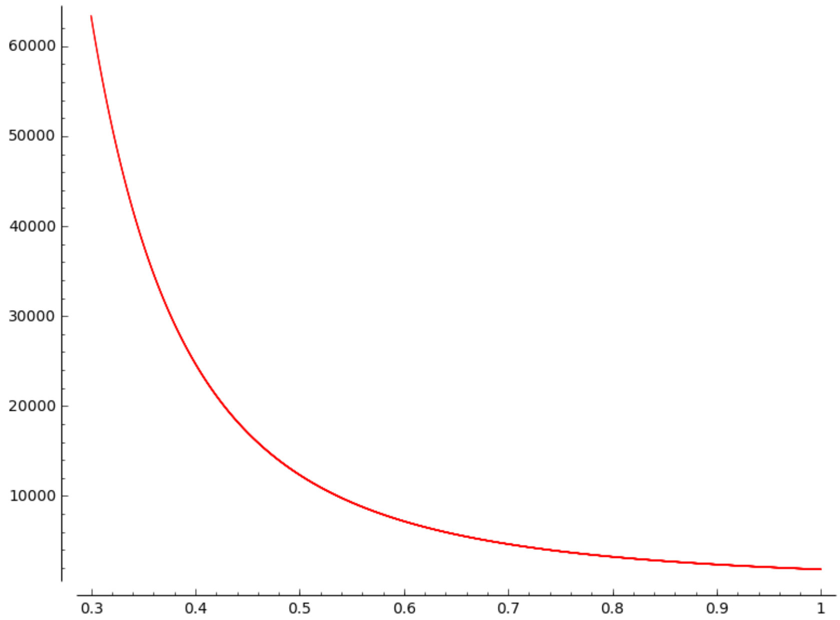
For any $0 < \alpha < \beta < 1$, the interval $[\alpha, \beta]$ is compact. Hence, by the extreme value theorem, $\mathcal{N}(a)$ has a maximum on $[\alpha, \beta]$.

Interpreting this in the context of our results, this maximum value is the $N$ independent of $a \in [\alpha, \beta]$. Hence for any $a$ in the said interval, and any polynomial $P(z)$ all of whose zeros are in the unit disk, with $a$ being one of the zeros, if the degree of $P(z)$ is greater than $N$, then Sendov's conjecture is true at $a$.

The figure below is a plot of the function $\mathcal{N}(a)$ for $a \in [0.3, 1]$.

**Remark 3.5.2.** *Although we have not shown this analytically, it turns out (experimentally) that* $\max\{\mathcal{N}_1(a), \mathcal{N}_2(a), \mathcal{N}_3(a)\} = \mathcal{N}_1(a)$*, hence we can find* $\mathcal{N}(a)$ *as* $a$ *approaches* 1*. Combining this with the fact that Sendov's conjecture is true at* 1*, we can replace the interval* $[\alpha, \beta]$ *with* $[\alpha, 1]$*.*

Figure 3.4: A plot of $\mathcal{N}(a)$ for $a \in [0.3, 1]$.




From the above figure (or by directly evaluating $\mathcal{N}(a)$ at $a = 0.3$), we arrive at the following result:

**Result**: Let $N = 63400$, $a \in [0.3, 1]$. Then Sendov's conjecture is true (at $a$) for *any* polynomial $P(z)$ with degree $n \geq N$ and all of whose zeroes are in the unit disk.

## Concluding remarks

We began this thesis with a discussion of some results from the literature on Sendov's conjecture, sampling a few known special cases. This was by no means an exhaustive list of all that is known about the conjecture. We only sampled a few of the results, most of which were proved using "classical" or direct methods within the scope of the *Theory of Polynomials*. The reader who goes on to consult the literature we pointed out further would soon find out that some of the novel approaches involved first translating the conjecture into a different area and then studying it using the techniques already established therein. Such approaches include *Variational Methods* and *Extremal Problems* [10], as well as the general theory of *Distributions of Zeroes of Entire Functions*. We ended the section with an elementary proof of a result of Rubinstein, which verified Sendov's conjecture for polynomials whose derivatives are bounded by their degree.

In the main chapter, we focused on the recent paper of Dégot. Closely following his treatment, we provided explicit and continuous versions of his degree bounds. These in turn enabled us to come up with a uniform bound $\mathcal{N}$ which works for any $a \in [\alpha, \beta] \subset (0, 1)$. On that note, we would like to bring the reader's attention to the following points:

- We would like a definitive result that would bridge the gaps $[0, \alpha)$ and $(\beta, 1]$. We are more inclined towards remarking that these gaps rather illustrate the limitation of this current approach, as opposed to the actual growth rate of the degree bounds as $a$ approaches the extremal points of the unit interval. However, as mentioned earlier, experimental evidence suggests that our approach still works as $a$ approaches 1, hence the main concern is as $a$ approaches 0.

- In our treatment, we leaned towards illustrating a method that could address Dégot's questions, as opposed to obtaining the sharpest results, more specifically:

- to obtain $\mathcal{N}_1(a)$ and $\mathcal{N}_2(a)$, we essentially replaced a term of $\mathcal{O}(\log n)$ with that of $\mathcal{O}(n^{\frac{1}{2}})$, which is much bigger.

- In coming up with $\mathcal{N}_0(a)$, we chose $\delta = \frac{a}{4}$ just for the convenience of calculations. In principle, one can investigate further to establish the optimal choice of $\delta$.

Therefore, through a more careful analysis, the function $\mathcal{N}(a)$ can still be considerably lowered to a sharper bound. However, it is unlikely that this approach alone would be successful for lowering the bound to values less than 100, since by construction the sharpness of our results is bounded below by the corresponding results from Dégot's paper. We would therefore like to consider more technically diverse approaches.

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